A remark on the monoidal structure of the category of $\mathbb{S}$-modules

Estanislao Herscovich

Abstract

The aim of this short note, only intended for pedagogical purposes, is to explicitly present a straightforward definition of the composite monoidal product in the category of $\mathbb{S}$-modules, that gives rise to operads. We essentially fill in a gap that is either somehow overlooked, or presented by enlarging the category of $\mathbb{S}$-modules to consider all species, *i.e.* the contraviariant functors from the grupoid of all finite sets with bijections as morphisms to a symmetric monoidal category. Nothing we present here is new or unknown to the experts.

1 Preliminaries

1.1 Notation

We will denote by $\mathbb{N}_0$ (resp., $\mathbb{N}$) the set of nonnegative (resp., positive) integers $\{0, 1, 2, \dots\}$ (resp., $\{1, 2, \dots\}$), $\mathbb{Z}$ the set of all integers $\{0, -1, 1, -2, 2, \dots\}$ and given $n', n'' \in \mathbb{Z}$ we denote by $[\![n', n'']\!]$ the set $\{n \in \mathbb{Z} : n' \leq n \leq n''\}$.

1.2 Basics on symmetric groups

We fix a field $\Bbbk$. Recall that, given $n \in \mathbb{N}_0$, $\mathbb{S}_n$ denotes the group of automorphisms of the set $[\![1, n]\!]$. Note that $\mathbb{S}_n$ has exactly $n!$ elements. In particular, $\mathbb{S}_0$ is the group formed by the empty function $\emptyset \to \emptyset$, and $\mathbb{S}_1$ is the group formed by the identity map of $\{1\}$.

Let $n \in \mathbb{N}$ and $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ with $m = m_1 + \dots + m_n$. Recall the map

$$\mathfrak{z}_{m_1,\dots,m_n} : \mathbb{S}_{m_1} \times \dots \times \mathbb{S}_{m_n} \to \mathbb{S}_m \tag{1.1}$$

sending $(\sigma_1, \dots, \sigma_n) \in \mathbb{S}_{m_1} \times \dots \times \mathbb{S}_{m_n}$ to the unique permutation $\sigma \in \mathbb{S}_m$ given by

$$\sigma\left(k + \sum_{i=1}^{j-1} m_i\right) = \sigma_j(k) + \sum_{i=1}^{j-1} m_i, \tag{1.2}$$

for all $j \in [\![1, n]\!]$ and $k \in [\![1, m_j]\!]$. The permutation σ is called the **ordered sum of the permutations** $\sigma_1, \dots, \sigma_n$. It is easy to see that $\mathfrak{z}_{m_1,\dots,m_n}$ is a morphism of groups, so it induces a morphism of $\Bbbk$-algebras

$$\Bbbk\mathbb{S}_{m_1} \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \Bbbk\mathbb{S}_{m_n} \to \Bbbk\mathbb{S}_m \tag{1.3}$$

for every field $\Bbbk$, sending $\sigma_1 \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \sigma_n$ to $\mathfrak{z}_{m_1,\dots,m_n}(\sigma_1, \dots, \sigma_n)$. We will also denote this morphism of $\Bbbk$-algebras by $\mathfrak{z}_{m_1,\dots,m_n}$. Moreover, if the n-tuple $(m_1, \dots, m_n)$ is clear, we will simply denote the previous maps by $\mathfrak{z}$.

Let $n \in \mathbb{N}$ and $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ with $m = m_1 + \dots + m_n$. Consider the map

$$\mathscr{b}_{m_1,\dots,m_n} : \mathbb{S}_n \to \mathbb{S}_m \tag{1.4}$$

sending $\tau \in \mathbb{S}_n$ to the unique permutation $\sigma \in \mathbb{S}_m$ given by

$$\sigma\left(k + \sum_{i=1}^{j-1} m_i\right) = k + \sum_{i=1}^{\tau(j)-1} m_{\tau^{-1}(i)}, \tag{1.5}$$

for all $j \in \llbracket 1, n \rrbracket$ and $k \in \llbracket 1, m_j \rrbracket$. The permutation σ is called the **block permutation of τ**.

