Renormalization in Quantum Field Theory (after R. Borcherds)[1]

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[1] From January, 27th, 2019. **Disclaimer:** if someone is aware of a better reference or proof of any statement, please let me know.

Abstract

The aim of this manuscript is to provide a complete and precise formulation of the renormalization picture for perturbative Quantum Field Theory (pQFT) on general curved spacetimes introduced by R. Borcherds in [10]. More precisely, we give a full proof of the free and transitive action of the group of renormalizations on the set of Feynman measures associated with a local precut propagator, and that such a set is nonempty if the propagator is further assumed to be manageable and of cut type. Even though we follow the general principles laid by Borcherds in [10], we have in many cases proceeded differently to prove his claims, and we have also needed to add some hypotheses to be able to prove the corresponding statements.

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Contents

Introduction

General description

The aim of this work is twofold:

(i) To give a complete and precise formulation of the renormalization picture for perturbative Quantum Field Theory (pQFT), following the point of view introduced by R. Borcherds in [10]. In particular, we explain in full detail the different objects he introduces, together with their algebraic and topological structures.

(ii) To give a complete proof of Theorems 15, 18, 20 and 21 in [10] about the free and transitive action of the group of renormalizations on the set of Feynman measures associated with a local propagator of cut type, as well as the existence of a Feynman measure associated with any such propagator. This is done in Theorems 6.0.8, 7.3.10, 7.4.2 and 7.5.2. For the existence we were obliged to impose a further condition on the propagator that does not appear in [10] and that we called *manageability* (see Definition 7.3.4). Without it we were unable to construct the needed Feynman measure, and we remark that this extra assumption is verified in the basic examples of scalar field theory or Dirac field theory on Minkowski, de Sitter and anti-de Sitter spacetime (see Remark 7.3.5).

Let us explain our motivations. The article [10] is really beautiful and full of ideas. However, it is very hard to read, because in many aspects it seems that the author has chosen to simplify or to avoid the corresponding explanations, and in others there are several inaccuracies or potentially misleading statements. For example, the author never gives a precise definition of the *support* of an element in $S\Gamma_c\omega SJ\Phi$ (where we follow the notation in [10]), and even if one may assume that the support of a homogeneous element $A \in S^m\Gamma_c\omega SJ\Phi$ should be a subset of M^m, he talks about the intersection of supports of homogeneous elements of different degrees (*e.g.* see the proof of Thm. 15) or the causal comparison between both (*e.g.* see Def. 9). Another example of such imprecision is when he states that $S\Gamma_c\omega SJ\Phi$ is a comodule over $\Gamma SJ\Phi$ in Lemma 14, whereas the precise structure is never explained, or when he introduces the notion of Feynman measure in his Def. 9, in which he claims that there is a map $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$ induced by the coaction $\omega SJ\Phi \to \omega SJ\Phi \otimes SJ\Phi$. We claim that such structure does not exist in general, and no induced map canonically exists, respectively (see Section 3.5). In order to circumvent this problem, we had considered instead another comodule over the same coalgebra, which retains enough information about the space $S\Gamma_c\omega SJ\Phi$ (*cf.* Sections 3.7-3.12 and Lemma 5.8.7). We may also add that we have specified some (minor) missing hypotheses in [10], such as the antisymmetric property for the causal relation on the definition of spacetime in his Def. 1, and we have made more precise the definition of cut propagator given in his Def. 7. See also Section 5.2 for some other technical differences with respect to the exposition of Borcherds. Despite of all of these facts, it is clear to us that all the statements in [10] are essentially correct. However, the technicality and the subtlety of the structures required to understand the objects presented in [10] is so involved, that we believe that the gaps left there are far from being automatically filled. Our humble intention is thus to clarify the technical details as well as the algebraic structures lurking behind to provide the corresponding precise statements with their proofs.

Let us also mention that even though our proofs are greatly inspired by those of [10], we have in many cases proceeded in a different way. Compare for example the proofs of Lemma 14, and Thm. 15 in [10], and those of Lemma 5.8.7 and Theorem 6.0.8. Concerning Thm. 18, 20 and 21 in [10], we follow the general philosophy established there, but the actual proofs of those results, given in Theorems 7.3.10, 7.4.2 and 7.5.2 in this manuscript, respectively, are more involved.

Concerning the novelty of the results presented here, we believe that the contents of Chapter 1, 2, and 4 are mostly well-known to the specialists, but we provide them for the convenience of the reader, since they come from rather scattered fields. We may only except Sections 2.2 and 2.3 from the previous list, because they contain material that is new as far as we know. Corollaries 1.2.9 and 1.3.5, as well as Theorems 4.3.10 and 4.5.6, also seem to be new, even though the latter is an adaptation of a result of N. V. Dang in [27]. Chapter 3 is a generalization of some well-known algebraic structures for vector spaces to a more involved situation dealing with locally convex spaces (and in particular with sections of vector bundles), so it is in our opinion somehow new. Chapter 5 is essentially based on the article [10] of Borcherds, but we provide many explanations that are absent in the mentioned paper. In particular, Sections 5.3 and 5.9 are somehow implicit in [10], as well as many results concerning propagators in Sections 5.4, 5.5 and 5.6. Moreover, since some of the results stated by Borcherds in [10] seem to be not completely correct or at least unclear (*e.g.* the existence of a coaction of $\Gamma SJ\Phi$ on the space of nonlocal Lagrangians $S\Gamma_c\omega SJ\Phi$ stated in his Lemma 14, or of the corresponding map $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$ induced by the coaction $\omega SJ\Phi \to \omega SJ\Phi \otimes SJ\Phi$ in his Def. 9 of Feynman measure), we deal in many situations with more involved algebraic structures than those in his exposition. For instance, we are forced to work with the tensor algebra instead of the symmetric algebra, and to show that (only) the final constructions of the theory depend on the class in the symmetric algebra of the considered elements. As stated before, this is however different from the considerations in [10], but also from previous expositions, *e.g.* [14], where the symmetric algebra was considered from the very beginning. A particular place where this difference is noticeable is in Section 5.6, where we construct from the propagator a Laplace pairing on a certain tensor algebra (and not on a symmetric algebra). The point that we cannot work with the symmetric algebra from the very beginning and we have to work with the tensor algebra instead is a reflection of the fact that we are dealing with two very different kind of tensors: one over k and another over $C^\infty(M)$, and they are not compatible if we intend to permute them. In algebraic terms, this is the shadow of the fact that the categories of modules over certain required symmetric constructions do not form a 2-monoidal category, a notion recalled in Definition 3.2.2, whereas the corresponding tensor algebras do fulfill the requirements of such definition (see Section 3.5). We also prove some results claimed in [10], e.g. Lemma 5.7.14. Finally, the proof of Thm. 15 of [10] that we give in Chapter 6 is completely new, whereas the proof of Thms. 18, 20 and 21 of [10] that we provide in Chapter 7 follows the general pattern established by Borcherds, even though we have added several hypotheses –that are physically reasonable– and we have ascertained some other intermediate steps that are not present in [10] in order to prove the mentioned results of [10].

As a final word, we would like to state that most of the results and ideas in this manuscript should not be completely new, for they profit from many of the usual constructions in pQFT, specially those presented by R. Brunetti and K. Fredenhagen in [19] (see also the work [52–54] of S. Hollands and R. Wald), which are in turn based on previous work by E. Stueckelberg and A. Petermann in [101], O. Steinmann in [97], and H. Epstein and V. Glaser in [35], to mention a few. The list of people who contributed to this field to boost our understanding and are thus relevant is so vast that we unfortunately have no sufficient space to mention them all. A modern and nice exposition on the subject that follows the previous lines may be found in [90]. We believe however that the point of view presented by Borcherds is somehow new and deserves attention, for it further extracts the algebraic features lurking behind. Furthermore, his point of view is more general than the usual ones, mentioned previously, and in particular, it can be used even if the spacetime is not assumed to be globally hyperbolic, *e.g.* for anti-de Sitter spacetime. Moreover, there is also no need to suppose any homogeneity condition nor any finiteness of a scaling degree of any kind for the distributions involved (see Remark 6.0.10).

Structure of the book

In Chapter 1 we provide the basic results on invariants and coinvariants, with special emphasis on the action of the symmetric group on a tensor product of locally convex spaces, either Hausdorff or not. We will focus particularly on bornological locally convex spaces and on their local completion –called convenient locally convex spaces– as well as on the bornological tensor product and its local complete version. The reason for doing so is the fact that the latter two symmetric monoidal categories are better behaved, and in particular, are closed and the (bornological or convenient) tensor products commutes with arbitrary colimits, which is not necessarily the case for the (completed) projective tensor product. We also remind the reader about some basic results on locally convex m-algebras and their locally convex modules, and of bornological algebras and their bornological locally convex modules. The latter also share the nice categorical properties of the categories of bornological or convenient locally convex spaces stated previously. Moreover, even though locally convex modules over Fréchet algebras are essentially enough to understand the spaces of sections (of compact support or not) of vector bundles over a manifold, their continuous duals (*i.e.* the spaces of distributions) are almost never locally convex modules but bornological locally convex modules. We believe that most of these results should be well-known among the experts.

In Chapter 2 we recall the basic results on vector bundles over a smooth manifold, from an algebraic point of view, but we also recall the natural topologies on the spaces of sections of such vector bundles. The only possible new piece of information may be provided in Section 2.2, where we study the sections of compact support, and the corresponding results in Section 2.3 concerning their topological properties (see Corollaries 2.3.7 and 2.3.10).

Chapter 3 seems to contain new information that is somehow implicit in [10]. It can be regarded as an extension to general manifolds of some of the algebraic structures studied by C. Brouder in [14] for the case of a QFT of dimension zero. We also recall the notion of *symmetric* 2*-monoidal category* (see Definition 3.2.2), the reasonable theory of bialgebras relative to such objects and comodules over such bialgebras. We introduce the new notion of *framed* symmetric 2-monoidal category, which is necessary for the definition of Laplace pairing in the sequel, and we prove that the category of modules over a certain tensor algebra construction on a commutative algebra A is a framed symmetric 2-monoidal category (see Proposition 3.3.9). The main pieces of information proved in this chapter and applied to the specific spaces we are interested in may be summarized as follows. Let A be a unitary Fréchet algebra, X a finitely generated projective Fréchet A-module, V a bornologically projective bornological locally convex A-module, $Y = V \otimes_A S_A X$, $\tilde{\Phi} A = \lim_{m \in \mathbb{N}} A^{\tilde{\otimes}_\beta m}$ and $\tilde{\Sigma} A = \lim_{m \in \mathbb{N}} \tilde{\Sigma}^m A$ be inverse limits in BLCS_{HD}, and $\underset{\sim}{T} A = \oplus_{m \in \mathbb{N}_0} A^{\tilde{\otimes}_\beta m}$ and $\underset{\sim}{\Sigma} A = \oplus_{m \in \mathbb{N}_0} \tilde{\Sigma}^m A$ be locally convex direct sum in BLCS_{HD}. Then,

(a) $S_A X$ is a canonical cocommutative unitary and noncounitary bialgebra in ${}_{\tilde{\Sigma} A}\mathfrak{BMod}$ and in ${}_{\underset{\sim}{\Sigma} A}\mathfrak{BMod}$ (see Fact 3.8.2);

(b) Y is the cofree comodule over $S_A X$ in ${}_A\mathfrak{BMod}$, so Y is *a fortiori* a comodule over $S_A X$ in ${}_{\tilde{\Sigma} A}\mathfrak{BMod}$ (see Porism 1.7.14);

(c) $\tilde{T}^+ S_A X$ has the induced structure of cocommutative noncounitary coalgebra in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$, and there is a morphism of noncounitary coalgebras from $\tilde{T}^+ S_A X$ to $S_A X$ in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$ (see Corollary 3.9.4);

(d) $\tilde{T} S_A X$ has the induced structure of cocommutative counitary coalgebra in ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ for the tensor product $\otimes_{\underset{\sim}{T} A}$ (see Corollary 3.9.2) and that structure forms also a cocommutative unitary and counitary bialgebra on $\tilde{T} S_A X$ relative to the symmetric 2-monoidal category ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ (see Corollary 3.9.3);

(e) $S_A^+ Y$ has a canonical structure of cocommutative noncounitary coalgebra in ${}^{S_A X}_{\tilde{\Sigma} A}\mathfrak{coBMod}$ (see Lemma 3.10.4);

(f) $\tilde{T}^+ Y$ is a conilpotent cofree noncounitary coalgebra in CLCS_{HD} with comodule structure over $\tilde{T}^+ S_A X$, so *a fortiori* over $S_A X$, in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$ (see Corollary 3.11.3);

(g) the canonical map $\tilde{T}^{+}Y \to T_A^{+}Y$ given by (3.12.4) is a morphism of comodules over $S_A X$ in ${}_{\tilde{\Phi}_A}\mathfrak{coCMod}$ (see Proposition 3.12.2).

In Chapter 4 we recall the basic definitions and constructions of distributions on manifolds. Most of the results are well-known but we provide them nonetheless for convenience of the reader and to establish the notation we shall later use. The only possible novelties are given by Theorem 4.3.10, where we show that the pull-back defined by L. Hörmander of distributions satisfying a certain wave front set condition coincides with the usual pull-back of functions if the distributions are regular –*i.e.* it is given by integrating against a continuous map– and by Theorem 4.5.6, where we show how to compute the wave front set of a product of distributions given by integrating a test function against a continuous function. This last result is however just a slight adaptation of one of N. V. Dang in [27], Thm. 3.1, which is in turn based on the proof of G. Eskin in [36], Thm. 14.3. Proposition 4.2.2 seems to be somehow new, specially for it provides an answer to a question posed in [64], Section I.5.8 (see Remark 4.2.4), even though it is based on already well-known results.

Chapter 5 presents the basic definitions of quantum field theory, following the exposition of Borcherds in [10] but also that of Brouder, B. Fauser, A. Frabetti, and R. Oeckl, where they explain the use of Laplace pairings in QFT (see [17]). Our definition of Laplace pairing is however more general than theirs, since we need to work with framed symmetric 2-monoidal categories, that we introduced in Definition 3.2.6: indeed, with exception of the coalgebra whose automorphisms define the group of renormalizations (see Definition 5.8.2) the algebraic objects constructed by Borcherds are not given by usual coalgebras, bialgebras or comodules over them in a symmetric monoidal category, but relative to framed symmetric 2-monoidal categories. As an example, when extending the propagator Δ to obtain a Laplace pairing in Section 5.6, the cocommutative unitary and counitary bialgebra on which it is defined is $\tilde{T}S_{C^\infty(M)}\Gamma(J^iE)$ (see Proposition 3.4.6), which is not a bialgebra in any symmetric monoidal category but relative to the symmetric 2-monoidal category ${}_{\underset{\sim}{T}C^\infty(M)}\mathfrak{CMod}$ of convenient locally convex modules over the convenient algebra $\underset{\sim}{T}C^\infty(M)$ introduced in 3.6.5. Most of the statements appearing in this chapter are based on those of Borcherds, but the underlying algebraic structures are fully explained by making use of the results of Chapter 3, which are new. We have also stated and proved some implicit results of [10], such as Lemma 5.7.14, and we have completed some results of Borcherds, such as Lemma 5.8.7. Furthermore, we have corrected some of (what we believe are) the weak points in [10]. More precisely, we recall that there are several results in that reference that seem to be not completely right, such as the existence of a coaction of $\Gamma SJ\Phi$ on the space of nonlocal Lagrangians $S\Gamma_c\omega SJ\Phi$ stated in his Lemma 14, or of the corresponding map $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$ induced by the coaction $\omega SJ\Phi \to \omega SJ\Phi \otimes SJ\Phi$ in his Def. 9 of Feynman measure (see Section 3.5). In order to solve these problems, we work with more delicate algebraic structures than those in [10]: the coaction of $\Gamma SJ\Phi$ on the space of nonlocal Lagrangians $S\Gamma_c\omega SJ\Phi$ mentioned by Borcherds is replaced by a coaction $\Gamma SJ\Phi$ on the tensor construction $T\Gamma_c\omega SJ\Phi$, whereas the group of renormalizations is still given by automorphisms of the symmetric coalgebra considered by Borcherds. Since automorphisms of the conilpotent cofree coalgebra TZ do not induce in general automorphisms of the corresponding conilpotent cocommutative cofree coalgebra SZ, this implies that the compatibility when dealing with both structures simultaneously is far from trivial. Moreover, one could guess that every situation involving both tensor products over k and over $C^\infty(M)$ forced to us to work with tensor constructions: this is for instance also the case when dealing with Laplace pairings constructed from propagators in Section 5.6,[1] or even when introducing the notion of Feynman measure associated with a local precut propagator in Definition 5.7.9. Fortunately, when combining all these structures, the final output of the Feynman measure only depends on the equivalence class of the element on the symmetric algebra, which is not the case for the intermediate steps. An example when this situation becomes really different from the usual arguments given in [10] is the fact that the group of renormalizations (or ultraviolet group, as called in [10]) acts on the sets of Feynman measures associated with a local precut propagator (compare the

[1] We recall that the results stated by Borcherds concerning the extension of propagators to Laplace pairings are in turn based on those given by Brouder, Fauser, Frabetti, and Oeckl in [17]. Since they only deal with tensor products over k, because they explain the main algebraic features of these particular part of pQFT in a somehow toy model, they can safely work with symmetric algebras instead of with tensor algebras.

proof of the first part of Thm. 15 in [10] and of the Proposition 6.0.5). The other possible completely new piece of information in this chapter may be given in Section 5.3, where we introduce and study the notion of support that is implicit in [10]. We have also provided in Section 5.9 the definitions of meromorphic families of propagators, Feynman measures and renormalizations, which are implicit in [10].

Chapter 6 gives the complete proof of the free and transitive action of the group of renormalizations on the set of Feynman measures associated with a local precut propagator. The proof of this result is decouple into two statements: Proposition 6.0.5 and Theorem 6.0.8. Even though we prove the statements appearing in [10], our methods are different.

Finally, in Chapter 7 we prove that the set of Feynman measures associated with a local propagator is nonempty if the latter is further assumed to be manageable and of cut type. Our proof follows the general yoga established in [10], but we have completed the gaps in that exposition, which also forced us to assume further hypotheses on the propagator: the manageability assumption introduced in Definition 7.3.4. *Grosso modo*, we have first proved that there is a meromorphic family of Feynman measures (see Theorem 7.3.10), which can be made holomorphic around the point we are interested in by means of a meromorphic family of renormalizations (see Theorem 7.4.2). The purported existence of a Feynman measure thus follows from evaluating the holomorphic family of Feynman measures at the point of interest (see Theorem 7.5.2).

As a general indication, we have added some comments on the left margin to indicate the level of novelty or difficulty of the results.

Prerequisites

As expected, the formulation of QFT makes intensive use of several branches of mathematics. Even though all the definitions and results that we used within this manuscript are recalled or referenced along the exposition, let us briefly mention the following suggested prerequisites, where we also provide a particular bibliographical source or sources as a guide:

(i) Differential geometry: differential manifolds, vector bundles, and jet bundles (see [77]);

(ii) Categorical algebra: basic category theory, including monoidal categories (see [61, 68, 74]);

(iii) Functional analysis: locally convex spaces, Fréchet and nuclear spaces (see [84, 95, 104]);

(iv) Algebra: basic algebraic structures, *e.g.* (co)algebras, (co)modules over them (see [75]);

(v) Global analysis: distributions on manifolds (see [46, 55, 107]).

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Chapter 1

Preliminaries on algebra and functional analysis

1.0.1. Throughout the manuscript k will be the field $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathbb{N}$ (resp., $\mathbb{N}_0$) the set of positive integers (resp., nonnegative positive integers), and given any real vector $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\bar{x} \gg 0$ means that $x_i > 0$ for all $i = 1, \dots, n$. Given a *nonunitary* (also called *nonunital*) commutative k-algebra A –*i.e.* A is not necessarily unitary–, a *module (over A)* (or an *A-module*) is a symmetric A-bimodule. We denote the category they form by ${}_A\mathrm{Mod}$. The A-dual $\mathrm{Hom}_A(X, A)$ of an A-module X will be denoted by $X^{\circledast}$, to distinguish it from the usual k-dual $X^* = \mathrm{Hom}_k(X, k)$. From now on, all unadorned tensor products and homomorphisms spaces will be over k, unless otherwise stated.

1.0.2. We recall that a commutative k-algebra A is said to have *enough idempotents* if there is a set $E \subseteq A$ formed by idempotents (*i.e.* $e^2 = e$ for all $e \in E$) that are orthogonal (*i.e.* $ee' = 0$ if $e, e' \in E$ satisfy that $e \neq e'$) and that $A = \oplus_{e \in E} Ae$. On the other hand, we say that A has a *set of local units* if there is a set $E \subseteq A$ formed by idempotents such that, for every finite set of elements $a_1, \dots, a_m \in A$, there exists $e \in E$ such that $ea_{m'}$ $(= a_{m'}e) = a_{m'}$ for all $m' = 1, \dots, m$ (*cf.* [1], Def. 1.1). Note that any commutative k-algebra with enough idempotents has a set of local units given by taking all the possible finite sums of the idempotents satisfying the former condition. A particular case is when the set of local units consists of just one element, *i.e.* if A is unitary. If the commutative k-algebra A has a set of local units E, any A-module X is further assumed to satisfy that $AX = X$, *i.e.* the action morphism $A \otimes_A X \to X$ is surjective (see [1], Section 1). This coincides with the property that, given any $x \in X$ there is $e \in E$ such that ex $(= xe) = x$ (see [1], Section 1), and in particular, it is equivalent to the usual unitary condition imposed on bimodules over a unitary algebra.

1.0.3. We remark that if A has a set of local units, then ${}_A\mathrm{Mod}$ is a symmetric monoidal category for the tensor product over A, the unit given by the object A, the usual structure maps, and the twist giving by the usual flip. Indeed, this holds because any module X (satisfying the previous unitary condition) over an algebra A with a set of local units is *firm* in the sense of D. Quillen, *i.e.* the action morphism $A \otimes_A X \to X$ (or $X \otimes_A A \to X$) is bijective (see [70], Prop. 5.9, for an elementary proof). Note that, if A is nonunitary, the category of all A-modules (not necessarily satisfying any kind of unitarity assumption) is in general only a nonunitary symmetric monoidal category (see Section 3.1). If A has enough idempotents, the assumption $AX = X$ on a module over A is tantamount to the fact that X is given by the internal direct sum $\oplus_{e \in E} eX$.

1.1 Invariants and coinvariants in pre-abelian categories having images

1.1.1. For the basics on category theory we refer the reader to [74], that we will follow. We remark that we shall strictly utilize the categorical terminology in this manuscript, *i.e.* words such as *monomorphism*, *epimorphism*, etc would always be used in the categorical sense. We recall that a *pre-abelian* k-linear category $\mathscr{A}$ is a category enriched over the category of k-vector spaces, having all finite products and coproducts (and they thus coincide, see [74], Cor. I.18.2), kernels and cokernes, and that this implies that $\mathscr{A}$ has finite intersections (see [74], Prop. I.8.1). Let $\mathscr{A}$ be a pre-abelian k-linear category having images. For example, this is satisfied if $\mathscr{A}$ is abelian (see [74], Lemma I.14.4), or $\mathscr{A}$ is a locally small k-linear category that is complete and cocomplete (see [74], I.10). We recall that a category is called *complete* (resp., *cocomplete*) if all small limits (resp., colimits) exist (see [74], II.2), and that the *image* of a morphism $f : A \to B$ is the smallest subobject $i : I \to B$ which f factors though, *i.e.* a monomorphism $i : I \to B$ for which there exists $\bar{f} : A \to I$ such that $f = i \circ \bar{f}$, and given any subobject $j : J \to B$ and any morphism $g : A \to J$ such that $f = j \circ g$, there is $i' : I \to J$ satisfying that $i = j \circ i'$ (see [74], I.10). This definition of image is in principle different from $\mathrm{Ker}(\mathrm{Coker}(f))$, that we are going to call *abelian image*. The two notions coincide however if one further assumes that $\mathscr{A}$ is *normal*, *i.e.* every monomorphism of $\mathscr{A}$ is the kernel of a map (see [74], Lemma I.14.4). There is of course the dual result, concerning the *coimage* of a morphism f defined in [74], I.10, and the *abelian coimage* $\mathrm{Coker}(\mathrm{Ker}(f))$. Given a diagram $\{(X_i, f_{i,j})_{i,j\in I}\}$ in a category $\mathscr{A}$, where $f_{i,j} : X_i \to X_j$ is a morphism in $\mathscr{A}$, for all $i, j \in I$, satisfying the usual compatibility assumption, we will denote the corresponding limit (resp., colimit) by

$$\mathscr{A}\text{-}\lim_{i\in I} X_i \quad (\text{resp., } \mathscr{A}\text{-}\operatorname{colim}_{i\in I} X_i),$$

or simply by

$$\lim_{i\in I} X_i \quad (\text{resp., } \operatorname{colim}_{i\in I} X_i)$$

if the category $\mathscr{A}$ is clear from the context. We may omit the subscript $i \in I$ if its is also clear.

We recall that the *sum* $\sum_{i\in I} A_i$ of a family $j_i : A_i \to A$ of subobjects of A is defined as the image of the induced morphism $j : \coprod_{i\in I} A_i \to A$. Let A be an object of $\mathscr{A}$, G a finite group and $\rho : G \to \mathrm{Aut}_{\mathscr{A}}(A)$ a group morphism. The space of invariants A^G is defined as the intersection $\cap_{g\in G} \mathrm{Ker}(\rho(g) - \mathrm{id}_A)$ and the space of coinvariants A/G is the cokernel of the subobject $\sum_{g\in G} \mathrm{Im}(\rho(g) - \mathrm{id}_A)$ of A. It is a trivial exercise to verify that A^G is an isomorphic subobject of A to the kernel of the map $\rho_p : A \to \prod_{g\in G} A$ whose g-th component is $\rho(g) - \mathrm{id}_A$, and A/G is an isomorphic quotient object of A to the cokernel of the map $\rho_c : \coprod_{g\in G} A \to A$ whose g-th component is $\rho(g) - \mathrm{id}_A$.

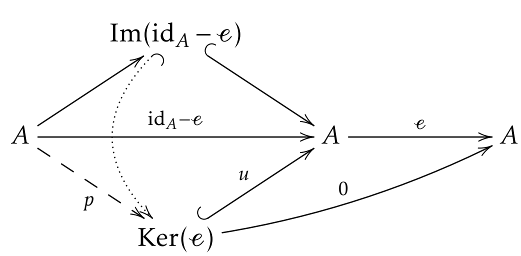
Trivial **1.1.2 Fact.** *Let $\mathscr{B}$ be any category with images and $f : A \to B$ be a morphism with decomposition $f = v \circ u$, where $v : I \to B$ is the image of f. If $i : A' \to A$ is a subobject of A, then the image of $f \circ i$ is a subobject of I.*

Proof. This is a direct consequence of the definition. □

1.1.3. We remark that the following result, well-known for the case of the category of abelian groups, also holds in this case. We show how to adapt the classical proof to apply for this situation (*cf.* [110], Prop. 6.1.10).

New? **1.1.4 Proposition.** *Let $\mathscr{A}$ be a pre-abelian k-linear category having images, A an object of $\mathscr{A}$, G a finite group, and $\rho : G \to \mathrm{Aut}_{\mathscr{A}}(A)$ a morphism of groups. Then, the natural map $A^G \to A/G$, given by the composition of the inclusion $A^G \to A$ and the canonical projection $A \to A/G$, is an isomorphism. Its inverse is given by the map $\bar{e} : A/G \to A^G$ induced by the endomorphism $e = (\sum_{g\in G} \rho(g))/\#(G) \in \mathrm{Hom}_{\mathscr{A}}(A, A)$.*

Proof. Consider $e = (\sum_{g\in G} \rho(g))/\#(G) \in \mathrm{Hom}_{\mathscr{A}}(A, A)$, which exists since $\mathscr{A}$ is k-linear (and k has characteristic zero). It is clearly an idempotent, and the same holds for $\mathrm{id}_A - e$. Hence, Prop. I.18.5 in [74] tells us that A is the coproduct of $\mathrm{Ker}(e)$ and $\mathrm{Ker}(\mathrm{id}_A - e)$. Moreover, $\mathrm{Im}(\mathrm{id}_A - e) \simeq \mathrm{Ker}(e)$ and $\mathrm{Im}(e) \simeq \mathrm{Ker}(\mathrm{id}_A - e)$ as subobjects of A. Indeed, let us briefly show the first one. Since $e \circ (\mathrm{id}_A - e) = 0$, there is a map $p : A \to \mathrm{Ker}(e)$ such that the following diagram commutes and by the definition of image it then gives a monomorphism $\mathrm{Im}(\mathrm{id}_A - e) \hookrightarrow \mathrm{Ker}(e)$ that further makes the diagram



commutative, where $u : \operatorname{Ker}(e) \to A$ is the kernel monomorphism of e. Hence $\operatorname{Im}(\mathrm{id}_A - e)$ is a subobject of $\operatorname{Ker}(e)$ inside of A. Note that $u : \operatorname{Ker}(e) \to A$ satisfies that $(\mathrm{id}_A - e) \circ u = u$. By Fact 1.1.2 we see that $\operatorname{Im}(u) = \operatorname{Ker}(e)$ is a subobject of $\operatorname{Im}(\mathrm{id}_A - e)$, which in turn implies the claim $\operatorname{Im}(\mathrm{id}_A - e) \simeq \operatorname{Ker}(e)$. Taking into account that the previous argument only depends on the idempotency of e, the isomorphism $\operatorname{Im}(e) \simeq \operatorname{Ker}(\mathrm{id}_A - e)$ of subojects of A follows by interchanging e and $\mathrm{id}_A - e$.

Since $(\rho(g) - \mathrm{id}_A) \circ e = 0$, for all $g \in G$, then $\rho_p \circ e$ also vanishes, and by the definition of image, there is a monomorphism $\operatorname{Im}(e) \hookrightarrow \cap_{g \in G} \operatorname{Ker}(\rho(g) - \mathrm{id}_A)$ of subobjects of A. On the other hand, from the identity $\nabla_A \circ \rho_p = \#(G).(\mathrm{id}_A - e)$, where $\nabla_A : \coprod_{g \in G} A \to A$ is the codiagonal morphism whose g-th component is the identity map of A and we are using the canonical isomorphism $\coprod_{g \in G} A \simeq \prod_{g \in G} A$, we see that, if $v : \cap_{g \in G} \operatorname{Ker}(\rho(g) - \mathrm{id}_A) \to A$ is the kernel monomorphism of ρ_p, then $(\mathrm{id}_A - e) \circ v$ vanishes, so $e \circ v = v$. Using again Fact 1.1.2, we conclude that $\operatorname{Im}(v) = \cap_{g \in G} \operatorname{Ker}(\rho(g) - \mathrm{id}_A)$ is a subobject of $\operatorname{Im}(e)$ inside of A, which in turn implies that they are isomorphic subobjects of A.

Finally, the obvious identity $e \circ (\rho(g) - \mathrm{id}_A) = 0$ for all $g \in G$ implies that $e \circ \rho_c = 0$, which in turn tells us that $\operatorname{Im}(\rho_c)$ is a subobject of $\operatorname{Ker}(e)$ inside of A. Since $\sum_{g \in G} \operatorname{Im}(\rho(g) - \mathrm{id}_A)$ is the image of the map $\rho_c : \coprod_{g \in G} A \to A$ defined before and $\#(G).(\mathrm{id}_A - e) = \rho_c \circ \Delta_A$, where $\Delta_A : A \to \prod_{g \in G} A$ is the diagonal map whose g-th component is the identity map of A and we are using the canonical isomorphism $\prod_{g \in G} A \simeq \coprod_{g \in G} A$, Fact 1.6.6 implies that $\operatorname{Im}(\mathrm{id}_A - e) \simeq \operatorname{Ker}(e)$ is a subobject of $\operatorname{Im}(\rho_c)$ inside A. As a consequence, $\sum_{g \in G} \operatorname{Im}(\rho(g) - \mathrm{id}_A)$ and $\operatorname{Ker}(e)$ are isomorphic subobjects of A. The statement now follows from the easy fact that $A \simeq \operatorname{Ker}(e) \oplus \operatorname{Ker}(\mathrm{id}_A - e)$ implies that $A^G \simeq \operatorname{Im}(e) \simeq A/\operatorname{Ker}(e) \simeq A/G$. □

1.1.5. *Remark.* Note that the category $\mathscr{A}$ was not assumed to be abelian in the previous proposition, so that the statement is not a "direct consequence" of the one given for modules.

We also want to caution the reader against automatically identifying invariants and coinvariants. Even though this is essentially safe when dealing with modules of certain type over algebras (for they usually form a category satisfying the assumptions of the previous proposition), it is not the case for algebras or bornological algebras. Indeed, if G is a finite group acting by algebra automorphisms on an algebra A, A^G is naturally a subalgebra of A, whereas A/G has no canonical structure of quotient algebra of A. This is the reason why the bornological algebras constructed in Chapter 3 are in terms of invariants, whereas the modules over them are either given as invariants or coinvariants.

1.1.6. We are interested in applying the previous proposition to the case where $G = \mathbb{S}_m$ is the symmetric group of m elements and $A = X^{\otimes m}$ is the tensor product of an object X in a symmetric monoidal category. The categories we would be interested in are given by (mostly complete) bornological locally convex vector spaces, or by (mostly complete) bornological locally convex modules over a Fréchet algebra. We will first examine the former case in Section 1.4, whereas the latter will be treated in Section 1.6.

1.2 Basics on locally convex spaces Well-known

1.2.1. For the basic definitions and results on topological vector spaces (TVS) and locally convex spaces (LCS) we refer the reader to [95], Ch. I-IV, or to [104], and also to the very clear exposition [84], where in particular all the results we need are nicely recalled, referenced and organized. Proposition 1.1.4 applies to the k-linear category of Hausdorff locally convex spaces LCS_{HD} as well to the category LCS of not necessarily Hausdorff ones, for they are (locally small) complete and cocomplete (see [84], Prop. 2.1.3, 2.1.5, 2.1.8, 3.1.3, and 3.1.4). Moreover, the fully faithful inclusion functor $\mathrm{i}_{HD} : \mathrm{LCS}_{HD} \to \mathrm{LCS}$ preserves limits and coproducts. We remark that the colimits are in general not preserved, because this is already not the case for cokernels. We caution the reader that that the terms image and coimage of a continuous linear map in the literature concerning LCS (and in particular in [84]) are what we have called abelian image and abelian coimage, resp. However, and despite the fact that the category LCS is neither normal nor conormal, it easily follows from the explicit description of the abelian images and abelian coimages for LCS given in [84], Prop.2.1.8, that they coincide with the definitions of image and coimage that we follow.

1.2.2. On the other hand, it is clear to verify that the image of a morphism f in LCS_{HD} coincides with its image in LCS (*i.e.* the set-theoretic image), whereas the abelian image is the closure in the codomain of the set-theoretic image with the induced topology of the codomain (see [84], Prop. 3.1.4). The coimage of a morphism in LCS_{HD} clearly coincides with the abelian coimage (see [84], Prop. 3.1.4). As an aside, if X is a Hausdorff LCS endowed with an action of a finite group G, by the proof of Proposition 1.1.4, $\sum_{g\in G}\mathrm{Im}(\rho(g)-\mathrm{id}_X)$ coincides with the kernel of $\mathrm{id}_X - e$, so it must be closed.

1.2.3. We also recall that, given two LCS X and Y, the usual tensor product space $X\otimes Y$ has a natural topology of LCS defined as the finest locally convex topology such that the bilinear map $X\times Y\to X\otimes Y$ is continuous (see [95], III.6.1-2), and it is called the *projective tensor product* $X\otimes_\pi Y$. It is Hausdorff if X and Y are (see [95], III.6.2). Equivalently, $X\otimes_\pi Y$ is the unique object in LCS representing the covariant functor $\mathrm{LCS}\to {}_k\mathrm{Mod}$ given by sending Z to the vector space $\mathscr{B}(X,Y;Z)$ of continuous bilinear maps $X\times Y\to Z$ (see [95], III.6.2).

1.2.4. On the other hand, since the k-linear category LCS^c_{HD} of complete locally convex spaces is also (locally small) complete and cocomplete (see [84], Prop. 4.1.6 and 4.1.8), Proposition 1.1.4 applies to it too. We recall that a locally convex space is said to be *complete* if it is Hausdorff and every Cauchy net is convergent (see [95], Prerequisites B.6, and I.1.4). The first of the previous references also tell us that the fully faithful inclusion functor $\mathrm{i}_c:\mathrm{LCS}^c_{HD}\to\mathrm{LCS}_{HD}$ preserves limits and coproducts, but it does not necessarily preserves cokernels. Moreover, the inclusion functor $\mathrm{i}_{HD}\circ\mathrm{i}_c$ has a left adjoint, called the *completion functor*, and that is denoted by $X\mapsto\hat{X}$ (see [84], Prop. 4.1.5). The unit of the adjunction will be denoted by $i_X: X\to\hat{X}$, for all LCS X. We caution the reader that the adjective complete is used with two rather different meanings: one for categories, and the other for locally convex spaces. It is trivial to verify that the abelian image (resp., coimage) of a morphism of LCS^c_{HD} described in [84], Prop. 4.1.8, satisfies as well the definitions of image (resp., coimage) in [74], so they coincide. We also note that, even though the categories LCS_{HD} and LCS are quasi-abelian (see [84], Prop. 2.1.11 and 3.1.8), LCS^c_{HD} is not, as proved in [84], Prop. 4.1.14. We have the following stronger result. We recall first that a category is said to be *semi-abelian* if it is pre-abelian and the morphism $\mathrm{Coker}(\mathrm{Ker}(f))\to\mathrm{Ker}(\mathrm{Coker}(f))$, induced by any morphism $f:A\to B$, is a monomorphism and an epimorphism.

New **1.2.5 Lemma.** *The category* LCS^c_{HD} *of complete (Hausdorff) LCS is not semi-abelian. In particular, it is not quasi-abelian.*

Proof. By [104], Ex. 5.3, there is an injective continuous linear map $f:X\to Y$ from a Hausdorff LCS X to a complete LCS Y such that the completion $\hat{f}:\hat{X}\to Y$ is not injective. Replacing Y by the closure of $f(X)$ inside of Y, which is complete, we may even assume that $\hat{f}$ is an epimorphism. Indeed, this follows from the fact the completion functor is left adjoint to the inclusion functor $\mathrm{LCS}^c_{HD}\to\mathrm{LCS}$, so the former preserves epimorphisms. By [30], Satz 9, there is also a complete LCS Z and a surjective continuous linear map $g:Z\to X$. Set $h=f\circ g:Z\to Y$. Then the induced morphism $\mathrm{Coker}(\mathrm{Ker}(h))\to\mathrm{Ker}(\mathrm{Coker}(h))$ is precisely $\hat{f}$, so it is not injective, and thus not a monomorphism (see [84], Prop. 4.1.8 and 4.1.10, to see how abelian images, abelian coimages, monomorphisms and epimorphisms in LCS^c_{HD} are). □

1.2.6. *Remark.* Even though the category LCS^c_{HD} is not semi-abelian, which is the usual property used in the homological algebra of locally convex spaces, it is a pre-abelian k-linear category having images, and in particular Proposition 1.1.4 holds for it.

1.2.7. Given two LCS X and Y, the completion of the projective tensor product $X\otimes_\pi Y$ is called the *completed projective tensor product*, and it is denoted by $X\hat{\otimes}_\pi Y$ (see [95], III.6.3). It is clear that $X\hat{\otimes}_\pi Y$ is the unique object in LCS^c_{HD} representing the covariant functor $\mathrm{LCS}^c_{HD}\to{}_k\mathrm{Mod}$ given by sending Z to the vector space $\mathscr{B}(X,Y;Z)$ of continuous bilinear maps $X\times Y\to Z$ (see 1.2.3). It is easy to prove that the map

$$X\hat{\otimes}_\pi Y \xrightarrow{\sim} \hat{X}\hat{\otimes}_\pi\hat{Y}, \tag{1.2.1}$$

given by the completion of $i_X\otimes i_Y$, where $i_X:X\to\hat{X}$ and $i_Y:Y\to\hat{Y}$ are the canonical morphisms, is an isomorphism of LCS, for all LCS X and Y.

1.2.8. We give the following application of Proposition 1.1.4. In particular it implies that the completion of the invariant space is isomorphic to the invariant of the completion, which is not immediate in its own, since, as recalled previously, the completion does not preserve monomorphisms.

1.2.9 Corollary. *Let $\mathscr{A} = \mathrm{LCS}_{HD}$ be the k-linear category of Hausdorff locally convex spaces, and let $\hat{\mathscr{A}} = \mathrm{LCS}^c_{HD}$ be the k-linear category of complete (Hausdorff) locally convex spaces. Suppose given an object X in $\mathscr{A}$ and an action $\rho : G \to \mathrm{Aut}_{\mathscr{A}}(X)$ of a finite group G. Then, ρ induces to a (unique) group homomorphism $\hat{\rho} : G \to \mathrm{Aut}_{\hat{\mathscr{A}}}(\hat{X})$ on the completion $\hat{X}$ of X such that $\hat{\rho}(g)|_X = \rho(g)$ for all $g \in G$. Moreover, the canonical morphisms $\widehat{X^G} \to \hat{X}^G$ and $\widehat{X/G} \to \hat{X}/G$ are isomorphisms in LCS^c_{HD}.* *Unknown? New?*

Proof. Given $g \in G$, the universal property of the completion (see [104], Thm. 5.2) tells us immediately that there exists unique linear maps $\hat{\rho}(g) : \hat{X} \to \hat{X}$ satisfying that $\hat{\rho}(g)|_X = \rho(g)$ for all $g \in G$. In fact, $\hat{\rho}(g)$ is precisely the completion $\widehat{\rho(g)}$ of $\rho(g)$. The identity $\hat{\rho}(g) \circ \hat{\rho}(g')|_X = \rho(g) \circ \rho(g') = \rho(gg') = \hat{\rho}(gg')|_X$ and the uniqueness of the family $\{\hat{\rho}(g)\}_{g\in G}$ implies that $\hat{\rho}(g) \circ \hat{\rho}(g') = \hat{\rho}(gg')$, for all $g, g' \in G$, so $\hat{\rho}$ is a group morphism. Note furthermore that the completion $\widehat{\rho_c}$ of the map $\rho_c : \oplus_{g\in G} X \to X$ whose g-th coordinate is $\rho(g) - \mathrm{id}_X$ is the only morphism $\hat{\rho}_c : \oplus_{g\in G} \hat{X} \to \hat{X}$ whose g-th coordinate is $\hat{\rho}(g) - \mathrm{id}_{\hat{X}}$. Thus, the cokernel of $\hat{\rho}_c$ is precisely $\hat{X}/G$, whereas $\widehat{\rho_c}$ is the completion of the morphism whose cokernel is X/G. By the universal property of the completion, the functor it determines is the left adjoint to the inclusion functor $\mathrm{i}_c : \mathrm{LCS}^c_{HD} \to \mathrm{LCS}_{HD}$, so it preserves in particular cokernels (see [74], Prop. II.12.1), which in turn implies that $\widehat{X/G} \to \hat{X}/G$ is an isomorphism.

We have now the commutative diagram

$$
\begin{array}{ccc}
\widehat{X^G} & \xrightarrow{\sim} & \widehat{X/G} \\
\downarrow & & \downarrow \wr \\
\hat{X}^G & \xrightarrow{\sim} & \hat{X}/G
\end{array}
$$

We have proved that the right vertical map is an isomorphism, and Proposition 1.1.4 tells us that the horizontal morphisms are isomorphisms as well. We conclude that the canonical map $\widehat{X^G} \to \hat{X}^G$ is an isomorphism, and the corollary is proved. □

1.3 Basic facts on symmetric monoidal categories

Well-known

1.3.1. Let $\mathscr{C}$ be a (locally small) complete and cocomplete k-linear symmetric monoidal category such that the tensor product $\otimes_{\mathscr{C}}$ commutes with countable colimits on each side. The unit of $\mathscr{C}$ will be typically denoted by $I_{\mathscr{C}}$, the *structure morphisms* by $a_{X,Y,Z} : (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} Z \to X \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Z)$, $\ell_X : I_{\mathscr{C}} \otimes_{\mathscr{C}} X \to X$ and $r_X : X \otimes_{\mathscr{C}} I_{\mathscr{C}} \to X$, and the *twist* by $\tau(X,Y) : X \otimes_{\mathscr{C}} Y \to Y \otimes_{\mathscr{C}} X$, for all objects X, Y, Z of $\mathscr{C}$. We refer the reader to [68], VII.1 and VII.7, or [61], Def. XI.2.1 and XIII.1.1, and XIII.1.5, for the corresponding axioms. If we do not impose the existence of a unit $I_{\mathscr{C}}$ and of the corresponding structure maps ℓ_X and r_X, then we will say that $\mathscr{C}$ is *nonunitary*. This is called a *semigroup category* in [37], 13.1. We recall that the usual definitions of non(co)unitary (co)algebra, nonunitary and noncounitary bialgebra, (co)module, etc. done for the category of vector spaces make sense in any nonunitary monoidal category $\mathscr{C}$, whereas the unitary and counitary versions make sense for any monoidal category $\mathscr{C}$ (*cf.* [68], VII). They will be called $\mathscr{C}$-(co)algebra (or (co)algebra in $\mathscr{C}$), $\mathscr{C}$-bialgebra, (co)module in $\mathscr{C}$, etc., (decorated with adjectives such as "(non)unitary" or "(non)counitary") to emphasize the underlying (nonunitary) monoidal category. The notions of (co)commutative (co)algebra –(co)unitary or not–, tensor product of (co)algebras, etc. need however further structure on $\mathscr{C}$, which is at least satisfied if $\mathscr{C}$ is symmetric. The category of nonunitary $\mathscr{C}$-algebras (resp., unitary $\mathscr{C}$-algebras) provided with the usual morphisms of algebras (resp., unitary algebras) will be denoted by $\mathrm{Alg}(\mathscr{C})$ (resp., $\mathrm{uAlg}(\mathscr{C})$), whereas that of noncounitary $\mathscr{C}$-coalgebras (resp., counitary $\mathscr{C}$-coalgebras) provided with the usual morphisms of coalgebras (resp., counitary coalgebras) will be denoted by $\mathrm{Cog}(\mathscr{C})$ (resp., $\mathrm{cCog}(\mathscr{C})$), to emphasize the monoidal category $\mathscr{C}$. The corresponding full subcategory of $\mathrm{Alg}(\mathscr{C})$ (resp., $\mathrm{uAlg}(\mathscr{C})$) formed by the commutative algebras will be be denoted by ${}^c\mathrm{Alg}(\mathscr{C})$ (resp., ${}^c\mathrm{uAlg}(\mathscr{C})$), whereas the full subcategory of $\mathrm{Cog}(\mathscr{C})$ (resp., $\mathrm{cCog}(\mathscr{C})$) formed by cocommutative coalgebras will be denoted by ${}^c\mathrm{Cog}(\mathscr{C})$ (resp., ${}^c\mathrm{cCog}(\mathscr{C})$). Moreover, given (unitary or not) $\mathscr{C}$-algebra A, its category of (left) modules over A in $\mathscr{C}$ will be denoted by ${}_A\mathrm{Mod}(\mathscr{C})$, where we assume that a module satisfies the usual unitary condition in case the algebra is unitary. Analogously, if C is a (counitary or not) $\mathscr{C}$-coalgebra, its category of (left) comodules over C in $\mathscr{C}$ will be denoted by ${}^C\mathrm{coMod}(\mathscr{C})$, where we assume that a comodule satisfies the usual counitary condition in case the coalgebra is counitary. We may incidentally omit the category $\mathscr{C}$ if it is clear from the context, or in the case when $\mathscr{C}$ is the category of vector spaces over k with the usual symmetric monoidal structure.

1.3.2.*Remark.* Note that the tensor product and direct sums of (co)commutative (co)unitary (co)algebras in a symmetric monoidal category $\mathscr{C}$ whose tensor product commutes with colimits on each side is also a (co)commutative (co)algebra in $\mathscr{C}$ (resp., (co)commutative (co)unitary (co)algebra in $\mathscr{C}$).

1.3.3. Some interesting examples of symmetric monoidal categories we may consider are LCS, LCS_{HD}, LCS^c_{HD} (see Section 1.2 for the definitions) or even the category ${}_A\mathrm{Mod}$ of modules over a commutative algebra A, where the first two categories are endowed with the projective tensor product $\otimes_\pi$ (see 1.2.3), the unit k and the twist is given by the usual flip, whereas the third one has the completed projective tensor product $\hat{\otimes}_\pi$ (see 1.2.7), the same unit, and the twist is the completion of that of LCS_{HD}. However, one major disadvantage of the first three categories is that the tensor product does not commute in general with direct sums on each side. It is for this reason that we may instead deal with the categories of bornological and of convenient vector spaces (see Section 1.4 for the definitions). Indeed, these categories are symmetric monoidal, where the first two are endowed with the bornological tensor product $\otimes_\beta$, the unit k and the twists given by the usual flip, whereas the last one is endowed with the convenient tensor product $\tilde{\otimes}_\beta$, the unit k and the twist defined as the unique continuous linear extension of the twist of BLCS_{HD}. As shown in (1.4.7) and (1.4.12), the tensor product of each of these category commutes with colimits on each side. Another important example of symmetric monoidal categories where the tensor product commutes with arbitrary colimits on each side is the category of (co)modules over a commutative and cocommutative unitary and counitary bialgebra B in $\mathscr{C}$, where the tensor product is given by the usual formula (see [75], Def. 1.8.2), and the unit and twist are the ones induced by those of $\mathscr{C}$.

1.3.4. Consider an object X in $\mathscr{C}$. Given $m \in \mathbb{N}_0$, $X^{\otimes_{\mathscr{C}} m}$ has a natural action of the symmetric group $\mathbb{S}_m$ of m elements (see [68], Thm. XI.1.1) so it makes sense to consider the spaces of invariants $\Sigma^m_{\mathscr{C}} X = (X^{\otimes_{\mathscr{C}} m})^{\mathbb{S}_m}$ and of coinvariants $S^m_{\mathscr{C}} X = X^{\otimes_{\mathscr{C}} m}/\mathbb{S}_m$. By Proposition 1.1.4 they are canonically isomorphic in $\mathscr{C}$.

Unknown? New? **1.3.5 Corollary.** *Let* LCS_{HD} *be the* k*-linear category of Hausdorff locally convex spaces, and let* LCS^c_{HD} *be the* k*-linear category of complete (Hausdorff) locally convex spaces. Suppose given an object* Y *in* LCS_{HD}*. Then, the canonical morphisms* $\widehat{(Y^{\otimes_\pi m})^{\mathbb{S}_m}} \to (\hat{Y}^{\hat{\otimes}_\pi m})^{\mathbb{S}_m}$ *and* $\widehat{(Y^{\otimes_\pi m})/\mathbb{S}_m} \to (\hat{Y}^{\hat{\otimes}_\pi m})/\mathbb{S}_m$ *are isomorphisms in* LCS^c_{HD}.

The proof is a direct application of Corollaries 1.2.9 together with property (1.2.1) of the completion of the projective tensor product.

1.3.6. We recall that a k-linear functor $F : \mathscr{C} \to \mathscr{C}'$ between two k-linear symmetric monoidal categories $\mathscr{C}$ and $\mathscr{C}'$ is called *lax symmetric monoidal* if it satisfies the same axioms as those given in [61], Def. XI.4.1 and XIII.3.6, except that the *coherence maps* (denoted there by $\varphi_0 : I_{\mathscr{C}'} \to F(I_{\mathscr{C}})$, where $I_{\mathscr{C}}$ is the unit of $\mathscr{C}$ and $I_{\mathscr{C}'}$ is the unit of $\mathscr{C}'$, and by $\varphi_2(X,Y) : F(X) \otimes_{\mathscr{C}'} F(Y) \to F(X \otimes_{\mathscr{C}} Y)$, for all X, Y objects of $\mathscr{C}$) are not required to be isomorphisms. We say that F is *oplax symmetric monoidal* if $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}'^{\mathrm{op}}$ is lax symmetric monoidal. In this case, the coherence maps are morphisms in $\mathscr{C}'$ of the form $\varphi_0 : F(I_{\mathscr{C}}) \to I_{\mathscr{C}'}$, where $I_{\mathscr{C}}$ is the unit of $\mathscr{C}$ and $I_{\mathscr{C}'}$ is the unit of $\mathscr{C}'$, and by $\varphi_2(X,Y) : F(X \otimes_{\mathscr{C}} Y) \to F(X) \otimes_{\mathscr{C}'} F(Y)$, for all X, Y objects of $\mathscr{C}$. Finally, a lax symmetric monoidal functor F is said to be *strong symmetric monoidal* (or just *symmetric monoidal*) if φ_0 and $\varphi_2(X,Y)$ are isomorphisms of $\mathscr{C}'$, for all X and Y objects of $\mathscr{C}$. If $\mathscr{C}$ and $\mathscr{C}'$ are nonunitary, a *nonunitary* lax (resp., oplax, strong) symmetric monoidal functor $F : \mathscr{C} \to \mathscr{C}'$ is given by the same definition as before but without imposing any constraint regarding the units of $\mathscr{C}$ and of $\mathscr{C}'$.

1.3.7 Lemma. *Let $(\mathscr{A},\otimes_{\mathscr{A}},I_{\mathscr{A}},\tau_{\mathscr{A}})$ and $(\mathscr{B},\otimes_{\mathscr{B}},I_{\mathscr{B}},\tau_{\mathscr{B}})$ be two k-linear closed symmetric monoidal categories, and let $F:\mathscr{A}\to\mathscr{B}$ and $G:\mathscr{B}\to\mathscr{A}$ be two k-linear functors such that F is oplax symmetric (with coherence morphisms φ_0 and φ_2), and there is an adjunction for the pair (F,G) given by the k-linear natural isomorphisms*

Probably well-known

$$\phi_{X,Y}:\mathrm{Hom}_{\mathscr{B}}\Big(F(X),Y\Big)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{A}}\Big(X,G(Y)\Big). \tag{1.3.1}$$

We will denote the internal homomorphism spaces of $\mathscr{A}$ and of $\mathscr{B}$ by $\mathcal{H}om_{\mathscr{A}}(-,-)$ and $\mathcal{H}om_{\mathscr{B}}(-,-)$, respectively. Assume that the coherence map $\varphi_0:F(I_{\mathscr{A}})\to I_{\mathscr{B}}$ is an isomorphism. Then, the following are equivalent

(i) F is strong monoidal;

(ii) there are isomorphisms

$$\Phi_{X,Y}:G\Big(\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big)\xrightarrow{\sim}\mathcal{H}om_{\mathscr{A}}\Big(X,G(Y)\Big) \tag{1.3.2}$$

in $\mathscr{A}$, natural in all objects X in $\mathscr{A}$ and Y in $\mathscr{B}$, such that the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{B}}\Big(F(X)\otimes_{\mathscr{B}}F(Z),Y\Big) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{B}}\Big(F(Z),\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}\Big(Z,G\Big(\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big)\Big) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{B}}\left(\varphi_2(X,Z),Y\right)} & & & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{B}}(Z,\Phi_{X,Y})} \\
\mathrm{Hom}_{\mathscr{B}}\Big(F\big(X\otimes_{\mathscr{A}}Z\big),Y\Big) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}\Big(X\otimes_{\mathscr{A}}Z,G(Y)\Big) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}\Big(Z,\mathcal{H}om_{\mathscr{A}}\big(X,G(Y)\big)\Big)
\end{array} \tag{1.3.3}$$

commutes for all objects X and Z in $\mathscr{A}$, and Y in $\mathscr{B}$, where the upper left and lower right horizontal isomorphisms are due to the adjunction between the tensor product and the internal homomorphism space, whereas the upper right and lower left horizontal ones follow from the adjunction (1.3.1) *between F and G.*

Moreover, the morphisms (1.3.2) *satisfy the following conditions. First, the diagram*

$$\begin{array}{ccc}
G\Big(\mathcal{H}om_{\mathscr{B}}\big(F(I_{\mathscr{A}}),Y\big)\Big) & \xrightarrow{\Phi_{I_{\mathscr{A}},Y}} & \mathcal{H}om_{\mathscr{A}}\big(I_{\mathscr{A}},G(Y)\big) \\
\Big\downarrow{\scriptstyle G\left(\mathcal{H}om_{\mathscr{B}}(\phi_0^{-1},Y)\right)} & & \Big\downarrow{\scriptstyle \wr} \\
G\Big(\mathcal{H}om_{\mathscr{B}}\big(I_{\mathscr{B}},Y\big)\Big) & \xrightarrow{\sim} & G(Y)
\end{array} \tag{1.3.4}$$

commutes for all objects Y in $\mathscr{B}$, where the right vertical isomorphism is induced by the adjunction between the tensor product and the internal homomorphism spaces in $\mathscr{A}$, and the lower horizontal isomorphism is given by the image under G of the isomorphism induced by the adjunction between the tensor product and the internal homomorphism spaces in $\mathscr{B}$. Finally, the isomorphism given as the composition of

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{B}}\big(F(X),Y\big) &\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{B}}\Big(I_{\mathscr{B}},\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{B}}\Big(F(I_{\mathscr{A}}),\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big)\\
&\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{A}}\Big(I_{\mathscr{A}},G\Big(\mathcal{H}om_{\mathscr{B}}\big(F(X),Y\big)\Big)\Big)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{A}}\Big(I_{\mathscr{A}},\mathcal{H}om_{\mathscr{A}}\big(X,G(Y)\big)\Big)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{A}}\Big(X,G(Y)\Big)
\end{aligned} \tag{1.3.5}$$

coincides with $\phi_{X,Y}$, where the first and last one are due to the adjunction between the tensor product and the internal homomorphism space, the second isomorphism is induced by φ_0, the third one is the adjunction $\phi_{I_{\mathscr{A}},Z}$ between F and G, with $Z=\mathcal{H}om_{\mathscr{B}}(F(X),Y)$, and the fourth one is $\mathrm{Hom}_{\mathscr{A}}(I_{\mathscr{A}},\Phi_{X,Y})$.

Proof. Assume that condition in (i) holds. Then, diagram (1.3.3) determines $\Phi_{X,Y}$ in terms of $\varphi_2(X,Z):F(X\otimes_{\mathscr{A}}Z)\to F(X)\otimes_{\mathscr{B}}F(Z)$, and establishes that one is an isomorphism if and only if the other is, because all the remaining arrows in diagram (1.3.3) are isomorphisms. The naturality of $\Phi_{X,Y}$ follows from that of the arrows in diagram (1.3.3), proving (ii). To prove the converse, note that diagram (1.3.3) implies that $\mathrm{Hom}_{\mathscr{B}}(\varphi_2(X,Z),Y)$ is a natural isomorphism in X, Y and Z, so, by the Yoneda Lemma, $\varphi_2(X,Z)$ is an isomorphism, for all X and Z, as was to be shown. Finally, the fact that F is strong monoidal with respect to the coherence morphism φ_0 implies that $\Phi_{X,Y}$ satisfies conditions (1.3.4) and (1.3.5). □

Well-known

1.4 Basics on bornological locally convex vector spaces

1.4.1. For the basics on bornological LCS we refer the reader to [95], II.8. A nice unpublished exposition on bornological spaces may be found in [85], whereas a more comprehensive one is given in [49, 50]. For the basics on locally complete LCS, we refer the reader to the fundamental book [82]. See also [41], Ch. 2, for a very nice source on such spaces, or the more recent and also very nice exposition [64], Ch. I. However, since some of these notions are not so well-known, we will briefly summarize those we need.

1.4.2. For later use, we recall that, given two subsets A and B of a vector space X, *A absorbs B* if there exists $c_0 \in k$ such that $B \subseteq cA$, for all $c \in k$ such that $|c| \geq |c_0|$. A subset A of the vector space X is called *radial (absorbing)* if it absorbs every finite subset of X, and it is called *balanced* (or *circled*) if $cA \subset A$ for all $c \in k$ such that $|c| \leq 1$. Given a balanced convex subset $B \subseteq X$, X_B denotes the vector subspace of X spanned by B provided with the seminorm μ_B defined as $\mu_B(x) = \inf\{\lambda \in \mathbb{R}_{\geq 0} : x \in \lambda.B\}$. Note that if $B \subseteq B'$, for two balanced convex subsets of X, then the natural inclusion mapping $X_B \to X_{B'}$ is a continuous linear map for the corresponding seminorms. This determines an inductive system $\{X_B\}_B$, where B runs over the balanced convex subsets of X. On the other hand, we recall that a *barrel* of a LCS X is a radial, balanced, convex, and closed subset of X. A LCS X is said to be *barreled* if every barrel is a (closed) neighborhood of the origin of X. We recall that the inductive limit (in the category LCS) of a system of barreled LCS is barreled (see [95], II.7.2, Cor. 2).

1.4.3. A LCS X is said to be *Schwartz*, if for every balanced, closed, convex neighborhood U of the origin of X, there exists a neighborhood V of the origin of X such that for every $c > 0$, the set V can be covered by finitely many translates of cU. We recall that the projective limit of a system of Schwartz LCS is Schwartz (see [57], Prop. 3.15.6), the quotient of a Schwartz LCS by a subspace is Schwartz (see [57], Prop. 3.15.7), and the strict inductive limit of a system of Schwartz LCS is Schwartz (see [57], Prop. 3.15.8).

1.4.4. A subset $B \subseteq X$ of a LCS is said to be *bounded* if given any neighborhood U of zero in X, there exists $\lambda \in k$ such that $B \subseteq \lambda U$. Note that if B is a bounded balanced convex subset of a LCS X, then the inclusion map $X_B \to X$ is continuous, and if X is further assumed to be Hausdorff, then the seminorm μ_B of X_B is in fact a norm (see [95], II.8.3). A (not necessarily continuous nor linear) map $f : X \to Y$ between two LCS is called *bounded* (or *bornological*) if the direct image $f(B)$ of any bounded subset of X is bounded in Y.[1] Note that any continuous linear map is bounded but the converse does not hold in general (see [95], I.5.4). Let us recall that a LCS X is *bornological* if any bounded linear map $f : X \to Y$ from X to any LCS Y is continuous. This is tantamount to say that every balanced convex subset of X that absorbs every bounded set of X is a neighborhood of zero in X (see [95], II.8, II.8.3 and Exercise II.18). Given any LCS X, let $\mathfrak{B}_X$ be the set of bounded sets of X. Then the set of locally convex topologies on X whose bounded sets are precisely $\mathfrak{B}_X$ has a supremum (for the inclusion). The resulting structure of LCS on X it determines is called the *bornological LCS associated with X*, and it is denoted by X_{born}. Equivalently, X_{born} is the colimit in LCS of the inductive system $\{X_B\}_B$, where B runs over all bounded balanced convex sets of X, and the morphisms as those explained in 1.4.2 (see [95], II.8.3). The identity map $X_{\mathrm{born}} \to X$ is clearly bounded, and its inverse is continuous. In consequence, a LCS X is bornological if and only if $X = X_{\mathrm{born}}$. We will denote the category of bornological LCS (resp., Hausdorff bornological LCS) provided with continuous (or, equivalently, bounded) linear maps between them by BLCS (resp., BLCS_{HD}). We will regard it as a full subcategory of the category LCS (resp., LCS_{HD}), and we will call the inclusion functor $\mathfrak{inc}$ (resp., $\mathfrak{inc}_{HD}$).

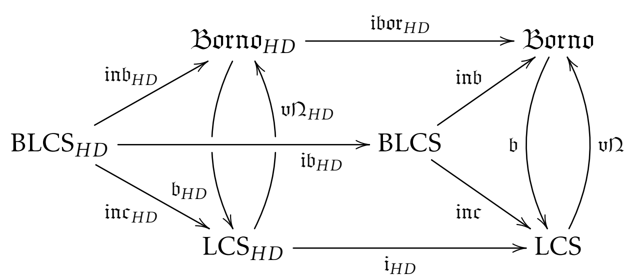
[1] The former is not the terminology of [95], only the latter.

Note that the inclusion functor $\mathfrak{inc}$ (resp., $\mathfrak{inc}_{HD}$) does not preserve limits, or even subobjects, because the subspace topology of vector subspace Y of a bornological LCS X is not necessarily bornological (see [95], Exercise IV.20). We will denote the canonical inclusion functor $\mathrm{BLCS}_{HD} \to \mathrm{BLCS}$ by $\mathfrak{ib}_{HD}$. It is clear that $\mathfrak{inc} \circ \mathfrak{ib}_{HD} = \mathfrak{i}_{HD} \circ \mathfrak{inc}_{HD}$, for the functor $\mathfrak{i}_{HD}$ defined in 1.2.1.

1.4.5. It will be useful to provide another description of bornological LCS (resp., Hausdorff bornological LCS). We recall that a *bornological space of convex type* (resp., *Hausdorff bornological space of convex type*) is a vector space X together with a family $\mathfrak{B}$ of subsets of X that contains all singletons, it is closed under finite unions, multiplication by positive scalars, convex hulls and descending inclusions (resp., and such that X_B is a normed space for all balanced convex $B \in \mathfrak{B}$). See [85], Def. 1.1 (resp., and 4.1). The elements of $\mathfrak{B}$ are called the *bounded* subsets of bornological space of convex type X. The definition of *bounded* (or *bornological*) linear map between bornological spaces of convex type is phrased in the same way as before, *i.e.* it is a linear map sending bounded sets to bounded sets. We denote by $\mathfrak{Borno}$ (resp., $\mathfrak{Borno}_{HD}$) the category of bornological spaces of convex type (resp., Hausdorff bornological spaces of convex type) provided with bounded linear maps, and we denote by $\mathfrak{ibor}_{HD} : \mathfrak{Borno}_{HD} \to \mathfrak{Borno}$ the inclusion functor. The category $\mathfrak{Borno}$ (resp., $\mathfrak{Borno}_{HD}$) is (locally small) quasi-abelian, complete and cocomplete (see [85], Prop. 1.8, 1.9, 4.10, and 4.12), and the inclusion functor $\mathfrak{ibor}_{HD}$ preserves limits and coproducts, but it does not necessarily preserves cokernels (see [85], Prop. 1.2, 1.5, 4.5, and 4.6).

1.4.6. There is a canonical functor $\mathfrak{v}\mathfrak{N} : \mathrm{LCS} \to \mathfrak{Borno}$ (resp., $\mathfrak{v}\mathfrak{N}_{HD} : \mathrm{LCS}_{HD} \to \mathfrak{Borno}_{HD}$) sending a bornological LCS X (resp., Hausdorff bornological LCS) to the same vector space X provided with its family of bounded sets $\mathfrak{B}_X$. We denote the composition of the inclusion functor (resp., $\mathfrak{inc}_{HD}$) $\mathfrak{inc}$ given in 1.4.4 and $\mathfrak{v}\mathfrak{N}$ (resp., $\mathfrak{v}\mathfrak{N}_{HD}$) by $\mathfrak{inb} : \mathrm{BLCS} \to \mathfrak{Borno}$ (resp., $\mathfrak{inb}_{HD} : \mathrm{BLCS}_{HD} \to \mathfrak{Borno}_{HD}$). Note that $\mathfrak{inb}$ (resp., $\mathfrak{inb}_{HD}$) is a fully faithful functor by the definition of bornological LCS. Moreover, define a functor $\mathfrak{b} : \mathfrak{Borno} \to \mathrm{LCS}$ (resp., $\mathfrak{b}_{HD} : \mathfrak{Borno}_{HD} \to \mathrm{LCS}_{HD}$) sending the bornological space of convex type (resp., Hausdorff bornological space of convex type) X provided with the family $\mathfrak{B}$ to the same vector space X provided with the locally convex topology generated by the basis of neighborhoods of zero formed by the balanced convex subsets U of X that absorb all elements $B \in \mathfrak{B}$.

1.4.7 Lemma. *The functors defined in 1.2.1, 1.4.4, 1.4.5, and 1.4.6 give the following commutative diagram* Well-known



(1.4.1)

where the functors that are not vertical are fully faithful, and the upward vertical functors are left adjoint to the corresponding downward vertical ones. Furthermore, $\mathfrak{b} \circ \mathfrak{v}\mathfrak{N}$ *(resp.,* $\mathfrak{b}_{HD} \circ \mathfrak{v}\mathfrak{N}_{HD}$*) factors through* $\mathfrak{inc}$ *(resp.,* $\mathfrak{inc}_{HD}$*), the functor it determines is precisely* $X \mapsto X_{\mathrm{born}}$*, which is the right adjoint of* $\mathfrak{inc}$ *(resp.,* $\mathfrak{inc}_{HD}$*). Analogously,* $\mathfrak{v}\mathfrak{N} \circ \mathfrak{b}$ *(resp.,* $\mathfrak{v}\mathfrak{N}_{HD} \circ \mathfrak{b}_{HD}$*) factors through* $\mathfrak{inb}$ *(resp.,* $\mathfrak{inb}_{HD}$*), and the functor it determines is the left adjoint of* $\mathfrak{inb}$ *(resp.,* $\mathfrak{inb}_{HD}$*).*

Proof. The only remaining identities to check are given by $\mathfrak{b} \circ \mathfrak{inb} = \mathfrak{inc}$ and $\mathfrak{b}_{HD} \circ \mathfrak{inb}_{HD} = \mathfrak{inc}_{HD}$. They are a direct consequence of the fact that every balanced convex subset of a bornological LCS that absorbs every bounded set is a neighborhood of zero (see 1.4.4). The mentioned adjunctions are a direct consequence of the definition of bornological LCS, and the fact that the identity map $X \to \mathfrak{v}\mathfrak{N}(\mathfrak{b}(X))$ (resp., $X \to \mathfrak{v}\mathfrak{N}_{HD}(\mathfrak{b}_{HD}(X))$) is bornological, for all bornological spaces of convex type (resp., Hausdorff bornological spaces of convex type) X, and the identity map $\mathfrak{b}(\mathfrak{v}\mathfrak{N}(Y)) \to Y$ (resp., $\mathfrak{b}_{HD}(\mathfrak{v}\mathfrak{N}_{HD}(Y)) \to Y$) is clearly continuous, for all LCS (resp., Hausdorff LCS) Y. The first part of the last two statement, concerning the bornological LCS associated with a LCS, is immediate. For the last part, note first that, by definition, the family of bounded sets of X_{born} coincides with the family of bounded sets of X, so $\mathfrak{vN} \circ \mathfrak{b} \circ \mathfrak{vN} = \mathfrak{vN}$ (resp., $\mathfrak{vN}_{HD} \circ \mathfrak{b}_{HD} \circ \mathfrak{vN}_{HD} = \mathfrak{vN}_{HD}$). Denote for the moment the functor on BLCS (resp., BLCS_{HD}) sending X to X_{born} by F, and define G as $F \circ \mathfrak{b}$ (resp., $F \circ \mathfrak{b}_{HD}$). We claim that $\mathfrak{vN} \circ \mathfrak{b} = \mathfrak{inb} \circ G$ (resp., $\mathfrak{vN}_{HD} \circ \mathfrak{b}_{HD} = \mathfrak{inb}_{HD} \circ G$). Indeed, from the previous comments we have

$$\mathfrak{inb} \circ G = \mathfrak{inb} \circ F \circ \mathfrak{b} = \mathfrak{vN} \circ \mathfrak{inc} \circ F \circ \mathfrak{b} = \mathfrak{vN} \circ \mathfrak{b} \circ \mathfrak{vN} \circ \mathfrak{b} = \mathfrak{vN} \circ \mathfrak{b},$$

where we have used in the last equality that $\mathfrak{vN} \circ \mathfrak{b} \circ \mathfrak{vN} = \mathfrak{vN}$, and similarly for the Hausdorff case. The fact that G is left adjoint of $\mathfrak{inb}$ (resp., $\mathfrak{inb}_{HD}$) is easily verified. □

Well-known **1.4.8 Corollary.** *The category* BLCS *(resp.,* BLCS_{HD}*) is (locally small) complete and cocomplete, and the inclusion functor* $\mathfrak{inc}$ *(resp.,* $\mathfrak{inc}_{HD}$*) inside of* LCS *(resp.,* LCS_{HD}*) preserves colimits and countable products, whereas the inclusion functor* $\mathfrak{inb}$ *(resp.,* $\mathfrak{inb}_{HD}$*) inside of* $\mathfrak{Borno}$ *(resp.,* $\mathfrak{Borno}_{HD}$*) preserves limits. As a consequence,* $\mathfrak{ib}_{HD}$ *preserves limits and coproducts.*

Proof. We prove the case for colimits, because the other is analogous. Since the inclusion functor $\mathfrak{inc}$ (resp., $\mathfrak{inc}_{HD}$) is a left adjoint it preserves colimits. Given an inductive system $\{X_j\}_{j\in J}$ in BLCS (resp., BLCS_{HD}), the cocompleteness of LCS (resp., LCS_{HD}) tells us that the colimit X of the inductive system $\{\mathfrak{inc}(X_j)\}_{j\in J}$ (resp., $\{\mathfrak{inc}_{HD}(X_j)\}_{j\in J}$) exists, and hence a simple diagrammatic argument shows that X_{born} is the colimit of $\{X_j\}_{j\in J} = \{(\mathfrak{inb}(X_j))_{\text{born}}\}_{j\in J}$ (resp., $\{X_j\}_{j\in J} = \{(\mathfrak{inb}_{HD}(X_j))_{\text{born}}\}_{j\in J}$). As the latter is in BLCS (resp., BLCS_{HD}), its cocompleteness follows. The last part follows from the fact that $\mathfrak{inc}$ (resp., $\mathfrak{inc}_{HD}$) is a left adjoint. The statement concerning countable products follows from [82], Cor. 6.2.11. The corollary is thus proved. □

1.4.9. There is yet another equivalent description of bornological LCS (resp., Hausdorff bornological LCS) in [41], Def. 2.4.2, (resp., Def. 2.4.2 and 2.5.3) under the name of *preconvenient space* (resp., *separated preconvenient space*). For the equivalence between the definitions recalled here and those of the mentioned reference, see [41], Thm. 2.4.3 and Thm. 2.5.2.

Well-known **1.4.10 Lemma.** *Let X be a pseudometrizable LCS. Then it is bornological.*

For a proof, see [95], II.8.1, which applies *verbatim*.

1.4.11. Let us first recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hausdorff LCS X is called *locally convergent* to $x \in X$ (resp., *locally Cauchy*) if there is a bounded balanced convex set $B \subseteq X$ such that $x \in X_B$, $\{x_n\}_{n\in\mathbb{N}}$ is included in X_B and it converges to x in (X_B, μ_B) (resp., $\{x_n\}_{n\in\mathbb{N}}$ is included in X_B and it is a Cauchy sequence in (X_B, μ_B)) (see [82], Def. 5.1.1), where (X_B, μ_B) is the seminormed space recalled in 1.4.2. The point $x \in X$ is called the *local limit* of the locally convergent sequence $\{x_n\}_{n\in\mathbb{N}}$. It is easy to see that any locally convergent (resp., locally Cauchy) sequence is convergent (resp., Cauchy) (see [82], Prop. 5.1.3). A Hausdorff locally convex space X is said to satisfy the *Mackey convergence condition* (resp., *strict Mackey condition*), or *M.c.c.* (resp., *s.M.c.*) for short, if every convergent sequence in X is locally convergent (resp., for every bounded set $A \subseteq X$ there is a bounded balanced closed convex set $B \subseteq X$ including A such that the topology on A induced by X coincides with the topology induced by X_B). It is clear that if X satisfies the s.M.c., then the M.c.c. also holds, and that any metrizable LCS satisfies the s.M.c. (see [82], Obs. 5.1.30). The M.c.c. (resp., s.M.c.) is stable under taking subspaces, and forming countable products and countable coproducts (see [82], Prop. 5.1.31), so in particular under countable limits. A subset $A \subseteq X$ is called *locally closed* if every local limit of a locally convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in A belongs to A, and the *local closure* of a subset A of X is defined as the intersection of all the locally closed subsets of X including A (see [82], Def. 5.1.14 and 5.1.18). The local closure of A clearly includes all the local limits of locally convergent sequences included in A, but the converse does not hold in general (see for instance [41], Example 6.3.1). We now recall that a Hausdorff locally convex space X is called *Mackey complete* (or *locally complete*) if each locally Cauchy sequence is locally convergent (see [82], Def. 5.1.5). Equivalently, X is locally complete if and only if given any bounded balanced convex set $B \subseteq X$, there is another bounded balanced convex set $B' \subseteq X$ such that $B \subseteq B'$ and the normed space $(X_{B'}, \mu_{B'})$ recalled in 1.4.2 is a Banach space. For the equivalence between both definitions see [82], Prop. 5.1.6 (or [41], Thm. 2.6.2, for other equivalent descriptions). The last definition immediately tells us that X is locally complete if and only if X_{born} is so. For this reason, define a *convenient* LCS to be a bornological locally complete LCS (see [41], Def. 2.6.3). Denote by CLCS_{HD} the category of convenient LCS provided with continuous (or, equivalently, bounded) linear maps. It is a full subcategory of BLCS_{HD} via the inclusion functor $\mathfrak{ib}_c : \mathrm{CLCS}_{HD} \to \mathrm{BLCS}_{HD}$. Furthermore, for any LCS X there is a locally complete LCS $\tilde{X}$ and a continuous linear map $i_X : X \to \tilde{X}$ such that, given any continuous linear map $f : X \to Y$ into a locally complete LCS Y, there exists a unique continuous linear map $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f} \circ i_X$. We call $(\tilde{X}, i_X)$ (or just $\tilde{X}$, for simplicity) the *local completion* of the LCS X. In more concrete terms, if $X \to \hat{X}$ is the usual completion of X, then $\tilde{X}$ can be constructed as the local closure of the image of X in $\hat{X}$ under the previous map, and $i_X : X \to \tilde{X}$ is the corestriction of $X \to \hat{X}$ (see [82], Prop. 5.1.21). As the reader could have noticed, we shall denote without any harm both maps $X \to \hat{X}$ and $X \to \tilde{X}$ by i_X. For the fact that the previous construction fulfills the universal property of the local completion mentioned previously, see [82], Prop. 5.1.25. Moreover, the local completion $\tilde{X}$ of a bornological LCS X is also bornological (see [82], Prop. 6.2.8). This implies that the inclusion functor $\mathfrak{ib}_{HD} \circ \mathfrak{ib}_c$ has as left adjoint the functor $X \mapsto \tilde{X}$, which is called in this case the *convenient completion* (*cf.* [41], Thm. 2.6.5). Hence, given a bornological LCS X and a convenient LCS Y, we have the isomorphism

$$\mathfrak{Hom}(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}(X, Y) \tag{1.4.2}$$

of vector spaces given by $\phi \mapsto \phi \circ i_X$, where $i_X : X \to \tilde{X}$ is the canonical map.

1.4.12 Fact. *Let X be a Hausdorff locally convex space, and let $(\hat{X}, i_X)$ be its completion. Assume that $i_X(X)$ is sequentially dense in $\hat{X}$ (i.e., every point of $\hat{X}$ is the limit of a convergent sequence in $i_X(X)$) and that $\hat{X}$ satisfies the M.c.c. Then, the local completion $\tilde{X}$ coincides with the completion $\hat{X}$.* *Well-known*

1.4.13.*Remark.* The main reason for using the local (or even the convenient) completion instead of the usual one for LCS is the fact that the completion of a bornological LCS is not necessarily bornological (see [106]). This can be regarded as somehow pathological because in all our situations of interest the bornological LCS would be given by a finite bornological tensor product of either spaces of sections (sometimes of compact support) of vector bundles, or of duals of them, for which the convenient completion coincides with the usual completion (see Propositions 2.3.13 and 4.2.2). However, the convenient completion fits better in the categorical picture presented here, justifying its treatment.

1.4.14. One says that a Hausdorff bornological space of convex type is *bornologically complete* if it fulfills precisely the second statement of the definition for the local completeness given in 1.4.11. This is precisely what is called a *complete* (convex) bornological space in [49], Ch. III, or a *complete* bornological space of convex type in [85], Section 5. However, we will not follow this terminology in order to avoid any confusion with the usual notion of completeness of LCS. Bornologically complete bornological spaces of convex type form a full subcategory $\mathfrak{Borno}^{bc}_{HD}$ of $\mathfrak{Borno}_{HD}$ that is (locally small) quasi-abelian, complete and cocomplete (see [85], Prop. 5.6). Denote the inclusion functor of $\mathfrak{Borno}^{bc}_{HD}$ inside of $\mathfrak{Borno}_{HD}$ by $\mathfrak{ibor}_c$. It preserves limits and colimits (see [85], Prop. 5.5 and 5.6). Furthermore, the inclusion functor $\mathfrak{ibor}_{HD} \circ \mathfrak{ibor}_c$ has a left adjoint, that we denote by $X \mapsto \widehat{X}$, and that it is called the *bornological completion* (see [85], Def. 5.10 and Prop. 5.11). In the previous reference, it is also showed that the bornological completion $\widehat{X}$ of a bornologically complete bornological space of convex type X coincides with X, and $\widehat{X}$ is given by the inductive limit in $\mathfrak{Borno}_{HD}$ of the system $\{\mathfrak{v}\Omega_{HD}(\widehat{X_B})\}_B$, where B runs over all bounded balanced convex subsets of X, $\widehat{X_B}$ denotes the Banach space defined as the usual completion of the seminormed vector space X_B, and the morphisms in the system are the image under $\mathfrak{v}\Omega_{HD}$ of the completion of those described in 1.4.2 for $\{X_B\}_B$. As a consequence,

$$\mathfrak{b}_{HD}(\widehat{X}) = \mathfrak{b}_{HD}\Big(\operatorname{colim} \mathfrak{v}\Omega_{HD}(\widehat{X_B})\Big) \simeq \operatorname{colim} \mathfrak{b}_{HD}\Big(\mathfrak{v}\Omega_{HD}(\widehat{X_B})\Big) = \operatorname{colim}(\widehat{X_B})_{\text{born}} = \operatorname{colim} \widehat{X_B},$$

where we have used that $\mathfrak{b}_{HD}$ preserves colimits (by Lemma 1.4.7), that $\mathfrak{b}_{HD}(\mathfrak{v}\Omega_{HD}(X)) = X_{\text{born}}$ for any Hausdorff LCS, and that any Banach space is bornological (see Lemma 1.4.10).

1.4.15. By definition, it is immediate that the image of $\mathfrak{inb}_{HD} \circ \mathfrak{ib}_c$ is included in $\mathfrak{Borno}^{bc}_{HD}$. Let us denote the induced functor by $\mathfrak{inb}_c : \mathrm{CLCS}_{HD} \to \mathfrak{Borno}^{bc}_{HD}$.