Remark 1.1. *It is easy to see that the block permutation defined in* (1.5) *is obtained as the unique permutation of $\llbracket 1, m \rrbracket$ given from identifying the latter with an ordered union of successive intervals of lengths $m_1, \dots, m_n$, respectively, applying τ to the n-tuple $(m_1, \dots, m_n)$, which gives $(m_{\tau^{-1}(1)}, \dots, m_{\tau^{-1}(n)})$, and re-identifying the latter tuple with the integral interval $\llbracket 1, m \rrbracket$, which is also regarded an ordered union of successive intervals of lengths $m_{\tau^{-1}(1)}, \dots, m_{\tau^{-1}(n)}$, respectively.*

The maps $\mathscr{b}_{m_1,\dots,m_n}$ are not morphisms of groups in general, but they satisfy the following nice property

$$\mathscr{b}_{m_{\tau^{-1}(1)},\dots,m_{\tau^{-1}(n)}}(\tau') \circ \mathscr{b}_{m_1,\dots,m_n}(\tau) = \mathscr{b}_{m_1,\dots,m_n}(\tau' \circ \tau), \tag{1.6}$$

for all $\tau, \tau' \in \mathbb{S}_n$, $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ and $n \in \mathbb{N}$ (this follows easily from the description in Remark 1.1). Moreover, the careful reader can verify the elementary identity

$$\begin{aligned} \mathscr{s}_{m_{\sigma^{-1}(1)},\dots,m_{\sigma^{-1}(n)}}(\tau_{\sigma^{-1}(1)}, \dots, \tau_{\sigma^{-1}(n)}) \circ \mathscr{b}_{m_1,\dots,m_n}(\sigma) \\ = \mathscr{b}_{m_1,\dots,m_n}(\sigma) \circ \mathscr{s}_{m_1,\dots,m_n}(\tau_1, \dots, \tau_n), \end{aligned} \tag{1.7}$$

for all $\tau_i \in \mathbb{S}_{m_i}$, $i \in \llbracket 1, n \rrbracket$, $\sigma \in \mathbb{S}_n$ and $n \in \mathbb{N}$.

1.3 Basics on modules in (symmetric) monoidal categories

An object X in a $\Bbbk$-linear category $\mathscr{C}$ (*i.e.* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a $\Bbbk$-vector space for all pairs of objects X and Y of $\mathscr{C}$ such that the composition is $\Bbbk$-bilinear) with a zero object 0 is said to be a **right $\Bbbk\mathbb{S}_n$-module** for $n \in \mathbb{N}_0$ if it is endowed with a morphism of $\Bbbk$-algebras $\rho_X : \Bbbk\mathbb{S}_n \to \mathrm{Hom}_{\mathscr{C}}(X, X)^{\mathrm{op}}$. Let X and Y be two right $\Bbbk\mathbb{S}_n$-modules in $\mathscr{C}$. A **morphism of right $\Bbbk\mathbb{S}_n$-modules** from X to Y is a morphism $f : X \to Y$ in $\mathscr{C}$ such that $f \circ \rho_X(\sigma) = \rho_Y(\sigma) \circ f$, for all $\sigma \in \Bbbk\mathbb{S}_n$. The composition of morphisms of right $\Bbbk\mathbb{S}_n$-modules is clearly a morphism of right $\Bbbk\mathbb{S}_n$-modules, and the identity of any right $\Bbbk\mathbb{S}_n$-module X in $\mathscr{C}$ is a morphism of right $\Bbbk\mathbb{S}_n$-modules. The definitions for left $\Bbbk\mathbb{S}_n$-modules, as well as bimodules over $\Bbbk\mathbb{S}_n$, are analogous. A left $\Bbbk\mathbb{S}_n$-module on an object X in $\mathscr{C}$ with structure morphism ρ_ℓ is in correspondence to a right $\Bbbk\mathbb{S}_n$-module on X with structure morphism ρ_r by means of $\rho_\ell(\sigma) = \rho_r(\sigma^{-1})$, for all $\sigma \in \mathbb{S}_n$.