Well-known **1.4.16 Fact.** *The category* CLCS_{HD} *is (locally small) complete and cocomplete. Furthermore, the inclusion functor* $\mathfrak{ib}_c : \mathrm{CLCS}_{HD} \to \mathrm{BLCS}_{HD}$ *preserves coproducts and limits, whereas the functor* $\mathfrak{inb}_c : \mathrm{CLCS}_{HD} \to \mathfrak{Borno}^{bc}_{HD}$ *preserves limits.*

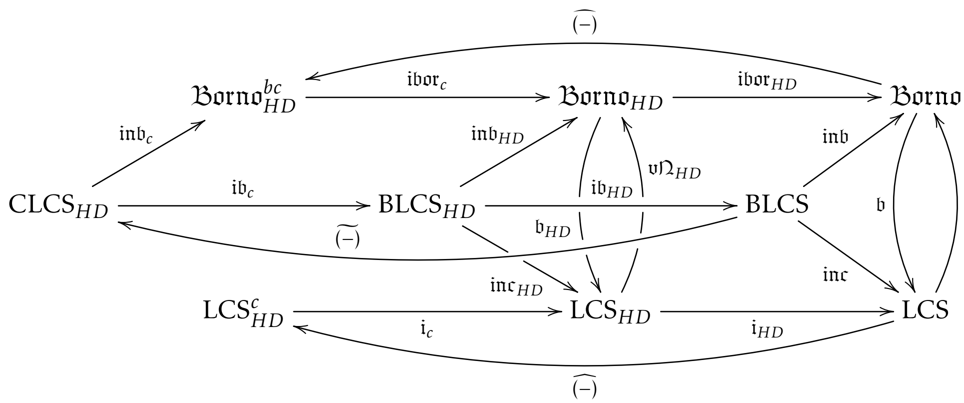
Proof. For the statement concerning the completeness and the fact that $\mathfrak{ib}_c$ preserves limits, we proceed as follows. By definition, a Hausdorff bornological LCS Y is convenient if and only if $\mathfrak{inb}_{HD}(Y)$ is bornologically complete. Given a projective family $\{X_j\}_{j\in J}$ in CLCS_{HD}, the limit X of $\{\mathfrak{ib}_c(X_j)\}_{j\in J}$ in BLCS_{HD} exists, due to the completeness of the latter category (see Corollary 1.4.8). The fact that $\mathfrak{inb}_{HD}$ preserves limits tells us that $\mathfrak{inb}_{HD}(X)$ is the limit of $\{\mathfrak{inb}_{HD}(\mathfrak{ib}_c(X_j))\}_{j\in J}$ in $\mathfrak{Borno}_{HD}$. Since $\mathfrak{inb}_{HD}(\mathfrak{ib}_c(X_j))$ is bornologically complete and $\mathfrak{ibor}_c$ preserves limits, we see that $\mathfrak{inb}_{HD}(X)$ is bornologically complete, which in turn implies that X is convenient, and the limit of $\{X_j\}_{j\in J}$. Moreover, $\mathfrak{ib}_c$ and $\mathfrak{ibor}_c$ preserve limits.

Let us now prove the statement concerning the cocompleteness. Given an inductive family $\{X_j\}_{j\in J}$, the cocompleteness of BLCS_{HD} tells us that $X = \operatorname{colim} \mathfrak{ib}_c(X_j)$ exists, and its convenient completion $\tilde{X}$ belongs to CLCS_{HD}. It is trivially verified that $\tilde{X}$ is the colimit of the family $\{X_j\}_{j\in J}$ in CLCS_{HD}. Finally, it remains to prove that $\mathfrak{ib}_c$ preserves coproducts. Taking into account that the coproduct in CLCS_{HD} is given as the convenient completion of the coproduct in BLCS_{HD}, and the functor $\mathfrak{inc}_{HD}$ in 1.4.4 preserves colimits, it suffices to shows that a coproduct of convenient LCS (taken in LCS_{HD}) is convenient. As we already know that a coproduct of bornological LCS is bornological, it suffices to show that the coproduct in LCS_{HD} of locally complete LCS is locally complete. This follows easily from the definition, taking into account that the bounded subsets of a coproduct are precisely of the form described in [85], Prop. 1.2.(a). □

Well-known **1.4.17 Fact.** *The coproduct of a family of objects in the category* BLCS *is convenient if and only if each member of the family is so.*

Proof. The if part was proved in Fact 1.4.16. For the converse, given a coproduct $\oplus_{j\in J} X_j$ of objects in BLCS that is convenient, and a fixed index $j_0 \in J$, take the cokernel of the continuous linear map $\oplus_{j\in J} X_j \to \oplus_{j\in J} X_j$ induced by the the canonical inclusion $X_{j'} \to \oplus_{j\in J} X_j$ for all $j' \in J \setminus \{j_0\}$ and the zero morphism $X_{j_0} \to \oplus_{j\in J} X_j$. This is precisely X_{j_0}, so the result follows. □

1.4.18. We may now complete the information conveyed in Lemma 1.4.7 and Corollary 1.4.8 by including in the following diagram the different complete and cocomplete categories we have recalled previously, together with the results stated in 1.4.14 and Fact 1.4.16,



(1.4.3)

where all the subdiagrams not involving the different completion functors $\widehat{(-)}$, $\widetilde{(-)}$ or $\overparen{(-)}$ are commutative, and all the functors that are not vertical and are not the previously mentioned completion functors are fully faithful. Moreover, all the horizontal functors preserve limits and coproducts, and $\mathfrak{ibor}_c$ also preserves colimits. We also recall that the downward vertical functors are left adjoint to the corresponding upward vertical functors, and that the upward sloping functors preserve limits, whereas the downward sloping functors preserve colimits and countable products.

1.4.19 Lemma. *Let X be a pseudometrizable (so bornological, by Lemma 1.4.10) LCS. Then, its convenient completion (or, equivalently, its local completion) $\tilde{X}$ is metrizable.* *Well-known*

Proof. By the construction of the local completion in [82], Def. 5.1.21, there is a canonical injective continuous map from the convenient completion of X to its usual completion. Furthermore, taking into account that the completion of a pseudometrizable LCS is metrizable, we conclude that $\tilde{X}$ has a topology finer than that of its image inside of $\hat{X}$, which is metrizable. Hence $\tilde{X}$ is metrizable, and the lemma follows. □

1.4.20 Lemma. *Let X be a pseudometrizable (so bornological, by Lemma 1.4.10) LCS. Then, X is convenient (or equivalently, locally complete) if and only if it is complete.* *Well-known*

For a proof, see [82], Prop. 5.1.4 and Obs. 5.1.22.

1.4.21 Corollary. *Let X be a pseudometrizable (so bornological, by Lemma 1.4.10) LCS. Then, its convenient completion (or, equivalently, its local completion) $\tilde{X}$ is isomorphic to its usual completion $\hat{X}$.* *Well-known*

This follows from Lemmas 1.4.19 and 1.4.20.

1.4.22. By Corollary 1.4.21 and [82], Prop. 5.1.25, we see that that a Hausdorff LCS X is locally complete if and only if given any continuous linear map $f : Y \to X$ from a normed space Y, there exists a continuous linear extension $\hat{f} : \hat{Y} \to X$ from the completion $\hat{Y}$ of Y, *i.e.* $\hat{f} \circ i_Y = f$, where $i_Y : Y \to \hat{Y}$ is the canonical map. Indeed, the only if part is precisely [82], Cor. 5.1.26, whereas the converse follows directly from taking $Y = X_B$, for any B bounded balanced convex subset of X. In particular, we see that any complete LCS is *a fortiori* locally complete.

1.4.23. Given two LCS X and Y, one defines the *bornological (projective) tensor product* $X \otimes_\beta Y$ as the LCS whose underlying vector space is the algebraic tensor product $X \otimes Y$ provided with the finest locally convex topology such that the canonical projection $X \times Y \to X \otimes Y$ is bounded, where $X \times Y$ has the usual product topology. Equivalently, $X \otimes_\beta Y$ is a representation of the covariant functor LCS $\to {}_k$Mod sending Z to the space $\mathfrak{B}_b(X, Y; Z)$ of bounded bilinear maps $X \times Y \to Z$. It is clear that $X \otimes_\beta Y$ is bornological (see [64], I.5.7). Moreover, it is easy to prove that the bornological tensor product $X \otimes_\beta Y$ is Hausdorff if both X and Y are so (see [41], Prop. 3.8.3), and it satisfies the usual associativity relation $(X \otimes_\beta Y) \otimes_\beta Z \simeq X \otimes_\beta (Y \otimes_\beta Z)$, the unit constraints $k \otimes_\beta X \simeq X \simeq X \otimes_\beta k$, and the commutativity $X \otimes_\beta Y \simeq Y \otimes_\beta X$, for all LCS X, Y, Z, and where the previous isomorphisms are only bornological (see [64], Thm. I.5.7). We remark that the topology on the bornological tensor product $X \otimes_\beta Y$ is in general finer than the usual projective tensor product $X \otimes_\pi Y$, *i.e.* the identity map

$$\mathfrak{c}_{X,Y} : X \otimes_\beta Y \to X \otimes_\pi Y \tag{1.4.4}$$

is continuous. It is further an isomorphism of LCS if both X and Y are pseudometrizable (see [64], Prop. I.5.8).

1.4.24. Given two bornological LCS X and Y, we will endow the space of continuous morphisms $\mathfrak{Hom}(X, Y)$ from X to Y with the finest locally convex topology whose bounded sets are the *equibounded sets of linear maps*, *i.e.* the subsets $\mathscr{F}$ of $\mathfrak{Hom}(X, Y)$ satisfying that for every bounded subset $B \subseteq X$, $\cup_{f \in \mathscr{F}} f(B)$ is a bounded subset of Y. This bornological LCS will be denoted by $\mathfrak{Hom}^{\mathfrak{b}}(X, Y)$. It is easy to prove that BLCS (resp., BLCS_{HD}) is a *closed* symmetric monoidal category, or, equivalently, there is an *internal homomorphism space* $\mathfrak{Hom}^{\mathfrak{b}}(-,-)$ for the symmetric monoidal category BLCS (resp., BLCS_{HD}), *i.e.* we have natural k-linear isomorphisms

$$\mathfrak{Hom}\Big(Y, \mathfrak{Hom}^{\mathfrak{b}}(X, Z)\Big) \xleftarrow{\sim} \mathfrak{Hom}(X \otimes_\beta Y, Z) \xrightarrow{\sim} \mathfrak{Hom}\Big(X, \mathfrak{Hom}^{\mathfrak{b}}(Y, Z)\Big), \tag{1.4.5}$$

for all objects X, Y, and Z in BLCS (resp., BLCS_{HD}), where the left map takes f to the morphism sending $y \in Y$ to the mapping $x \mapsto f(x \otimes y)$, and the right map takes g to the morphism sending $x \in X$ to the mapping $y \mapsto f(x \otimes y)$ (see [41], Prop. 3.8.1 and 3.8.3, and the comments after). By a general categorical argument, the isomorphisms in (1.4.5) are even of bornological LCS if all the homomorphisms spaces are internal. Indeed, the proof of the isomorphism between the internal versions of the first two terms in (1.4.5) follows from the chain of isomorphisms of vector spaces given by

$$\mathfrak{Hom}\Big(W, \mathfrak{Hom}^{\mathrm{b}}\big(Y, \mathfrak{Hom}^{\mathrm{b}}(X,Z)\big)\Big) \xleftarrow{\sim} \mathfrak{Hom}\big(Y \otimes_\beta W, \mathfrak{Hom}^{\mathrm{b}}(X,Z)\big) \xleftarrow{\sim} \mathfrak{Hom}\big(X \otimes_\beta (Y \otimes_\beta W), Z\big)$$
$$\xrightarrow{\sim} \mathfrak{Hom}\big((X \otimes_\beta Y) \otimes_\beta W, Z\big) \xrightarrow{\sim} \mathfrak{Hom}\big(W, \mathfrak{Hom}^{\mathrm{b}}(X \otimes_\beta Y, Z)\big), \tag{1.4.6}$$

for all W in BLCS (resp., BLCS_{HD}). By the adjunction between the bornological tensor functor $\otimes_\beta$ and the internal homomorphism $\mathfrak{Hom}^{\mathrm{b}}(-,-)$, we obtain the isomorphisms in BLCS (resp., BLCS_{HD}) of the form

$$\operatorname*{colim}_{j\in J}\big(X \otimes_\beta Y_j\big) \simeq X \otimes_\beta \Big(\operatorname*{colim}_{j\in J} Y_j\Big), \tag{1.4.7}$$

for any bornological LCS (resp., Hausdorff bornological LCS) X and any system $\{Y_j : j \in J\}$ of bornological LCS (resp., Hausdorff bornological LCS) and bounded (or continuous) linear maps, where the previous colimits are taken in the category BLCS (resp., BLCS_{HD}) of bornological LCS (resp., Hausdorff bornological LCS). This shows that the category BLCS (resp., BLCS_{HD}) provided with the tensor product $\otimes_\beta$, the unit k and the usual flip is a symmetric monoidal category, such that the tensor product commutes with colimits in that category. Moreover, the same adjunction gives the isomorphisms

$$\mathfrak{Hom}^{\mathrm{b}}(\operatorname{colim} X_j, Y) \simeq \lim \mathfrak{Hom}^{\mathrm{b}}(X_j, Y) \quad \text{and} \quad \mathfrak{Hom}^{\mathrm{b}}(X, \lim Y_j) \simeq \lim \mathfrak{Hom}^{\mathrm{b}}(X, Y_j) \tag{1.4.8}$$

in BLCS (resp., BLCS_{HD}), for all bornological LCS (resp., Hausdorff bornological LCS) X and Y, and all systems $\{X_j\}_{j\in J}$ and $\{Y_j\}_{j\in J}$ of bornological LCS (resp., Hausdorff bornological LCS) (*cf.* [41], Prop. 3.8.5).

1.4.25. Note that the LCS $\mathfrak{Hom}^{\mathrm{b}}(X,Y)$ does not necessarily coincide with the LCS $\mathfrak{Hom}_b(X,Y)$ given by the vector space $\mathfrak{Hom}(X,Y)$ endowed with the topology of uniform convergence on bounded sets, *i.e.* the locally convex topology on the space $\mathfrak{Hom}(X,Y)$ generated by the local base formed by the sets $\mathscr{F}_{B,U} = \{f \in \mathfrak{Hom}(X,Y) : f(B) \subseteq U\}$, for all $B \subseteq X$ bounded and U neighborhood of zero in Y (see [95], III.3.1-2, for the definition). The topology on $\mathfrak{Hom}_b(X,Y)$ in case $Y = k$ is called the *strong topology* on the space of continuous linear functionals on X (see [95], IV.5), and we shall denote it by X'_b, in order to distinguish it from $X' = \mathfrak{Hom}^{\mathrm{b}}(X,k)$. To avoid any possible ambiguity on the topology considered on the space of continuous linear functionals of a bornological LCS X (and in particular to distinguish it from the strong topology), we will call the former topology *bornologically strong*. For an example when X'_b is not bornological, so it does not coincide with $X' = \mathfrak{Hom}^{\mathrm{b}}(X,k)$, see [95], Exercise IV.13.

1.4.26. By definition of the topology of uniform convergence on bounded sets of $\mathfrak{Hom}(X,Y)$, we see that its bounded sets are precisely the equibounded subsets of the vector space of continuous linear maps from X to Y (see [95], III.3.3), *i.e.* the bounded sets of $\mathfrak{Hom}^{\mathrm{b}}(X,Y)$ and of $\mathfrak{Hom}_b(X,Y)$ coincide. As a consequence, $\mathfrak{Hom}^{\mathrm{b}}(X,Y)$ is the bornological space associated with the LCS $\mathfrak{Hom}_b(X,Y)$, *i.e.* $\mathfrak{Hom}^{\mathrm{b}}(X,Y) = (\mathfrak{Hom}_b(X,Y))_{\mathrm{born}}$, and, in particular, $X' = (X'_b)_{\mathrm{born}}$. Under the assumption that X is metrizable, X'_b is bornological if and only if it is barreled, and in particular this holds for any normable space X, for any reflexive Fréchet LCS X or for every metrizable LCS X whose strong dual is separable (see [95], IV.6.6). In all these situations we have thus $X' = X'_b$, so we do not have to distinguish the two. We recall that, if X and Y are two complete LCS such that X is nuclear, then the canonical map Given any nuclear complete LCS X and any complete LCS Y, then the canonical map

$$\delta'_{X,Y} : X \hat{\otimes}_\pi Y \xrightarrow{\sim} \mathfrak{Hom}_e(Y'_\tau, X) \tag{1.4.9}$$

induced by sending $x \otimes y$ to the continuous linear map $\delta'_{X,Y}(x\otimes y)(f) = f(y)x$, for all $f \in Y'$, $x \in X$ and $y \in Y$, is an isomorphism of LCS (see [104], Prop. 50.4), where Y'_τ denotes the continuous dual of Y provided with the *Mackey topology*, *i.e.* the one given by uniform convergence on the convex balanced and weakly compact sets of Y, and $\mathfrak{Hom}_e(Y',X)$ denotes the space $\mathfrak{Hom}_e(Y',X)$ provided with the topology of uniform convergence on the equicontinuous subsets of Y' (see [104], Prop. 42.2 and the paragraph before it). Moreover, is if X and Y are two complete LCS such that X is barreled and X'_b is nuclear and complete, then the canonical map

$$\mathfrak{d}_{X,Y} : X'_b \hat{\otimes}_\pi Y \xrightarrow{\sim} \mathfrak{Hom}_b(X,Y) \tag{1.4.10}$$

induced by sending $\lambda \otimes y$ to $x \mapsto \lambda(x)y$, for all $x \in X$, $y \in Y$ and $\lambda \in X'_b$, is an isomorphism of LCS (see [104], Prop. 50.5).

1.4.27 Lemma. *Let $X = \operatorname{colim} X_m$ be an* (LF)-space, *i.e. a strict inductive limit of a sequence of Fréchet LCS $\{X_m\}_{m\in\mathbb{N}}$ where $X_m \to X_{m+1}$ is an embedding of LCS (see* [95], *II.6.3). If, for all $m \in \mathbb{N}$, X_m is* distinguished, *i.e. the strong dual $(X_m)'_b$ is barreled, then X'_b is barreled and bornological. In particular, X'_b coincides with X'.* Well-known

Proof. For the first part, see [57], Thm. 3.16.2. The last part follows from the first one and the comments in 1.4.26. □

1.4.28. Finally, given two LCS X and Y, the *convenient (projective) tensor product* $X\tilde{\otimes}_\beta Y$ is defined as the convenient completion of the bornological tensor product $X \otimes_\beta Y$. By taking the convenient completion of the identities satisfied by the bornological tensor product, we conclude that the category CLCS_{HD} provided with the tensor product $\tilde{\otimes}_\beta$, the unit k and the convenient completion of the usual flip is a symmetric monoidal category.

1.4.29. Let X be a bornological LCS X and Y be a convenient LCS Y. Then, the internal homomorphism space $\mathfrak{Hom}^{\mathrm{b}}(X, \mathfrak{ib}_c(Y))$ of BLCS is also convenient (see [41], Prop. 3.6.3). Hence, CLCS_{HD} is a closed symmetric monoidal category for the same internal homomorphism space as in BLCS, *i.e.* we have natural k-linear isomorphisms

$$\mathfrak{Hom}\Big(Y, \mathfrak{Hom}^{\mathrm{b}}(X,Z)\Big) \xleftarrow{\sim} \mathfrak{Hom}(X\tilde{\otimes}_\beta Y, Z) \xrightarrow{\sim} \mathfrak{Hom}\Big(X, \mathfrak{Hom}^{\mathrm{b}}(Y,Z)\Big), \tag{1.4.11}$$

for all objects X, Y, and Z in CLCS_{HD}, with the same morphisms as in (1.4.5) (see [41], Thm. 3.8.4). Using the same proof as in (1.4.6), the isomorphisms in (1.4.11) are even of convenient LCS if all the homomorphisms spaces are internal. As in the case of bornological LCS described 1.4.24, the adjunction between the convenient tensor functor $\tilde{\otimes}_\beta$ and the internal homomorphism $\mathfrak{Hom}^{\mathrm{b}}(-,-)$ gives us the isomorphisms

$$\operatorname*{colim}_{j\in J}\Big(X\tilde{\otimes}_\beta Y_j\Big) \simeq X\tilde{\otimes}_\beta\Big(\operatorname*{colim}_{j\in J} Y_j\Big), \tag{1.4.12}$$

as well as

$$\mathfrak{Hom}^{\mathrm{b}}(\operatorname{colim} X_j, Y) \simeq \lim \mathfrak{Hom}^{\mathrm{b}}(X_j, Y) \quad \text{and} \quad \mathfrak{Hom}^{\mathrm{b}}(X, \lim Y_j) \simeq \lim \mathfrak{Hom}^{\mathrm{b}}(X, Y_j) \tag{1.4.13}$$

of convenient LCS, for all convenient LCS X and Y, and all systems $\{X_j\}_{j\in J}$ and $\{Y_j\}_{j\in J}$ of convenient LCS, where the colimits and limits are computed in CLCS_{HD}.

1.4.30. Since the convenient completion functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ preserves colimits, which in particular means that the colimits in CLCS_{HD} are given as the convenient completion of the colimits computed in BLCS, by taking the convenient completion of (1.4.7) we see that the convenient tensor product also commutes with colimits of that category, *i.e.*

$$\begin{gathered}\mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{j\in J}\Big(X\tilde{\otimes}_\beta Y_j\Big) \simeq X\tilde{\otimes}_\beta\Big(\mathrm{BLCS}\text{-}\operatorname*{colim}_{j\in J} Y_j\Big)\\ \Big(\text{resp., } \mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{j\in J}\Big(X\tilde{\otimes}_\beta Y_j\Big) \simeq X\tilde{\otimes}_\beta\Big(\mathrm{BLCS}_{HD}\text{-}\operatorname*{colim}_{j\in J} Y_j\Big)\Big),\end{gathered} \tag{1.4.14}$$

for any bornological LCS (resp., Hausdorff bornological LCS) X and any system $\{Y_j : j \in J\}$ of bornological LCS (resp., Hausdorff bornological LCS).

1.4.31 Lemma. *Let X and Y be two bornological LCS. Then, $X\tilde{\otimes}_\beta Y \simeq \tilde{X}\tilde{\otimes}_\beta \tilde{Y}$. In other words, the functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ is strong monoidal.* Probably well-known

Proof. As X and Y are bornological, they are the colimit in LCS of $\{X_B\}_B$ and $\{Y_B\}_{B'}$, where B and B' run over all bounded balanced convex subsets of X and Y, respectively (see 1.4.4). Since the functor $\mathfrak{inc}$ preserves colimits, then one can equivalently take the previous colimits in BLCS. Hence,

$$
\begin{aligned}
X\tilde{\otimes}_\beta Y &\simeq \widetilde{\Big(\mathrm{BLCS}\text{-}\operatorname*{colim}_B X_B\Big)\otimes_\beta\Big(\mathrm{BLCS}\text{-}\operatorname*{colim}_{B'} Y_{B'}\Big)} \simeq \widetilde{\mathrm{BLCS}\text{-}\operatorname*{colim}_{B,B'}\Big(X_B\otimes_\beta Y_{B'}\Big)} \simeq \mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{B,B'}\widetilde{\Big(X_B\otimes_\beta Y_{B'}\Big)} \\
&\simeq \mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{B,B'}\widehat{\Big(X_B\otimes_\pi Y_{B'}\Big)} \simeq \mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{B,B'}\Big(\widehat{X}_B\hat{\otimes}_\pi\widehat{Y}_{B'}\Big) \simeq \mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{B,B'}\Big(\widetilde{X}_B\tilde{\otimes}_\beta\widetilde{Y}_{B'}\Big) \\
&\simeq \Big(\mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_B\widetilde{X}_B\Big)\tilde{\otimes}_\beta\Big(\mathrm{CLCS}_{HD}\text{-}\operatorname*{colim}_{B'}\widetilde{Y}_{B'}\Big) \simeq \widetilde{\Big(\mathrm{BLCS}\text{-}\operatorname*{colim}_B X_B\Big)}\tilde{\otimes}_\beta\widetilde{\Big(\mathrm{BLCS}\text{-}\operatorname*{colim}_{B'} Y_{B'}\Big)} = \tilde{X}\tilde{\otimes}_\beta\tilde{Y},
\end{aligned}
$$

where we have used that the bornological tensor products commutes with colimits on each side in the second isomorphism, that the convenient completion functor preserves colimits in the third and penultimate isomorphisms, the fact that (1.4.4) is an isomorphism for pseudometrizable spaces and Corollary 1.4.21 in the fourth and sixth isomorphisms, the easy identity (1.2.1) for any pair of LCS X and Y in the fifth isomorphism –which means that the functor $\widehat{(-)} : \mathrm{LCS} \to \mathrm{LCS}^c_{HD}$ is strong monoidal–, and that the convenient tensor product commutes with colimits on each side in the seventh isomorphism. Hence, the functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ is strong monoidal, and the lemma is proved. □

New Expected **1.4.32 Corollary.** *The isomorphism in* (1.4.2) *is in fact of convenient LCS, i.e. given any bornological LCS X and any convenient convenient Y,* (1.4.2) *induces an isomorphism of the form*

$$\mathfrak{Hom}^\flat(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}^\flat(X, Y). \tag{1.4.15}$$

Proof. This is a direct consequence of the fact that the functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ is strong monoidal (see Lemma 1.4.31) and Lemma 1.3.7. □

Unknown? New? **1.4.33 Lemma.** *The inclusion functors* $\mathfrak{inc}$ *and* $\mathfrak{inc}_{HD}$ *in diagram* (1.4.3) *are oplax symmetric monoidal, whereas* $\mathfrak{i}_c$ *and* $\mathfrak{ib}_c$ *are lax symmetric monoidal, and the inclusion functors* $\mathfrak{i}_{HD}$ *and* $\mathfrak{ib}_{HD}$ *are strong symmetric monoidal. Moreover, the completion functors* $\widehat{(-)}$ *and* $\widetilde{(-)}$ *are strong symmetric monoidal. The structure map* φ_0 *is the identity of k in all cases, whereas the structure map* $\varphi_2(X,Y)$ *is given by* (1.4.4) *for the first two functors, by the canonical map from $X\otimes_\pi Y$ to $X\hat{\otimes}_\pi Y$ for the third, by the canonical map from $X\otimes_\beta Y$ to $X\tilde{\otimes}_\beta Y$ for the fourth, by the identity for the fifth and sixth functors, and by a canonical isomorphism for the last two (see Lemma 1.4.31 and* (1.2.1)*).*

This is easy to verify.

1.4.34. Since the completed projective tensor product $X\hat{\otimes}_\pi Y$ is a complete LCS, it is locally complete by 1.4.22, so the continuous (and thus bounded) linear map $X\otimes_\beta Y \to X\hat{\otimes}_\pi Y$ defined as the composition of the continuous linear map (1.4.4) and the canonical map $i_{X\otimes_\pi Y}$ from the projective tensor product to its completion induces in turn a continuous linear map

$$\tilde{\mathfrak{c}}_{X,Y} : X\tilde{\otimes}_\beta Y \to X\hat{\otimes}_\pi Y. \tag{1.4.16}$$

By Corollary 1.4.21 and 1.4.23, it is an isomorphism if both X and Y are pseudometrizable.

1.4.35. We recall that, given two LCS X and Y their *inductive tensor product* $X\otimes_i Y$ is the vector space $X\otimes Y$ provided with the finest locally convex topology for which the canonical bilinear map $X\times Y \to X\otimes Y$ is separately continuous (see [95], III.6.5). In other words, $X\otimes_i Y$ gives a representation of the covariant functor from LCS to ${}_k\mathrm{Mod}$ sending Z to the vector space $\mathfrak{B}(X,Y;Z)$ of separately continuous bilinear maps $X\times Y \to Z$. It is clear that $X\otimes_i Y$ is Hausdorff if X and Y are so. Its completion $X\hat{\otimes}_i Y$ is called the *completed inductive tensor product*. It is clear that $X\hat{\otimes}_i Y$ is a representation of the covariant functor from LCS^c_{HD} to ${}_k\mathrm{Mod}$ sending Z to the vector space $\mathfrak{B}(X,Y;Z)$ of separately continuous bilinear maps $X\times Y \to Z$.

Suppose X and Y are bornological. Then, $\mathfrak{B}_b(X,Y;Z) \subseteq \mathfrak{B}(X,Y;Z)$, for any LCS Z, because any bounded bilinear map $X\times Y \to Z$ is separately bounded, so separately continuous, for X and Y are bornological. Hence, the LCS topology on $X\otimes_i Y$ is finer than that of $X\otimes_\beta Y$ and the identity map

$$\mathfrak{h}_{X,Y} : X\otimes_i Y \to X\otimes_\beta Y \tag{1.4.17}$$

is continuous.

1.4.36. Let X, Y and Z be three LCS. Define $\mathfrak{B}_h(X,Y;Z)$ as the vector space of hypocontinuous bilinear maps $X \times Y \to Z$. We recall that a bilinear map $\phi : X \times Y \to Z$ is said to be *hypocontinuous* if given any pair of bounded sets $B' \subseteq X$ and $B'' \subseteq Y$, the set of maps $\{{}_x\phi : Y \to Z\}_{x \in B'}$ and $\{\phi_y : X \to Z\}_{y \in B''}$ are equicontinuous, where ${}_x\phi(y) = \phi(x,y) = \phi_y(x)$, for all $(x,y) \in X \times Y$. It is clear that $\mathfrak{B}_h(X,Y;Z)$ is a subspace of $\mathfrak{B}(X,Y;Z)$. Moreover, $\mathfrak{B}_h(X,Y;Z)$ is a subspace of the space $\mathfrak{B}_b(X,Y;Z)$ of bounded bilinear maps. Indeed, let $B \subseteq X \times Y$ be a bounded set. Hence, there exist $B' \subseteq X$ and $B'' \subseteq Y$ such that $B \subseteq B' \times B''$ (see [95], I.5.5). Given any hypocontinuous bilinear map $\phi : X \times Y \to Z$, the family of maps $\{{}_x\phi : Y \to Z\}_{x \in B'}$ is equicontinuous, so the Cor. in [95], III.4.1, tells us that $\{{}_x\phi : Y \to Z\}_{x \in B'}$ is a bounded subset of the LCS $\mathfrak{Hom}(Y,Z)$ provided with the topology of bounded convergence (see [95], III.3.2, Ex. d), which in turn means that $\cup_{x \in B'} {}_x\phi(B'') \subseteq Z$ is bounded (see [95], III.3.3). As a consequence, $\phi(B) \subseteq Z$ is bounded, and $\phi \in \mathfrak{B}_b(X,Y;Z)$.

1.4.37 Fact. *Let X, Y and Z be three LCS. We consider the LCS structure on $\mathfrak{B}_h(X,Y;Z)$ given by the topology of uniform convergence on all sets of the form $B' \times B''$, where $B' \subseteq X$ and $B'' \subseteq Y$ are bounded sets. Then, if Y is barreled, there is an isomorphism* *Well-known*

$$\mathfrak{B}_h(X,Y;Z) \to \mathfrak{Hom}_b\Big(X, \mathfrak{Hom}_b(Y,Z)\Big) \tag{1.4.18}$$

of LCS sending ϕ to the map $x \mapsto {}_x\phi$, where ${}_x\phi(y) = \phi(x,y)$, for all $x \in X$ and $y \in Y$.

Proof. This follows from [65], §40.2(3) and §40.4(7). □

1.4.38 Lemma. *Let us assume that X and Y are barreled bornological LCS. Then, given any LCS Z, we have the identities $\mathfrak{B}(X,Y;Z) = \mathfrak{B}_h(X,Y;Z) = \mathfrak{B}_b(X,Y;Z)$ between the spaces of separately continuous, hypocontinuous and bounded bilinear maps, respectively. In particular, $\mathfrak{h}_{X,Y}$ is an isomorphism of LCS. On the other hand, assume that X and Y are given as strict inductive limits in* LCS *of sequences $\{X_j\}_{j \in \mathbb{N}}$ and $\{Y_{j'}\}_{j' \in \mathbb{N}}$ of nuclear Fréchet LCS, respectively. Then, X and Y are clearly barreled and bornological, the convenient completion of $X \otimes_\beta Y$ coincides with its completion, and the completion of $\mathfrak{h}_{X,Y}$ induces thus an isomorphism of LCS* *Probably well-known*

$$\hat{\mathfrak{h}}_{X,Y} : X \hat{\otimes}_i Y \to X \tilde{\otimes}_\beta Y. \tag{1.4.19}$$

Proof. We start with the first statement. Let Z be any LCS. Since X and Y are bornological we have the inclusion $\mathfrak{B}_b(X,Y;Z) \subseteq \mathfrak{B}_b(X,Y;Z)$ (see 1.4.35), whereas the inclusion $\mathfrak{B}_h(X,Y;Z) \subseteq \mathfrak{B}_b(X,Y;Z)$ always holds (see 1.4.36), which means that $\mathfrak{B}_h(X,Y;Z) \subseteq \mathfrak{B}_b(X,Y;Z) \subseteq \mathfrak{B}(X,Y;Z)$. The fact that X and Y are barreled LCS implies moreover that the space $\mathfrak{B}(X,Y;Z)$ of separately continuous bilinear maps coincides with the space $\mathfrak{B}_h(X,Y;Z)$ of hypocontinuous bilinear maps from $X \times Y$ to Z (see [95], III.5.2), so the first part of the lemma follows.

Let us prove the second part of the lemma. Since the inductive limit of barreled (resp., bornological) spaces is barreled (resp., bornological), and every Fréchet space is barreled and bornological, X and Y are clearly barreled and bornological (see [95], II.7.2, Cor. 1, and II.7.2, Cor. 1). By the first part of the lemma $\mathfrak{h}_{X,Y}$ is an isomorphism, so its completion

$$\hat{\mathfrak{h}}_{X,Y} : X \hat{\otimes}_i Y \to \widehat{(X \otimes_\beta Y)} \tag{1.4.20}$$

is also an isomorphism of LCS. It suffices to show that the codomain is isomorphic to $X \tilde{\otimes}_\beta Y$. Using that the bornological tensor product commutes with inductive limits of BLCS on each side and the fact that the inclusion functor $\mathfrak{inc} : \mathrm{BLCS} \to \mathrm{LCS}$ preserves colimits, we see that $X \otimes_\beta Y$ is the inductive limit in LCS of the sequence $\{X_j \otimes_\beta Y_j\}_{j \in \mathbb{N}}$. The LCS X_j and Y_j being metrizable implies that $X_j \otimes_\beta Y_j \simeq X_j \otimes_\pi Y_j$, whereas the fact that they are nuclear implies that $X_j \hat{\otimes}_\pi Y_j \to X_{j+1} \hat{\otimes}_\pi Y_{j+1}$ is a embedding, *i.e.* an injective continuous linear map that induces an homeomorphism with its image regarded as a subspace of $X_{j+1} \hat{\otimes}_\pi Y_{j+1}$ (see [104], Prop. 43.7 and Thm. 50.1). Moreover, the commutativity of the diagram

$$\begin{array}{ccc} X_j \otimes_\pi Y_j & \longrightarrow & X_{j+1} \otimes_\pi Y_{j+1} \\ \big\downarrow & & \big\downarrow \\ X_j \hat{\otimes}_\pi Y_j & \longrightarrow & X_{j+1} \hat{\otimes}_\pi Y_{j+1} \end{array}$$

and the fact that the vertical maps are embeddings (see [95], II.4.1), as well as the lower horizontal map, implies that the upper horizontal map is an embedding. As a consequence, $X \otimes_\beta Y$ is the strict inductive limit in LCS (or BLCS) of the sequence $\{X_j \otimes_\pi Y_j\}_{j\in\mathbb{N}}$. Since the convenient completion functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ preserves colimits – which in particular means that the colimit in CLCS_{HD} is given by the convenient completion of the colimit taken in BLCS– we have that $X\tilde{\otimes}_\beta Y$ is the convenient completion of the strict inductive limit in LCS (or BLCS) of the sequence $\{X_j\tilde{\otimes}_\pi Y_j\}_{j\in\mathbb{N}}$, or equivalently $\{X_j\hat{\otimes}_\pi Y_j\}_{j\in\mathbb{N}}$. Since the strict inductive of a sequence of complete bornological LCS is complete and bornological, and every complete bornological LCS is convenient, $X\tilde{\otimes}_\beta Y$ is the strict inductive limit in LCS (or BLCS) of the sequence $\{X_j\hat{\otimes}_\pi Y_j\}_{j\in\mathbb{N}}$. Analogously, we have the isomorphisms

$$\widehat{X \otimes_\beta Y} \simeq \widehat{\mathrm{LCS}\text{-}\operatorname*{colim}_{j\in\mathbb{N}}\Big(X_j \otimes_\pi Y_j\Big)} \simeq \mathrm{LCS}^c_{HD}\text{-}\operatorname*{colim}_{j\in\mathbb{N}}\widehat{\Big(X_j \otimes_\pi Y_{j'}\Big)} \simeq \mathrm{LCS}\text{-}\operatorname*{colim}_{j\in\mathbb{N}}\widehat{\Big(X_j \otimes_\pi Y_{j'}\Big)} \simeq X\tilde{\otimes}_\beta Y$$

of LCS, where we have used in the second isomorphism that the completion functor $\widehat{(-)} : \mathrm{LCS} \to \mathrm{LCS}^c_{HD}$ preserves colimits, and in the third isomorphism that the strict inductive limit of a sequence of complete LCS is complete. The last isomorphism follows from the comments in the previous paragraph. The lemma is thus proved. □

Unknown? New? **1.4.39 Corollary.** *Let $\mathscr{A} = \mathrm{BLCS}_{HD}$ be the k-linear category of Hausdorff bornological locally convex spaces, and let $\tilde{\mathscr{A}} = \mathrm{CLCS}_{HD}$ be the k-linear category of convenient locally convex spaces. Suppose given an object X in $\mathscr{A}$ and an action $\rho : G \to \mathrm{Aut}_{\mathscr{A}}(X)$ of a finite group G. Then, ρ induces to a (unique) group homomorphism $\tilde{\rho} : G \to \mathrm{Aut}_{\tilde{\mathscr{A}}}(\tilde{X})$ on the convenient completion $\tilde{X}$ of X such $\tilde{\rho}(g)|_X = \rho(g)$ for all $g \in G$. Furthermore, the canonical morphisms $\widetilde{X^G} \to \tilde{X}^G$ and $\widetilde{X/G} \to \tilde{X}/G$ are isomorphisms in CLCS_{HD}.*

The proof is the same as that of Corollary 1.2.9 by changing LCS_{HD} and LCS^c_{HD} by BLCS_{HD} and CLCS_{HD}, respectively, and completion by convenient completion, and taking into account Lemma 1.4.31.

Unknown? New? **1.4.40 Corollary.** *Let BLCS_{HD} denote the k-linear category of Hausdorff bornological locally convex spaces, and let CLCS_{HD} be the k-linear category of (Hausdorff) convenient locally convex spaces. Suppose given an object Y in BLCS_{HD}. Then, the canonical morphisms $\widetilde{(Y^{\otimes_\beta m})^{\mathbb{S}_m}} \to (\tilde{Y}^{\tilde{\otimes}_\beta m})^{\mathbb{S}_m}$ and $\widetilde{(Y^{\otimes_\pi m})/\mathbb{S}_m} \to (\tilde{Y}^{\tilde{\otimes}_\beta m})/\mathbb{S}_m$ are isomorphisms in CLCS_{HD}.*

The proof is a direct application of Corollary 1.4.39, together with the property given in Lemma 1.4.31 for the convenient completion of bornological tensor products.

Well-known 1.5 Basic facts on (co)algebras and (co)modules in symmetric monoidal categories

1.5.1. We present next some basic facts about (co)algebras and (co)modules over them in any k-linear symmetric monoidal category.

Well-known **1.5.2 Fact.** *Let $\mathscr{C}$ and $\mathscr{C}'$ be two symmetric monoidal categories and let $F : \mathscr{C} \to \mathscr{C}'$ be an oplax symmetric monoidal functor between them. Then,*

(i) if (C,Δ,η) is a counitary $\mathscr{C}$-coalgebra, $(F(C),\varphi_2(C,C)\circ F(\Delta),\varphi_0\circ F(\eta))$ is a counitary $\mathscr{C}'$-coalgebra;

(ii) if (C,Δ,η) is a counitary $\mathscr{C}$-coalgebra and (V,δ) is a right (resp., left) $\mathscr{C}$-comodule over C, $(F(V),\varphi_2(V,C)\circ F(\delta))$ (resp., $(F(V),\varphi_2(C,V)\circ F(\delta))$) is a right (resp., left) $\mathscr{C}'$-comodule over $(F(C),\varphi_2(C,C)\circ F(\Delta),\varphi_0\circ F(\eta))$.

The analogous results hold for noncounitary coalgebras and comodules over them, if the functor F is only assumed to be nonunitary oplax symmetric monoidal, and the categories $\mathscr{C}$ and $\mathscr{C}'$ are nonunitary.

The proof is immediate from the definitions, and well-known in the literature. Of course, we also have the dual result involving lax symmetric monoidal functors, algebras and their modules.

1.5.3. Let $\mathscr{C}$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product $\otimes_{\mathscr{C}}$ commutes with colimits on each side. Following the indications for the category of vector spaces given in 1.0.2 and 1.0.3, a (commutative) nonunitary algebra A in $\mathscr{C}$ has *enough idempotents* if there is a family $\{A_e\}_{e\in E}$ of (commutative) unitary algebras in $\mathscr{C}$, A is the coproduct of $\{A_e\}_{e\in E}$ in $\mathscr{C}$, and the restriction of the product of A to $A_e \otimes_{\mathscr{C}} A_{e'}$ is the product of A_e if $e = e'$, and zero else. We suppose that the decomposition of A is part of the data. In this case, a *right* (resp., *left*) *module* over a commutative nonunitary A (in $\mathscr{C}$) with enough idempotents E will always be an object X of $\mathscr{C}$ provided with a decomposition $X = \oplus_{e\in E} X_e$ such that X_e is a unitary module over A_e, and the restriction of the right (resp., left) action of A on X to $X_{e'} \otimes_{\mathscr{C}} A_e$ (resp., $A_e \otimes_{\mathscr{C}} X_{e'}$) is that of A_e on X_e if $e = e'$, and zero else. As usual, the term module over a commutative algebra will always mean symmetric bimodule over that algebra.

1.5.4. For later use we state the following notation. If A is an not necessarily commutative and nonunitary algebra in $\mathscr{C}$ with product μ_A, we denote by

$$\mu_A^{(m)} : A^{\otimes_{\mathscr{C}} m} \to A \tag{1.5.1}$$

the morphism defined recursively by $\mu_A^{(1)} = \mathrm{id}_A$, and $\mu_A^{(m+1)} = \mu_A^{(m)} \circ (\mu_A \otimes_{\mathscr{C}} \mathrm{id}_A^{\otimes_{\mathscr{C}}(m-1)})$ for all $m \in \mathbb{N}$.

1.5.5. Let $B = \oplus_{e\in E} B_e$ be a (commutative) nonunitary algebra in $\mathscr{C}$ with enough idempotents E. Suppose that A has also a counitary coalgebra structure given by Δ_B and ϵ_B. We say that the counitary coalgebra structure is *compatible* with the family of idempotents E, if there is monoid structure with product m and unit 1 on E satisfying that $\mathcal{m}_e = m^{-1}(\{e\})$ is finite for all $e \in E$, the restriction

$$\Delta_B|_{B_{m(e,e')}} : B_{m(e,e')} \to B \otimes_{\mathscr{C}} B$$

of Δ_B to $B_{m(e,e')}$ factors through the inclusion $\oplus_{(f,f')\in \mathcal{m}_{m(e,e')}} B_f \otimes_{\mathscr{C}} B_{f'} \to B \otimes_{\mathscr{C}} B$, and the restriction of ϵ_B to B_e vanishes for all $e \neq 1$. We see that Δ_B is thus determined by a family of maps in $\mathscr{C}$ of the form

$$\Delta_{(e,e')} : B_{m(e,e')} \to B_e \otimes_{\mathscr{C}} B_{e'}. \tag{1.5.2}$$

We shall suppose in this case that the monoid structure on E is part of the data. A (commutative) counitary bialgebra B in $\mathscr{C}$ with enough idempotents $(E, m, 1)$ is a (commutative) nonunitary algebra in $\mathscr{C}$ with enough idempotents E with a compatible structure of counitary coalgebra such that $\epsilon : B_1 \to I_{\mathscr{C}}$ and (1.5.2) are morphisms of unitary algebras in $\mathscr{C}$.

1.5.6.*Remark.* The definition of algebra with enough idempotents is (nonfunctorially) equivalent to the notion of algebra in the symmetric monoidal Turaev category introduced by S. Caenepeel and M. De Lombaerde (see [20], 2.1), based on previous work of V. Turaev (see the late published book [105]). Furthermore, our notion of compatible coalgebra and bialgebra can be regarded as a locally finite version of the coalgebras and bialgebras in the symmetric monoidal Turaev category. The definition of module we follow is however rather different.

1.5.7 Lemma. *Let $\mathscr{C}$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side. Let A be a commutative nonunitary algebra A in $\mathscr{C}$ that has enough idempotents. Consider the category ${}_A\mathrm{Mod}(\mathscr{C})$ of modules over A in $\mathscr{C}$, and given two objects X and Y in ${}_A\mathrm{Mod}(\mathscr{C})$ define $X \otimes_A Y$ as the cokernel of the map*

$$X \otimes_{\mathscr{C}} A \otimes_{\mathscr{C}} Y \to X \otimes_{\mathscr{C}} Y \tag{1.5.3}$$

given by $\rho_X \otimes_{\mathscr{C}} \mathrm{id}_Y - \mathrm{id}_X \otimes_{\mathscr{C}} \rho'_Y$, where $\rho_X : X \otimes_{\mathscr{C}} A \to X$ and $\rho'_Y : A \otimes_{\mathscr{C}} Y \to Y$ are the corresponding actions. We recall that we are assuming that $\rho_X = \rho'_X \circ \tau(X, A)$, for all X, i.e. the term module over a commutative algebra means symmetric bimodule. The action of A on $X \otimes_A Y$ is defined by the map

$$A \otimes_{\mathscr{C}} (X \otimes_A Y) \to X \otimes_A Y \tag{1.5.4}$$

induced by $\rho'_X \otimes_{\mathscr{C}} \mathrm{id}_Y$. Then, ${}_A\mathrm{Mod}(\mathscr{C})$ is a symmetric monoidal category for the tensor product $X \otimes_A Y$, the unit A and the twist induced by that of $\mathscr{C}$. Moreover, the inclusion functor ${}_A\mathrm{Mod}(\mathscr{C}) \to \mathscr{C}$ preserves limits and colimits, and, in particular, the tensor product $\otimes_A$ commutes with colimits in ${}_A\mathrm{Mod}(\mathscr{C})$ on each side.

Proof. We verify first that the action (1.5.4) is well-defined. Indeed, we have the diagram

$$
\begin{array}{ccc}
A\otimes_{\mathscr{C}} X\otimes_{\mathscr{C}} A\otimes_{\mathscr{C}} Y & \xrightarrow{\rho'_X\otimes_{\mathscr{C}}\mathrm{id}_{A\otimes_{\mathscr{C}}Y}} & X\otimes_{\mathscr{C}} A\otimes_{\mathscr{C}} Y \\
{\scriptstyle \mathrm{id}_A\otimes_{\mathscr{C}}(\rho_X\otimes_{\mathscr{C}}\mathrm{id}_Y-\mathrm{id}_X\otimes_{\mathscr{C}}\rho'_Y)}\downarrow & & \downarrow{\scriptstyle \rho_X\otimes_{\mathscr{C}}\mathrm{id}_Y-\mathrm{id}_X\otimes_{\mathscr{C}}\rho'_Y} \\
A\otimes_{\mathscr{C}} X\otimes_{\mathscr{C}} Y & \xrightarrow{\rho'_X\otimes_{\mathscr{C}}\mathrm{id}_Y} & X\otimes_{\mathscr{C}} Y \\
\downarrow & & \downarrow \\
A\otimes_{\mathscr{C}}(X\otimes_A Y) & \overset{\rho'_{X\otimes_A Y}}{\dashrightarrow} & X\otimes_A Y
\end{array}
$$

and the commutativity of the upper square and the fact that the columns are cokernels tell us that the dashed arrow exists and makes the lower square commute. This latter map is (1.5.4). The fact that the twist of $\mathscr{C}$ induces a twist on ${}_A\mathrm{Mod}(\mathscr{C})$ follows from a similar argument.

It remains to show that the structure morphisms $\ell : A\otimes_A X \to X$ and $r : X\otimes_A A \to X$ giving the left and right units of ${}_A\mathrm{Mod}(\mathscr{C})$ are isomorphisms, for the obvious structure maps are clear. Let E be the fixed set of enough idempotents of A. The definition of ${}_A\mathrm{Mod}(\mathscr{C})$, any of its object X is decomposed as an internal direct sum (in $\mathscr{C}$) of the form $\oplus_{e\in E}X_e$, and analogously, any morphism $f : X \to X'$ between two objects X and X' is the direct sum (in $\mathscr{C}$) $\oplus_{e\in E} f|_{X_e}$, where $f|_{X_e} : X_e \to X'_e$, for all $e \in E$. In particular, f is an isomorphism in ${}_A\mathrm{Mod}(\mathscr{C})$ if and only if $f|_{X_e}$ is an isomorphism in ${}_{A_e}\mathrm{Mod}(\mathscr{C})$. The result follows by applying the previous remark to the structure morphisms ℓ and r in ${}_{A_e}\mathrm{Mod}(\mathscr{C})$, and the fact that $\ell|_{X_e}$ and $r|_{X_e}$ are clearly isomorphisms, because A_e is unitary for all $e \in E$.

The fact that inclusion functor ${}_A\mathrm{Mod}(\mathscr{C}) \to \mathscr{C}$ preserves limits holds in general, without any hypothesis on the tensor product of $\mathscr{C}$, and we leave it to the reader as an easy exercise. The inclusion functor ${}_A\mathrm{Mod}(\mathscr{C}) \to \mathscr{C}$ preserving colimits is a direct consequence of the fact that the tensor product of $\mathscr{C}$ commutes with colimits on each side. Finally, given an inductive family $\{X_j\}_{j\in J}$ in ${}_A\mathrm{Mod}(\mathscr{C})$ and an A-module Y in $\mathscr{C}$, we have

$$
\begin{aligned}
\big(\operatorname{colim} X_j\big)\otimes_A Y &\simeq \operatorname{Coker}\Big(\big(\operatorname{colim} X_j\big)\otimes_{\mathscr{C}} A\otimes_{\mathscr{C}} Y \to \big(\operatorname{colim} X_j\big)\otimes_{\mathscr{C}} Y\Big)\\
&\simeq \operatorname{Coker}\Big(\operatorname{colim}\big(X_j\otimes_{\mathscr{C}} A\otimes_{\mathscr{C}} Y\big)\to \operatorname{colim}\big(X_j\otimes_{\mathscr{C}} Y\big)\Big)\\
&\simeq \operatorname{colim}\operatorname{Coker}\Big(\big(X_j\otimes_{\mathscr{C}} A\otimes_{\mathscr{C}} Y\big)\to\big(X_j\otimes_{\mathscr{C}} Y\big)\Big)\simeq \operatorname{colim}\big(X_j\otimes_A Y\big),
\end{aligned}
$$

where we have used that the inclusion functor ${}_A\mathrm{Mod}(\mathscr{C}) \to \mathscr{C}$ preserves colimits (so we have not distinguished the colimits for each category), taking cokernels (a particular type of colimit) commutes with taking colimits, since colimits commute with colimits, and the tensor product of $\mathscr{C}$ commutes with colimits on each side. The lemma is thus proved. □

Mostly well-known

1.6 Basics on locally m-convex algebras and their locally convex modules

1.6.1. We recall that a *locally m-convex algebra* is a nonunitary k-algebra A provided with a Hausdorff topology defined by an arbitrary family $\{q_i\}_{i\in I}$ of submultiplicative real seminorms, *i.e.* $q_i : A \to \mathbb{R}_{\geq 0}$ satisfies that $q_i(a+b) \leq q_i(a)+q_i(b)$ and $q_i(\lambda a) = |\lambda| q_i(a)$, and $q_i(ab) \leq q_i(a)q_i(b)$, for all $i \in I$, $\lambda \in k$ and $a, b \in A$ (see [72], Def. 2.1). We say that a locally m-convex algebra is *complete* if its underlying LCS is complete. Notice that a locally m-convex algebra A is *a fortiori* a nonunitary monoid in the symmetric monoidal category LCS_{HD}, provided with the projective tensor product $\otimes_\pi$, because the product $A\times A \to A$ is (jointly) continuous. It is clear that if A is a locally m-convex algebra, then the *opposite* algebra A^{op} is as well. If A is complete, then it is naturally a monoid in the symmetric monoidal category LCS^{c}_{HD}, provided with the completed projective tensor product $\hat{\otimes}_\pi$. If A is a locally convex m-algebra, its completion $\hat{A}$ (as a LCS) is a naturally complete locally m-convex algebra and the natural map $A \to \hat{A}$ is a morphism of locally m–convex algebras. Moreover, the identity $\widehat{A^{\rm op}} = \hat{A}^{\rm op}$ is an isomorphism of locally m-convex algebras and $\hat{A}$ is commutative if A is so. All of the locally m-convex algebras in this manuscript will be assumed to be commutative.

1.6.2. Following [76], we say that a locally m-convex algebra A is called a *Fréchet algebra* if the underlying Hausdorff LCS is a Fréchet space. They are called $\mathscr{F}$-algebras in [72] (see Def. 4.1). A *morphism* of locally convex m-algebras (or of Fréchet algebras) is a morphism of algebras that is continuous for the underlying topologies. A locally m-convex algebra is called *unitary* if there is a morphism of locally m-convex algebras $\eta_A : k \to A$, where k has the usual locally m-convex algebra structure. More generally, we say that a locally m-convex algebra A has *enough idempotents* if there is a set E of orthogonal idempotents such that A is the internal direct sum (in the category LCS_{HD}) $\oplus_{e\in E} Ae$ (see 1.5.3). Given unitary locally m-convex algebras A and A', a morphism $f : A \to A'$ of locally m-convex algebras is called *unitary* if $f \circ \eta_A = \eta_{A'}$.

1.6.3. We recall that a *left locally convex module* X over a locally m-convex algebra A is a left module X over A such that X is provided with a LCS structure satisfying that the left action map $A \times X \to X$ is continuous, where $A \times X$ has the product topology.[2] We remark that we do not assume the LCS to be Hausdorff in the previous definition. As usual, if A has a unit, one assumes that it acts trivially on any left locally convex A-module X. More generally, if A has a set of enough idempotents E, we assume that X is the internal direct sum (in the category LCS) $\oplus_{e\in E} eX$ (see 1.5.3). A left locally convex module over a locally m-convex algebra A is said to be a *Fréchet left module* if the underlying LCS is Fréchet (so in particular, it is Hausdorff). A *morphism* of left locally convex modules over a locally m-convex algebra A is a morphism between the underlying left modules over A that is continuous for the corresponding topologies. We denote the category of left locally convex A-modules provided with the previous morphisms by ${}_A\mathfrak{Mod}$. Given X and Y two left locally convex modules over A, we denote by $\mathfrak{Hom}_A(X,Y)$ the corresponding space of morphisms, in order to distinguish it from $\mathrm{Hom}_A(X,Y)$ where no topology is involved. We recall that ${}_A\mathfrak{Mod}$ is complete and cocomplete, where the limits and colimits are given by those in ${}_A\mathrm{Mod}$ endowed with the topologies constructed in LCS. As noted in [76], 6.1, the completion $\hat{X}$ of the underlying LCS of a left locally convex module X is a left locally convex module over $\hat{A}$, so *a fortiori* over A, and it satisfies the usual property

$$\mathfrak{Hom}_{\hat{A}}(\hat{X},Y) \xrightarrow{\sim} \mathfrak{Hom}_A(\hat{X},Y) \xrightarrow{\sim} \mathfrak{Hom}_A(X,Y), \tag{1.6.1}$$

for every complete left locally convex module Y over A, where the first map is the inclusion and the second mapping is given by $\phi \mapsto \phi \circ i_X$, where $i_X : X \to \hat{X}$ is the canonical map. By changing A for $A^{\rm op}$ in the previous definitions, we obtain the corresponding notions for *right locally convex A-modules*. As usual, if A is commutative, by *locally convex A-module* we will mean a symmetric left and right locally convex A-module. In this case, it is clear that ${}_A\mathfrak{Mod}$ is naturally identified with the category of locally convex A-modules.

1.6.4.*Remark.* Let A be a locally convex m-algebra and X be a left locally convex A-module. The usual dual X^* of the left A-module X has a natural structure of right A-module by means of $(\lambda \cdot a)(x) = \lambda(a \cdot x)$, for all $x \in X$, $a \in A$ and $\lambda \in X^*$. Moreover, the vector subspace X' of X^* given by the continuous linear functionals is clearly a right A-submodule. It is however not necessarily a right locally convex A-module for the strong topology. For example, the space of distributions $C_c^\infty(\mathbb{R})'$ provided with the strong topology is not a locally convex $C^\infty(\mathbb{R})$-module (see [60], 4.6). We will consider for this reason another structure on X' (see 1.7.4 and Corollary 1.7.8).

1.6.5. We first recall that the coproduct of a family $\{X_a\}_{a\in A}$ in the category LCS is the direct sum $\oplus_{a\in A} X_a$ provided with the final locally convex topology for the family of canonical inclusions $X_a \to \oplus_{a\in A} X_a$ for all $a \in A$ (see [84], Prop. 2.1.5). The previous construction is Hausdorff if each member of the family $\{X_a\}_{a\in A}$ is, giving the coproduct in the category LCS_{HD} (see [84], Prop. 3.1.3, or [95], II.6.1) Moreover, the coproduct in the category LCS of an arbitrary family of LCS is complete if and only if each summand is so (see [95], II.6.2). It is an easy exercise to show that if the family is further assumed to be finite and its members are Fréchet spaces, the coproduct is Fréchet as well. In particular, given a Fréchet algebra A and $m \in \mathbb{N}$,

[2] This definition is not the weakest possible, and it coincides with the so-called *left module with jointly continuous action*.

the left A-module A^m with the coproduct topology in the category LCS_{HD} is a Fréchet left module over A. Moreover, a submodule (resp., closed submodule) of a left locally convex module (resp., Fréchet left module) over A is also a left locally convex module (resp., Fréchet left module). For more details on these properties see [22], Ch. 2, or [76], Ch. 6.

Trivial **1.6.6 Fact.** *If X is a left locally convex module over a locally m-convex algebra A and $\{x_1,\dots,x_s\}$ is a finite set of elements of X, then the map $A^s \to X$ given by $(a_1,\dots,a_s) \mapsto a_1x_1 + \cdots + a_sx_s$ is continuous, where A^s has the topology of the coproduct in the category LCS_{HD}.*

The proof immediately follows from the definition of left locally convex module.

Probably well-known but no reference! **1.6.7 Proposition** (Automatic continuity)**.** *Let A be a unitary Fréchet algebra. Then, given $r \in \mathbb{N}$, the left A-module A^r has a unique structure of Fréchet left module. Moreover, if X is a finitely generated left A-module, then any two structures of Fréchet left modules on X coincide. Given a left locally convex module Y over A, any morphism $f : X \to Y$ of the underlying left A-modules is continuous, so a morphism of left locally convex modules.*

Proof. The first statement follows from the second, so it suffices to prove the latter. Assume that X has a Fréchet left module structure over A. Consider any surjective morphism of left A-modules $\rho : A^r \to X$ for some $r \in \mathbb{N}$. Fact 1.6.6 tells us that the map $\rho : A^r \to X$ is in fact continuous, and we consider the final locally convex topology on X given by this map. We denote by $X^\sharp$ this Fréchet left module. Hence, we have a morphism $X^\sharp \to X$ of Fréchet left modules given by the identity map of the underlying vector spaces. By the Open Mapping theorem (see [95], III.2.2), it is an isomorphism of Fréchet left modules. This proves the required statement.

Let us now show the last part of the proposition. Since the topology of X coincides with the final locally convex topology of the surjective map $\rho : A^r \to X$, and thus a morphism of the underlying left A-modules $f : X \to Y$ is continuous if and only if $f \circ \rho$ is so, Fact 1.6.6 implies the result. □

1.6.8. We will say that a left locally convex A-module X over a locally m-convex algebra A is *topologically projective* if it is a projective object in the category ${}_A\mathfrak{Mod}$ (see [74], II.14). We recall that any direct summand of a topologically projective left locally convex A-module is also topologically projective, and the direct sum of topologically projective left locally convex A-modules is so (see [74], Prop. II.14.3), where we recall that the previous constructions are done in the category ${}_A\mathfrak{Mod}$.

Easy **1.6.9 Fact.** *Let A be a unitary locally m-convex algebra and let X be a locally convex A-module. Then X is topologically projective if and only if it is a direct summand of a coproduct $A^{(J)}$ in the category ${}_A\mathfrak{Mod}$, for some set J. In particular, if A is further assumed to be Fréchet, then X is complete.*

Proof. The proof follows the same pattern as the one given for usual left A-modules, with the only possible exception of the existence of an epimorphism in ${}_A\mathfrak{Mod}$ of the form $A^{(J)} \to X$ for every left locally convex A-module X. Since the epimorphisms of ${}_A\mathfrak{Mod}$ are precisely its surjective morphisms, it suffices to show that any (surjective) A-linear map $f : A^{(J)} \to X$ is continuous, where $A^{(J)}$ has the locally convex coproduct topology. The latter statement is immediate, because f is continuous if and only if $f \circ u_j$ is so for all $j \in J$, where $u_j : A \to A^{(J)}$ is the canonical j-th inclusion, and $f \circ u_j$ is continuous by Fact 1.6.6. □

New? **1.6.10 Lemma.** *Let A be a unitary Fréchet algebra and let X be a finitely generated left locally convex A-module that is projective as a left A-module. Then X is topologically projective. In particular, A is topologically projective.*

Proof. Since the epimorphisms of ${}_A\mathfrak{Mod}$ are exactly its surjective morphisms, and $\mathfrak{Hom}_A(X,-) = \mathrm{Hom}_A(X,-)$ (due to Proposition 1.6.7), we see that X is topologically projective. □

1.6.11. Following [76], 6.2, we recall that if X and Y are a right and left locally convex modules over a locally m-convex algebra A, resp., then the usual tensor product $X \otimes_A Y$ of the underlying right and left A-modules, resp., has a structure of LCS given as the final locally convex topology for the canonical surjective map $X \otimes_\pi Y \to X \otimes_A Y$, where $X \otimes_\pi Y$ denotes the usual tensor product $X \otimes Y$ over k provided with the projective tensor product topology. Equivalently, $X \otimes_A Y$ gives a representation of the covariant functor $\mathrm{LCS} \to {}_k\mathrm{Mod}$ sending a LCS Z to the vector space $\mathscr{B}^A(X,Y;Z)$ of (jointly) continuous bilinear and

A-balanced maps $X \times Y \to Z$ (see [78], Prop. 1, (ii), for the Hausdorff case, the proof being the same for the non Hausdorff one). We recall that a map $\phi : X \times Y \to Z$ is A-balanced if $\phi(xa, y) = \phi(x, ay)$, for all $a \in A$, If A is commutative, the tensor product $X \otimes_A Y$ also gives a representation of the covariant functor ${}_A\mathfrak{Mod} \to {}_k\text{Mod}$ sending a locally convex A-module Z to the vector space of (jointly) continuous A-bilinear and A-balanced maps $X \times Y \to Z$ (see [78], Cor. 3, for the Hausdorff case, the proof being the same for the non Hausdorff one). Recall that, if A is unitary and commutative, ${}_A\mathfrak{Mod}$ is a symmetric monoidal category with the projective tensor product $\otimes_A$, the unit A, and the twist given by the usual flip. Note that $X \otimes_A Y$ is not necessarily Hausdorff, even if X and Y are. In the case of the category ${}_A\mathfrak{Mod}_{HD}$ of Hausdorff locally convex A-modules, the (Hausdorff) tensor product $X \otimes_A^{HD} Y$ is defined as the (unique) Hausdorff quotient of $X \otimes_A Y$ (see [22], proof of Thm. II.1.3). However, we will not need to consider the Hausdorff tensor product in this manuscript. Analogously, the completion of $X \otimes_A Y$ will be denoted by $X \hat{\otimes}_A Y$. The complete LCS $X \otimes_A Y$ gives a representation of the covariant functor $\text{LCS}^c_{HD} \to {}_k\text{Mod}$ sending a complete LCS Z to the vector space of (jointly) continuous bilinear and A-balanced maps $X \times Y \to Z$ (see [22], Thms. II.1.3 and II.1.8). We have the canonical isomorphisms

$$X \hat{\otimes}_A Y \xrightarrow{\sim} \hat{X} \hat{\otimes}_A \hat{Y} \xrightarrow{\sim} \hat{X} \hat{\otimes}_{\hat{A}} \hat{Y} \tag{1.6.2}$$

of LCS, where the first map is the completion of $i_X \otimes i_Y : X \otimes_A Y \to \hat{X} \otimes_A \hat{Y}$, with $i_X : X \to \hat{X}$ and $i_Y : X \to \hat{Y}$ the canonical maps, and the second map is the completion of the canonical projection $\hat{X} \otimes_A \hat{Y} \to \hat{X} \otimes_{\hat{A}} \hat{Y}$ (see [76], Prop. 6.6). If A is commutative, then the previous isomorphisms are of locally convex A-modules (or $\hat{A}$-modules). We denote by ${}_A\mathfrak{Mod}^c$ the full subcategory of ${}_A\mathfrak{Mod}$ given by the left locally convex A-modules that are complete LCS. Again, if A is unitary and commutative, it is a symmetric monoidal category with the completed tensor product $\hat{\otimes}_A$, the unit $\hat{A}$, and the twist given by the completion of the twist of ${}_A\mathfrak{Mod}$.

1.6.12. Let X be a finitely generated topologically projective left locally convex module over a unitary locally m-convex algebra A. As a consequence, X is a direct summand of a direct sum of a finite number number of copies of A in ${}_A\mathfrak{Mod}$. As the tensor product $\otimes_A$ in ${}_A\mathfrak{Mod}$ commutes with finite coproducts, we get that $\hat{A} \otimes_A X$ is a complete left locally convex $\hat{A}$-module and the map of complete left locally convex $\hat{A}$-modules of the form

$$\hat{A} \otimes_A X \xrightarrow{\sim} \hat{X} \tag{1.6.3}$$

induced by the canonical map $i_X : X \to \hat{X}$ and the left action of $\hat{A}$ on $\hat{X}$ is an isomorphism. Unfortunately, the previous result cannot be expected for any topologically projective locally convex module. Indeed, one main drawback of the tensor products $\otimes_A$ and $\hat{\otimes}_A$ considered in this section is that they do not necessarily commute with arbitrary direct sums, for this is already the case in the category LCS and LCS^c_{HD} (see 1.3.3). For this reason we are mostly going to consider the bornological (resp., convenient) version of the theory, recalled in the next section, whereas the projective tensor product (resp., completed projective tensor product) will only serve us to prove properties of the former.

1.7 Basics on bornological algebras and their bornological locally convex modules

Mostly well-known

1.7.1. As BLCS_{HD} is a symmetric monoidal category, the notion of *bornological algebra* (resp., *commutative*, *unitary bornological algebra*) is immediate. Indeed, such an object would be just a monoid (resp., commutative, unitary monoid) in BLCS_{HD}, where we recall that we are using the bornological tensor product $\otimes_\beta$. The definition of *morphism* of bornological algebras is standard, as well as the *opposite bornological algebra* A^{op} of a bornological algebra A. Analogously, given two bornological algebras A and A', their bornological tensor product $A \otimes_\beta A'$ has a structure of bornological algebra by means of $(a_1 \otimes a'_1)(a_2 \otimes a'_2) = a_1 a_2 \otimes a'_1 a_2$, for all $a_i \in A$ and $a'_i \in A'$, with $i = 1, 2$. The analogous definition of a *convenient algebra* (resp., *commutative*, *unitary convenient algebra*) are immediately obtained, as well the notion of *morphism* of convenient algebras, of *opposite convenient algebra* A^{op} of a convenient algebra A, or of convenient tensor product $A \tilde{\otimes}_\beta A'$ of convenient algebras, by considering instead the symmetric monoidal category CLCS_{HD}, provided with the convenient tensor product $\tilde{\otimes}_\beta$. This is precisely the notion considered in [108], even though it is called bornological algebra there. By Lemma 1.4.31, we see that the convenient completion $\tilde{A}$ of a bornological algebra A gives a convenient algebra, and $\tilde{A}$ is commutative if A is so. Moreover, the identity $\widetilde{A^{\mathrm{op}}} = \tilde{A}^{\mathrm{op}}$ is an isomorphism of convenient algebras, and the isomorphism

$$A\tilde{\otimes}_\beta A' \simeq \tilde{A}\tilde{\otimes}_\beta \tilde{A}'$$

of bornological LCS given in Lemma 1.4.31 is an isomorphism of convenient algebras. We remark that any locally m-convex algebra structure on a bornological (resp., convenient) LCS A is naturally a bornological (resp., convenient) algebra, by means of (1.4.4) (resp., (1.4.16)), because the functor $\mathfrak{inc}_{HD}$ is lax symmetric monoidal (see Lemma 1.4.33 and Fact 1.5.2). We shall call them *bornological locally m-convex algebras* (resp., *convenient locally m-convex algebras*).

1.7.2. Let A be a bornological algebra. A *left bornological locally convex module* X over A is a bornological locally convex space X provided with a structure of a left A-module such that the action $A \times X \to X$ is bounded. This is tantamount to saying that the induced map $A \otimes_\beta X \to X$ is continuous. Note that we do not assume the LCS X to be Hausdorff in the previous definition. A *right bornological locally convex module* over A is a left bornological locally convex module over the opposite bornological algebra A^{op}. As usual, if A is commutative, by *bornological locally convex module* over A we will mean a symmetric left and right bornological locally convex A-module. If A has a unit, one assumes that it acts trivially on any left or right bornological locally convex A-module X, and if A has a set of enough idempotents E, we assume that X is the internal direct sum (in the category BLCS) $\oplus_{e\in E} eX$ (see 1.5.3). As in the previous section, a bornological locally convex module over A is said to be *Fréchet* if the underlying LCS is Fréchet (so in particular, it is Hausdorff). A *morphism* of left (resp., right) bornological locally convex modules over the bornological algebra A is a morphism between the underlying modules over A that is bounded (or continuous) for the corresponding topologies. Given two left bornological locally convex A-modules X and Y, we denote by $\mathfrak{Hom}_A(X,Y)$ the corresponding space of morphisms. We denote the category of left bornological locally convex A-modules provided with the previous morphisms by ${}_A\mathfrak{BMod}$. If A is commutative, this category is naturally identified with the category of bornological locally convex modules over A.

1.7.3. Given a right and a left bornological locally convex modules X and Y, resp., over a bornological algebra A, then the usual tensor product $X \otimes_A Y$ of the underlying A-modules has a structure of bornological LCS defined as the final locally convex topology for the canonical surjective map $X \otimes_\beta Y \to X \otimes_A Y$ given as the cokernel of the map $X \otimes_\beta A \otimes_\beta Y \to X \otimes_\beta Y$ sending $x \otimes a \otimes y$ to $xa \otimes y - x \otimes ay$. Since the quotient of a bornological space is bornological (*cf.* [95], II.8.2, Cor. 1), then the previous locally convex topology on $X \otimes_A Y$ is bornological. Equivalently, $X \otimes_A Y$ gives a representation of the covariant functor BLCS $\to {}_k$Mod sending a bornological LCS Z to the vector space $\mathfrak{B}_b^A(X,Y;Z)$ of bounded bilinear and A-balanced maps $X \times Y \to Z$ (see [64], Lemma I.5.21, (2), for the Hausdorff case, the proof being the same for the non Hausdorff one). It is clear that, if X is a right $A \otimes_\beta B^{\mathrm{op}}$-module, and Y is a left $A \otimes_\beta C^{\mathrm{op}}$-module, where A, B and C are bornological algebras, then $X \otimes_A Y$ has a canonical structure of left $B \otimes_\beta C^{\mathrm{op}}$-module via $a(x \otimes y)c = (ax) \otimes (yc)$, for $a \in A$, $c \in C$, $x \in X$ and $y \in Y$ (see [64], Lemma I.5.21, (3)). In particular, if A is commutative, $X \otimes_A Y$ is naturally a bornological locally convex A-module, and the tensor product $X \otimes_A Y$ also gives a representation of the covariant functor ${}_A\mathfrak{BMod} \to {}_k$Mod sending a locally convex A-module Z to the vector space of bounded A-bilinear and A-balanced maps $X \times Y \to Z$ (see [78], Cor. 3, for the Hausdorff case, the proof being the same for the non Hausdorff one). Note that, if A is unitary and commutative, ${}_A\mathfrak{BMod}$ is a symmetric monoidal category with the bornological tensor product $\otimes_A$, the unit A, and the twist given by the usual flip. Notice also that $X \otimes_A Y$ is not necessarily Hausdorff, even if X and Y are. In the case of the category ${}_A\mathfrak{BMod}_{HD}$ of Hausdorff bornological locally convex A-modules, the (Hausdorff) tensor product $X \otimes_A^{HD} Y$ is defined as the (unique) Hausdorff quotient of $X \otimes_A Y$, which is clearly bornological (see [78], Section 3). However, we will not need to consider the bornological Hausdorff tensor product in this manuscript.

1.7.4. The category ${}_A\mathfrak{BMod}$ is (locally small) complete and cocomplete, where the limits and colimits are given by those in ${}_A$Mod endowed with the bornological locally convex topologies constructed in BLCS. We will endow $\mathfrak{Hom}_A(X,Y)$ with the bornological locally convex topology associated with the subspace topology of $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ inside of $\mathfrak{Hom}^{\mathrm{b}}(X,Y)$ where X and Y are regarded as bornological LCS (resp., Hausdorff bornological LCS), and it will be denoted by $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$. Assume furthermore that X is a left bornological locally convex $A\otimes_\beta B^{\mathrm{op}}$-module, and Y is a left bornological locally convex $A\otimes_\beta C^{\mathrm{op}}$-module, where A, B and C are bornological algebras. In this case, $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ has an obvious structure of $B\otimes_\beta C^{\mathrm{op}}$-module, for the usual action $(b\cdot f\cdot c)(x)=f(xb)c$, for all $x\in X$, $f\in\mathfrak{Hom}_A(X,Y)$, $b\in B$ and $c\in C$.

1.7.5 Lemma. *Let A, B and C be bornological algebras, and let X be a left bornological locally convex $A\otimes_\beta B^{\mathrm{op}}$-module, and Y a left bornological locally convex $A\otimes_\beta C^{\mathrm{op}}$-module. Then, the usual space of continuous and A-linear morphisms from X to Y with the bornological locally convex topology associated with the subspace topology of $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ inside of $\mathfrak{Hom}^{\mathrm{b}}(X,Y)$ defined previously is a left bornological locally convex module over $B\otimes_\beta C^{\mathrm{op}}$. In particular, if A is a commutative bornological algebra, then $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ is a bornological locally convex A-module.* *New Expected*

Proof. The action of B on X and of C on Y are clearly continuous, so $b\cdot f\cdot c$ is an element of $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$. The left action of $B\otimes_\beta C^{\mathrm{op}}$ on $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ is given by the map

$$B\otimes_\beta C^{\mathrm{op}}\otimes_\beta \mathfrak{Hom}^{\mathrm{b}}_A(X,Y)\to\mathfrak{Hom}^{\mathrm{b}}_A(X,Y) \tag{1.7.1}$$

sending $b\otimes c\otimes f$ to the map $(b\cdot f\cdot c)(x)=f(xb)c$, for all $x\in X$, $f\in\mathfrak{Hom}_A(X,Y)$, $b\in B$ and $c\in C$. It suffices to show that (1.7.1) is continuous. In order to prove this, note that the map

$$X\otimes_\beta B\otimes_\beta C^{\mathrm{op}}\otimes_\beta \mathfrak{Hom}^{\mathrm{b}}_A(X,Y)\to Y \tag{1.7.2}$$

sending $x\otimes b\otimes c\otimes f$ to $f(xb)c$ is a morphism of bornological LCS, for it is given as the composition of maps involving the action of B on X, the action of C on Y and the evaluation map $\mathrm{ev}_{X,Y}: X\otimes_\beta\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)\to Y$. They are clearly bounded, so continuous, because all the involved LCS are bornological. By the adjunction between the bornological tensor functor $\otimes_\beta$ and the internal homomorphism $\mathfrak{Hom}^{\mathrm{b}}(-,-)$, (1.7.2) is tantamount to the morphism

$$B\otimes_\beta C^{\mathrm{op}}\otimes_\beta \mathfrak{Hom}^{\mathrm{b}}_A(X,Y)\to\mathfrak{Hom}^{\mathrm{b}}(X,Y)$$

of bornological LCS and since its image is included in $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$, we conclude that the latter space is a bornological locally convex $B\otimes_\beta C^{\mathrm{op}}$-module. If A is commutative, $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ is thus canonically a locally convex A-module. The lemma is thus proved. □

1.7.6 Lemma. *Let A, B and C be bornological algebras. Let X be a left bornological locally convex $A\otimes_\beta B^{\mathrm{op}}$-module, Y be a left bornological locally convex $B\otimes_\beta C^{\mathrm{op}}$-module and Z a left bornological locally convex $A\otimes_\beta C^{\mathrm{op}}$-module. Then, we have the isomorphisms* *New Expected*

$$\mathfrak{Hom}_A(X\otimes_B Y,Z)\xrightarrow{\sim}\mathfrak{Hom}_B\big(Y,\mathfrak{Hom}^{\mathrm{b}}_A(X,Z)\big)\quad\textit{and}\quad \mathfrak{Hom}_{C^{\mathrm{op}}}(X\otimes_B Y,Z)\xrightarrow{\sim}\mathfrak{Hom}_{B^{\mathrm{op}}}\big(X,\mathfrak{Hom}^{\mathrm{b}}_{C^{\mathrm{op}}}(Y,Z)\big) \tag{1.7.3}$$

of vector spaces, where the first map sends $\phi\in\mathfrak{Hom}_A(X\otimes_B Y,Z)$ to the mapping whose value at $y\in Y$ is $\phi_y(x)=\phi(x\otimes_B y)$, whereas the second sends $\phi\in\mathfrak{Hom}_{C^{\mathrm{op}}}(X\otimes_B Y,Z)$ to the map whose value at $x\in X$ is ${}_x\phi(y)=\phi(x\otimes_B y)$.

Proof. We will prove the lemma for the first map in (1.7.3), since the proof for the other is analogous. Recall that the first map in (1.7.3) is the restriction of the isomorphism

$$\mathrm{Hom}_A(X\otimes_B Y,Z)\xrightarrow{\sim}\mathrm{Hom}_B\big(Y,\mathrm{Hom}_A(X,Z)\big) \tag{1.7.4}$$

given by the same expression (see [3], Prop. 20.6), and whose inverse sends ψ in the codomain of (1.7.4) to the map $x\otimes_B y\mapsto\psi(y)(x)$, for all $x\in X$ and $y\in Y$. If $\phi\in\mathfrak{Hom}_A(X\otimes_B Y,Z)$, then the map $\phi_y: X\mapsto Z$ given by $\phi_y(x)=\phi(x\otimes_B y)$ is clearly bounded, because the associated bilinear map $(x,y)\mapsto\phi(x\otimes_B y)$ is by definition so and the bounded sets of $X\times Y$ are precisely the subsets of products $B'\times B''$ of bounded sets $B'\subseteq X$ and $B''\subseteq Y$. Hence, (1.7.4) induces a mapping

$$\mathfrak{Hom}_A(X\otimes_B Y,Z)\to\mathrm{Hom}_B\big(Y,\mathfrak{Hom}_A(X,Z)\big). \tag{1.7.5}$$

Furthermore, the same argument as before tells us that the map $Y \to \mathfrak{Hom}^{\mathfrak{b}}_A(X,Z)$ given by $y \mapsto \phi_y$ is bounded, so it induces the first map in (1.7.3). The fact that the inverse of (1.7.4) is also bounded (so continuous) follows from the same argument. The lemma is thus proved. □

1.7.7. Let A be a commutative bornological algebra. A direct consequence of the Lemma 1.7.6 is that ${}_A\mathfrak{BMod}$ is a closed symmetric monoidal category for the internal homomorphism space $\mathfrak{Hom}^{\mathfrak{b}}_A(-,-)$, *i.e.* we have natural k-linear isomorphisms

$$\mathfrak{Hom}_A\Big(Y, \mathfrak{Hom}^{\mathfrak{b}}_A(X,Z)\Big) \xleftarrow{\sim} \mathfrak{Hom}_A(X \otimes_A Y, Z) \xrightarrow{\sim} \mathfrak{Hom}_A\Big(X, \mathfrak{Hom}^{\mathfrak{b}}_A(Y,Z)\Big), \tag{1.7.6}$$

for all objects X, Y, and Z in ${}_A\mathfrak{BMod}$, where the left map takes f to the morphism sending $y \in Y$ to the mapping $x \mapsto f(x \otimes_A y)$, and the right map takes g to the morphism sending $x \in X$ to the mapping $y \mapsto f(x \otimes_A y)$ (see [78], Cor. 3). By a general categorical argument, the isomorphisms in (1.7.6) are even of bornological LCS and also A-linear if all the homomorphisms spaces are internal (see (1.4.6)). By the adjunction between the bornological tensor functor $\otimes_A$ and the internal homomorphism $\mathfrak{Hom}^{\mathfrak{b}}_A(-,-)$, we obtain the isomorphisms in ${}_A\mathfrak{BMod}$ of the form

$$\operatorname*{colim}_{j \in J}\Big(X \otimes_A Y_j\Big) \simeq X \otimes_A \Big(\operatorname*{colim}_{j \in J} Y_j\Big), \tag{1.7.7}$$

for any bornological locally convex A-module X and any system $\{Y_j : j \in J\}$ of bornological locally convex A-modules and bounded (or continuous) A-linear maps, where the previous colimits are taken in the category ${}_A\mathfrak{BMod}$. This shows that the category ${}_A\mathfrak{BMod}$ provided with the tensor product $\otimes_A$, the unit A and the usual flip is a symmetric monoidal category, such that the tensor product commutes with colimits in that category. Moreover, the same adjunction gives the isomorphisms

$$\mathfrak{Hom}^{\mathfrak{b}}_A(\operatorname{colim} X_j, Y) \simeq \lim \mathfrak{Hom}^{\mathfrak{b}}_A(X_j, Y) \quad \text{and} \quad \mathfrak{Hom}^{\mathfrak{b}}_A(X, \lim Y_j) \simeq \lim \mathfrak{Hom}^{\mathfrak{b}}_A(X, Y_j) \tag{1.7.8}$$

in ${}_A\mathfrak{BMod}$, for all bornological locally convex A-modules X and Y, and all systems $\{X_j\}_{j \in J}$ and $\{Y_j\}_{j \in J}$ of bornological locally convex A-modules.