From now, we work on a $\Bbbk$-linear symmetric monoidal category $(\mathscr{C}, \otimes, \mathbf{1}, \tau_{\mathscr{C}})$. We recall that this means $(\mathscr{C}, \otimes, \mathbf{1}, \tau_{\mathscr{C}})$ is a symmetric monoidal category such that the underlying category $\mathscr{C}$ is $\Bbbk$-linear (with zero object) and the bifunctors $\otimes$ and $\tau_{\mathscr{C}}$ are $\Bbbk$-(bi)linear (see [4], VII). We recall that this implies that, given $n \in \mathbb{N}_0$ and an object X in $\mathscr{C}$, the object $X^{\otimes n}$ in $\mathscr{C}$ given by $X \otimes \dots \otimes X$ (with n factors X) has a natural left action of $\Bbbk\mathbb{S}_n$ (see [4], Thm. XI.1.1). Given $X_1, \dots, X_n$ objects in $\mathscr{C}$ and $\sigma \in \mathbb{S}_n$, we will denote the morphism induced from the symmetric braiding $\tau_{\mathscr{C}}$ of $\mathscr{C}$ by

$$\tau_{X_1,\dots,X_n}(\sigma) : X_1 \otimes \dots \otimes X_n \to X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(n)}. \tag{1.8}$$

For instance, in the category of vector spaces $\tau_{X_1,\dots,X_n}(\sigma)$ sends $x_1 \otimes \cdots \otimes x_n \in X_1 \otimes \cdots \otimes X_n$ to $x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$. If $X_1 = \cdots = X_n = X$, we will denote $\tau_{X_1,\dots,X_n}(\sigma)$ simply by $\tau_{X,n}(\sigma)$.

If X and Y are right $\Bbbk\mathbb{S}_n$-modules with structure morphisms ρ_X and ρ_Y, respectively, then $X \otimes Y$ is also a right $\Bbbk\mathbb{S}_n$-module by means of $\rho_{X\otimes Y}(\sigma) = \rho_X(\sigma) \otimes \rho_Y(\sigma)$, for all $\sigma \in \mathbb{S}_n$. Assume from now on that the category $\mathscr{C}$ has arbitrary colimits, such that the tensor product $\otimes$ commutes with coproducts and filtered coproducts. Moreover, if X is a right $\Bbbk\mathbb{S}_n$-module, we define the **space of coinvariants** $X_{\mathbb{S}_n}$ as the colimit of the system $\{p_\sigma : X \to X_\sigma : \sigma \in \mathbb{S}_n\}$ in $\mathscr{C}$, where $p_\sigma = \rho_X(\sigma) - \mathrm{id}_X : X \to X$, for all $\sigma \in \mathbb{S}_n$.

Given $n \in \mathbb{N}_0$, we define the object $\mathscr{S}_n$ in $\mathscr{C}$ as the coproduct $\coprod_{\sigma\in\mathbb{S}_n} X_\sigma$, where $X_\sigma = \mathbf{1}$ is the unit of $\mathscr{C}$ for all $\sigma \in \mathbb{S}_n$. We define a bimodule structure over $\Bbbk\mathbb{S}_n$ on $\mathscr{S}_n$ via the following $\Bbbk$-linear maps $\rho_\ell : \Bbbk\mathbb{S}_n \to \mathrm{Hom}_{\mathscr{C}}(\mathscr{S}_n, \mathscr{S}_n)$ and $\rho_r : \Bbbk\mathbb{S}_n \to \mathrm{Hom}_{\mathscr{C}}(\mathscr{S}_n, \mathscr{S}_n)^{\mathrm{op}}$. Given $\sigma \in \mathbb{S}_n$, $\rho_\ell(\sigma)$ (resp., $\rho_r(\sigma)$) is the unique morphism $\mathscr{S}_n \to \mathscr{S}_n$ whose restriction to $X_{\sigma'}$ is the composition of the identity $X_{\sigma'} = \mathbf{1} \to \mathbf{1} = X_{\sigma\sigma'}$ (resp., $X_{\sigma'} = \mathbf{1} \to \mathbf{1} = X_{\sigma'\sigma}$) of $\mathbf{1}$ in $\mathscr{C}$ and the canonical map $X_{\sigma'\sigma} \to \coprod_{\tau\in\mathbb{S}_n} X_\tau = \mathscr{S}_n$ (resp., $X_{\sigma\sigma'} \to \coprod_{\tau\in\mathbb{S}_n} X_\tau = \mathscr{S}_n$), for all $\sigma' \in \mathbb{S}_n$. It is easy to see that ρ_ℓ gives a structure of left module, ρ_r gives a structure of right module, and $\rho_r(\sigma) \circ \rho_\ell(\sigma') = \rho_\ell(\sigma') \circ \rho_r(\sigma)$ for all $\sigma, \sigma' \in \mathbb{S}_n$. By abuse of notation, we will denote the $\Bbbk\mathbb{S}_n$-bimodule $\mathscr{S}_n$ in $\mathscr{C}$ by $\Bbbk\mathbb{S}_n$. Moreover, we denote by $\iota : \mathbf{1} \to \Bbbk\mathbb{S}_n$ the composition of the identity $\mathbf{1} \to \mathbf{1} = X_e$ and the inclusion of X_e into $\Bbbk\mathbb{S}_n$, where $e \in \mathbb{S}_n$ is the neutral element.