New Expected **1.7.8 Corollary.** *Let X be a bornological locally convex A-module over a bornological algebra A. Recall that X' is the bornological LCS $\mathfrak{Hom}^{\mathfrak{b}}(X,k)$, which has a canonical structure of right A-module via $(f \cdot a)(x) = f(ax)$, for all $x \in X$, $f \in X'$ and $a \in A$. Then X' is a right bornological locally convex module over A.*

Proof. This is a direct consequence of Lemma 1.7.5. □

New Expected **1.7.9 Fact.** *Let A be a unitary bornological algebra and let X be a left bornological locally convex A-module. Then, we have the isomorphisms*

$$A \otimes_A X \xrightarrow{\sim} X \quad \text{and} \quad \mathfrak{Hom}^{\mathfrak{b}}_A(A, X) \xrightarrow{\sim} X \tag{1.7.9}$$

of left bornological locally convex A-modules, where the first map sends $a \otimes x$ to ax, whereas the second sends $\phi \in \mathfrak{Hom}_A(A,X)$ to $\phi(1_A)$.

Proof. The maps in (1.7.9) are clearly bounded, so continuous. The inverse maps are $x \mapsto 1_A \otimes x$ and $x \mapsto (a \mapsto ax)$, respectively, which are clearly well-defined. Moreover, they are also evidently bounded, and the statement follows. □

1.7.10. Let A be a locally m-convex algebra whose underlying LCS is bornological. We denote the associated bornological algebra also by A. A particular example of bornological module over the bornological algebra A is obtained from any locally convex A-module X whose underlying LCS is bornological via the map (1.4.4). The same holds for any morphism $f : X \to Y$ of locally convex A-modules whose underlying LCS structure are bornological. This induces a functor

$$\mathfrak{res} : \mathrm{BLCS} \cap {}_A\mathfrak{Mod} \to {}_A\mathfrak{BMod}, \tag{1.7.10}$$

where the intersection denotes the full subcategory of ${}_A\mathfrak{Mod}$ formed by the objects whose underlying LCS is bornological. The domain category is cocomplete, and the colimits are the same as those in ${}_A\mathfrak{Mod}$, because the colimit in LCS of bornological LCS is bornological. Furthermore, the functor $\mathfrak{res}$ preserves colimits, for they are computed in precisely the same way on both sides of (1.7.10).

1.7.11. Analogously, a *left* (resp., *right*) *convenient locally convex module* X over a bornological A is a left (resp., right) bornological locally convex module over A whose underlying LCS is convenient. In particular, the left (resp., right) action induces a continuous linear map $A\tilde{\otimes}_\beta X \to X$ (resp., $X\tilde{\otimes}_\beta A \to X$). Let us denote by ${}_A\mathfrak{CMod}$ the full subcategory of ${}_A\mathfrak{BMod}$ formed by the left convenient locally convex A-modules. As usual, if A commutative, by *convenient locally convex module over A* we will always mean a symmetric left and right convenient locally convex module. The category of convenient locally convex module over a commutative convenient algebra A is thus naturally identified with ${}_A\mathfrak{CMod}$. If A is a bornological algebra and X is a left (resp., right) bornological locally convex module over A, Lemma 1.4.31 immediately tells us that $\tilde{X}$ is a convenient locally convex module over the convenient algebra $\tilde{A}$. Moreover, by the same result, if $f : X \to Y$ is a morphism of left (resp., right) bornological locally convex modules over A, then $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism of left (resp., right) bornological locally convex modules over $\tilde{A}$. In other words, the functor $\widetilde{(-)} : \mathrm{BLCS} \to \mathrm{CLCS}_{HD}$ naturally induces a functor

$$\widetilde{(-)} : {}_A\mathfrak{BMod} \to {}_{\tilde{A}}\mathfrak{CMod}. \tag{1.7.11}$$

The latter is clearly a left adjoint to the inclusion ${}_{\tilde{A}}\mathfrak{CMod} \to {}_A\mathfrak{BMod}$, *i.e.* we have the canonical isomorphisms

$$\mathfrak{Hom}_{\tilde{A}}(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}_A(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}_A(X, Y) \tag{1.7.12}$$

of vector spaces, for every left bornological locally convex module X over A and left convenient locally convex module Y over A, where the first map is the inclusion and the second is given by $\phi \mapsto \phi \circ i_X$, where $i_X : X \to \tilde{X}$ is the canonical map.

1.7.12. The category ${}_A\mathfrak{CMod}$ is complete and cocomplete, where the limits and colimits are given by those in CLCS_{HD} endowed with the clear A-module structures. Indeed, note that the functor $\mathfrak{ib}_{HD} \circ \mathfrak{ib}_c :$ $\mathrm{CLCS}_{HD} \to \mathrm{BLCS}$ preserves limits as well as the inclusion functor $\mathrm{BLCS} \to {}_k\mathrm{Mod}$. It is easy to verify that the usual A-module structure on the limit computed in ${}_k\mathrm{Mod}$ together with the bornological locally convex topology computed in BLCS thus gives the limit of a system in ${}_A\mathfrak{CMod}$. Analogously, for the result concerning colimits, one notices that the colimit of a system in CLCS_{HD} is given as the convenient completion of the colimit in BLCS of the image of the system under the functor $\mathfrak{ib}_{HD} \circ \mathfrak{ib}_c$. Since the forgetful functor $\mathrm{BLCS} \to {}_k\mathrm{Mod}$ preserves colimits, and the colimit of a system of left A-modules has a natural structure of left A-module, one notices that the colimit of a system in CLCS_{HD} has a natural structure of convenient locally convex module over $\tilde{A} = A$, due to (1.7.11).

1.7.13. Given X and Y a left and right bornological locally convex modules over a bornological algebra A, resp., then the convenient completion of the bornological LCS $X \otimes_A Y$ will be denoted by $X\tilde{\otimes}_A Y$. Equivalently, $X\tilde{\otimes}_A Y$ is given as a representation of the covariant functor $\mathrm{CLCS}_{HD} \to {}_k\mathrm{Mod}$ sending a convenient LCS Z to the vector space $\mathfrak{B}^A_b(X, Y; Z)$ of bounded bilinear and A-balanced maps $X \times Y \to Z$ (see [21], Lemma 2.7). In case A is commutative, $X\tilde{\otimes} :_A Y$ has a natural structure of convenient locally convex module. It it is easy to show in this case that the convenient tensor product $X\tilde{\otimes}_A Y$ also gives a representation of the covariant functor ${}_A\mathfrak{CMod} \to {}_k\mathrm{Mod}$ sending a locally convex A-module Z to the vector space of bounded A-bilinear and A-balanced maps $X \times Y \to Z$ (*cf.* 1.7.3). Again, if A is unitary and commutative, ${}_A\mathfrak{CMod}$ is a symmetric monoidal category with the convenient tensor product $\tilde{\otimes}_A$, the unit $\tilde{A}$, and the twist given by the convenient completion of the twist of ${}_A\mathfrak{BMod}$. Since the functor (1.7.11) is a left adjoint, it preserves colimits. Hence, using that the tensor product $\otimes_A$ in ${}_A\mathfrak{BMod}$ preserves colimits, we get that

$${}_A\mathfrak{CMod}\text{-}\operatorname*{colim}_{j\in J}\Big(X\tilde{\otimes}_\beta Y_j\Big) \simeq X\tilde{\otimes}_\beta\Big({}_A\mathfrak{BMod}\text{-}\operatorname*{colim}_{j\in J} Y_j\Big). \tag{1.7.13}$$

1.7.14. *Porism.* A typical example we will use of lax symmetric monoidal functor is the following. Let A' and A be two commutative unitary bornological (resp., convenient) algebras and let $f : A' \to A$ be a morphism of bornological (resp., convenient) algebras (*e.g.*, $A' = k$ and f is the unit map). Then, Fact 1.5.2 implies that the functor from ${}_A\mathfrak{BMod}$ (resp., ${}_A\mathfrak{CMod}$) to ${}_{A'}\mathfrak{BMod}$ (resp., ${}_{A'}\mathfrak{CMod}$) given by restriction of scalars for the symmetric monoidal structures recalled in 1.7.3 (resp., 1.7.13), and the obvious coherence maps, is lax symmetric monoidal. In particular, the forgetful functor from ${}_A\mathfrak{BMod}$ (resp., ${}_A\mathfrak{CMod}$) to BLCS (resp., CLCS_{HD}), where A is any commutative unitary bornological (resp., convenient) algebra, provided with the obvious coherence maps, is lax symmetric monoidal.

If f is surjective, the restriction of previous scalars functor is further nonunitary symmetric monoidal. As a consequence, Lemma 1.5.2 tells us that, if C is a nonunitary coalgebra in ${}_A\mathfrak{BMod}$ (resp., ${}_A\mathfrak{CMod}$), and V is a comodule over C in ${}_A\mathfrak{BMod}$ (resp., ${}_A\mathfrak{CMod}$), then we may also regard C as a nonunitary coalgebra in ${}_{A'}\mathfrak{BMod}$ (resp., ${}_{A'}\mathfrak{CMod}$), and V as a comodule over C in ${}_{A'}\mathfrak{BMod}$ (resp., ${}_{A'}\mathfrak{CMod}$).

New Expected **1.7.15 Lemma.** *Let A be a bornological algebra, and let X and Y be a right and left bornological locally convex A-modules. Then, we have the natural isomorphisms*

$$X \tilde{\otimes}_A Y \xrightarrow{\sim} \tilde{X} \tilde{\otimes}_A \tilde{Y} \xrightarrow{\sim} \tilde{X} \tilde{\otimes}_{\tilde{A}} \tilde{Y} \tag{1.7.14}$$

of convenient LCS, where the first map is the convenient completion of $i_X \otimes i_Y : X \otimes_A Y \to \tilde{X} \otimes_A \tilde{Y}$, with $i_X : X \to \tilde{X}$ and $i_Y : X \to \tilde{Y}$ the canonical maps, and the second map is the convenient completion of the canonical projection $\tilde{X} \otimes_A \tilde{Y} \to \tilde{X} \otimes_{\tilde{A}} \tilde{Y}$. These isomorphisms are even of convenient locally convex modules over A (or $\tilde{A}$) if A is commutative, and as a consequence, the functor (1.7.11) *is strong monoidal in that case.*

Proof. It suffices to show that the induced maps

$$\mathfrak{Hom}(\tilde{X} \tilde{\otimes}_{\tilde{A}} \tilde{Y}, Z) \to \mathfrak{Hom}(\tilde{X} \tilde{\otimes}_A \tilde{Y}, Z) \to \mathfrak{Hom}(X \tilde{\otimes}_A Y, Z) \tag{1.7.15}$$

are isomorphisms of vector spaces, for all convenient LCS Z. This is tantamount to proving that

$$\mathfrak{Hom}(\tilde{X} \otimes_{\tilde{A}} \tilde{Y}, Z) \to \mathfrak{Hom}(\tilde{X} \otimes_A \tilde{Y}, Z) \to \mathfrak{Hom}(X \otimes_A Y, Z)$$

are isomorphisms, for all convenient LCS Z, or that the maps

$$\mathfrak{B}_b^{\tilde{A}}(\tilde{X}, \tilde{Y}; Z) \to \mathfrak{B}_b^{A}(\tilde{X}, \tilde{Y}; Z) \to \mathfrak{B}_b^{A}(X, Y; Z) \tag{1.7.16}$$

are isomorphisms of vector spaces, where the first mapping is the inclusion and the second is given by restricting a bilinear form on $\tilde{X} \times \tilde{Y}$ to $X \times Y$. By Lemma 1.4.31, we know that the last map is the restriction of the isomorphism

$$\mathfrak{B}_b(\tilde{X}, \tilde{Y}; Z) \to \mathfrak{B}_b(X, Y; Z),$$

also induced by the restriction. To prove that both mappings in (1.7.16) are isomorphisms, it suffices to show that given any bilinear and A-balanced map $\phi : X \times Y \to Z$, its unique extension to a bilinear map $\tilde{\phi} : \tilde{X} \times \tilde{Y} \to Z$ is $\tilde{A}$-balanced. We show first it is A-balanced. For any $a \in A$, the bilinear map $r_a : \tilde{X} \times \tilde{Y} \to Z$ given by $(x,y) \mapsto \tilde{\phi}(xa,y) - \tilde{\phi}(x,ay)$ is clearly bounded, for it is obtained from $\tilde{\phi}$ and the action of A on X and Y. Its restriction to $X \times Y$ is given by $(x,y) \mapsto \phi(xa,y) - \phi(x,ay)$, which vanishes because ϕ is A-balanced, so r_A also vanishes, by Lemma 1.4.31, which means that $\tilde{\phi}$ is A-balanced. We finally prove that $\tilde{\phi}$ is $\tilde{A}$-balanced. For any $x \in \tilde{X}$ and $y \in \tilde{Y}$, consider the linear map $s_{x,y} : \tilde{A} \to Z$ given by $a \mapsto \tilde{\phi}(xa,y) - \tilde{\phi}(x,ay)$. It is clearly bounded, for $\tilde{\phi}$ is so as well as the action of $\tilde{A}$ on $\tilde{X}$ and $\tilde{Y}$. Its restriction to A is the zero map, because ϕ is A-balanced, which in turn implies that $s_{x,y}$ also vanishes, by definition of the convenient completion. Hence, $\tilde{\phi}$ is $\tilde{A}$-balanced. The lemma is thus proved. □

New Expected **1.7.16 Corollary.** *Let A be a commutative bornological algebra. The isomorphisms in* (1.7.12) *are in fact of convenient locally convex A-modules, i.e. they induce isomorphisms of the form*

$$\mathfrak{Hom}^{\flat}_{\tilde{A}}(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}^{\flat}_{A}(\tilde{X}, Y) \xrightarrow{\sim} \mathfrak{Hom}^{\flat}_{A}(X, Y). \tag{1.7.17}$$

Proof. This is a direct consequence of the fact that (1.7.11) is strong monoidal (see Lemma 1.7.15) and Lemma 1.3.7. □

1.7.17 Fact. *Let A be a convenient algebra. Given a left bornological locally convex A-module X and a left convenient locally convex A-module Y, the internal homomorphism space $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ of ${}_A\mathfrak{BMod}$, where Y is regarded as a bornological locally convex A-module, is a convenient LCS. If A is commutative, $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ is thus convenient locally convex A-module.* New Expected

Proof. We will modify the proof of [41], Prop. 3.6.3, that corresponds precisely to our statement for the case $A=k$. Note that $\mathfrak{Hom}^{\mathrm{b}}_A(X,Y)$ is given by the intersection of the kernels of the maps $\mathrm{ev}_{ax}-a\,\mathrm{ev}_x : \mathfrak{Hom}^{\mathrm{b}}(X,Y)\to Y$, for all $x\in X$ and $a\in A$, where $\mathrm{ev}_x : \mathfrak{Hom}^{\mathrm{b}}(X,Y)\to Y$ is the mapping sending f to $f(x)$, and $a\,\mathrm{ev}_x : \mathfrak{Hom}^{\mathrm{b}}(X,Y)\to Y$ sends f to $af(x)$. They are continuous, so the kernels are locally closed, because the inverse image of a locally closed set under a continuous linear map is locally closed (see [82], Lemma 5.1.23, (ii)) and the set formed by the origin of Y is clearly locally closed. The result follows from [82], Prop. 5.1.20, (ii). □

1.7.18 Lemma. *Let A, B and C be convenient algebras. Let X be a left convenient locally convex $A\otimes_\beta B^{\mathrm{op}}$-module, Y be a left convenient locally convex $B\otimes_\beta C^{\mathrm{op}}$-module and Z a left convenient locally convex $A\otimes_\beta C^{\mathrm{op}}$-module. Then, we have the isomorphisms* New Expected

$$\mathfrak{Hom}_A(X\tilde{\otimes}_B Y,Z)\xrightarrow{\sim}\mathfrak{Hom}_B\Big(Y,\mathfrak{Hom}^{\mathrm{b}}_{C^{\mathrm{op}}}(X,Z)\Big)\quad\textit{and}\quad\mathfrak{Hom}_{C^{\mathrm{op}}}(X\tilde{\otimes}_B Y,Z)\xrightarrow{\sim}\mathfrak{Hom}_{B^{\mathrm{op}}}\Big(X,\mathfrak{Hom}^{\mathrm{b}}_{C^{\mathrm{op}}}(Y,Z)\Big),\tag{1.7.18}$$

of vector spaces, where the first map sends $\phi\in\mathfrak{Hom}_A(X\tilde{\otimes}_B Y,Z)$ to the mapping whose value at $y\in Y$ is $\phi_y(x)=\phi(x\otimes y)$, whereas the second sends $\phi\in\mathfrak{Hom}_{C^{\mathrm{op}}}(X\tilde{\otimes}_B Y,Z)$ to the map whose value at $x\in X$ is ${}_x\phi(y)=\phi(x\otimes y)$.

Proof. This is proved by the same argument as the one for Lemma 1.7.6, but using the universal property of the convenient tensor product of convenient locally convex modules instead (see 1.7.13). □

1.7.19. Assume A is a commutative convenient algebra. Then, Lemma 1.7.18 tells us that the symmetric monoidal category CLCS_{HD} is closed for the same internal homomorphism space as in BLCS, *i.e.* we have natural k-linear isomorphisms

$$\mathfrak{Hom}_A\Big(Y,\mathfrak{Hom}^{\mathrm{b}}(X,Z)\Big)\xleftarrow{\sim}\mathfrak{Hom}_A(X\tilde{\otimes}_A Y,Z)\xrightarrow{\sim}\mathfrak{Hom}\Big(X,\mathfrak{Hom}^{\mathrm{b}}_A(Y,Z)\Big),\tag{1.7.19}$$

for all objects X, Y, and Z in ${}_A\mathfrak{CMod}$, with the same morphisms as in (1.4.5). Using the same proof as in (1.4.6), the isomorphisms in (1.7.19) are even of convenient locally convex A-modules if all the homomorphisms spaces are internal. As in the case of bornological LCS described 1.4.24, the adjunction between the convenient tensor functor $\tilde{\otimes}_A$ and the internal homomorphism $\mathfrak{Hom}^{\mathrm{b}}_A(-,-)$ gives us the isomorphism

$$\operatorname*{colim}_{j\in J}\Big(X\tilde{\otimes}_A Y_j\Big)\xrightarrow{\sim}X\tilde{\otimes}_A\Big(\operatorname*{colim}_{j\in J}Y_j\Big)\tag{1.7.20}$$

as well as

$$\mathfrak{Hom}^{\mathrm{b}}_A(\operatorname{colim}X_j,Y)\simeq\lim\mathfrak{Hom}^{\mathrm{b}}_A(X_j,Y)\quad\text{and}\quad\mathfrak{Hom}^{\mathrm{b}}_A(X,\lim Y_j)\simeq\lim\mathfrak{Hom}^{\mathrm{b}}_A(X,Y_j)\tag{1.7.21}$$

of convenient locally convex A-modules, for all convenient locally convex A-modules X and Y, and all systems $\{X_j\}_{j\in J}$ and $\{Y_j\}_{j\in J}$ of convenient locally convex A-modules, where the colimits and limits are computed in ${}_A\mathfrak{CMod}$.

1.7.20. We will say that a bornological locally convex module X over a bornological algebra A is *bornologically projective* if it is a projective object in the category ${}_A\mathfrak{BMod}$. The properties mentioned in 1.6.8 for topologically projectives modules over a locally m-convex algebra, as well as the first part of Fact 1.6.9, clearly hold for bornologically projective modules over a bornological algebra.

New Expected **1.7.21 Fact.** *Let A be a bornological locally m-convex algebra and let X be a topologically projective over the locally m-convex algebra A. Then, X is a bornological LCS and the induced bornological locally convex module structure on X is bornologically projective.*

Proof. The statement that X is bornological follows from the fact that X is a direct summand of a direct sum of copies of A in ${}_A\mathfrak{Mod}$, so *a fortiori* in LCS, and the underlying LCS of A is bornological. To prove the last part, note that the coproduct $A^{(I)}$ in ${}_A\mathfrak{Mod}$, which is clearly a bornological LCS since the functor $\mathfrak{inc}$: BLCS → LCS preserves coproducts, coincides with the coproduct $A^{(I)}$ in ${}_A\mathfrak{BMod}$, because the latter coproduct is the image of the former one under the functor (1.7.10), which preserves colimits. Moreover, since X is a direct summand of $A^{(I)}$ in ${}_A\mathfrak{Mod}$, applying the functor (1.7.10) we get that X is a direct summand of $A^{(I)}$ in${}_A\mathfrak{BMod}$. Since A is bornologically projective (as a bornological locally convex module over itself), and direct summands of direct sums of bornologically projectives are bornologically projectives, the result follows. □

1.7.22. Given two bornological locally convex modules X and Y over a commutative unitary bornological algebra A that are bornologically projective, their bornological tensor product $X \otimes_A Y$ is also bornologically projective. This follows directly from the fact X and Y are direct summands of a direct sum of copies of A in ${}_A\mathfrak{BMod}$, and the tensor product $\otimes_A$ in ${}_A\mathfrak{BMod}$ commutes with coproducts.

1.7.23. Let A be a unitary bornological algebra and let X be any left bornological locally convex A-module that is bornologically projective. Then, the convenient completion of the action of A on X induces an isomorphism

$$\tilde{A} \otimes_A X \xrightarrow{\sim} \tilde{X}. \tag{1.7.22}$$

Indeed, by (1.7.12), it suffices to show that the functor ${}_A\mathfrak{BMod} \to {}_{\tilde{A}}\mathfrak{CMod}$ given by $X \mapsto \tilde{A} \otimes_A X$ is a left adjoint to the inclusion functor ${}_{\tilde{A}}\mathfrak{CMod} \to {}_A\mathfrak{BMod}$. We first note that $\tilde{A} \otimes_A X$ is indeed convenient, for X being a direct summand of a coproduct $A^{(I)}$ in ${}_A\mathfrak{BMod}$ implies that $\tilde{A} \otimes_A X$ is a direct summand of a coproduct $\tilde{A} \otimes_A A^{(I)} \simeq \tilde{A}^{(I)}$ in ${}_{\tilde{A}}\mathfrak{CMod}$. Moreover, let Y be any convenient locally convex $\tilde{A}$-module. Then, (1.7.22) follows from the chain of isomorphisms

$$\mathfrak{Hom}_{\tilde{A}}(\tilde{A} \otimes_A X, Y) \xrightarrow{\sim} \mathfrak{Hom}_A\Big(X, \mathfrak{Hom}^{\mathfrak{b}}_{\tilde{A}}(\tilde{A}, Y)\Big) \xrightarrow{\sim} \mathfrak{Hom}_A(X, Y)$$

of vector spaces, where the first map is given by Lemma 1.7.6, and the second one is due to Fact 1.7.9.

Chapter 2

Preliminaries on vector bundles

2.1 The Serre-Swan theorem and its enhancements

Well-known results

2.1.1. For the following definitions we refer the reader to basic texts on differential geometry, such as [77], which we specially recommend. Throughout this manuscript M, N, ... will denote finite dimensional connected smooth manifolds (or *smooth manifolds* for short). They are assumed to be Hausdorff and to satisfy the second axiom of countability, so they are in particular paracompact. We shall denote by $C^\infty(M)$ the k-algebra of smooth functions from M to the (real manifold) k. Let E be a smooth finite dimensional k-vector bundle (or *vector bundle* for short) over M. Then the space of smooth global sections $\Gamma(E)$ (or $\Gamma(M,E)$ if we want to emphasize the base space) of E over M is clearly a $C^\infty(M)$-module for the usual left and right actions. We will also use the space of continuous global sections $\Gamma^0(E)$ formed by the set of continuous maps $\sigma : M \to E$ satisfying that $\sigma(p) \in E_p$, for all $p \in M$. It is also a $C^\infty(M)$-module, with the same actions as $\Gamma(E)$.

2.1.2. We recall that given two vector bundles over M, there exist the *tensor product bundle* $E \otimes_k F$ and the *homomorphism vector bundle* $\mathrm{Hom}_k(E,F)$ (see [77], 11.35 and 11.37). The fiber of the former is $E_p \otimes_k F_p$ whereas the fiber of the latter is $\mathrm{Hom}_k(E_p,F_p)$, for all $p \in M$. Since the field k is fixed we shall usually write in this situation unadorned tensor products of the form $E \otimes F$ and unadorned homomorphism spaces $\mathrm{Hom}(E,F)$, as well as for the corresponding fibers $E_p \otimes F_p$ and $\mathrm{Hom}(E_p,F_p)$. This will not cause any harm because the objects involved are only vector bundles. As usual, if $F = M \times k$ is the trivial line bundle, we write E^* instead of $\mathrm{Hom}(E,F)$.

2.1.3 Theorem (Serre-Swan theorem)**.** *Let* Vect_M *be the category of vector bundles over* M*. Consider the map* *Well-known*

$$\Gamma : \mathrm{Vect}_M \to {}_{C^\infty(M)}\mathrm{Mod}$$

sending a vector bundle to its space of global sections. Then, the image of Γ *lies inside the subcategory of finitely generated projective* $C^\infty(M)$*-modules, i.e* $\Gamma(E)$ *is a finitely generated projective* $C^\infty(M)$*-module for all vector bundles* E *over* M*, and moreover* Γ *gives an equivalence between that subcategory and* Vect_M*.*

See [77], Thm. 11.29 and 11.32, for the case $k = \mathbb{R}$. The proof for $k = \mathbb{C}$ is identical.

2.1.4 Theorem. *Let* E *and* F *be two vector bundles over* M*. Consider the canonical map of* $C^\infty(M)$*-modules* *Well-known*

$$\xi_{E,F} : \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) \to \Gamma(E \otimes F)$$

induced by the mapping $\Gamma(E) \times \Gamma(F) \to \Gamma(E \otimes F)$ *sending* (σ, σ') *to the section of* $E \otimes F$ *given by* $p \mapsto \sigma(p) \otimes \sigma'(p)$*, for* $p \in M$*. Then* $\xi_{E,F}$ *is an isomorphism of* $C^\infty(M)$*-modules.*

Furthermore, the natural map

$$\chi_{E,F} : \Gamma\Big(\mathrm{Hom}(E,F)\Big) \to \mathrm{Hom}_{C^\infty(M)}\Big(\Gamma(E), \Gamma(F)\Big)$$

defined as $\chi_{E,F}(\lambda)(\sigma)(p) = (\lambda(p))(\sigma(p))$*, where* $\lambda \in \Gamma(\mathrm{Hom}(E,F))$*,* $\sigma \in \Gamma(E)$ *and* $p \in M$*, is an isomorphism of* $C^\infty(M)$*-modules.*

The case $k = \mathbb{R}$ is given in [77], Thm. 11.39, and the statement for $k = \mathbb{C}$ is its clear consequence.

We recall that given a morphism $f : M \to N$ of smooth manifolds and a vector bundle F over N, there exists the *pull-back bundle* f^*F on M, and its fiber is $f^*F_p = F_{f(p)}$, for all $p \in M$ (see [77], 10.16). The relation between the space of global sections of f^*F and of F is given in the next result.

Well-known **2.1.5 Theorem.** *Let* $f : M \to N$ *be a morphism of smooth manifolds and* F *a vector bundle over* N*. Then* f *induces a morphism of* k*-algebras* $f^* : C^\infty(N) \to C^\infty(M)$ *sending* g *to* $g \circ f$*, for all* $g \in C^\infty(N)$*. Consider the canonical map of* $C^\infty(M)$*-modules*

$$\theta_{f,F} : C^\infty(M) \otimes_{C^\infty(N)} \Gamma(F) \to \Gamma(f^*F)$$

that sends $h \otimes_{C^\infty(N)} \sigma$ *to the section of* $\Gamma(f^*F)$ *given by* $p \mapsto h(p)\sigma(f(p))$*, for all* $p \in M$*, where* $h \in C^\infty(M)$ *and* $\sigma \in \Gamma(F)$*. Then* $\theta_{f,F}$ *is an isomorphism of* $C^\infty(M)$*-modules.*

The case $k = \mathbb{R}$ is given in [77], Thm. 11.54, and the case $k = \mathbb{C}$ follows directly from it.

2.1.6. Let M and N be two smooth manifolds and E and F be two vector bundles over M and N, respectively. Recall that the *external tensor product* $E \boxtimes F$ of E and F is the vector bundle over $M \times N$ given by the tensor product $\pi_M^* E \otimes \pi_N^* F$ of vector bundles over $M \times N$, where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the canonical projections. Note that the fiber $(E \boxtimes F)_{p,q}$ of $E \boxtimes F$ at a point $(p,q) \in M \times N$ is $E_p \otimes F_q$.

Well-known **2.1.7 Corollary.** *Let* M *and* N *be two smooth manifolds and* E *and* F *be two vector bundles over* M *and* N*, respectively. Then, the canonical map of* $C^\infty(M \times N)$*-modules*

$$\Gamma(E) \otimes_{C^\infty(M)} C^\infty(M \times N) \otimes_{C^\infty(N)} \Gamma(F) \to \Gamma(E \boxtimes F)$$

sending $\sigma \otimes_{C^\infty(M)} h \otimes_{C^\infty(N)} \sigma'$ *to the section* $(p,q) \mapsto h(p,q)(\sigma(p) \otimes \sigma'(q))$*, for all* $(p,q) \in M \times N$*, is an isomorphism.*

This is a direct consequence of Theorem 2.1.5.

2.1.8. Consider two vector bundles E over M and F over N, and an isomorphism $f : M \to N$ of smooth manifolds. We recall that a morphism $t : E \to F$ of vector bundles over f induces a map of $C^\infty(N)$-modules

$$t_* : \Gamma(M,E) \to \Gamma(N,F) \tag{2.1.1}$$

of the form $\sigma \mapsto t \circ \sigma \circ f^{-1}$.

Some new results

2.2 Sections of compacts support

Well-known **2.2.1 Theorem.** *The ideal* $C_c^\infty(M)$ *of* $C^\infty(M)$ *given by the functions of compact support is a projective* $C^\infty(M)$*-module. Moreover, if* $K \subseteq M$ *is compact, the subset* $C_K^\infty(M)$ *of* $C_c^\infty(M)$ *formed by the functions whose support is included in* K*, is a finitely generated projective* $C^\infty(M)$*-module.*

Proof. See the Corollary in [38] for $k = \mathbb{R}$. The proof for $k = \mathbb{C}$ is identical. The last statement is proved by precisely the same argument as the one used for the first implication of the main Theorem in [38]. □

2.2.2. We will state some consequences of the previous result. Given a vector bundle E over M (resp., and $K \subseteq M$ is a compact subset of M), denote by $\Gamma_c(E)$ (resp., $\Gamma_K(E)$) the space of smooth global sections (resp., whose support is included in K) of compact support of E. It will be denoted by $\Gamma_c(M,E)$ (resp., $\Gamma_K(M,E)$) if we want to emphasize the base space. They are $C^\infty(M)$-submodules of $\Gamma(E)$.

New **2.2.3 Corollary.** *Let* E *be a vector bundle over* M *(resp., and* $K \subseteq M$ *is a compact subset of* M*). Then* $\Gamma_c(E)$ *(resp.,* $\Gamma_K(E)$*) is a projective* $C^\infty(M)$*-module (resp., finitely generated projective* $C^\infty(M)$*-module).*

Proof. Since E be a vector bundle over M, Theorem 2.1.3 tells us that there exists another vector bundle E' such that $E \oplus E'$ is isomorphic to a trivial vector bundle. Suppose that $E \oplus E' \simeq M \times k^m$ for some $m \in \mathbb{N}$. In this case it is trivial to see that $\Gamma_c(E \oplus E') \simeq C_c^\infty(M)^m$ for the obvious isomorphism of $C^\infty(M)$-modules given by $\sigma \mapsto \pi_2 \circ \sigma$, for $\sigma \in \Gamma_c(E)$ and $\pi_2 : M \times k^m \to k^m$ the canonical projection. Taking into account that the functor Γ_c commutes with finite direct sums, we get that

$$\Gamma_c(E) \oplus \Gamma_c(E') \simeq \Gamma_c(E \oplus E') \simeq \Gamma_c(M \times k^m) = C_c^\infty(M)^m.$$

Hence, $\Gamma_c(E)$ is a direct summand of a projective $C^\infty(M)$-module, so a projective $C^\infty(M)$-module as well. The proof for $\Gamma_K(E)$ is analogous. □

2.2.4 Proposition. *Let E and F be two vector bundles over M. The canonical map $\xi_{E,F}$ in Theorem 2.1.4 induces an isomorphism of $C^\infty(M)$-modules* New

$$\bar{\xi}_{E,F} : \Gamma(E) \otimes_{C^\infty(M)} \Gamma_c(F) \to \Gamma_c(E \otimes F). \tag{2.2.1}$$

On the other hand, $\xi_{E,F}$ also induces the isomorphism of $C^\infty(M)$-modules

$$\hat{\xi}_{E,F} : \Gamma_c(E) \otimes_{C^\infty(M)} \Gamma_c(F) \to \Gamma_c(E \otimes F). \tag{2.2.2}$$

Proof. We consider the first part of the statement. Let us start with the case where F is the trivial 1-dimensional bundle $M \times k$. By means of the isomorphism of vector bundles $E \otimes F \simeq E$, we see that (2.2.1) is tantamount to prove that the map

$$\bar{\xi}_E : \Gamma(E) \otimes_{C^\infty(M)} C_c^\infty(M) \to \Gamma_c(E)$$

induced by the right action on $\Gamma(E)$ is an isomorphism of $C^\infty(M)$-modules. Moreover, $\bar{\xi}_E$ is the corestriction to $\Gamma_c(E)$ of the map given by tensoring the inclusion map $C_c^\infty(M) \to C^\infty(M)$ on the left with $\Gamma(E)$ over $C^\infty(M)$. Since $\Gamma(E)$ is a projective $C^\infty(M)$-module by Theorem 2.1.3, it is flat, so $\bar{\xi}_E$ is injective. It is also surjective. Indeed, given any section σ of compact support K of E, consider a smooth function of compact support $f \in C_c^\infty(M)$ satisfying that $f(p) = 1$ for all $p \in K$. Such a function exists by a usual argument using a partition of the unity (see [76], Lemma 1.5). Then $\bar{\xi}_E(\sigma \otimes_{C^\infty(M)} f) = \sigma$.

The fact that (2.2.1) is an isomorphism follows from the commutativity of the diagram

$$\begin{array}{ccc}
\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) \otimes_{C^\infty(M)} C_c^\infty(M) & \xrightarrow{\xi_{E,F} \otimes_{C^\infty(M)} \mathrm{id}_{C_c^\infty(M)}} & \Gamma(E \otimes F) \otimes_{C^\infty(M)} C_c^\infty(M) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\Gamma(E)} \otimes_{C^\infty(M)} \bar{\xi}_F} & & \Big\downarrow {\scriptstyle \bar{\xi}_{E \otimes F}} \\
\Gamma(E) \otimes_{C^\infty(M)} \Gamma_c(F) & \xrightarrow{\quad \bar{\xi}_{E,F} \quad} & \Gamma_c(E \otimes F)
\end{array}$$

To prove the second part of the statement we show first that the map

$$\hat{\xi}_F : C_c^\infty(M) \otimes_{C^\infty(M)} \Gamma_c(F) \to \Gamma_c(F)$$

induced by the left action on $\Gamma_c(F)$ is an isomorphism of $C^\infty(M)$-modules. Since $\hat{\xi}_F$ is given by tensoring the inclusion map $C_c^\infty(M) \to C^\infty(M)$ with $\Gamma_c(F)$ over $C^\infty(M)$, and $\Gamma_c(F)$ is a projective $C^\infty(M)$-module by Corollary 2.2.3 (so flat), $\hat{\xi}_F$ is injective. The surjectivity is proved by the same argument as the one given for $\bar{\xi}_F$. The general situation follows from the commutative diagram

$$\begin{array}{ccc}
C_c^\infty(M) \otimes_{C^\infty(M)} \Gamma(E) \otimes_{C^\infty(M)} \Gamma_c(F) & \xrightarrow{\mathrm{id}_{C_c^\infty(M)} \otimes_{C^\infty(M)} \bar{\xi}_{E,F}} & C_c^\infty(M) \otimes_{C^\infty(M)} \Gamma_c(E \otimes F) \\
\Big\downarrow {\scriptstyle (\bar{\xi}_E \circ \tau) \otimes_{C^\infty(M)} \mathrm{id}_{\Gamma_c(F)}} & & \Big\downarrow {\scriptstyle \hat{\xi}_{E \otimes F}} \\
\Gamma_c(E) \otimes_{C^\infty(M)} \Gamma_c(F) & \xrightarrow{\quad \hat{\xi}_{E,F} \quad} & \Gamma_c(E \otimes F)
\end{array}$$

where $\tau : C_c^\infty(M) \otimes_{C^\infty(M)} \Gamma(E) \to \Gamma(E) \otimes_{C^\infty(M)} C_c^\infty(M)$ is the usual flip. The proposition is thus proved. □

New **2.2.5 Proposition.** *Let E and F be two vector bundles over M. The canonical map $\chi_{E,F}$ in Theorem 2.1.4 induces an isomorphisms of $C^\infty(M)$-modules*

$$\bar{\chi}_{E,F} : \Gamma_c\Big(\operatorname{Hom}(E,F)\Big) \to \operatorname{Hom}_{C^\infty(M)}\Big(\Gamma(E),\Gamma_c(F)\Big). \tag{2.2.3}$$

On the other hand, the obvious inclusion

$$\operatorname{Hom}_{C^\infty(M)}\Big(\Gamma_c(E),\Gamma_c(F)\Big) \to \operatorname{Hom}_{C^\infty(M)}\Big(\Gamma_c(E),\Gamma(F)\Big) \tag{2.2.4}$$

is an isomorphism of $C^\infty(M)$-modules, and there is a canonical map of $C^\infty(M)$-modules

$$\hat{\chi}_{E,F} : \operatorname{Hom}_{C^\infty(M)}\Big(\Gamma(E),\Gamma(F)\Big) \to \operatorname{Hom}_{C^\infty(M)}\Big(\Gamma_c(E),\Gamma_c(F)\Big) \tag{2.2.5}$$

induced by sending $\lambda \in \operatorname{Hom}_{C^\infty(M)}(\Gamma(E),\Gamma(F))$ to its restriction $\lambda|_{\Gamma_c(E)}$. Moreover, $\hat{\chi}_{E,F}$ is injective.

Proof. Given any pair of $C^\infty(M)$-modules X and Y, consider the canonical map

$$\upsilon_{X,Y} : X^{\circledast} \otimes_{C^\infty(M)} Y \to \operatorname{Hom}_{C^\infty(M)}(X,Y),$$

sending $\lambda \otimes_{C^\infty(M)} y$ to the map $x \mapsto \lambda(x)y$, for $x \in X$, $y \in Y$, and $\lambda \in X^{\circledast}$. It is an isomorphism if X is finitely generated and projective (see [3], Prop. 20.10), which clearly holds for $\Gamma(E)$. The statement for $\bar{\chi}_{E,F}$ now follows from the fact that (2.2.3) is the composition of

$$\bar{\xi}^{-1}_{\operatorname{Hom}(E,F)} : \Gamma_c\Big(\operatorname{Hom}(E,F)\Big) \to \Gamma\Big(\operatorname{Hom}(E,F)\Big) \otimes_{C^\infty(M)} C^\infty_c(M),$$

the map

$$\upsilon^{-1}_{\Gamma(E),\Gamma(F)} \otimes_{C^\infty(M)} \operatorname{id}_{C^\infty_c(M)} : \Gamma\Big(\operatorname{Hom}(E,F)\Big) \otimes_{C^\infty(M)} C^\infty_c(M) \to \Gamma(E)^{\circledast} \otimes_{C^\infty(M)} \Gamma(F) \otimes_{C^\infty(M)} C^\infty_c(M),$$

the mapping $\operatorname{id}_{\Gamma(E)^{\circledast}} \otimes_{C^\infty(M)} \bar{\xi}_F$ and $\upsilon_{\Gamma(E),\Gamma_c(F)}$.

The fact that the inclusion (2.2.4) is surjective is clear. Indeed, given any $\lambda \in \operatorname{Hom}_{C^\infty(M)}(\Gamma_c(E),\Gamma(F))$ and $\sigma \in \Gamma_c(E)$, taking $f \in C^\infty_c(M)$ such that $f(p) = 1$ for all p in the support of σ we see that $\lambda(\sigma) = \lambda(f\sigma) = f\lambda(\sigma)$, so $\lambda(\sigma)$ has compact support.