Let X be a right $\Bbbk\mathbb{S}_n$-module and Y a left $\Bbbk\mathbb{S}_n$-module in $\mathscr{C}$. We define $X \otimes_{\Bbbk\mathbb{S}_n} Y$ as the space of coinvariants $(X \otimes Y)_{\mathbb{S}_n}$ of the right $\Bbbk\mathbb{S}_n$-module $X \otimes Y$. If Y is a $\Bbbk\mathbb{S}_n$-$\Bbbk\mathbb{S}_m$-bimodule, then the right $\Bbbk\mathbb{S}_m$-module structure on Y induces a structure of right $\Bbbk\mathbb{S}_m$-module on $X \otimes_{\Bbbk\mathbb{S}_n} Y$. In the concrete category of $\Bbbk$-vector spaces (as well as graded vector spaces, complexes of vector spaces, etc) $X \otimes_{\Bbbk\mathbb{S}_n} Y$ is exactly the usual tensor product over $\Bbbk\mathbb{S}_n$.

2 $\mathbb{S}$-modules

We recall that an **$\mathbb{S}$-module** S is a sequence $(S(n))_{n\in\mathbb{N}_0}$ such that $S(n)$ is a right $\Bbbk\mathbb{S}_n$-module in $\mathscr{C}$ for all $n \in \mathbb{N}_0$. Moreover, a **morphism of $\mathbb{S}$-modules** $S \to S'$ is a sequence of $f = (f(n))_{n\in\mathbb{N}_0}$ such that $f(n) : S(n) \to S'(n)$ is a morphism of right $\Bbbk\mathbb{S}_n$-modules in $\mathscr{C}$ for all $n \in \mathbb{N}_0$. We define the **composition** of two morphisms $f : S \to S'$ and $g : S' \to S''$ of $\mathbb{S}$-modules as the sequence $(g(n) \circ f(n))_{n\in\mathbb{N}_0}$. We will denote the composition of g and f by $g \circ f$. It is clear that $\mathbb{S}$-modules with the previous morphisms and compositions form a category, that we will denote by Mod-$\mathbb{S}$.

Assume that the category $\mathscr{C}$ has arbitrary products and coproducts. Then, the category Mod-$\mathbb{S}$ also has direct products and coproducts. Indeed, given an arbitrary family $\{S_i : i \in I\}$ of $\mathbb{S}$-modules, the product $\prod_{i\in I} S_i$ is the $\mathbb{S}$-module whose n-th component is the right $\Bbbk\mathbb{S}_n$-module

$$\left(\prod_{i\in I} S_i\right)(n) = \prod_{i\in I} S_i(n),$$

for all $n \in \mathbb{N}_0$, where the latter product is in $\mathscr{C}$, and the coproduct $\coprod_{i\in I} S_i$ is the $\mathbb{S}$-module whose n-th component is the right $\Bbbk\mathbb{S}_n$-module

$$\left(\coprod_{i\in I} S_i\right)(n) = \bigoplus_{i\in I} S_i(n),$$

for all $n \in \mathbb{N}_0$, where $\oplus$ denotes the coproduct in $\mathscr{C}$.