Finally, the canonical map $\hat{\chi}_{E,F}$ is also well-defined, since the previous argument also shows that given any $\lambda \in \operatorname{Hom}_{C^\infty(M)}(\Gamma(E),\Gamma(F))$ and $\sigma \in \Gamma_c(E)$, then $\lambda(\sigma)$ has compact support. Let $\hat{\chi}_E : \Gamma(E)^{\circledast} \to \Gamma_c(E)^{\circledast}$ be the map given by $\lambda \mapsto \lambda|_{\Gamma_c(E)}$. Then, $\hat{\chi}_{E,F}$ coincides with $\hat{\chi}_E \otimes_{C^\infty(M)} \operatorname{id}_{\Gamma(F)}$. Since $\Gamma(F)$ is projective (so flat), it suffices to show that $\hat{\chi}_E$ is injective. If $E = M \times k$ is the trivial line bundle, then $\hat{\chi}_E$ gives the map $C^\infty(M) \to C^\infty_c(M)^{\circledast}$ sending f to the functional $g \mapsto gf$, for $f \in C^\infty(M)$ and $g \in C^\infty_c(M)$. This map is obviously injective, because $f \neq 0$, implies that there is $p \in M$ such that $f(p) \neq 0$, so $gf \neq 0$ for any function $g \in C^\infty_c(M)$ such that $g(p) \neq 0$. The case of a trivial bundle $E = M \times k^r$ follows from the previous considerations, because in this case $\hat{\chi}_E$ is the direct sum of the r copies of the morphisms $\hat{\chi}_{M\times k}$ considered for the case of the line bundle. Analogously, the case of a general vector bundle E holds, because if F is another vector bundle such that $E \oplus F$ is a trivial vector bundle $M \times k^r$, then $\hat{\chi}_E \oplus \hat{\chi}_F$ coincides with $\hat{\chi}_{M\times k^r}$. □

2.2.6. We finally remark that, given an isomorphism $t : E \to F$ of vector bundles over an isomorphism $f : M \to N$ of smooth manifolds, the morphism (2.1.1) restricts to a map of $C^\infty(N)$-modules

$$t_* : \Gamma_c(M,E) \to \Gamma_c(N,F). \tag{2.2.6}$$

2.3 Topologies on the spaces of sections

2.3.1. For any smooth manifold M of dimension n, the k-algebra $C^\infty(M)$ has a structure of locally m-convex algebra given by the so-called *usual topology* defined as follows. Let $\{K_\ell\}_{\ell\in\mathbb{N}_0}$ be a countable collection of compact subsets of M such that $M = \cup_{\ell \in \mathbb{N}_0} K_\ell^\circ$ and K_ℓ is included in a chart $(U_{i_\ell}, \phi_{i_\ell})$ of the atlas of M. For $\ell, m \in \mathbb{N}_0$, define

$$p_{\ell,m}(f) = \sup_{x \in \phi_{i_\ell}(K_\ell)} \sup_{\bar{a} \in \mathbb{N}^n_{0,\leq m}} \left| 2^m \frac{\partial^\alpha (f \circ \phi_{i_\ell}^{-1})}{\partial x^\alpha}(x) \right|, \tag{2.3.1}$$

where $\mathbb{N}^n_{0,\leq m}$ is the subset of $\mathbb{N}_0^n$ formed by the elements $\bar{a} = (a_1, \dots, a_n)$ such that $|\bar{a}| = a_1 + \dots + a_n \leq m$, and $f \in C^\infty(M)$. This family of real seminorms induces a structure of locally m-convex algebra on $C^\infty(M)$ and it is in fact a Fréchet algebra (see [69], IV.4.(2)). Furthermore, Eq. (4.19) in [69], IV.4, tells us that the family $\{C^\infty(M) \to C^\infty(U_{i_\ell})\}_{\ell \in \mathbb{N}_0}$ of maps given by restriction induces an isomorphism

$$C^\infty(M) \simeq \varprojlim C^\infty(U_{i_\ell}) \tag{2.3.2}$$

of locally m-convex algebras. The previous isomorphism together with [95], III.7.4, and [104], Cor. after Thm. 51.5, imply that $C^\infty(M)$ is a nuclear space. Moreover, $C^\infty(M)$ is also a Schwartz space, for any nuclear space is also a Schwartz space (see [51], Ch. III, Section 3:1-2, Rk. 4).

2.3.2 Theorem. *Let M be a smooth manifold. Then, there exists a unique Fréchet algebra structure on $C^\infty(M)$.* *Well-known*

This is a direct consequence of [72], Thm. 14.2.

2.3.3. It can be verified directly from the previous family of seminorms that the map $f^* : C^\infty(N) \to C^\infty(M)$ induced by a morphism $f : M \to N$ of smooth manifolds is a morphism of Fréchet algebras.

2.3.4.*Remark.* Note that, in the case $A = C^\infty(M)$, $X = \Gamma(E)$, and $Y = \Gamma_c(E)$, the dual space $X^\circledast$ has the unique Fréchet topology coming from the identification $\Gamma(E)^\circledast \simeq \Gamma(E^*)$ (see Theorem 2.1.4), and $Y^\circledast$ has the locally convex topology coming from the identification $\Gamma_c(E)^\circledast \simeq \Gamma_c(E^*)$ (see Proposition 2.2.5).

2.3.5. Given any smooth manifold M of dimension n and a vector bundle E of rank r over M, the $C^\infty(M)$-module $\Gamma(E)$ has the structure of locally convex module for the *usual topology*, which we now recall. Choose $\{K_\ell\}_{\ell \in \mathbb{N}_0}$ a countable collection of compact subsets of M such that $M = \cup_{\ell \in \mathbb{N}_0} K_\ell^\circ$ and K_ℓ is included in an open set U_{i_ℓ} that is part of a chart $(U_{i_\ell}, \phi_{i_\ell})$ of the atlas of M and of a trivialization $(U_{i_\ell}, \psi_{i_\ell})$ of the vector bundle E. For $\ell, m \in \mathbb{N}_0$, define

$$p_{\ell,m}(\sigma) = \sup_{x \in \phi_{i_\ell}(K_\ell)} \sup_{\bar{a} \in \mathbb{N}^n_{0,\leq m}} \sup_{s=1,\dots,r} \left| 2^m \frac{\partial^\alpha (\pi_s \circ \psi_{i_\ell} \circ \sigma \circ \phi_{i_\ell}^{-1})}{\partial x^\alpha}(x) \right|, \tag{2.3.3}$$

where $\mathbb{N}^n_{0,\leq m}$ was defined after equation (2.3.1), $\pi_s : U_{i_\ell} \times k^r \to k$ is the canonical projection onto the s-th component of k^r, and $\sigma \in \Gamma(E)$. The same proof as the one for the seminorms (2.3.1) can be used to prove that this family gives a structure of locally convex module on $\Gamma(E)$ over $C^\infty(M)$ and it is furthermore a Fréchet module. Note that $\Gamma(E)$ is a nuclear space, because it is a direct summand of a finite direct sum of copies of the nuclear space $C^\infty(M)$ (see [95], III.7.4), and, as a consequence, a Schwartz space (see 2.3.1).

2.3.6 Corollary. *Let M be a smooth manifold and consider the unique Fréchet algebra structure on $C^\infty(M)$. Then, given any vector bundle E over M, the space of global sections $\Gamma(E)$ has a unique structure of Fréchet module over $C^\infty(M)$. Furthermore, if F is another vector bundle over M, any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ is a morphism of Fréchet modules.* *Well-known*

This is a direct consequence of Proposition 1.6.7, and it roughly states that more or less the topology on the spaces of global sections is somehow redundant.

2.3.7 Corollary. *Let M be a smooth manifold and consider the unique Fréchet algebra structure on $C^\infty(M)$. Then $\Gamma(E)$ is a topologically projective Fréchet $C^\infty(M)$-module. If $K \subseteq M$ is a compact subset, then $\Gamma_K(E)$ is a topologically projective Fréchet $C^\infty(M)$-module as well.* *Probably new*

This is a direct consequence of Lemma 1.6.10 together with Theorem 2.1.3, and Corollaries 2.2.3 and 2.3.6.

Well-known **2.3.8 Theorem.** *Let M be a smooth manifold. Then the structure sheaf $\mathscr{O}_M$ of M is a sheaf of Fréchet algebras. Moreover, the functor from the category of sheaves of Fréchet modules over $\mathscr{O}_M$ to the category of Fréchet modules over $C^\infty(M)$ given by taking global sections is an equivalence. It is also a monoidal functor for the completed tensor products on both sides.*

This is a special case of [76], Thm. A.3, and whose last statement follows from the comments about tensor products on p. 155. We refer the reader to [76], App. A, for the definition of sheaf of locally convex (resp., Fréchet) modules over a sheaf of locally convex m-algebras.

2.3.9. Let E be a vector bundle over M. Given any compact $K \subseteq M$, the definition of the seminorms (2.3.3) imply that the vector subspace $\Gamma_K(E)$ of $\Gamma(E)$ formed by the sections whose support is contained in K is closed. It is also a $C^\infty(M)$-submodule of $\Gamma(E)$, so a Fréchet module over $C^\infty(M)$ as well. Furthermore, by [95], III.7.4, it is a nuclear space, which, together with the completeness, imply that it is in particular semi-reflexive by [95], III.7.2, Cor. 2, and IV.5.5. Since a Fréchet space is barreled (see [95], II.7.1, Cor.), [95], Thm. IV.5.6, implies that $\Gamma_K(E)$ is reflexive. Furthermore, it is also bornological, because it is Fréchet (see [95], II.8.1), and Schwartz, for it is a subspace of the Schwartz space $\Gamma(E)$. All of the previous comments in this paragraph apply to the Fréchet nuclear space $\Gamma(E)$ to show that it is barreled, bornological and reflexive. Note that $\Gamma_c(E)$ is the union of all $C^\infty(M)$-modules $\Gamma_K(E)$, for all compact subsets $K \subseteq M$. We define the final locally convex topology on $\Gamma_c(E)$ for the family of inclusions

$$\Big\{\Gamma_K(E) \to \Gamma_c(E) : K \subseteq M \text{ compact}\Big\}. \tag{2.3.4}$$

It is clear that $\Gamma_c(E)$ is a locally convex module over $C^\infty(M)$, and by its very definition, $\Gamma_c(E)$ is an (LF)-space, so barreled (see [95], II.7.2, Cor. 2) and bornological (see [95], II.8.2, Cor. 2). It is complete and reflexive, because each space $\Gamma_K(E)$ is so (see [95], II.6.6, and IV.5.8). As the manifold M satisfies the second axiom of countability, the family of compacts K in (2.3.4) can be taken to be countable, and in particular $\Gamma_c(E)$ is also nuclear (see [95], III. 7.4, Cor.), so in particular it is a Schwartz space (see 2.3.1). Moreover, applying the previous ideas to the trivial line bundle $M \times k$ tells us that $C_c^\infty(M)$ is a locally convex module over $C^\infty(M)$ satisfying the same conditions.

New **2.3.10 Corollary.** *Let M be a smooth manifold and consider the unique Fréchet algebra structure on $C^\infty(M)$. Given two vector bundles E and F over M, consider the locally convex $C^\infty(M)$-modules $\Gamma_c(E)$ and $\Gamma_c(F)$ defined before, and let $f : \Gamma_c(E) \to \Gamma_c(F)$ be any $C^\infty(M)$-linear map. Then f is continuous. Furthermore, $\Gamma_c(E)$ is topologically projective.*

Proof. Since any $C^\infty(M)$-linear map $f : \Gamma_c(E) \to \Gamma_c(F)$ is continuous if and only if its restriction to $\Gamma_K(E)$ is continuous, and the latter space is a finitely generated locally convex $C^\infty(M)$-module, Proposition 1.6.7 tells us that $f|_{\Gamma_K(E)}$ is continuous.

To prove the last statement, recall that $\Gamma_c(E)$ is a projective $C^\infty(M)$-module by Corollary 2.2.3. This implies that there exists morphisms of $C^\infty(M)$-modules $h : \Gamma_c(E) \to C^\infty(M)^{(J)}$ and $g : C^\infty(M)^{(J)} \to \Gamma_c(E)$, for some set J, where we remark that $C^\infty(M)^{(J)}$ has the locally convex coproduct topology, such that $g \circ h = \mathrm{id}_{\Gamma_c(E)}$. By the arguments in 2.3.9, h is continuous if and only if $h|_{\Gamma_K(E)}$ is continuous for all compact subsets $K \subseteq M$. As $\Gamma_K(E)$ is a finitely generated Fréchet module over $C^\infty(M)$, Proposition 1.6.7 implies $h|_{\Gamma_K(E)}$ is continuous. Moreover, by the definition of the locally convex coproduct topology g is continuous if and only if $g \circ u_j : C^\infty(M) \to \Gamma_c(E)$ is so for all $j \in J$, where $u_j : C^\infty(M) \to C^\infty(M)^{(J)}$ is the canonical j-th inclusion. By Proposition 1.6.7, the composition $g \circ u_j$ is also continuous, and the corollary follows. □

Well-known **2.3.11 Lemma.** *Let M and N be two manifolds. Then, the canonical inclusions $C^\infty(M) \otimes_\pi C^\infty(N) \to C^\infty(M \times N)$ induces an isomorphism of locally m-convex algebras $C^\infty(M)\hat{\otimes}_\pi C^\infty(N) \to C^\infty(M \times N)$. Since both $C^\infty(M)$ and $C^\infty(N)$ are metrizable (and thus bornological) LCS, we also obtain an isomorphism of bornological algebras $C^\infty(M)\tilde{\otimes}_\beta C^\infty(N) \to C^\infty(M \times N)$.*

Proof. Let us start with the first isomorphism, and assume that M and N are are open subsets of Euclidean spaces. The first isomorphism is in this case given by [104], Thm. 51.6, (51.4). Furthermore, the canonical inclusion $C^\infty(M) \otimes_\pi C^\infty(N) \to C^\infty(M \times N)$ restricts to a morphism between the associated inverse systems of the form (2.3.2). Taking into account that the inverse limits are determined by taking any covering of the involved manifolds, and the comparison morphisms between the systems are isomorphisms by the previous comments, the general case follows.

For the last part, we first note that $C^\infty(M) \otimes_\pi C^\infty(N) \simeq C^\infty(M) \otimes_\beta C^\infty(N)$ via the canonical morphism (1.4.4), because the spaces are metrizable (see 1.4.23). Moreover, since the previous tensor product is also metrizable, its completion and its convenient completion coincide by Corollary 1.4.21, and the last result follows. □

2.3.12 Lemma. *Let M and N be two manifolds. Then, the canonical inclusion $C_c^\infty(M) \otimes_\beta C_c^\infty(N) \to C^\infty(M \times N)$ induces an isomorphism of bornological algebras $C_c^\infty(M) \tilde{\otimes}_\beta C_c^\infty(N) \to C_c^\infty(M \times N)$.* *Well-known*

Proof. Recall that both $C_c^\infty(M)$ and $C_c^\infty(N)$ are strict inductive limits in LCS (or BLCS) of sequences of Fréchet bornological algebras of the form (2.3.4). By the arguments given in the proof of Lemma 1.4.38, $C_c^\infty(M) \tilde{\otimes}_\beta C_c^\infty(N)$ is the strict inductive limit in LCS of a sequence of terms of the form $C_K^\infty(M) \hat{\otimes}_\pi C_{K'}^\infty(N)$, with $K \subseteq M$ and $K' \subseteq N$ two compact subsets. The result now follows from [104], Thm. 51.6, (51.5), and the definition of $C_c^\infty(M \times N)$. □

2.3.13 Proposition. *Let M and N be smooth manifolds, and E and F be two vector bundles over M and N, respectively. Then, there is a canonical isomorphism of bornological locally convex $C^\infty(M \times N)$-modules of the form* *Probably well-known but no reference!*

$$\kappa : \Gamma(E) \tilde{\otimes}_\beta \Gamma(E) \to \Gamma(E \boxtimes F) \tag{2.3.5}$$

which restricts to an isomorphism

$$\bar{\kappa} : \Gamma_c(E) \tilde{\otimes}_\beta \Gamma_c(F) \to \Gamma_c(E \boxtimes F). \tag{2.3.6}$$

We suspect that this result should be well-known.

Proof. Corollary 2.1.7 tells us that $\Gamma(E \boxtimes F)$ is canonically isomorphic (as $C^\infty(M \times N)$-modules) to

$$\Gamma(E) \otimes_{C^\infty(M)} C^\infty(M \times N) \otimes_{C^\infty(N)} \Gamma(F), \tag{2.3.7}$$

whereas Corollary 2.2.4 together with (2.3.7) imply that $\Gamma_c(E \boxtimes F)$ is canonically isomorphic (as $C^\infty(M \times N)$-modules) to

$$\Gamma(E) \otimes_{C^\infty(M)} C_c^\infty(M \times N) \otimes_{C^\infty(N)} \Gamma(F). \tag{2.3.8}$$

The inclusion mapping $C^\infty(M) \otimes_\beta C^\infty(N) \to C^\infty(M \times N)$ is in fact the convenient completion of its domain by Lemma 2.3.11. Using (1.7.22) with $A = C^\infty(M) \otimes C^\infty(N)$ (so $\tilde{A} \simeq C^\infty(M \times N)$) and X given by $\Gamma(E) \otimes_\beta \Gamma(F)$, we get that the bornological locally convex $C^\infty(M \times N)$-module (2.3.7) is isomorphic to the convenient completion $\Gamma(E) \tilde{\otimes}_\beta \Gamma(F)$. The isomorphism κ is the one induced by the map from $\Gamma(E) \otimes_\beta \Gamma(F)$ to (2.3.7) defined as

$$\sigma \otimes \sigma' \mapsto \sigma \otimes_{C^\infty(M)} 1_{M \times N} \otimes_{C^\infty(N)} \sigma', \tag{2.3.9}$$

where $1_{M \times N}$ is the constant unit map on $M \times N$. Indeed, the map (2.3.9) is bilinear separately continuous by Proposition 1.6.7, so it is jointly continuous (see [95], III.5.1), for $\Gamma(E)$ and $\Gamma(F)$ are Fréchet and then barreled (see [95], II.7.2, Cor. 2), and thus extends to a morphism of locally convex $C^\infty(M \times N)$-modules (2.3.5) by the universal property of the convenient completion. The proof for $\bar{\kappa}$ is precisely the same using that the inclusion mapping $C_c^\infty(M) \otimes_\beta C_c^\infty(N) \to C_c^\infty(M \times N)$ is also the convenient completion of its domain by Lemma 2.3.12. □

2.3.14. *Remark.* Combining the isomorphisms (2.3.6) together with Lemma 1.4.38, we obtain the isomorphism

$$\Gamma_c(E) \hat{\otimes}_i \Gamma_c(F) \to \Gamma_c(E \boxtimes F). \tag{2.3.10}$$

of LCS explained in [47], II.3.3, Example 4, pp. 82–85. Moreover, from the arguments appearing in precisely the same example of the previous reference (note that one can always construct a continuous norm on the space of sections of compact support of a vector bundle by using a Riemannian metric on it) we see that the canonical continuous linear map

$$\tilde{\varepsilon}_{\Gamma_c(E),\Gamma_c(F)} : \Gamma_c(E)\tilde{\otimes}_\beta\Gamma_c(F) \to \Gamma_c(E)\hat{\otimes}_\pi\Gamma_c(F) \tag{2.3.11}$$

is bijective, but not an isomorphism of LCS unless the manifolds are compact. Using some results of Grothendieck stated *loc. cit.*, we will provide a proof of a slightly more general version of this result in Lemma 4.2.3

2.3.15.*Remark.* Since the spaces of all sections of the vector bundles E and F over the manifold M are finitely generated topologically projective modules over the Fréchet algebra $C^\infty(M)$, one may use (1.6.3) instead of (1.7.22) in the previous proof, to obtain that the inclusion $C^\infty(M)\otimes_\pi C^\infty(N) \to C^\infty(M\times N)$ induces an isomorphism

$$\kappa : \Gamma(E)\hat{\otimes}_\pi\Gamma(E) \to \Gamma(E\boxtimes F) \tag{2.3.12}$$

of locally convex modules over the locally m-convex algebra $C^\infty(M)$. This is precisely the same map as (2.3.5), for the metrizability of the LCS $\Gamma(E)$ and $\Gamma(E)$ implies that the bornological (resp., convenient) tensor product coincides with the projective tensor product (resp., completed projective tensor product) (see 1.4.34).

2.3.16. Given $\sigma \in \Gamma(E)$ and $\sigma' \in \Gamma(F)$, we shall denote the image of $\sigma\otimes\sigma'$ under the previous map κ by $\sigma\boxtimes\sigma'$, and call it the *external tensor product* of the sections σ and σ'. The same applies to sections $\sigma \in \Gamma_c(E)$ and $\sigma' \in \Gamma_c(F)$ by means of the map $\bar{\kappa}$.

2.3.17. We summarize the topological information about the spaces of smooth sections. Let E and F be two vector bundles (of finite rank) over smooth manifolds M and N, respectively. Then,

(a) $C^\infty(M)$ has a unique structure of locally m-convex algebra that is Fréchet (see Theorem 2.3.2);

(b) the finitely generated projective $C^\infty(M)$-module $\Gamma(E)$ has a unique structure of Fréchet locally convex module over $C^\infty(M)$ and any morphism of $C^\infty(M)$-modules from $\Gamma(E)$ to $\Gamma(F)$ is continuous (see Corollary 2.3.6);

(c) the $C^\infty(M)$-module $\Gamma_c(E)$ is projective and it is a strict countable inductive limit of the Fréchet locally convex $C^\infty(M)$-modules $\Gamma_K(E)$, where K runs over the (a countable exhaustive collection of) compact subsets of M, so an (LF)-space (see Corollary 2.3.7 and 2.3.9), and any morphism of $C^\infty(M)$-modules from $\Gamma_c(E)$ to $\Gamma_c(F)$ is continuous (see Corollary 2.3.10);

(d) as LCS $\Gamma(E)$ and $\Gamma_K(E)$ are Fréchet (so barreled and bornological), nuclear, thus reflexive (see 2.3.1, 2.3.5, and 2.3.9), and Schwartz; whereas $\Gamma_c(E)$ is a strict countable inductive limit of spaces of the form $\Gamma_K(E)$, for K running over a countable exhausting family of compacts subsets of M, which in turn implies that $\Gamma_c(E)$ is an (LF)-space (so barreled, bornological, and complete) that is nuclear, reflexive and Schwartz (see 2.3.9);

(e) $C^\infty(M\times N) \simeq C^\infty(M)\tilde{\otimes}_\beta C^\infty(N)$ is an isomorphism of locally m-convex algebras (see Lemma 2.3.11);

(f) there are natural isomorphisms $\Gamma(E\boxtimes F) \simeq \Gamma(E)\tilde{\otimes}_\beta\Gamma(F)$ and $\Gamma_c(E\boxtimes F) \simeq \Gamma_c(E)\tilde{\otimes}_\beta\Gamma_c(F)$ of bornological locally convex modules over $C^\infty(M\times N)$ (see Proposition 2.3.13).

Chapter 3

Some results on tensor products

New

3.1 The tensor and symmetric coalgebra and the cofree comodule in a symmetric monoidal category

Well-known

3.1.1. Let $\mathscr{C}$ be a k-linear symmetric monoidal category that is complete and cocomplete and whose tensor product commutes with colimits on each side. We recall that the subcoalgebra $\Sigma_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}_0} \Sigma^m_{\mathscr{C}} X$ (resp., $\Sigma^+_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}} \Sigma^m_{\mathscr{C}} X$) of the tensor coalgebra $T_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}_0} X^{\otimes_{\mathscr{C}} m}$ (resp., $T^+_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}} X^{\otimes_{\mathscr{C}} m}$) provided with the coproduct given by deconcatenation is cofree in the category of conilpotent cocommutative coaugmented $\mathscr{C}$-coalgebras (resp., conilpotent cocommutative noncounitary $\mathscr{C}$-coalgebras). This holds for any field k, not necessarily of characteristic zero. Analogously, the symmetric construction (resp., reduced symmetric construction) $S_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}_0} S^m_{\mathscr{C}} X$ (resp., $S^+_{\mathscr{C}} X = \oplus_{m\in\mathbb{N}} S^m_{\mathscr{C}} X$) has a canonical structure of (resp., noncounitary) coaugmented $\mathscr{C}$-coalgebra (resp., noncounitary $\mathscr{C}$-coalgebra) given by

$$\Delta(x_1 \dots x_m) = \sum_{(I,J)\in \mathrm{Par}_0(m,2)} x_I \otimes x_J \left(\text{resp., } \Delta(x_1 \dots x_m) = \sum_{(I,J)\in \mathrm{Par}(m,2)} x_I \otimes x_J\right), \tag{3.1.1}$$

where $\mathrm{Par}_0(m,2)$ is the set of pairs (I,J), where $I,J \subseteq \{1,\dots,m\}$, $I\cap J = \emptyset$ and $I \cup J = \{1,\dots,m\}$ (resp., $\mathrm{Par}(m,2)$ is the set of pairs (I,J), where $I,J \subseteq \{1,\dots,m\}$ are nonempty sets, $I\cap J = \emptyset$ and $I \cup J = \{1,\dots,m\}$), and $x_I = x_{i_1} \otimes \dots \otimes x_{i_{m'}}$, for $I = \{i_1 < \dots < i_{m'}\}$ (*cf.* [67], 1.2.9). We recall that $S^0_{\mathscr{C}} X$ indicates the unit of the symmetric monoidal category, and $S_{\mathscr{C}} X$ is coaugmented by means of the trivial inclusion of the unity of the monoidal category $\mathscr{C}$ into $S_{\mathscr{C}} X$. Of course the previous identities in (3.1.1) should be understood as indicating the corresponding action of the symmetric group $\mathbb{S}_m$ on the tensor factors of $S^m_{\mathscr{C}} X$, and not on elements of that object, and $x_\emptyset$ indicates the coaugmentation of $S_{\mathscr{C}} X$.

3.1.2. Moreover, the coalgebra $S_{\mathscr{C}} X$ (resp., $S^+_{\mathscr{C}} X$) is isomorphic to $\Sigma_{\mathscr{C}} X$ (resp., $\Sigma^+_{\mathscr{C}} X$), with the isomorphism given by the map from $S_{\mathscr{C}} X$ (resp., $S^+_{\mathscr{C}} X$) to $\Sigma_{\mathscr{C}} X$ (resp., $\Sigma^+_{\mathscr{C}} X$) whose m-th component $S^m_{\mathscr{C}} X \to \Sigma^m_{\mathscr{C}} X$ is just $m!\bar{e}$, where $\bar{e} : S^m_{\mathscr{C}} X \to \Sigma^m_{\mathscr{C}} X$ is the morphism stated in Proposition 1.1.4 corresponding to the object $X^{\otimes_{\mathscr{C}} m}$ under the action of $\mathbb{S}_m$. This means that, for any conilpotent cocommutative coaugmented $\mathscr{C}$-coalgebra (resp., conilpotent cocommutative noncounitary $\mathscr{C}$-coalgebra) C, the map $\mathrm{Hom}_{{}^c\mathrm{Cog}_\bullet}(C, S_{\mathscr{C}} X) \to \mathrm{Hom}_{\mathscr{C}}(C,X)$ (resp., $\mathrm{Hom}_{{}^c\mathrm{Cog}}(C, S^+_{\mathscr{C}} X) \to \mathrm{Hom}_{\mathscr{C}}(C,X)$) given by composing with the projection $p_1 : S_{\mathscr{C}} X \to X$ (resp., $p_1 : S^+_{\mathscr{C}} X \to X$) is an isomorphism, where the first homomorphism space is in the category of cocommutative coaugmented $\mathscr{C}$-coalgebras (resp., cocommutative noncounitary $\mathscr{C}$-coalgebras). The inverse map sends $g \in \mathrm{Hom}_{\mathscr{C}}(C,X)$ to

$$\sum_{m\in\mathbb{N}_0} \frac{1}{m!} \pi_{X,m} \circ \bar{g}^{\otimes_{\mathscr{C}} m} \circ \Delta^{(m)}_{\bar{C}} \circ \pi^{\otimes_{\mathscr{C}} m} \left(\text{resp., } \sum_{m\in\mathbb{N}} \frac{1}{m!} \pi_{X,m} \circ g^{\otimes_{\mathscr{C}} m} \circ \Delta^{(m)}_{C}\right), \tag{3.1.2}$$

where $\bar{C} = C/\mathrm{Im}(\eta_C)$ is the quotient of C by its coaugmentation η_C, $\pi : C \to \bar{C}$ the canonical projection, $\Delta_{\bar{C}}$ is the noncounitary coproduct of $\bar{C}$ induced by that of C, $\Delta^{(m)}_{\bar{C}} = (\Delta^{(m-1)}_{\bar{C}} \otimes_{\mathscr{C}} \mathrm{id}_{\bar{C}}) \circ \Delta_{\bar{C}}$, for $m \geq 2$, $\Delta^{(1)}_{\bar{C}} = \mathrm{id}_{\bar{C}}$, $\bar{g} : \bar{C} \to S^{+}_{\mathscr{C}}X$ is the map induced by g, $\Delta^{(0)}_{\bar{C}} \circ \pi^{\otimes_{\mathscr{C}} 0} = \epsilon_C$, $\bar{g}^{\otimes_{\mathscr{C}} 0}$ is the coaugmentation of $S_{\mathscr{C}}X$, and $\pi_{X,m} : X^{\otimes_{\mathscr{C}} m} \to S^{m}_{\mathscr{C}}X$ is the canonical projection. To shorten the notation in (3.1.2) we will write $\bar{\Delta}^{(m)}_{C}$ instead of $\Delta^{(m)}_{\bar{C}} \circ \pi^{\otimes_{\mathscr{C}} m}$, and we will usually omit the morphism $\pi_{X,m}$ and the bars on the morphism g as well. For a nice proof of the fact that (3.1.2) gives the purported inverse of the previous map $\mathrm{Hom}_{{}^{c}\mathrm{Cog}_{\bullet}}(C, S_{\mathscr{C}}X) \to \mathrm{Hom}_{\mathscr{C}}(C, X)$ (resp., $\mathrm{Hom}_{{}^{c}\mathrm{Cog}}(C, S^{+}_{\mathscr{C}}X) \to \mathrm{Hom}_{\mathscr{C}}(C, X)$) in the particular case of the category of super vector spaces, we refer the reader to [4], Thm. III.2.1. The general proof is analogous.

3.1.3. It is direct to prove that, given $C = S_{\mathscr{C}}X$ (resp., $C = S^{+}_{\mathscr{C}}X$), an endomorphism f of the counitary (resp., noncounitary) $\mathscr{C}$-coalgebra C is an isomorphism if and only if $p_1 \circ f \circ i_1$ is an isomorphism in the monoidal category, where i_1 denotes the canonical monomorphism $X \to C$. The only nontrivial implication of that statement follows from writing down the expressions $p_1 \circ f \circ g \in \mathrm{Hom}_{\mathscr{C}}(C, X)$ for any two endomorphisms f and g of C, noticing that $f \circ g = \mathrm{id}_C$ if and only if $p_1 \circ f \circ g = p_1$, and realizing in this case that one of the endomorphisms can be recursively written in terms of the other if the precomposition of the latter with i_1 and its postcomposition with p_1 is an isomorphism of $\mathscr{C}$. Furthermore, $S_{\mathscr{C}}X$ (resp., $S^{+}_{\mathscr{C}}X$) is a unitary and counitary (resp., nonunitary and noncounitary) $\mathscr{C}$-bialgebra for the usual commutative algebra structure. The unit of $S_{\mathscr{C}}X$ is given by the canonical inclusion of the unit of $\mathscr{C}$, which is just $S^{0}_{\mathscr{C}}X$, inside of $S_{\mathscr{C}}X$. We will refer to this unitary and counitary (resp., nonunitary and noncounitary) $\mathscr{C}$-bialgebra structure on $S_{\mathscr{C}}X$ (resp., $S^{+}_{\mathscr{C}}X$) as *canonical*.

3.1.4. If C denotes a counitary $\mathscr{C}$-coalgebra and Y is any object of the monoidal symmetric category of $\mathscr{C}$, the right comodule $\mathrm{Cof}(Y, C) = Y \otimes_{\mathscr{C}} C$ in $\mathscr{C}$ provided with coaction induced by the coproduct of C is cofree, *i.e.* for any right comodule Z over C in $\mathscr{C}$, the map $\mathrm{Hom}^{C}(Z, \mathrm{Cof}(Y, C)) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ given by composing with $\mathrm{id}_Y \otimes_{\mathscr{C}} \epsilon_C$ is an isomorphism, where ϵ_C is the counit of C, and the first homomorphism space is in the category of right C-comodules in $\mathscr{C}$. The inverse is just given by sending $f \in \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ to $(f \otimes_{\mathscr{C}} \mathrm{id}_C) \circ \rho_Z$, where $\rho_Z : Z \to Z \otimes_{\mathscr{C}} C$ is the coaction of Z. Suppose that C is also coaugmented by means of η_C. It is an direct to prove that, given $Z = \mathrm{Cof}(Y, C)$, an endomorphism f of the C-comodule Z in $\mathscr{C}$ is an isomorphism if and only if $(\mathrm{id}_Y \otimes_{\mathscr{C}} \epsilon_C) \circ f \circ (\mathrm{id}_Y \otimes_{\mathscr{C}} \eta_C)$ is an isomorphism of Y in the underlying monoidal category $\mathscr{C}$. The proof is similar to the analogous property mentioned for symmetric coalgebras in $\mathscr{C}$. The same results hold for left comodules.

3.1.5. Let us introduce the following terminology. If $\{X_\ell : \ell \in \mathcal{L}\}$ is a finite family of (different) objects of $\mathscr{C}$. We denote by $X_{\ell_1} \circledS_{\mathscr{C}} \dots \circledS_{\mathscr{C}} X_{\ell_q}$ the image of $X_{\ell_1} \otimes_{\mathscr{C}} \cdots \otimes_{\mathscr{C}} X_{\ell_q}$ under the canonical projection of $(\oplus_{\ell \in \mathcal{L}} X_\ell)^{\otimes_{\mathscr{C}} q}$ onto $(\oplus_{\ell \in \mathcal{L}} X_\ell)^{\otimes_{\mathscr{C}} q}/\mathbb{S}_q$. By Proposition 1.1.4, the latter is isomorphic with the object given by the invariants $((\oplus_{\ell \in \mathcal{L}} X_\ell)^{\otimes_{\mathscr{C}} q})^{\mathbb{S}_q}$. Note that the tensor product $X \circledS_{\mathscr{C}} \dots \circledS_{\mathscr{C}} X$ of q equal factors is just $S^{q}_{\mathscr{C}}X$.

3.1.6. In general, we will omit the subscript $\mathscr{C}$ from the tensor products $\otimes_{\mathscr{C}}$ and $\circledS_{\mathscr{C}}$, the space of invariants $\Sigma^{m}_{\mathscr{C}}X$ or the space of coinvariants $S^{m}_{\mathscr{C}}X$, as well the from the tensor algebra, etc. if it is clear from the context what symmetric monoidal category we are dealing with, and we will write thus $\otimes$, $\circledS$, $\Sigma^m X$, $S^m X$, etc. This will also be typically the case when $\mathscr{C}$ is the category of vector spaces over k. For the category of modules over a commutative k-algebra A, we will usually add A as subscript, *e.g.* $\otimes_A$, $\circledS_A$, $S^m_A X$, $\Sigma^m_A X$. On the other hand, in many situations we will simultaneously consider several tensor products on the same category $\mathscr{C}$, e.g. $\otimes$ and $\boxtimes$, for which we will prefer to use the symbol of the corresponding tensor product as subscript of T, S or Σ to distinguish them.

3.2 More on symmetric monoidal categories and bialgebras over them

3.2.1. The following definition can be found in [2], 6.1.

Recent **3.2.2 Definition.** *A* double monoidal category *is a tuple* $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$, *where* $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$ *and* $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$

are monoidal categories. It will be called symmetric *if it is further provided with a natural morphism* τ *such that* $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ *is a symmetric monoidal category. A* 2-monoidal category *is a double monoidal category together with a natural morphism*

$$\mathrm{sh}_{A,B,C,D} : (A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D) \to (A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D) \tag{3.2.1}$$

in $\mathscr{C}$ *and three morphisms*

$$\mu_{\boxtimes} : I_{\otimes} \boxtimes_{\mathscr{C}} I_{\otimes} \to I_{\otimes}, \quad \Delta_{\otimes} : I_{\boxtimes} \to I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes}, \text{ and } \nu : I_{\boxtimes} \to I_{\otimes}, \tag{3.2.2}$$

in $\mathscr{C}$ *such that the following conditions hold. We will denote the associativity isomorphism and the left and right units of* $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$ *by* $a^{\boxtimes}_{X,Y,Z}$, $\ell^{\boxtimes}_X$ *and* $r^{\boxtimes}_X$*, respectively, and those of* $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$ *by* $a^{\otimes}_{X,Y,Z}$, $\ell^{\otimes}_X$ *and* $r^{\otimes}_X$*, respectively.*

(i) Given $X_1, Y_1, X_2, Y_2, X_3, Y_3$ *objects in* $\mathscr{C}$*, the following diagrams*

$$\begin{array}{ccc}
 & \big((X_1 \otimes_{\mathscr{C}} Y_1) \boxtimes_{\mathscr{C}} (X_2 \otimes_{\mathscr{C}} Y_2)\big) \boxtimes_{\mathscr{C}} (X_3 \otimes_{\mathscr{C}} Y_3) & \\
\swarrow{\scriptstyle \mathrm{sh}_{X_1,Y_1,X_2,Y_2} \boxtimes_{\mathscr{C}} \mathrm{id}_{X_3 \otimes_{\mathscr{C}} Y_3}} & & \searrow{\scriptstyle a^{\boxtimes}_{X_1 \otimes_{\mathscr{C}} Y_1, X_2 \otimes_{\mathscr{C}} Y_2, X_3 \otimes_{\mathscr{C}} Y_3}} \\
\big((X_1 \boxtimes_{\mathscr{C}} X_2) \otimes_{\mathscr{C}} (Y_1 \boxtimes_{\mathscr{C}} Y_2)\big) \boxtimes_{\mathscr{C}} (X_3 \otimes_{\mathscr{C}} Y_3) & & (X_1 \otimes_{\mathscr{C}} Y_1) \boxtimes_{\mathscr{C}} \big((X_2 \otimes_{\mathscr{C}} Y_2) \boxtimes_{\mathscr{C}} (X_3 \otimes_{\mathscr{C}} Y_3)\big) \\
\downarrow{\scriptstyle \mathrm{sh}_{X_1 \boxtimes_{\mathscr{C}} X_2, Y_1 \boxtimes_{\mathscr{C}} Y_2, X_3, Y_3}} & & \downarrow{\scriptstyle \mathrm{id}_{X_1 \otimes_{\mathscr{C}} Y_1} \boxtimes_{\mathscr{C}} \mathrm{sh}_{X_2,Y_2,X_3,Y_3}} \\
\big((X_1 \boxtimes_{\mathscr{C}} X_2) \boxtimes_{\mathscr{C}} X_3\big) \otimes_{\mathscr{C}} \big((Y_1 \boxtimes_{\mathscr{C}} Y_2) \boxtimes_{\mathscr{C}} Y_3\big) & & (X_1 \otimes_{\mathscr{C}} Y_1) \boxtimes_{\mathscr{C}} \big((X_2 \boxtimes_{\mathscr{C}} X_3) \otimes_{\mathscr{C}} (Y_2 \boxtimes_{\mathscr{C}} Y_3)\big) \\
\searrow{\scriptstyle a^{\boxtimes}_{X_1,X_2,X_3} \otimes_{\mathscr{C}} a^{\boxtimes}_{Y_1,Y_2,Y_3}} & & \swarrow{\scriptstyle \mathrm{sh}_{X_1,Y_1,X_2 \boxtimes_{\mathscr{C}} X_3, Y_2 \boxtimes_{\mathscr{C}} Y_3}} \\
 & \big(X_1 \boxtimes_{\mathscr{C}} (X_2 \boxtimes_{\mathscr{C}} X_3)\big) \otimes_{\mathscr{C}} \big(Y_1 \boxtimes_{\mathscr{C}} (Y_2 \boxtimes_{\mathscr{C}} Y_3)\big) &
\end{array} \tag{3.2.3}$$

and

$$\begin{array}{ccc}
 & \big((X_1 \boxtimes_{\mathscr{C}} Y_1) \otimes_{\mathscr{C}} (X_2 \boxtimes_{\mathscr{C}} Y_2)\big) \otimes_{\mathscr{C}} (X_3 \boxtimes_{\mathscr{C}} Y_3) & \\
\nearrow{\scriptstyle \mathrm{sh}_{X_1,X_2,Y_1,Y_2} \otimes_{\mathscr{C}} \mathrm{id}_{X_3 \boxtimes_{\mathscr{C}} Y_3}} & & \searrow{\scriptstyle a^{\otimes}_{X_1 \boxtimes_{\mathscr{C}} Y_1, X_2 \boxtimes_{\mathscr{C}} Y_2, X_3 \boxtimes_{\mathscr{C}} Y_3}} \\
\big((X_1 \otimes_{\mathscr{C}} X_2) \boxtimes_{\mathscr{C}} (Y_1 \otimes_{\mathscr{C}} Y_2)\big) \otimes_{\mathscr{C}} (X_3 \boxtimes_{\mathscr{C}} Y_3) & & (X_1 \boxtimes_{\mathscr{C}} Y_1) \otimes_{\mathscr{C}} \big((X_2 \boxtimes_{\mathscr{C}} Y_2) \otimes_{\mathscr{C}} (X_3 \boxtimes_{\mathscr{C}} Y_3)\big) \\
\uparrow{\scriptstyle \mathrm{sh}_{X_1 \otimes_{\mathscr{C}} X_2, X_3, Y_1 \otimes_{\mathscr{C}} Y_2, Y_3}} & & \uparrow{\scriptstyle \mathrm{id}_{X_1 \boxtimes_{\mathscr{C}} Y_1} \otimes_{\mathscr{C}} \mathrm{sh}_{X_2,X_3,Y_2,Y_3}} \\
\big((X_1 \otimes_{\mathscr{C}} X_2) \otimes_{\mathscr{C}} X_3\big) \boxtimes_{\mathscr{C}} \big((Y_1 \otimes_{\mathscr{C}} Y_2) \otimes_{\mathscr{C}} Y_3\big) & & (X_1 \boxtimes_{\mathscr{C}} Y_1) \otimes_{\mathscr{C}} \big((X_2 \otimes_{\mathscr{C}} X_3) \boxtimes_{\mathscr{C}} (Y_2 \otimes_{\mathscr{C}} Y_3)\big) \\
\searrow{\scriptstyle a^{\otimes}_{X_1,X_2,X_3} \boxtimes_{\mathscr{C}} a^{\otimes}_{Y_1,Y_2,Y_3}} & & \nearrow{\scriptstyle \mathrm{sh}_{X_1, X_2 \otimes_{\mathscr{C}} X_3, Y_1, Y_2 \otimes_{\mathscr{C}} Y_3}} \\
 & \big(X_1 \otimes_{\mathscr{C}} (X_2 \otimes_{\mathscr{C}} X_3)\big) \boxtimes_{\mathscr{C}} \big(Y_1 \otimes_{\mathscr{C}} (Y_2 \otimes_{\mathscr{C}} Y_3)\big) &
\end{array} \tag{3.2.4}$$

commute.