Given a finite sequence $S_1, \dots, S_\ell$ of $\mathbb{S}$-modules, with $\ell \in \mathbb{N}$, we define their **Cauchy tensor product** $S_1 \otimes \cdots \otimes S_\ell$ as the $\mathbb{S}$-module given by

$$\begin{aligned} &\Big(S_1 \otimes \cdots \otimes S_\ell\Big)(m) \\ &= \bigoplus_{\substack{(m_1,\dots,m_\ell) \in \mathbb{N}_0^\ell \\ m_1+\cdots+m_\ell = m}} \bigg(\Big(S_1(m_1) \otimes \cdots \otimes S_\ell(m_\ell)\Big) \otimes_{\Bbbk\mathbb{S}_{m_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_\ell}} \Bbbk\mathbb{S}_m\bigg), \end{aligned} \tag{2.1}$$

where $\Bbbk\mathbb{S}_m$ is considered as a left module over $\Bbbk\mathbb{S}_{m_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_\ell}$ via $\mathfrak{z}_{m_1,\dots,m_\ell}$ in the direct summand indexed by $(m_1, \dots, m_\ell)$, and $S_1(m_1) \otimes \cdots \otimes S_\ell(m_\ell)$ is a right module over $\Bbbk\mathbb{S}_{m_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_\ell}$ with the diagonal action. The structure of right module of the Cauchy tensor product over $\Bbbk\mathbb{S}_m$ is given from the coproduct in $\mathscr{C}$ of the right modules over $\Bbbk\mathbb{S}_m$ for every $(m_1, \dots, m_\ell) \in \mathbb{N}_0^\ell$ such that $m_1 + \cdots + m_\ell = m$.

The following result is now a simple exercise, that we leave to the reader.

Lemma 2.1. *The Cauchy tensor product $\otimes$ defines a monoidal product on the category $\mathbb{S}$-modules, whose unit is the unique $\mathbb{S}$-module $\mathbf{1}_C$ such that $\mathbf{1}_C(n) = 0$ for all $n \in \mathbb{N}$ and $\mathbf{1}_C(0)$ is the unit $\mathbf{1}$ of the symmetric monoidal category $\mathscr{C}$.*

Let S be an $\mathbb{S}$-module, and denote by $S^{\otimes n}$ the Cauchy tensor product $S \otimes \cdots \otimes S$ of n factors S. The aim of the rest of this section is to explain that the monoidal structure on the category $\mathbb{S}$-Mod described in the previous lemma has a natural symmetric braiding induced from the symmetric braiding $\tau_\mathscr{C}$ of $\mathscr{C}$. From the comments about symmetric monoidal categories in the second paragraph of Subsection 1.3, this implies in turn that the Cauchy tensor product $S^{\otimes n}$ also has an left action of $\mathbb{S}_n$. Since it will be more convenient for the definition of operad, we will in fact explicitly describe this latter left action of $\mathbb{S}_n$.

Let $n \in \mathbb{N}$, $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ and $\sigma \in \mathbb{S}_n$. Set $m_1 + \cdots + m_\ell = m$. Given $\mathbb{S}$-modules $S_1, \dots, S_n$, define

$$\begin{aligned} \mathfrak{p}_{m_1,\dots,m_n}(\sigma) : &\Big(S_1(m_1) \otimes \cdots \otimes S_n(m_n)\Big) \otimes_{\Bbbk\mathbb{S}_{m_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_n}} \Bbbk\mathbb{S}_m \\ \to &\Big(S_{\sigma^{-1}(1)}(m_{\sigma^{-1}(1)}) \otimes \cdots \otimes S_{\sigma^{-1}(n)}(m_{\sigma^{-1}(n)})\Big) \otimes_{\Bbbk\mathbb{S}_{m_{\sigma^{-1}(1)}} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_{\sigma^{-1}(n)}}} \Bbbk\mathbb{S}_m \end{aligned} \tag{2.2}$$