(ii) The triple $(I_\otimes, \mu_\boxtimes, \nu)$ is a unitary algebra in the monoidal category $(\mathscr{C}, \boxtimes_\mathscr{C}, I_\boxtimes)$, and the triple $(I_\boxtimes, \Delta_\otimes, \nu)$ is a counitary coalgebra in the monoidal category $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes)$.

(iii) For any pair of objects X and Y in $\mathscr{C}$, the diagrams

$$\begin{array}{ccc} I_\boxtimes \boxtimes_\mathscr{C} (X \otimes_\mathscr{C} Y) & \xrightarrow{\Delta_\otimes \boxtimes_\mathscr{C} \mathrm{id}_{X\otimes_\mathscr{C} Y}} & (I_\boxtimes \otimes_\mathscr{C} I_\boxtimes) \boxtimes_\mathscr{C} (X \otimes_\mathscr{C} Y) \\ \wr\downarrow \ell^\boxtimes_{X\otimes_\mathscr{C} Y} & & \downarrow \mathrm{sh}_{I_\boxtimes, I_\boxtimes, X, Y} \\ X \otimes_\mathscr{C} Y & \xrightarrow[\sim]{(\ell^\boxtimes_X \otimes_\mathscr{C} \ell^\boxtimes_Y)^{-1}} & (I_\boxtimes \boxtimes_\mathscr{C} X) \otimes_\mathscr{C} (I_\boxtimes \boxtimes_\mathscr{C} Y) \end{array} \qquad \begin{array}{ccc} (X \otimes_\mathscr{C} Y) \boxtimes_\mathscr{C} I_\boxtimes & \xrightarrow{\mathrm{id}_{X\otimes_\mathscr{C} Y} \boxtimes_\mathscr{C} \Delta_\otimes} & (X \otimes_\mathscr{C} Y) \boxtimes_\mathscr{C} (I_\boxtimes \otimes_\mathscr{C} I_\boxtimes) \\ \wr\downarrow r^\boxtimes_{X\otimes_\mathscr{C} Y} & & \downarrow \mathrm{sh}_{X, Y, I_\boxtimes, I_\boxtimes} \\ X \otimes_\mathscr{C} Y & \xrightarrow[\sim]{(r^\boxtimes_X \otimes_\mathscr{C} r^\boxtimes_Y)^{-1}} & (X \boxtimes_\mathscr{C} I_\boxtimes) \otimes_\mathscr{C} (Y \boxtimes_\mathscr{C} I_\boxtimes) \end{array}$$

and

$$\begin{array}{ccc} (I_\otimes \otimes_\mathscr{C} X) \boxtimes_\mathscr{C} (I_\otimes \otimes_\mathscr{C} Y) & \xrightarrow[\sim]{\ell^\otimes_X \boxtimes_\mathscr{C} \ell^\otimes_Y} & X \boxtimes_\mathscr{C} Y \\ \downarrow \mathrm{sh}_{I_\otimes, X, I_\otimes, Y} & & \wr\downarrow (\ell^\otimes_{X\boxtimes_\mathscr{C} Y})^{-1} \\ (I_\otimes \boxtimes_\mathscr{C} I_\otimes) \otimes_\mathscr{C} (X \boxtimes_\mathscr{C} Y) & \xrightarrow{\mu_\boxtimes \otimes_\mathscr{C} \mathrm{id}_{X\boxtimes_\mathscr{C} Y}} & I_\otimes \otimes_\mathscr{C} (X \boxtimes_\mathscr{C} Y) \end{array} \qquad \begin{array}{ccc} (X \otimes_\mathscr{C} I_\otimes) \boxtimes_\mathscr{C} (Y \otimes_\mathscr{C} I_\otimes) & \xrightarrow[\sim]{r^\otimes_X \boxtimes_\mathscr{C} r^\otimes_Y} & X \boxtimes_\mathscr{C} Y \\ \downarrow \mathrm{sh}_{X, I_\otimes, Y, I_\otimes} & & (r^\otimes_{X\boxtimes_\mathscr{C} Y})^{-1} \downarrow\wr \\ (X \boxtimes_\mathscr{C} Y) \otimes_\mathscr{C} (I_\otimes \boxtimes_\mathscr{C} I_\otimes) & \xrightarrow{\mathrm{id}_{X\boxtimes_\mathscr{C} Y} \otimes_\mathscr{C} \mu_\boxtimes} & (X \boxtimes_\mathscr{C} Y) \otimes_\mathscr{C} I_\otimes \end{array}$$

commute.

Moreover, we call a 2-monoidal category $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes, \boxtimes_\mathscr{C}, I_\boxtimes, \mathrm{sh})$ symmetric *if the monoidal category $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes)$ is provided with a symmetric braiding τ such that $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes, \tau)$ is a symmetric monoidal category, $(I_\boxtimes, \Delta_\otimes, \nu)$ is a cocommutative counitary coalgebra in $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes, \tau)$, and for any objects A, B, C and D in $\mathscr{C}$ the diagram*

$$\begin{array}{ccc} (A \otimes_\mathscr{C} B) \boxtimes_\mathscr{C} (C \otimes_\mathscr{C} D) & \xrightarrow{\mathrm{sh}_{A,B,C,D}} & (A \boxtimes_\mathscr{C} C) \otimes_\mathscr{C} (B \boxtimes_\mathscr{C} D) \\ \downarrow \tau(A,B)\otimes\tau(C,D) & & \downarrow \tau(A\boxtimes_\mathscr{C} C, B\boxtimes_\mathscr{C} D) \\ (B \otimes_\mathscr{C} A) \boxtimes_\mathscr{C} (D \otimes_\mathscr{C} C) & \xrightarrow{\mathrm{sh}_{B,A,D,C}} & (B \boxtimes_\mathscr{C} D) \otimes_\mathscr{C} (A \boxtimes_\mathscr{C} C) \end{array} \tag{3.2.5}$$

commutes.[1]

3.2.3. *Remark.* A double monoidal category $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes, \boxtimes_\mathscr{C}, I_\boxtimes)$ with structure morphisms (3.2.1) and (3.2.2) is 2-monoidal if either of the following equivalent conditions hold:

(a) the functors $\otimes_\mathscr{C} : (\mathscr{C}, \boxtimes_\mathscr{C}, I_\boxtimes) \times (\mathscr{C}, \boxtimes_\mathscr{C}, I_\boxtimes) \to (\mathscr{C}, \boxtimes_\mathscr{C}, I_\boxtimes)$ and $I_\otimes : \mathscr{k} \to (\mathscr{C}, \boxtimes_\mathscr{C}, I_\boxtimes)$ are lax monoidal,

(b) the functors $\boxtimes_\mathscr{C} : (\mathscr{C}, \otimes_\mathscr{C}, I_\otimes) \times (\mathscr{C}, \otimes_\mathscr{C}, I_\otimes) \to (\mathscr{C}, \otimes_\mathscr{C}, I_\otimes)$ and $I_\boxtimes : \mathscr{k} \to (\mathscr{C}, \otimes_\mathscr{C}, I_\otimes)$ are oplax monoidal,

where $\mathscr{k}$ indicates the k-linear monoidal category with one object whose space of endomorphisms is k. The proof just follows from writing down the definitions (see [2], Prop. 6.4). We may equivalently rephrase (b) (resp., (a)) as saying that a 2-monoidal category is a pseudomonoid in the monoidal 2-category $\mathscr{opl}(\mathrm{Cat})$ (resp., $\mathscr{l}(\mathrm{Cat})$) whose 0-cells are monoidal categories, whose 1-cells are oplax (resp., lax) monoidal functors, and whose 2-cells are monoidal natural transformations (see [2], Prop. 6.73).

3.2.4. *Remark.* Let $(\mathscr{C}, \otimes_\mathscr{C}, I_\otimes, \tau, \boxtimes_\mathscr{C}, I_\boxtimes, \mathrm{sh})$ be a symmetric 2-monoidal category, and $X_1, Y_1, \dots, X_m, Y_m$ a collection of objects in $\mathscr{C}$ for $m \in \mathbb{N}$. We have the morphism

$$\mathrm{sh}_{X_1, Y_1, \dots, X_m, Y_m} : (X_1 \otimes_\mathscr{C} Y_1) \boxtimes_\mathscr{C} \cdots \boxtimes_\mathscr{C} (X_m \otimes_\mathscr{C} Y_m) \to (X_1 \boxtimes_\mathscr{C} \cdots \boxtimes_\mathscr{C} X_m) \otimes_\mathscr{C} (Y_1 \boxtimes_\mathscr{C} \cdots \boxtimes_\mathscr{C} Y_m) \tag{3.2.6}$$

[1] Note that our definition of symmetric 2-monoidal category coincides with the notion of $\otimes_\mathscr{C}$-symmetric 2-monoidal category in [2], Def. 6.5. Since in all of our major examples $\boxtimes_\mathscr{C}$ will not be braided, we drop the explicit mention of the tensor product $\otimes_\mathscr{C}$ for simplicity.

in $\mathscr{C}$ given by $\mathrm{sh}_{X_1,Y_1} = \mathrm{id}_{X_1 \otimes_{\mathscr{C}} Y_1}$, and

$$\mathrm{sh}_{X_1,Y_1,\dots,X_m,Y_m} = \mathrm{sh}_{X_1,Y_1,\dots,X_{m-2},Y_{m-2},X_{m-1}\boxtimes X_m,Y_{m-1}\boxtimes Y_m} \circ \big(\mathrm{id}_{(X_1\otimes_{\mathscr{C}} Y_1)\boxtimes_{\mathscr{C}}\cdots\boxtimes_{\mathscr{C}}(X_{m-2}\otimes_{\mathscr{C}} Y_{m-2})} \boxtimes_{\mathscr{C}} \mathrm{sh}_{X_{m-1},Y_{m-1},X_m,Y_m}\big),$$

for $m \geq 2$. By item (i) in Definition 3.2.2 we may equivalently define (3.2.6) by applying successively (3.2.1) in a different order, since all of them coincide (see [2], 6.2). We shall usually omit the subscripts of the map (3.2.6), because they will be clear from the context.

3.2.5. For simplicity, we will denote a symmetric 2-monoidal category by the data $(\mathscr{C}, \otimes_{\mathscr{C}}, I_\otimes, \tau, \boxtimes_{\mathscr{C}}, I_\boxtimes, \mathrm{sh})$, omitting the other structure, because we will follow the notation of the previous definition.

3.2.6 Definition. *A symmetric 2-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_\otimes, \tau, \boxtimes_{\mathscr{C}}, I_\boxtimes, \mathrm{sh})$ is called* framed *if there is a symmetric monoidal category $(\mathscr{C}', \boxtimes_{\mathscr{C}'}, I'_\boxtimes, \tau')$ and a faithful functor $F : \mathscr{C} \to \mathscr{C}'$ that is symmetric lax monoidal for the symmetric monoidal structure $(\mathscr{C}, \otimes_{\mathscr{C}}, I_\otimes, \tau)$, with coherence morphisms φ_0 and φ_2, and it is strong monoidal for $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_\boxtimes)$, with coherence isomorphisms ψ_0 and ψ_2, such that* New

$$\begin{array}{ccc}
F(A)\boxtimes_{\mathscr{C}'}F(B)\boxtimes_{\mathscr{C}'}F(C)\boxtimes_{\mathscr{C}'}F(D) & \xrightarrow{\varphi_2(A,B)\boxtimes_{\mathscr{C}'}\varphi_2(C,D)} & F(A\otimes_{\mathscr{C}}B)\boxtimes_{\mathscr{C}'}F(C\otimes_{\mathscr{C}}D) \\
\downarrow{\scriptstyle \mathrm{id}_{F(A)}\boxtimes_{\mathscr{C}'}\tau'_\boxtimes(F(B),F(C))\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(D)}} & & \downarrow{\scriptstyle \psi_2(A\otimes_{\mathscr{C}}B,C\otimes_{\mathscr{C}}D)\ \wr} \\
F(A)\boxtimes_{\mathscr{C}'}F(C)\boxtimes_{\mathscr{C}'}F(B)\boxtimes_{\mathscr{C}'}F(D) & & F\big((A\otimes_{\mathscr{C}}B)\boxtimes_{\mathscr{C}}(C\otimes_{\mathscr{C}}D)\big) \\
\wr\downarrow{\scriptstyle \psi_2(A,C)\boxtimes_{\mathscr{C}'}\psi_2(B,D)} & & \downarrow{\scriptstyle F(\mathrm{sh}_{A,B,C,D})} \\
F(A\boxtimes_{\mathscr{C}}C)\boxtimes_{\mathscr{C}'}F(B\boxtimes_{\mathscr{C}}D) & \xrightarrow{\varphi_2(A\boxtimes_{\mathscr{C}}C,D\boxtimes_{\mathscr{C}}D)} & F\big((A\boxtimes_{\mathscr{C}}C)\otimes_{\mathscr{C}}(B\boxtimes_{\mathscr{C}}D)\big)
\end{array} \tag{3.2.7}$$

commutes for all objects A, B, C, and D of $\mathscr{C}$, as well as

$$\begin{array}{ccc}
I'_\boxtimes\boxtimes_{\mathscr{C}'}I'_\boxtimes & \xrightarrow[\sim]{\ell'_{I'_\boxtimes}} & I'_\boxtimes \\
\downarrow{\scriptstyle \varphi_0\boxtimes_{\mathscr{C}'}\varphi_0} & & \\
F(I_\otimes)\boxtimes_{\mathscr{C}'}F(I_\otimes) & & \downarrow{\scriptstyle \varphi_0} \\
\downarrow{\scriptstyle \psi_2(I_\otimes,I_\otimes)\ \wr} & & \\
F(I_\otimes\boxtimes_{\mathscr{C}}I_\otimes) & \xrightarrow{F(\mu_\boxtimes)} & F(I_\otimes)
\end{array}
\qquad
\begin{array}{ccc}
I'_\boxtimes & \xrightarrow{\varphi_0} & \\
\downarrow{\scriptstyle \psi_0\ \wr} & & F(I_\otimes) \\
F(I_\boxtimes) & \xrightarrow{F(\nu)} &
\end{array}
\qquad
\begin{array}{ccc}
I'_\boxtimes & \xrightarrow[\sim]{\psi_0} & F(I_\boxtimes) \\
\downarrow{\scriptstyle (\ell'_{I'_\boxtimes})^{-1}\ \wr} & & \\
I'_\boxtimes\boxtimes_{\mathscr{C}'}I'_\boxtimes & & \downarrow{\scriptstyle F(\Delta_\otimes)} \\
\downarrow{\scriptstyle \psi_0\boxtimes_{\mathscr{C}'}\psi_0\ \wr} & & \\
F(I_\boxtimes)\boxtimes_{\mathscr{C}'}F(I_\boxtimes) & \xrightarrow{\varphi_2(I_\boxtimes,I_\boxtimes)} & F(I_\boxtimes\otimes_{\mathscr{C}}I_\boxtimes)
\end{array} \tag{3.2.8}$$

where we denote the left and right unit isomorphisms of $\mathscr{C}'$ by ℓ'_Y and r'_Y, respectively.

3.2.7. *Examples.* (a) Any symmetric monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ can be regarded as a symmetric 2-monoidal category where both symmetric monoidal structures coincide, the unit $I_{\mathscr{C}}$ has the obvious unitary algebra and counitary coalgebra structures, and $\mathrm{sh}_{A,B,C,D}$ is given by $\mathrm{id}_A \otimes_{\mathscr{C}} \tau(B,C) \otimes_{\mathscr{C}} \mathrm{id}_D$.

(b) Let $(\mathscr{C}', \otimes_{\mathscr{C}'}, I_{\mathscr{C}'}, \tau')$ be a symmetric monoidal category that is complete and cocomplete and whose tensor product commutes with colimits on each side, and let B be a commutative unitary and counitary bialgebra. We denote its coproduct by Δ_B. Then, the category of modules ${}_B\mathrm{Mod}(\mathscr{C}')$ over B has two natural structures of monoidal category:

(i) the first structure is the one described in Lemma 1.5.7, that we denote by $X \otimes_B Y$. This is the standard symmetric monoidal structure considered in commutative algebra.

(ii) given two objects X and Y in ${}_B\mathrm{Mod}(\mathscr{C}')$ set $X \boxtimes Y = X \otimes_{\mathscr{C}'} Y$, and the action of B on $X \boxtimes Y$ is defined via Δ_B, *i.e.* it is the map

$$B \otimes_{\mathscr{C}'} (X \boxtimes Y) \to X \boxtimes Y$$

given by the composition of $\Delta_B \otimes_{\mathscr{C}'} \mathrm{id}_X \otimes_{\mathscr{C}'} \mathrm{id}_Y$, $\mathrm{id}_B \otimes_{\mathscr{C}'} \tau'(B,X) \otimes_{\mathscr{C}'} \mathrm{id}_Y$ and $\rho_X \otimes_{\mathscr{C}'} \rho_Y$. The unit is in this case $I_{\mathscr{C}'}$, which is a B-module via the counit of B. This category is also symmetric for the twist of $\mathscr{C}'$ (*i.e.* the twist is a morphism of B-modules) if Δ_B is cocommutative. This is the standard monoidal structure considered in Hopf algebra theory.

The category ${}_B\mathrm{Mod}(\mathscr{C}')$ is thus provided with the two previous monoidal structures, the first of which is symmetric. This gives us another example of symmetric double monoidal category. It is however not necessarily a 2-monoidal category, and we refer the reader to Section 3.3 for a case where it is. Moreover, the inclusion functor ${}_B\mathrm{Mod}(\mathscr{C}') \to \mathscr{C}'$ is lax symmetric monoidal for the structure described in (i) (for the structure morphisms φ_0 given by the unit of B and $\varphi_2(X,Y) : X \otimes_{\mathscr{C}'} Y \to X \otimes_B Y$ in $\mathscr{C}'$ defined as the cokernel of (1.5.3)), and it is strong monoidal for the structure defined in (ii).

(c) The previous example also works if B is only assumed to be a commutative counitary bialgebra with enough idempotents $(E, m, 1)$ (as defined in 1.5.3 and 1.5.5), and the category of modules ${}_B\mathrm{Mod}(\mathscr{C}')$ over B is precisely the one recalled in 1.5.3. We recall that ${}_B\mathrm{Mod}(\mathscr{C}')$ is a symmetric monoidal category for the tensor product $\otimes_B$ (see Lemma 1.5.7). We only have to notice that the tensor products given in (i) and (ii) are well-defined, *i.e.* they have a decomposition of the form $\oplus_{e\in E}(X \otimes_B Y)_e$ and $\oplus_{e\in E}(X \boxtimes Y)_e$. This is indeed so by defining $(X \otimes_B Y)_e = X_e \otimes_B Y_e$ and $(X \boxtimes Y)_e = \oplus_{(e',e'')\in m_e} X_{e'} \otimes_{\mathscr{C}'} Y_{e''}$.

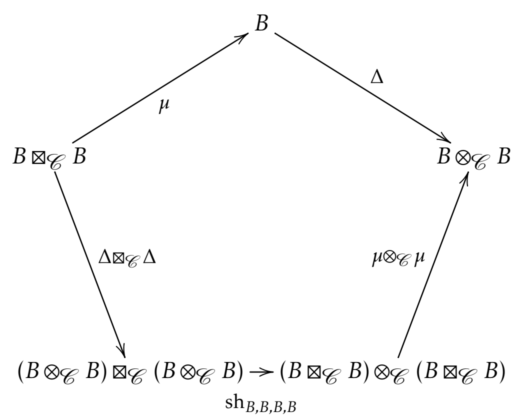
3.2.8. The following definition can be found in [2], 6.5, under the name of *bimonoid*.

Recent **3.2.9 Definition.** *Consider a 2-monoidal category given by* $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \boxtimes_{\mathscr{C}}, I_{\boxtimes}, \mathrm{sh})$. *A* unitary and counitary bialgebra *relative to the 2-monoidal category is an object B of $\mathscr{C}$ with the following structure:*

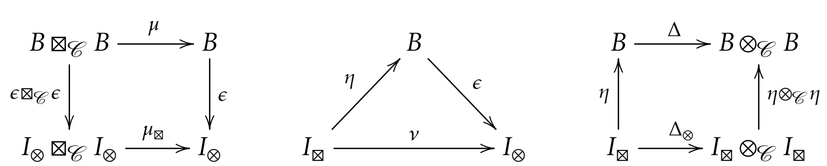
(i) a unitary algebra structure (B, μ, η) with respect to $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_{\boxtimes})$;

(ii) a counitary coalgebra structure (B, Δ, ϵ) with respect to $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$;

such that the following diagrams



and



(3.2.9)

commute.

3.2.10. *Remark.* If the symmetric 2-monoidal category is that of Example 3.2.7, (a), a unitary and counitary bialgebra relative to it coincides with the usual definition of unitary and counitary bialgebra in $\mathscr{C}$.

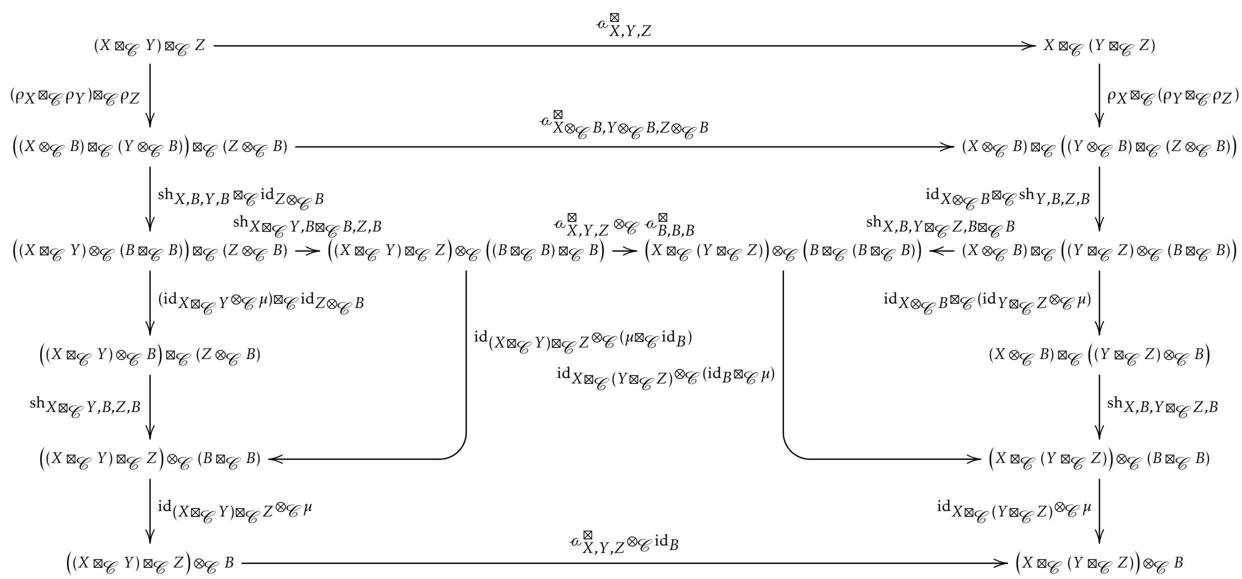
Recent **3.2.11 Definition.** *Assume the same hypotheses of Definition 3.2.9, and let B be a unitary and counitary bialgebra relative to the 2-monoidal category. A* comodule *over B is a comodule over the counitary coalgebra structure on B in the symmetric monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes})$. Given two right comodules X and Y over B with coactions ρ_X and ρ_Y, their tensor product $X \boxtimes_{\mathscr{C}} Y$ has a natural structure of comodule over B of the form*

$$X \boxtimes_{\mathscr{C}} Y \to (X \boxtimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} B$$

defined as the composition of $\rho_X \boxtimes_{\mathscr{C}} \rho_Y$, $\mathrm{sh}_{X,B,Y,B}$ and $\mathrm{id}_{X\boxtimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \mu$. Moreover, $I_\boxtimes$ has a structure of right B-module given by $(\mathrm{id}_{I_\boxtimes} \otimes_{\mathscr{C}} \eta) \circ \Delta_\otimes$. There is a similar definition for left comodules.

3.2.12 Proposition ([2], Prop. 6.41)**.** *Assume the same hypotheses of Definition 3.2.9, and let B be a unitary and counitary bialgebra relative to the 2-monoidal category. The category of right comodules over B is monoidal for the tensor product given in Definition 3.2.11, and the unit $I_\boxtimes$. The same holds for the category of left B-comodules.* *Recent*

Proof. We shall first show that the associativity isomorphism $a^{\boxtimes}_{X,Y,Z} : (X \boxtimes_{\mathscr{C}} Y) \boxtimes_{\mathscr{C}} Z \to X \boxtimes_{\mathscr{C}} (Y \boxtimes_{\mathscr{C}} Z)$ of the monoidal category $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_\boxtimes)$ is a morphism of B-comodules. This follows from the commutativity of the diagram



Indeed, the upper square commutes by the naturality of the associativity morphism, the second square commutes by the naturality of the map sh, as well as the two lateral squares, and the lower subdiagram commutes due to the associativity of μ. Since the composition of the morphisms of each of the external columns is the corresponding coaction of its domain, the statement follows.

We will now prove that the multiplication by the left unit $\ell^{\boxtimes}_X : I_\boxtimes \boxtimes_{\mathscr{C}} X \to X$ of the category $(\mathscr{C}, \boxtimes_{\mathscr{C}}, I_\boxtimes)$ is a morphism of B-modules. The case of the right unit is analogous. In order to prove it, note that the following diagram

$$
\begin{array}{ccc}
I_{\boxtimes} \boxtimes_{\mathscr{C}} X & & \\
\Big\downarrow {\scriptstyle \mathrm{id}_{I_{\boxtimes}} \boxtimes_{\mathscr{C}} \rho_X} & & \\
I_{\boxtimes} \boxtimes_{\mathscr{C}} (X \otimes_{\mathscr{C}} B) & & \\
\Big\downarrow {\scriptstyle \Delta_{\otimes} \boxtimes_{\mathscr{C}} \mathrm{id}_{X \otimes_{\mathscr{C}} B}} & & \\
(I_{\boxtimes} \otimes_{\mathscr{C}} I_{\boxtimes}) \boxtimes_{\mathscr{C}} (X \otimes_{\mathscr{C}} B) & \xrightarrow{(\mathrm{id}_{I_{\boxtimes}} \otimes_{\mathscr{C}} \eta) \boxtimes_{\mathscr{C}} \mathrm{id}_{X \otimes_{\mathscr{C}} B}} & (I_{\boxtimes} \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (X \otimes_{\mathscr{C}} B) \\
\Big\downarrow {\scriptstyle \mathrm{sh}_{I_{\boxtimes}, I_{\boxtimes}, X, B}} & & \Big\downarrow {\scriptstyle \mathrm{sh}_{I_{\boxtimes}, B, X, B}} \\
(I_{\boxtimes} \boxtimes_{\mathscr{C}} X) \otimes_{\mathscr{C}} (I_{\boxtimes} \boxtimes_{\mathscr{C}} B) & \xrightarrow{\mathrm{id}_{I_{\boxtimes} \boxtimes_{\mathscr{C}} X} \otimes_{\mathscr{C}} (\eta \boxtimes_{\mathscr{C}} \mathrm{id}_B)} & (I_{\boxtimes} \boxtimes_{\mathscr{C}} X) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} B) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{I_{\boxtimes} \boxtimes_{\mathscr{C}} X} \otimes_{\mathscr{C}} \ell^{\boxtimes}_B} & & \Big\downarrow {\scriptstyle \mathrm{id}_{I_{\boxtimes} \boxtimes_{\mathscr{C}} X} \otimes_{\mathscr{C}} \mu} \\
(I_{\boxtimes} \boxtimes_{\mathscr{C}} X) \otimes_{\mathscr{C}} B & =\!=\!=\!=\!= & (I_{\boxtimes} \boxtimes_{\mathscr{C}} X) \otimes_{\mathscr{C}} B \\
\Big\downarrow {\scriptstyle \ell^{\boxtimes}_X \otimes_{\mathscr{C}} \mathrm{id}_B} & & \\
X \otimes_{\mathscr{C}} B & &
\end{array}
$$

is commutative. Indeed, the upper square commutes by the naturality of sh, whereas the lower square commutes by the fact that B is a unitary algebra. By the first diagram of condition (iii) in Definition 3.2.2, the composition of the morphisms in the left vertical column is precisely $\ell_{X \otimes^{\boxtimes}_{\mathscr{C}} B} \circ (\mathrm{id}_{I_{\boxtimes}} \boxtimes_{\mathscr{C}} \rho_X)$, whereas the composition of the morphisms of the rightmost path going from top to bottom gives $(\ell^{\boxtimes}_X \otimes_{\mathscr{C}} \mathrm{id}_B) \circ \rho_{I_{\boxtimes} \boxtimes_{\mathscr{C}} X}$. This proves the claim. The proposition is thus proved. □

3.3 Some useful constructions

3.3.1. For the next example, we will make use of the tensor construction TX and the symmetric construction ΣX recalled in 3.1.1, and we recall that we follow the conventions stated in 1.3.1. Let A be a commutative unitary algebra in a k-linear complete and cocomplete symmetric monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$, whose tensor product commutes with colimits on each side. Let $TA = \oplus_{m \in \mathbb{N}_0} A^{\otimes_{\mathscr{C}} m}$ be the tensor construction and let $\Sigma A = \oplus_{m \in \mathbb{N}_0} \Sigma^m A$ be the symmetric construction, where the 0-th component is just $I_{\mathscr{C}}$ (see the notation in 1.3.4). As recalled in 3.1.1, TA has a canonical structure of counitary coalgebra in $\mathscr{C}$, whose coproduct is given by deconcatenation, and ΣA is a cocommutative counitary subcoalgebra of TA. To simplify the treatment, we shall assume that $\mathscr{C}$ is concrete, but all the manipulations with elements can be easily translated into operations on the corresponding objects that they belong to.

3.3.2. For any $m \in \mathbb{N}_0$, define the map $\imath_m : \Sigma^m A \to \Sigma A$ (resp., $\hbar_m : A^{\otimes_{\mathscr{C}} m} \to TA$) given by the canonical inclusion, and the morphism $p_m : \Sigma A \to \Sigma^m A$ (resp., $\mathfrak{s}_m : TA \to A^{\otimes_{\mathscr{C}} m}$) given by the canonical projection. The latter is the morphism in $\mathscr{C}$ induced by the collection of maps $p'_{m'} : \Sigma^{m'} A \to \Sigma^m A$ (resp., $\mathfrak{s}'_{m'} : A^{\otimes_{\mathscr{C}} m'} \to A^{\otimes_{\mathscr{C}} m}$) defined as the identity if $m = m'$ and zero else.

3.3.3. Consider the following exceptional structure of commutative nonunitary algebra on TA defined as follows. Given elements $\sigma \in A^{\otimes_{\mathscr{C}} m}$ and $\sigma' \in A^{\otimes_{\mathscr{C}} m'}$, their product is zero if $m \neq m'$, and it is the usual *tensorwise product* of the algebra $A^{\otimes_{\mathscr{C}} m}$ induced by that of A if $m = m'$, *i.e.* $(a_1|\dots|a_m)(a'_1|\dots|a'_m) = (a_1 a'_1)|\dots|(a_m a'_m)$, where we have used bar symbols instead of tensors. This also defines a unitary algebra structure on $A^{\otimes_{\mathscr{C}} m}$ in the category $\mathscr{C}$, for all $m \in \mathbb{N}_0$. When translating this definition in categorical nonsense, one should notice that this notation is understood as applying a shuffle permutation and then a m-th tensor power of the product of A.

We claim first that this algebra structure on TA is compatible with the coproduct given by deconcatenation and the counit $\mathfrak{s}_0$, *i.e.* TA is a nonunitary and counitary bialgebra in $\mathscr{C}$ for those structures. Indeed, given $\sigma \in A^{\otimes_{\mathscr{C}} m}$ and $\sigma' \in A^{\otimes_{\mathscr{C}} m'}$, their product vanishes if $m \neq m'$, and one easily sees that the product of $\Delta(\sigma)$ and $\Delta(\sigma')$ is also zero. Suppose thus that $m = m'$, $\sigma = a_1|\dots|a_m$, and $\sigma' = a'_1|\dots|a'_m$. A combinatorial argument shows that

$$\begin{aligned}\Delta(\sigma\sigma') &= \Delta\Big((a_1a'_1)|\dots|(a_ma'_m)\Big) = \sum_{j=0}^{m}(a_1a'_1)|\dots|(a_ja'_j)\otimes(a_{j+1}a'_{j+1})|\dots|(a_ma'_m)\\ &= \Big(\sum_{j=0}^{m}(a_1|\dots|a_j)\otimes(a_{j+1}|\dots|a_m)\Big)\Big(\sum_{j=0}^{m}(a'_1|\dots|a'_j)\otimes(a'_{j+1}|\dots|a'_m)\Big) = \Delta(\sigma)\Delta(\sigma').\end{aligned}$$

The fact that the counit $\mathscr{s}_0$ is a morphism of algebras for the given product is trivial, so TA is a nonunitary and counitary bialgebra. It is clearly commutative, for A is so. Since the action of the permutation group $\mathbb{S}_m$ on m letters on $A^{\otimes_{\mathscr{C}} m}$ is clearly by automorphisms of algebras (for the tensor-wise product of $A^{\otimes_{\mathscr{C}} m}$), the invariant space $\Sigma^m A$ is a subalgebra of $A^{\otimes_{\mathscr{C}} m}$, and ΣA is thus a subalgebra of TA. Taking into account that TA is a bialgebra and ΣA is a subcoalgebra of TA, ΣA is *a fortiori* a bialgebra as well. It is a commutative and cocommutative nonunitary and counitary subbialgebra of TA. To stress this particular choice of bialgebra structures, we shall denote the latter by ${}^{\mu}TA$ and the former by ${}^{\mu}\Sigma A$, and we will call them *induced*.

3.3.4 Lemma. *The bialgebras ${}^{\mu}TA$ and ${}^{\mu}\Sigma A$ defined before have enough idempotents $\{1_A^{\otimes m}\}_{m\in\mathbb{N}_0}$ (as defined in 1.5.5), where the latter form a monoid with unit for the obvious product $1_A^{\otimes m}1_A^{\otimes m'} = 1_A^{\otimes(m+m')}$ and the unit $1_A^{\otimes 0}$. As a consequence, the respective categories of modules (in $\mathscr{C}$) over ${}^{\mu}TA$ and ${}^{\mu}\Sigma A$ have a symmetric monoidal structure for the usual tensor product over the corresponding algebra.* Easy

Proof. This is a straightforward verification. The last part follows from Lemma 1.5.7. □

3.3.5. It is easy to verify that $\mathscr{i}_m$ and $\mathscr{p}_m$ (resp., $\mathscr{h}_m$ and $\mathscr{s}_m$) are morphisms of nonunitary algebras for all $m \in \mathbb{N}_0$, and that $\mathscr{i}_m$ (resp., $\mathscr{h}_m$) is in fact a morphism of ΣA-modules (resp., TA-modules) in $\mathscr{C}$, where $\Sigma^m A$ (resp., $A^{\otimes_{\mathscr{C}} m}$) has the structure of ΣA-module (resp., TA-module) in $\mathscr{C}$ by means of $\mathscr{p}_m$ (resp., $\mathscr{s}_m$).

Given any $m \in \mathbb{N}_0$, the canonical projection $\mathscr{s}_m$ of ${}^{\mu}TA$ onto $A^{\otimes_{\mathscr{C}} m}$ induces an fully faithful functor ${}_{A^{\otimes_{\mathscr{C}} m}}\mathrm{Mod}(\mathscr{C}) \to {}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$. The same holds for ${}^{\mu}\Sigma A$ instead of ${}^{\mu}TA$ and $\Sigma^m A$ instead of $A^{\otimes_{\mathscr{C}} m}$. Moreover, the inclusion of ${}^{\mu}\Sigma A$ inside of ${}^{\mu}TA$ induces a functor ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C}) \to {}_{{}^{\mu}\Sigma A}\mathrm{Mod}(\mathscr{C})$, and the same happens for the inclusion of $\Sigma^m A$ inside of $A^{\otimes_{\mathscr{C}} m}$. They form the following commutative diagram of functors

$$\begin{array}{ccccc} {}_{A^{\otimes_{\mathscr{C}} m}}\mathrm{Mod}(\mathscr{C}) & \hookrightarrow & {}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C}) & \searrow & \\ \downarrow & & \downarrow & & \mathscr{C} \\ {}_{\Sigma^m A}\mathrm{Mod}(\mathscr{C}) & \hookrightarrow & {}_{{}^{\mu}\Sigma A}\mathrm{Mod}(\mathscr{C}) & \nearrow & \end{array} \tag{3.3.1}$$

where the composition of any collection that arrives at $\mathscr{C}$ is the corresponding forgetful functor, the horizontal functors are fully faithful and strong symmetric monoidal, where we consider the tensor products over the corresponding algebras, and the left vertical functor is the identity if $m = 1$.

3.3.6. From now on we denote either the bialgebra ${}^{\mu}TA$ or ${}^{\mu}\Sigma A$ simply by B, and B^m denotes $A^{\otimes_{\mathscr{C}} m}$ or $\Sigma^m A$, respectively. Since ${}^{\mu}TA$ or ${}^{\mu}\Sigma A$ are commutative counitary bialgebras with enough idempotents by Lemma 3.3.4, Example 3.2.7, (c), tells us that the category ${}_B\mathrm{Mod}(\mathscr{C})$ of B-modules in $\mathscr{C}$ (recalled in 1.5.3) is naturally a symmetric double monoidal category. We shall show that it is also 2-monoidal in case $B = {}^{\mu}TA$ (see Definition 3.2.9).

3.3.7. From now on, fix $B = {}^{\mu}TA$. We will typically not write the (faithful) inclusion functor of ${}_B\mathrm{Mod}(\mathscr{C})$ in $\mathscr{C}$, so we will usually regard a B-module $X = \oplus_{m\in\mathbb{N}_0} X_m$, where $X_m = 1_A^{\otimes m}X$, simply as an object of $\mathscr{C}$. Under this identification, the tensor functor $\otimes_{\mathscr{C}}$ is precisely $\boxtimes$. Consider the map

$$\mathrm{sh}_{X,X',Y,Y'} : (X \otimes_B X') \boxtimes (Y \otimes_B Y') \to (X \boxtimes Y) \otimes_B (X' \boxtimes Y') \tag{3.3.2}$$

in $\mathscr{C}$ given as follows. Since each B-module is a direct sum in ${}_B\mathrm{Mod}(\mathscr{C})$ of homogeneous components and all the tensor products commute with colimits on each side (see Lemma 1.5.7), we may assume $X = X_m$, $X' = X'_{m'}$, $Y = Y_{m''}$ and $Y' = Y'_{m'''}$, for $m, m', m'', m''' \in \mathbb{N}_0$. As the tensor product $\otimes_B$ clearly satisfies that $Z_m \otimes_B Z'_{m'}$ vanishes if $m \neq m'$, for any pair of B-modules Z and Z', we define (3.3.2) as zero if $m \neq m'$ or $m'' \neq m'''$. Suppose now that $m = m'$ and $m'' = m'''$, and define the morphism

$$(X \otimes_{\mathscr{C}} X') \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Y') \to (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} (X' \otimes_{\mathscr{C}} Y') \tag{3.3.3}$$

in $\mathscr{C}$ given by $\mathrm{id}_X \otimes_{\mathscr{C}} \tau(X', Y) \otimes_{\mathscr{C}} \mathrm{id}_{Y'}$. Consider the following diagram

$$\begin{array}{ccc}
 & & (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} B \otimes_{\mathscr{C}} (X' \otimes_{\mathscr{C}} Y') \\
 & \nearrow^{T} & \Big\downarrow {\scriptstyle \mathrm{id}_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \Delta \otimes_{\mathscr{C}} \mathrm{id}_{X' \otimes_{\mathscr{C}} Y'}} \\
\begin{array}{c}(X \otimes_{\mathscr{C}} B \otimes_{\mathscr{C}} X') \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Y') \\ \oplus \\ (X \otimes_{\mathscr{C}} X') \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} B \otimes_{\mathscr{C}} Y')\end{array} & & (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} B \otimes_{\mathscr{C}} B \otimes_{\mathscr{C}} (X' \otimes_{\mathscr{C}} Y') \\
{\scriptstyle \begin{array}{c}(\rho_X \otimes_{\mathscr{C}} \mathrm{id}_{X'} - \mathrm{id}_X \otimes_{\mathscr{C}} \rho'_{X'}) \otimes_{\mathscr{C}} \mathrm{id}_{Y \otimes_{\mathscr{C}} Y'} \\ \oplus \\ \mathrm{id}_{X \otimes_{\mathscr{C}} X'} \otimes_{\mathscr{C}} (\rho_Y \otimes_{\mathscr{C}} \mathrm{id}_{Y'} - \mathrm{id}_Y \otimes_{\mathscr{C}} \rho'_{Y'})\end{array}} \Big\downarrow & & \Big\downarrow {\scriptstyle \rho_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \mathrm{id}_{X' \otimes_{\mathscr{C}} Y'} - \mathrm{id}_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \rho'_{X' \otimes_{\mathscr{C}} Y'}} \\
(X \otimes_{\mathscr{C}} X') \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Y') & \xrightarrow{\mathrm{id}_X \otimes_{\mathscr{C}} \tau(X', Y) \otimes_{\mathscr{C}} \mathrm{id}_{Y'}} & (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} (X' \otimes_{\mathscr{C}} Y') \\
\Big\downarrow & & \Big\downarrow \\
(X \otimes_B X') \boxtimes (Y \otimes_B Y') & \stackrel{\mathrm{sh}_{X, X', Y, Y'}}{\dashrightarrow} & (X \boxtimes Y) \otimes_B (X' \boxtimes Y')
\end{array} \tag{3.3.4}$$

where we leave to the reader the verification that the lower vertical arrows are the cokernels of the composition of the vertical morphisms above it, where the map T is the sum given by

$$\begin{aligned}
&\Big(\mathrm{id}_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \mathscr{x}_{B, m''} \otimes_{\mathscr{C}} \mathrm{id}_{X' \otimes_{\mathscr{C}} Y'}\Big) \circ \Big(\mathrm{id}_X \otimes_{\mathscr{C}} \tau(B, Y) \otimes_{\mathscr{C}} \mathrm{id}_{X' \otimes_{\mathscr{C}} Y'}\Big) \circ \Big(\mathrm{id}_{X \otimes_{\mathscr{C}} B} \otimes_{\mathscr{C}} \tau(X', Y) \otimes_{\mathscr{C}} \mathrm{id}_{Y'}\Big) \\
&\oplus \Big(\mathrm{id}_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \mathscr{y}_{B, m} \otimes_{\mathscr{C}} \mathrm{id}_{X' \otimes_{\mathscr{C}} Y'}\Big) \circ \Big(\mathrm{id}_{X \otimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \tau(X', B) \otimes_{\mathscr{C}} \mathrm{id}_{Y'}\Big) \circ \Big(\mathrm{id}_X \otimes_{\mathscr{C}} \tau(X', Y) \otimes_{\mathscr{C}} \mathrm{id}_{B \otimes_{\mathscr{C}} Y'}\Big),
\end{aligned}$$

and the maps $\mathscr{x}_{B, m''} : B \to B$ and $\mathscr{y}_{B, m} : B \to B$ send b to $b \otimes 1_A^{\otimes m''}$ and to $1_A^{\otimes m} \otimes b$, respectively. Moreover, the upper part of the diagram is commutative, as the reader may easily check. Then, (3.3.2) is the dashed line induced by (3.3.3). It is not difficult but rather lengthy to check that the previous map is of B-modules. Furthermore, the naturality of (3.3.2) follows in turn from the naturality of the diagram (3.3.4). Since in this case the coherence maps φ_2 for the inclusion of $({}_B\mathrm{Mod}, \otimes_B, B)$ inside of $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}})$ are given by epimorphisms, the commutativity of (3.2.3) and (3.2.4) follows from the axioms of the symmetric monoidal category $\mathscr{C}$ (see [68], Thm. XI.1.1).