as the unique morphism induced by the morphism

$$\Big(S_1(m_1) \otimes \cdots \otimes S_n(m_n)\Big) \otimes \Bbbk\mathbb{S}_m \to \Big(S_{\sigma^{-1}(1)}(m_{\sigma^{-1}(1)}) \otimes \cdots \otimes S_{\sigma^{-1}(n)}(m_{\sigma^{-1}(n)})\Big) \otimes \Bbbk\mathbb{S}_m \tag{2.3}$$

in $\mathscr{C}$ given by $\tau_{X_1,\dots,X_n}(\sigma) \otimes \rho_\ell(\mathfrak{b}_{m_1,\dots,m_n}(\sigma))$, where $\tau_{X_1,\dots,X_n}(\sigma)$ was defined in (1.8), ρ_ℓ denotes the left action of $\Bbbk\mathbb{S}_m$ on itself and $\mathfrak{b}_{m_1,\dots,m_n}$ was defined in (1.4). Identity (1.7) tells us that (2.3) indeed induces the morphism (2.2), for every $\sigma \in \mathbb{S}_n$. Moreover, $\mathfrak{p}_{m_1,\dots,m_n}(\sigma)$ is by definition a morphism of right $\Bbbk\mathbb{S}_m$-modules.

For $m \in \mathbb{N}_0$ and $\sigma \in \mathbb{S}_n$, let

$$\mathfrak{p}_m(\sigma) : (S_1 \otimes \cdots \otimes S_n)(m) \to (S_{\sigma^{-1}(1)} \otimes \cdots \otimes S_{\sigma^{-1}(n)})(m) \tag{2.4}$$

be the unique morphism in $\mathscr{C}$ whose restriction to the direct summand

$$\Big(S_1(m_1) \otimes \cdots \otimes S_n(m_n)\Big) \otimes_{\Bbbk\mathbb{S}_{m_1} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_n}} \Bbbk\mathbb{S}_m$$

of $(S_1 \otimes \cdots \otimes S_n)(m)$ is the composition of $\mathfrak{p}_{m_1,\dots,m_n}(\sigma)$ and the inclusion of

$$\Big(S_{\sigma^{-1}(1)}(m_{\sigma^{-1}(1)}) \otimes \cdots \otimes S_{\sigma^{-1}(n)}(m_{\sigma^{-1}(n)})\Big) \otimes_{\Bbbk\mathbb{S}_{m_{\sigma^{-1}(1)}} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk\mathbb{S}_{m_{\sigma^{-1}(n)}}} \Bbbk\mathbb{S}_m$$

inside of $(S_{\sigma^{-1}(1)}\otimes\cdots\otimes S_{\sigma^{-1}(n)})(m)$, for $(m_1,\dots,m_n)\in\mathbb{N}_0^n$ with $m_1+\cdots+m_n=m$. It is clear that $\varkappa_m(\sigma)$ is a morphism of right $\Bbbk\mathbb{S}_m$-modules, for all $\sigma\in\mathbb{S}_n$. This gives a morphism of $\mathbb{S}$-modules

$$\varkappa(\sigma) : S_1\otimes\cdots\otimes S_n \to S_{\sigma^{-1}(1)}\otimes\cdots\otimes S_{\sigma^{-1}(n)}, \tag{2.5}$$

given by $\varkappa(\sigma)(m)=\varkappa_m(\sigma)$ for all $\sigma\in\mathbb{S}_n$ and $m\in\mathbb{N}_0$.

The careful reader can easily verify the following result, which makes intensive use of (1.6).

Lemma 2.2. *The morphism $\mathbb{S}$-modules $\varkappa(\sigma) : S_1\otimes S_2\to S_2\otimes S_1$ given by (2.5) for $n=2$ and σ the unique transposition of $\mathbb{S}_2$ defines a symmetric braiding on the monoidal category of $\mathbb{S}$-modules in Lemma 2.1.*

As we recalled in the second paragraph of Subsection 1.3, the previous lemma implies that, given $n\in\mathbb{N}_0$ and an $\mathbb{S}$-module S, $S^{\otimes n}$ has a natural left action of $\Bbbk\mathbb{S}_n$. More concretely, if $S_1=\cdots=S_n=S$, (1.6) tells us that the map

$$\varkappa_m : \Bbbk\mathbb{S}_n \to \mathrm{Hom}_{\text{-}\Bbbk\mathbb{S}_m}\Big(\big(S^{\otimes n}\big)(m), \big(S^{\otimes n}\big)(m)\Big)$$

sending $\sigma\in\mathbb{S}_n$ to $\varkappa_m(\sigma)$ is a morphism of $\Bbbk$-algebras, where the latter space is the one given by all morphisms of right $\Bbbk\mathbb{S}_n$-modules. Furthermore, the $\Bbbk$-linear map

$$\varkappa : \Bbbk\mathbb{S}_n \to \mathrm{Hom}_{\mathrm{Mod}\text{-}\mathbb{S}}\big(S^{\otimes n}, S^{\otimes n}\big) \tag{2.6}$$

sending $\sigma\in\mathbb{S}_n$ to $\varkappa(\sigma)$ given in (2.5) is a morphism of $\Bbbk$-algebras, where the latter morphism space is the one given by the morphisms of $\mathbb{S}$-modules. We summarize the previous discussion in the following result, which is implicit in several sources on operads.

Proposition 2.3. *Given $n\in\mathbb{N}$ and an $\mathbb{S}$-module S, the $\mathbb{S}$-module $S^{\otimes n}$ has a natural structure of left $\Bbbk\mathbb{S}_n$-module in the category $\mathbb{S}$-modules via the morphism of $\Bbbk$-algebras $\varkappa$ defined in (2.6).*

Remark 2.4. *The previous result is implicit in [3], 5.1.4 and 5.1.5, where the authors essentially present it in small examples with $n=2$. On the other hand, other classical sources such as [5] or [1] provide a composite tensor product by enlarging the category of $\mathbb{S}$-modules to consider the category of all species, i.e. all contravariant functors from the grupoid of all finite sets with bijective maps to the category $\mathscr{C}$. Even though this latter point of view provides a more robust framework to work on, which is very good for discussing general properties, it considerably increases the size of the objects to study, which might cause some trouble when trying to decode what they actually mean (cf. the incorrect definition of operad in [5], Def. 1.4, 2, p. 42). We present this down-to-earth construction of the left action of $\mathbb{S}_n$ on $S^{\otimes n}$, which is key in the next definition, to help avoiding some of these difficulties.*

The previous result allows us to define the **composite tensor product** $S'\odot S$ of two $\mathbb{S}$-modules as the coproduct of $\mathbb{S}$-modules

$$S'\odot S = S'_0\oplus\bigoplus_{n\in\mathbb{N}}\Big(S'(n)\otimes_{\Bbbk\mathbb{S}_n}\big(S^{\otimes n}\big)\Big), \tag{2.7}$$

where S'_0 is the unique $\mathbb{S}$-module satisfying $S'_0(0)=S'(0)$ and $S'_0(n)=0$ for all $n\in\mathbb{N}$, and the left module structure over $\Bbbk\mathbb{S}_n$ on $S^{\otimes n}$ is given by Lemma 2.3. The $\mathbb{S}$-module structure of each direct summand of (2.7) indexed by $n\in\mathbb{N}$ exists by Lemma 2.3, since the left action of $\Bbbk\mathbb{S}_n$ on $S^{\otimes n}$ commutes with the corresponding right actions.

Let S and S' be two $\mathbb{S}$-modules. Given $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, let $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ such that $m = m_1 + \cdots + m_n$. Define the morphism

$$\iota_{m_1,\dots,m_n} : S'(n) \otimes \Big(S(m_1) \otimes \cdots \otimes S(m_n) \Big) \to \big(S' \odot S \big)(m) \tag{2.8}$$

given by the composition of the map $\mathrm{id}_{S'(n)\otimes(S(m_1)\otimes\cdots\otimes S(m_n))} \otimes \iota$, where $\iota : \mathbf{1} \to \Bbbk\mathbb{S}_m$ was defined in Subsection 1.3, the canonical inclusion

$$\begin{aligned} &S'(n) \otimes \Big(S(m_1) \otimes \cdots \otimes S(m_n) \Big) \otimes \Bbbk\mathbb{S}_m \\ &\qquad \to S'(n) \otimes \Bigg(\bigoplus_{\substack{(m_1,\dots,m_\ell) \in \mathbb{N}_0^\ell \\ m_1 + \cdots + m_\ell = m}} \Big(S(m_1) \otimes \cdots \otimes S(m_n) \Big) \otimes \Bbbk\mathbb{S}_m \Bigg), \end{aligned}$$

the canonical projections

$$S'(n) \otimes \Bigg(\bigoplus_{\substack{(m_1,\dots,m_\ell) \in \mathbb{N}_0^\ell \\ m_1 + \cdots + m_\ell = m}} \Big(S(m_1) \otimes \cdots \otimes S(m_n) \Big) \otimes \Bbbk\mathbb{S}_m \Bigg) \to S'(n) \otimes \big(S^{\otimes n} \big)(m)$$

and

$$S'(n) \otimes \big(S^{\otimes n} \big)(m) \to S'(n) \otimes_{\Bbbk\mathbb{S}_n} \big(S^{\otimes n} \big)(m)$$

and the canonical inclusion of the latter space into $\big(S' \odot S \big)(m)$.

The following result is now a simple exercise, that we leave to the reader.

Lemma 2.5. *The composite tensor product $\odot$ defines a monoidal product on the category $\mathbb{S}$-modules, whose unit is the unique $\mathbb{S}$-module $\mathbf{1}_{\mathbb{S}}$ such that $\mathbf{1}_{\mathbb{S}}(n) = 0$ for all $n \in \mathbb{N}_0 \setminus \{1\}$ and $\mathbf{1}_{\mathbb{S}}(1)$ is the unit $\mathbf{1}$ of the symmetric monoidal category $\mathscr{C}$.*

Remark 2.6. *The composite tensor product is not symmetric.*

Definition 2.7. *An **operad** in the $\Bbbk$-linear symmetric monoidal category $\mathscr{C}$ is a unitary monoid in the monoidal category of $\mathbb{S}$-modules endowed with the composite tensor product $\odot$. More concretely, an operad is a triple (S, γ, η), where S is an $\mathbb{S}$-module, $\gamma : S \odot S \to S$ and $\eta : \mathbf{1}_{\mathbb{S}} \to S$ are morphisms of $\mathbb{S}$-modules such that the diagrams*

$$\begin{tikzcd} S \odot S \odot S \arrow[r, "\gamma\odot\mathrm{id}_S"] \arrow[d, "\mathrm{id}_S\odot\gamma"'] & S \odot S \arrow[d, "\gamma"] \\ S \odot S \arrow[r, "\gamma"] & S \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & S \odot S \arrow[d, "\gamma"] & \\ \mathbf{1}_{\mathbb{S}} \odot S \arrow[ur, "\eta\odot\mathrm{id}_S"] \arrow[r, equal] & S & S \odot \mathbf{1}_{\mathbb{S}} \arrow[ul, "\mathrm{id}_S\odot\eta"'] \arrow[l, equal] \end{tikzcd}$$

commute, where the equalities are the canonical isomorphisms in the monoidal category of $\mathbb{S}$-modules.

The following result is a direct verification, and shows that our definition of composite monoidal product is the one we wanted.

Lemma 2.8. *Let S be an $\mathbb{S}$-module and let $\gamma : S \odot S \to S$ and $\eta : \mathbf{1}_{\mathbb{S}} \to S$ be morphisms of $\mathbb{S}$-modules. Given $n \in \mathbb{N}$ and $(m_1, \dots, m_n) \in \mathbb{N}_0^n$, define the morphism*

$$\gamma_{n,m_1,\dots,m_n} : S(n) \otimes S(m_1) \otimes \cdots \otimes S(m_n) \to S(m)$$

in $\mathscr{C}$ as the composition of (2.8) and γ. Then (S, γ, η) is an operad if and only if the maps $\gamma_{n,m_1,\dots,m_n}$ for $n \in \mathbb{N}$ and $(m_1, \dots, m_n) \in \mathbb{N}_0^n$ satisfy the axioms in Def. 1.1 of [2].

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