3.3.8. In this case the map ν of Definition 3.2.2, (3.2.2), is the canonical inclusion of $I_{\mathscr{C}}$ inside of ${}^{\mu}TA$ as the zeroeth component, the product $\mu_{\boxtimes}$ of ${}^{\mu}TA$ is the usual concatenation product, and the coproduct $\Delta_{\otimes}$ of $I_{\mathscr{C}}$ is the canonical isomorphism $I_{\mathscr{C}} \simeq I_{\mathscr{C}} \otimes_B I_{\mathscr{C}}$. We leave to the reader the easy verification of the commutativity of the remaining diagrams in that definition. Moreover, by definition, from the inclusion of $({}_B\mathrm{Mod}, \otimes_B, B)$ inside of $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}})$ we see that the former symmetric 2-monoidal category is framed. The conclusion of the two previous paragraph is the following result.

New **3.3.9 Proposition.** *Let A be a commutative unitary algebra in a k-linear complete and cocomplete symmetric monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$, whose tensor product commutes with colimits on each side. Let $B = {}^{\mu}TA$ be the commutative counitary bialgebra in $\mathscr{C}$ with enough idempotents defined in 3.3.3. Then, the double monoidal category structure defined on ${}_B\mathrm{Mod}(\mathscr{C})$ is framed symmetric 2-monoidal for the morphism* (3.3.2), *and the unitary algebra structure on ${}^{\mu}TA$ and the counitary coalgebra structure on $I_{\mathscr{C}}$ defined in 3.3.8 (see Definition 3.2.2).*

3.4 Some important morphisms

3.4.1. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, and let A be a commutative unitary algebra in $\mathscr{C}$. We consider thus the induced bialgebras ${}^{\mu}TA$ and ${}^{\mu}\Sigma A$ defined in the previous section, and we recall that the symmetric monoidal category ${}_{B}\mathrm{Mod}(\mathscr{C})$ is naturally a symmetric double monoidal category for the morphism (3.3.2), where B is one of the two previous bialgebras. We will be mainly interested in the cases $\mathscr{C}$ = BLCS, BLCS or CLCS$_{HD}$. Given ${}^{\mu}TA$-modules $X_1, Y_1, \dots, X_m, Y_m$ (in $\mathscr{C}$), Remark 3.2.4 tells us that we also have the morphism

$$\mathrm{sh} : (X_1 \otimes_{{}^{\mu}TA} Y_1) \boxtimes \cdots \boxtimes (X_m \otimes_{{}^{\mu}TA} Y_m) \to (X_1 \boxtimes \cdots \boxtimes X_m) \otimes_{{}^{\mu}TA} (Y_1 \boxtimes \cdots \boxtimes Y_m) \tag{3.4.1}$$

of ${}^{\mu}TA$-modules (in $\mathscr{C}$). This holds in particular if $X_1, Y_1, \dots, X_m, Y_m$ are A-modules (in $\mathscr{C}$), since then they may be regarded as ${}^{\mu}TA$-modules via the canonical projection ${}^{\mu}TA \to A$. Note also that, given ${}^{\mu}TA$-modules $X_1, Y_1, \dots, X_m, Y_m$ (in $\mathscr{C}$), they may be regarded as ${}^{\mu}\Sigma A$-modules via the canonical inclusion ${}^{\mu}\Sigma A \to {}^{\mu}TA$.

3.4.2 Fact. *Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, and let A be a commutative unitary algebra in $\mathscr{C}$. Given any ${}^{\mu}TA$-module X (in $\mathscr{C}$) and $m \in \mathbb{N}$, the subobject $(X^{\otimes_{\mathscr{C}} m})^{\mathbb{S}_m}$ of $X^{\otimes_{\mathscr{C}} m}$ is naturally a submodule over ${}^{\mu}\Sigma A$, where we regard the ${}^{\mu}TA$-module $X^{\otimes_{\mathscr{C}} m} = X^{\boxtimes m}$ as an ${}^{\mu}\Sigma A$-module via the inclusion ${}^{\mu}\Sigma A \to {}^{\mu}TA$ (see* (3.3.1)*). We are going to denote this ${}^{\mu}\Sigma A$-module by $\Sigma^m_{\boxtimes} X$.* Easy

This is a trivial verification.

3.4.3 Lemma. *Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, and let A be a commutative unitary algebra in $\mathscr{C}$. Given any A-module X (in $\mathscr{C}$) and $m \in \mathbb{N}$, the space of coinvariants $S^m X$, i.e. the quotient object $(X^{\otimes_{\mathscr{C}} m})/\mathbb{S}_m$ of $X^{\otimes_{\mathscr{C}} m}$ in the symmetric monoidal category $\mathscr{C}$, is a naturally module over ${}^{\mu}\Sigma A$ (in $\mathscr{C}$), that we are going to denote by $S^m_{\boxtimes} X$. Moreover, the canonical isomorphism map $(X^{\otimes_{\mathscr{C}} m})^{\mathbb{S}_m} \to (X^{\otimes_{\mathscr{C}} m})/\mathbb{S}_m$ in the category $\mathscr{C}$ is also ${}^{\mu}\Sigma A$-linear, so it gives an isomorphism $\Sigma^m_{\boxtimes} X \to S^m_{\boxtimes} X$ of ${}^{\mu}\Sigma A$-modules.* Expected

Proof. Since $S^m X$ is the quotient of $X^{\otimes_{\mathscr{C}} m}$ by the sum of the images $\mathrm{Im}(\varsigma - \mathrm{id}_{X^{\otimes_{\mathscr{C}} m}})$ for all $\varsigma \in \mathbb{S}_m$, it suffices to check that this subobject is a submodule of $X^{\otimes_{\mathscr{C}} m}$ over ${}^{\mu}\Sigma A$. This follows from

$$\begin{aligned}
&\Big(\varsigma^{-1} \cdot (x_1 \otimes \cdots \otimes x_m) - x_1 \otimes \cdots \otimes x_m\Big)\Big(\sum_{\tau \in \mathbb{S}_m} a_{\tau(1)} \otimes \cdots \otimes a_{\tau(m)}\Big) \\
&= (x_{\varsigma(1)} \otimes \cdots \otimes x_{\varsigma(m)} - x_1 \otimes \cdots \otimes x_m)\Big(\sum_{\tau \in \mathbb{S}_m} a_{\tau(1)} \otimes \cdots \otimes a_{\tau(m)}\Big) \\
&= \sum_{\tau \in \mathbb{S}_m} \Big(\Big((x_{\varsigma(1)} a_{\tau(1)}) \otimes \cdots \otimes (x_{\varsigma(m)} a_{\tau(m)})\Big) - \Big((x_1 a_{\tau(1)}) \otimes \cdots \otimes (x_m a_{\tau(m)})\Big)\Big),
\end{aligned}$$

which coincides with

$$\begin{aligned}
&\sum_{\tau' \in \mathbb{S}_m} \Big((x_{\varsigma(1)} a_{\tau'\varsigma(1)}) \otimes \cdots \otimes (x_{\varsigma(m)} a_{\tau'\varsigma(m)})\Big) - \sum_{\tau' \in \mathbb{S}_m} \Big((x_1 a_{\tau'(1)}) \otimes \cdots \otimes (x_m a_{\tau'(m)})\Big) \\
&= \varsigma^{-1} \cdot \Big(\sum_{\tau' \in \mathbb{S}_m} \Big((x_1 a_{\tau'(1)}) \otimes \cdots \otimes (x_m a_{\tau'(m)})\Big)\Big) - \sum_{\tau' \in \mathbb{S}_m} \Big((x_1 a_{\tau'(1)}) \otimes \cdots \otimes (x_m a_{\tau'(m)})\Big),
\end{aligned}$$

where we have defined $\tau' = \tau\varsigma^{-1}$ in the first sum, and $\tau' = \tau$ in the second one. Finally, the last statement is a direct consequence of the previous one. □

3.4.4. If $X = X_1 = \cdots = X_m$ and $Y = Y_1 = \cdots = Y_m$, the morphism (3.4.1) gives us a canonical morphism of ${}^{\mu}TA$-modules (in $\mathscr{C}$)

$$\mathrm{shs} : (X \otimes_{{}^{\mu}TA} Y)^{\boxtimes m} \to X^{\boxtimes m} \otimes_{{}^{\mu}TA} Y^{\boxtimes m}. \tag{3.4.2}$$

Furthermore, suppose Y has a commutative product μ_Y in the symmetric monoidal category ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ of ${}^{\mu}TA$-modules inside the symmetric monoidal category $\mathscr{C}$. Since the inclusion functor ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ inside of $\mathscr{C}$ is lax symmetric monoidal, we may regard Y as a commutative algebra in $\mathscr{C}$, so we will write $\mu_Y : Y^{\boxtimes 2} \to Y$. Then, there is a morphism ${}^{\mu}TA$-modules (in $\mathscr{C}$)

$$\mathrm{shc} : (X \otimes_{{}^{\mu}TA} Y)^{\boxtimes m} \to X^{\boxtimes m} \otimes_{{}^{\mu}TA} Y \tag{3.4.3}$$

given by the composition (3.4.2) and $\mathrm{id}_{X^{\boxtimes m}} \otimes_{{}^{\mu}TA} \mu_Y^{(n)}$, where $\mu_Y^{(n)} : Y^{\boxtimes m} \to Y$ was defined in (1.5.1).

3.4.5. The previous statements, namely Fact 3.4.2, and Lemma 3.4.3, as well the existence of the maps (3.4.1), (3.4.2) and (3.4.3) also trivially hold in the case $m = 0$, because $X^{\otimes_{\mathscr{C}} m} = \Sigma^m_{\mathscr{C}} X = S^m_{\mathscr{C}} X = I_{\mathscr{C}}$.

Easy **3.4.6 Proposition.** *Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, let A be a commutative unitary algebra in $\mathscr{C}$ and let (C, Δ, ϵ) be a counitary coalgebra in the symmetric monoidal category ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ of ${}^{\mu}TA$-modules in $\mathscr{C}$ provided with the tensor $\otimes_{{}^{\mu}TA}$ (see Lemma 1.5.7). Recall the symmetric 2-monoidal category structure on ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ stated in Proposition 3.3.9. Define ${}_{\Delta}TC = \oplus_{m \in \mathbb{N}_0} C^{\boxtimes m}$ with*

(i) a unitary algebra structure in the monoidal ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ provided with the tensor product $\boxtimes$, for the usual concatenation product using the tensor product $\boxtimes$ and the unit given by canonical inclusion of $I_{\mathscr{C}} = C^{\boxtimes 0}$ inside of ${}_{\Delta}TC$;

(ii) a counitary coalgebra structure in the monoidal ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ provided with the tensor product $\otimes_{{}^{\mu}TA}$, for the counit whose restriction to $C^{\boxtimes m}$ is given by the composition of $\epsilon^{\boxtimes m}$ and the canonical inclusion $\hbar_m$ defined in 3.3.2, and for the coproduct whose restriction to $C^{\boxtimes m}$ is the composition of $\Delta^{\boxtimes m}$, the canonical map (3.4.2), and the canonical map $\hbar_m \otimes_{{}^{\mu}TA} \hbar_m$.

Then, ${}_{\Delta}TC$ is a unitary and counitary bialgebra relative to the symmetric 2-monoidal category structure given on ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$. Moreover, ${}_{\Delta}TC$ is cocommutative (with respect to τ) if C is so.

The proof is a trivial but lengthy verification.

Easy **3.4.7 Proposition.** *Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\otimes}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, let A be a commutative unitary algebra in $\mathscr{C}$, and let (C, Δ, ϵ) be a counitary coalgebra in the symmetric monoidal category ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ provided with the tensor $\otimes_{{}^{\mu}TA}$ (see Lemma 1.5.7). Recall the symmetric 2-monoidal category structure on ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ described in Proposition 3.3.9, and the bialgebra ${}_{\Delta}TC = \oplus_{m \in \mathbb{N}_0} C^{\boxtimes m}$ relative to this symmetric 2-monoidal category (see Proposition 3.4.6). Let Y be a (left) comodule over C with (left) coaction ρ in the symmetric monoidal category ${}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C})$ provided with the tensor $\otimes_{{}^{\mu}TA}$. Define ${}_{\rho}TY = \oplus_{n \in \mathbb{N}_0} Y^{\boxtimes m}$. Then, ${}_{\rho}TY$ is a (left) comodule over the bialgebra ${}_{\Delta}TC$ relative to the symmetric 2-monoidal category mentioned previously for the coaction whose restriction to $Y^{\boxtimes m}$ is given by the composition of $\rho^{\boxtimes m}$, (3.4.2), and the canonical map from $C^{\boxtimes m} \otimes_{{}^{\mu}TA} Y^{\boxtimes m}$ to ${}_{\Delta}TC \otimes_{{}^{\mu}TA} {}_{\rho}TY$.*

This is a direct consequence of Proposition 3.2.12 and the fact that the canonical inclusion $C \to {}_{\Delta}TC$ is a morphism of coalgebras in the symmetric monoidal category $({}_{{}^{\mu}TA}\mathrm{Mod}(\mathscr{C}), \otimes_{{}^{\mu}TA}, {}^{\mu}TA)$.

3.4.8.*Remark.* Note that the explicit expressions of the coproduct and counit of ${}_{\Delta}TC$, and of the coaction of ${}_{\rho}TY$, follow essentially the same pattern as the ones used for defining the coproduct and the counit of a tensor product of counitary coalgebras over a fixed base field, and the coaction of a tensor product of comodules over a unitary and counitary bialgebra over a fixed field, respectively.

3.5 Two caveats

3.5.1. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, I_{\mathscr{C}}, \tau)$ be a k-linear complete and cocomplete symmetric monoidal category whose tensor product commutes with colimits on each side, and let A be a commutative unitary algebra in $\mathscr{C}$. We want to state the following two caveats. For simplicity, let us assume that $\mathscr{C}$ is the category of vector spaces over the field k with the usual symmetric monoidal structure, but these comments apply to more general situations as well.

3.5.2. First, contrary to what trivially occurs in the case if $A = k$, we are completely unable to induce a morphism of the form

$$\mathrm{shs}' : S^m_{\boxtimes}(X \otimes_A Y) \to S^m_{\boxtimes}(X) \otimes_{^{\mu}\Sigma A} S^m_{\boxtimes}(Y) \tag{3.5.1}$$

from (3.4.1) (for $m \geq 2$). Furthermore, we believe there is no canonical choice of such a morphism is possible if $\Sigma A \neq TA$. Indeed, already the case $m = 2$ poses a challenge. For example, supposing that X and Y are A-modules, (3.4.1) sends an element $(x \otimes y)|(x' \otimes y') + (x' \otimes y')|(x \otimes y)$ to $(x|x') \otimes (y|y') + (x'|x) \otimes (y'|y)$, where the bars indicate tensor over k and $\otimes$ is the tensor product over A for the element of the domain and $A^{\otimes 2}$ for the one of the image. Hence, (3.5.1) should send $(x \otimes y) \circledS (x' \otimes y')$ to $(x \circledS x') \otimes (y \circledS y')$, where $\circledS$ indicates the symmetric tensor over k and $\otimes$ is the tensor product over A for the element of the domain and over $\Sigma^2 A$ for the one of the image. However, since $(xa \otimes y) \circledS (x' \otimes y') = (x \otimes ay) \circledS (x' \otimes y')$ should be sent to both $(xa \circledS x') \otimes (y \circledS y')$ and $(x \circledS x') \otimes (ay \circledS y')$, and these two elements do not coincide, because $a|1 \in A^{\boxtimes 2} \setminus \Sigma^2 A$, the previous expression for (3.5.1) is not well-defined. One would also be tempted to consider else a tensor product over $^{\mu}TA$ on the codomain of (3.5.1), but this would not be defined either because $S^m_{\boxtimes}(X)$ is not a module over $A^{\boxtimes m}$. In particular, we do not see how the map alluded in the last paragraph of [10], Def. 9, can exist.

3.5.3. Second, suppose further Y has a commutative product μ_Y in the symmetric monoidal category $_A\mathrm{Mod}$ of A-modules inside the symmetric monoidal category $\mathscr{C}$. One would also be tempted to say that the map (3.4.1) can be lifted to a morphism

$$\mathrm{shc}' : S^m_{\boxtimes}(X \otimes_A Y) \to S^m_{\boxtimes}(X) \otimes_{^{\mu}\Sigma A} Y \tag{3.5.2}$$

of $^{\mu}\Sigma A$-modules such that

$$\begin{array}{ccccc}
S^m_{\boxtimes}(X \otimes_A Y) & \xrightarrow{\mathrm{shc}'} & S^m_{\boxtimes}(X) \otimes_{^{\mu}\Sigma A} Y & & \\
 & & & \searrow^{\widehat{e} \otimes_{^{\mu}\Sigma A} \mathrm{id}_Y} & \\
\Big\downarrow{\scriptstyle \mathrm{shs}|_{S^m_{\boxtimes}(X \otimes_A Y)}} & & & & X^{\boxtimes m} \otimes_{^{\mu}\Sigma A} Y \\
 & & & \nearrow_{\mathrm{can}} & \\
X^{\boxtimes m} \otimes_{^{\mu}TA} Y^{\boxtimes m} & \xrightarrow{\mathrm{id}_{X^{\boxtimes m}} \otimes_{^{\mu}\Sigma A} \mu_Y^{(m)}} & X^{\boxtimes m} \otimes_{^{\mu}TA} Y & &
\end{array}$$

commutes, where $\mu_Y^{(n)} : Y^{\boxtimes m} \to Y$ was defined in (1.5.1), $\widehat{e} : S^m_{\boxtimes} X \to X^{\boxtimes m}$ is the canonical inclusion, and the map can is the one induced by the identity of $X^{\boxtimes m} \boxtimes Y$. In more concrete terms, shc$'$ would be given by

$$(x_1 \otimes y_1) \circledS \dots \circledS (x_m \otimes y_m) \mapsto (x_1 \circledS \dots \circledS x_m) \otimes (y_1 \dots y_m),$$

where $\otimes$ indicates $\otimes_{^{\mu}\Sigma A}$, and we denote the product of Y simply by juxtaposition. The existence of this map is weaker than that of (3.5.1), for the good definition of the latter trivially would imply that of the former. However, the map (3.5.2) is not well-defined either. Indeed, for $m = 2$, $(x_1 a \otimes y_1) \circledS (x_2 \otimes y_2) = (x_1 \otimes a y_1) \circledS (x_2 \otimes y_2)$ should be sent to both $(x_1 a \circledS x_2) \otimes (y_1 y_2)$ and to

$$\frac{1}{2}(x_1 a \circledS x_2 + x_1 \circledS x_2 a) \otimes (y_1 y_2) = (x_1 \circledS x_2) \frac{1}{2}(a|1 + 1|a) \otimes (y_1 y_2) = (x_1 \circledS x_2) \otimes (a y_1 y_2),$$

which are clearly different.

3.5.4. *Example.* We will give a simple example showing that the previous image elements do not necessarily coincide. Let us first note that they coincide if and only if

$$(x_1 a \circledS x_2 - x_1 \circledS x_2 a) \otimes y = 0, \tag{3.5.3}$$

where $x_1, x_2 \in X$, $a \in A$ and $y \in Y$. Suppose $Y = A$, and let $X = A.e_1 \oplus A.e_2$ be a free A-module with basis $\{e_1, e_2\}$. As a consequence, $S^2 X \simeq \Sigma^2 X \simeq \Sigma^2 A \oplus \Sigma^2 A \oplus (A.e_1 \otimes A.e_2)$, as $\Sigma^2 A$-modules, and thus,

$$S^2_{\boxtimes} X \otimes_{^{\mu}\Sigma A} Y = S^2_{\boxtimes} A \otimes_{\Sigma^2 A} Y \simeq \big(\Sigma^2 A \oplus \Sigma^2 A \oplus (A.e_1 \otimes A.e_2)\big) \otimes_{\Sigma^2 A} A \simeq A \oplus A \oplus \big((A.e_1 \otimes A.e_2) \otimes_{\Sigma^2 A} A\big).$$

Set $x_j = e_j$, for $j = 1,2$, and $y = 1_A$ in (3.5.3). Suppose further that $A = k[x]$ is the polynomial algebra in one indeterminate, and choose $a = x$. We will show that (3.5.3) does not hold in this case. Recall that $Y = k[x]$ is a module over $A^{\otimes 2} = k[x_1,x_2]$ via the canonical projection onto A given by $x_j \mapsto x$, for $j = 1,2$. Moreover, a classical result in invariant theory tells us that $\Sigma^2 A = k[x_1,x_2]^{\mathbb{S}_2} = k[s_1,s_2]$, where $s_1 = x_1 + x_2$, and $s_2 = x_1 x_2$. Hence, $Y = k[x]$ is isomorphic to $k[s_1,s_2]/(s_1^2 - 4s_2)$ as $k[s_1,s_2]$-modules, so

$$(A.e_1 \otimes A.e_2) \otimes_{\Sigma^2 A} A = k[x_1,x_2] \otimes_{k[s_1,s_2]} k[x] \simeq k[x_1,x_2] \otimes_{k[s_1,s_2]} k[s_1,s_2]/(s_1^2 - 4s_2) \simeq k[x_1,x_2]/(x_1 - x_2)^2,$$

so the element $(e_1 x \circledS e_2 - e_1 \circledS e_2 x) \otimes 1$ given in (3.5.3) is identified under the previous chain of isomorphisms to $x_1 - x_2$, which is clearly different from zero.

3.5.5. As a striking consequence of the previous paragraph, and following the notation of [10], we do not understand how the author obtains a comodule structure on $S\Gamma_c \omega SJ\Phi$ over $\Gamma SJ\Phi$ (see Lemma 14 in the mentioned reference), or, translated into our notation, a comodule structure on $S^+_\boxtimes Y$ over $S_A X$, where $Y = V \otimes_A S_A X$, $V = \Gamma_c(\mathrm{Vol}(M))$ is a bornologically projective bornological A-module, and $X = \Gamma(J^i E)$ is a finitely generated projective Fréchet module over $A = C^\infty(M)$ (see Definition 5.1.13). Indeed, the "natural" candidate given by the previous expression (3.5.2) is unfortunately ill-defined in general, even though it provides the correct expression if $A = k$. Note that $S_A X$ is a coalgebra in the symmetric monoidal category ${}_A\mathfrak{CMod}$, but has no canonical coalgebra structure in CLCS_{HD} (because the inclusion functor of the former inside of the latter is lax symmetric monoidal, not oplax), so the natural symmetric monoidal category to regard the coalgebra structure $S_A X$ which may also allow to consider $S^+_\boxtimes Y$ as a possible comodule in it over $S_A X$ could be thus ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$ or ${}_{\underset{\sim}{\Sigma}A}\mathfrak{CMod}$ (for the tensor product $\otimes_{\underset{\sim}{T}A}$ or $\otimes_{\underset{\sim}{\Sigma}A}$, respectively), where $S_A X$ becomes a $\underset{\sim}{T}A$-module via the canonical projection map $\underset{\sim}{T}A \to A$, and analogously for $\underset{\sim}{\Sigma}A$. Even though the category ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$ would be a nice choice, for it has a nice compatibility between the two tensor products $\boxtimes$ and $\otimes_{\underset{\sim}{T}A}$ (precisely because this category can be endowed with the structure of a symmetric 2-monoidal category), the symmetric construction $S^+_\boxtimes Y$ does not belong to ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$. The second choice of category to work in, *i.e.* ${}_{\underset{\sim}{\Sigma}A}\mathfrak{CMod}$, does not work either, meaning that (3.5.2) is not well-defined. This is a shadow of the fact that symmetric double monoidal category structure on ${}_{\underset{\sim}{\Sigma}A}\mathfrak{CMod}$ is not 2-monoidal.

3.5.6. Of course, in order to avoid the problem raised by the lack of good definition of (3.5.2), the reader may propose the possibility to refrain from considering altogether symmetric constructions of the form $S^+_\boxtimes Y$, and to work instead with the better behaved objects $T^+_\boxtimes Y$, which are clearly comodules over the coalgebra $S_A X$ in the symmetric 2-monoidal category structure defined on ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$. However, the symmetric construction $S^+_\boxtimes Y$ is not a real choice: it is imposed by physical motivations since this symmetric construction represents time-ordered products, which are undoubtedly commutative. We will learn how to circumvent this problem in the next chapters, but let us just mention that it would involve working with the tensor construction $T^+_\boxtimes Y$ when we need to consider such a well-defined coaction over $S_A X$, and then checking that at the end of the day the final result (which involves not only the coaction, but also several other operations) is invariant under permutations, so inducing a well-defined map on $S^+_\boxtimes Y$.

3.6 A particular case

3.6.1. We now apply the rather general statements of Sections 3.3 and 3.4 to the more concrete case when $\mathscr{C} = \mathrm{BLCS}$ or $\mathscr{C} = \mathrm{CLCS}_{HD}$. They will be handy in Section 3.9, and specially for its use in the extension of propagators in Section 5.6 and the introduction of Feynman measures in Section 5.7 (see in particular Definition 5.7.9, (3)).

Easy **3.6.2 Corollary.** *Let A be a bornological algebra, and let $X_1, \dots, X_m$ be bornological locally convex A-modules, for $m \in \mathbb{N}$. Then,*

(i) $A^{\otimes_\beta m}$ is a bornological algebra for the usual tensor-wise product $(a_1 \otimes \cdots \otimes a_m)(a'_1 \otimes \cdots \otimes a'_m) = (a_1 a'_1) \otimes \cdots \otimes (a_m a'_m)$, and the same holds for its convenient completion $A^{\tilde{\otimes}_\beta m}$;

(ii) $X_1 \otimes_\beta \cdots \otimes_\beta X_m$ *is a bornological locally convex module over* $A^{\otimes_\beta m}$ *for the corresponding* tensor-wise action $(a_1 \otimes \cdots \otimes a_m)(x_1 \otimes \cdots \otimes x_m) = (a_1 x_1) \otimes \cdots \otimes (a_m x_m)$. *By taking its convenient completion,* $X_1 \tilde{\otimes}_\beta \ldots \tilde{\otimes}_\beta X_m$ *is a module over* $A^{\tilde{\otimes}_\beta m}$.

Proof. The part concerning bornological modules is a direct consequence of the comments in Section 3.3 for the case $\mathscr{C}$ = BLCS, whereas the case of convenient modules follows from the fact that the convenient completion $\widetilde{(-)}$ is a strong symmetric monoidal functor (see Lemma 1.4.31), together with Corollary 1.3.5. □

3.6.3 Corollary. *We assume the same hypotheses of Corollary 3.6.2.* *Easy*

(i) *The subalgebra* $(A^{\otimes_\beta m})^{\mathbb{S}_m}$ *of* $(A^{\otimes_\beta m})$, *which we denote by* $\Sigma^m A$, *is a commutative bornological subalgebra of* $A^{\otimes_\beta m}$, *and thus* $X_1 \otimes_\beta \cdots \otimes_\beta X_m$ *is a bornological locally convex module over* $\Sigma^m A$.

(ii) *We denote by* $\tilde{\Sigma}^m A$ *the bornological algebra given as the convenient completion of* $\Sigma^m A$. *There is an isomorphism of bornological algebras*

$$\widetilde{(A^{\otimes_\beta m})^{\mathbb{S}_m}} \simeq (\tilde{A}^{\tilde{\otimes}_\beta m})^{\mathbb{S}_m}.$$

Moreover, $X_1 \tilde{\otimes}_\beta \ldots \tilde{\otimes}_\beta X_m$ *is a bornological locally convex* $\tilde{\Sigma}^m A$*-module.*

(iii) $(X^{\otimes_\beta m})^{\mathbb{S}_m}$ *is a bornological locally convex submodule of* $X^{\otimes_\beta m}$ *over* $\Sigma^m A$, *and by taking the convenient completion* $(X^{\tilde{\otimes}_\beta m})^{\mathbb{S}_m}$ *is a bornological locally convex submodule of* $X^{\tilde{\otimes}_\beta m}$ *over* $\tilde{\Sigma}^m A$.

Proof. Again, the part concerning bornological modules is a direct consequence of the comments in Section 3.3 and Fact 3.4.2 for the case $\mathscr{C}$ = BLCS, whereas the case of convenient modules follows from the fact that the convenient completion $\widetilde{(-)}$ is a strong symmetric monoidal functor (see Lemma 1.4.31), together with Corollary 1.3.5. □

3.6.4 Corollary. *Let* A *be a bornological algebra,* X *a bornological locally convex* A*-module, and* $m \in \mathbb{N}$. *The space of coinvariants* $S^m X$ *in the symmetric monoidal category* BLCS *is a naturally bornological locally convex module over* $\Sigma^m A$. *Furthermore, if* X *is convenient and we denote by* $\tilde{S}^m X$ *the space of coinvariants* $X^{\tilde{\otimes}_\beta m}/\mathbb{S}_m$ *(in the category* CLCS_{HD}*), then* $\tilde{S}^m X$ *is a convenient locally convex module over* $\tilde{\Sigma}^m A$. *New*

As a consequence, the canonical bijective map $(X^{\otimes_\beta m})^{\mathbb{S}_m} \to S^m X$ *in the category* BLCS *is also* $\Sigma^m A$*-linear, and the same result holds for the conveniently completed versions.*

Proof. This is a direct consequence of Lemma 3.4.3, for $\mathscr{C}$ = BLCS or $\mathscr{C} = \mathrm{CLCS}_{HD}$. □

3.6.5. In case $\mathscr{C} = \mathrm{CLCS}_{HD}$, we will denote ${}^\mu TA$ by $\underset{\sim}{T}A$, ${}^\mu \Sigma A$ by $\underset{\sim}{\Sigma}A$, and the previous morphisms (3.4.1) and (3.4.2) by $\widetilde{\mathrm{sh}}$ and $\widetilde{\mathrm{shs}}$, respectively. Note that $\underset{\sim}{T}A$ and $\underset{\sim}{\Sigma}A$ are however never metrizable (see [95], Ch. I, Exercise 9), so they are not a Fréchet algebra, even if A is.

3.6.6.*Convention.* Unless otherwise stated, each time we consider tensor products of bornological algebras and bornological locally convex modules over them, we will mean the tensor-wise product and tensor-wise action.

3.6.7. We want to study further the case $A = C^\infty(M)$ and $X = \Gamma(E)$, where M is a smooth manifold and E a vector bundle over M. In this case, since both A and X are metrizable LCS, the bornological tensor product $\otimes_\beta$ coincides with the usual projective tensor product $\otimes_\pi$ (see 1.4.23), as well as the convenient tensor product and the completed projective tensor product (see 1.4.34). Even though the geometric picture is somehow lost when considering their symmetric convenient tensor products, because there is in general no smooth manifold associated with $\tilde{\Sigma}^m A$ (see [109]), we may still consider the isomorphic convenient locally convex $\tilde{\Sigma}^m A$-module $(X^{\tilde{\otimes}_\beta m})^{\mathbb{S}_m}$, for the latter is a convenient locally convex $\tilde{\Sigma}^m A$-submodule of $X^{\tilde{\otimes}_\beta m}$, which is isomorphic to $\Gamma(E^{\boxtimes m})$ by Proposition 2.3.13. This allows us to canonically consider elements of $\tilde{S}^m X$ as sections of $E^{\boxtimes m}$ over M^m. In particular, given an independent family $\{F_\ell : \ell \in \mathcal{L}\}$ of vector bundles over M

and a tuple $\bar{\ell} = (\ell_1, \dots, \ell_q)$ of elements of $\mathcal{L}$, the elements of $\Gamma(F_{\ell_1}) \circledS \dots \circledS \Gamma(F_{\ell_q})$ can be regarded as sections of the bundle $E(\bar{\ell})^{\boxtimes q}$ over M^q, where

$$E(\bar{\ell}) = E(F_{\ell_1}, \dots, F_{\ell_q}) = \bigoplus_{l \in \{\ell_1, \dots, \ell_q\}} F_l \tag{3.6.1}$$

is a vector bundle over M. Equivalently, the elements of $\Gamma(F_{\ell_1}) \circledS \dots \circledS \Gamma(F_{\ell_q})$ can be considered as sections of $E(\ell)^{\boxtimes q}$ that are invariant under the action of $\mathbb{S}_q$ given by sending a section σ to $\tilde{\varsigma}_{M^q} \circ \sigma \circ \varsigma_{M^q}^{-1}$, where $\tilde{\varsigma} : E(\bar{\ell})^{\boxtimes q} \to E(\bar{\ell})^{\boxtimes q}$ is the natural morphism of vector bundles over the map $\varsigma_{M^q} : M^q \to M^q$ defined as $(p_1, \dots, p_q) \mapsto (p_{\varsigma^{-1}(1)}, \dots, p_{\varsigma^{-1}(q)})$. We have the analogous definition for the space of sections of compact support.

3.6.8. The next algebraic structures appearing in the rest of the chapter are generalizations of those considered by C. Brouder for the case of QFT of dimension zero in [14] (see also [103]).

3.7 Two inverse limit constructions

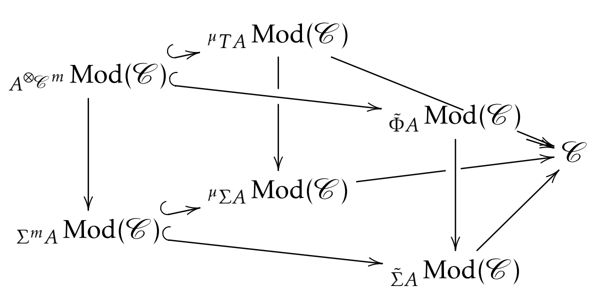
3.7.1. Let A be a commutative unitary algebra in a symmetric monoidal category $\mathscr{C}$ that is complete and cocomplete and whose tensor product commutes with colimits on each side. For every $m \in \mathbb{N}$ define the morphism $\mu_{A,\mathrm{sym}}^{(m)} : \Sigma^m A \to A$ in $\mathscr{C}$ as the restriction of (1.5.1) to $\Sigma^m A$. Since A is unitary, (1.5.1) is an epimorphism for all $m \in \mathbb{N}$. The commutativity of A tells that (1.5.1) is invariant under the symmetrization map e introduced in Proposition 1.1.4. As a consequence, $\mu_{A,\mathrm{sym}}^{(m)}$ is an epimorphism for all $m \in \mathbb{N}$.

New **3.7.2 Definition.** *Define $\tilde{\Sigma} A$ (resp., $\tilde{\Phi} A$) as the inverse limit in $\mathscr{C}$ of the system $\{\mu_{A,\mathrm{sym}}^{(m)} : \Sigma^m A \to A\}_{m \in \mathbb{N}}$ (resp., $\{\mu_A^{(m)} : A^{\otimes_{\mathscr{C}} m} \to A\}_{m \in \mathbb{N}}$) considered before. Given $m \in \mathbb{N}$, we denote by $q_m : \tilde{\Sigma} A \to \Sigma^m A$ (resp., $r_m : \tilde{\Phi} A \to A^{\otimes_{\mathscr{C}} m}$) the associated morphisms. The very definition of inverse limit implies that it is a commutative algebra, and that q_m (resp., r_m) is an epimorphism in $\mathscr{C}$ that is a morphism of algebras in $\mathscr{C}$ for all $m \in \mathbb{N}$, because $\mu_{A,\mathrm{sym}}^{(m)}$ (resp., $\mu_A^{(m)}$) is also. Furthermore, $\tilde{\Sigma} A$ (resp., $\tilde{\Phi} A$) is unitary.*

We will need these constructions in Sections 3.9, 3.11 and 3.12, and specially for their use in renormalization theory from Section 5.8 on.

3.7.3. Note that $\mu_{A,\mathrm{sym}}^{(m)} \circ q_m$ (resp., $\mu_A^{(m)} \circ r_m$) is independent of $m \in \mathbb{N}$, by definition of inverse limit. We shall denote this morphism of unitary algebras in $\mathscr{C}$ by q (resp., r). Moreover, by definition of inverse limit, we have a morphism of unitary algebras in $\mathscr{C}$ of the form $\tilde{\Sigma} A \to \tilde{\Phi} A$, induced by the family of monomorphisms of algebras $\{\Sigma^m A \to A^{\otimes_{\mathscr{C}} m}\}_{m \in \mathbb{N}}$.

3.7.4. Given $m \in \mathbb{N}$, the canonical projection $q_m : \tilde{\Sigma} A \to \Sigma^m A$ (resp., $r_m : \tilde{\Phi} A \to A^{\otimes_{\mathscr{C}} m}$) induces a fully faithful functor ${}_{\Sigma^m A}\mathrm{Mod}(\mathscr{C}) \to {}_{\tilde{\Sigma} A}\mathrm{Mod}(\mathscr{C})$ (resp., ${}_{A^{\otimes_{\mathscr{C}} m}}\mathrm{Mod}(\mathscr{C}) \to {}_{\tilde{\Phi} A}\mathrm{Mod}(\mathscr{C})$). The morphism $\tilde{\Sigma} A \to \tilde{\Phi} A$ of unitary algebras in $\mathscr{C}$ induces a functor ${}_{\tilde{\Phi} A}\mathrm{Mod}(\mathscr{C}) \to {}_{\tilde{\Sigma} A}\mathrm{Mod}(\mathscr{C})$. Combining these functors with those considered in (3.3.1), we have the following commutative diagram



(3.7.1)

where the composition of any collection of functors arriving at $\mathscr{C}$ is the corresponding forgetful functor, the hook-shaped functors are fully faithful and strong symmetric monoidal, for the tensor products over the corresponding algebras, and the leftmost vertical functor is the identity if $m = 1$. In particular, we may equivalently regard the symmetric monoidal category ${}_{A^{\otimes_{\mathscr{C}} m}}\mathrm{Mod}(\mathscr{C})$ inside of ${}_{\tilde{\Phi}A}\mathrm{Mod}(\mathscr{C})$ or ${}_{\mu TA}\mathrm{Mod}(\mathscr{C})$, and the analogous comments hold for ${}_{\Sigma^m A}\mathrm{Mod}(\mathscr{C})$.

3.7.5. In case $\mathscr{C} = \mathrm{BLCS}_{HD}$, and taking into account that the inverse limit of a countable projective system in BLCS_{HD} of metrizable bornological LCS is the inverse limit of the same system computed in LCS_{HD}, $\tilde{\Sigma}A$ (resp., $\tilde{\Phi}A$) is metrizable if A is metrizable, for the index set of the projective family is countable. Moreover, if $\mathscr{C} = \mathrm{CLCS}_{HD}$, $\tilde{\Sigma}A$ (resp., $\tilde{\Phi}A$) is a unitary Fréchet algebra, if A is so (by Corollary 1.4.21).

3.7.6.*Remark.* In [14], Brouder considered essentially that $M = \{x_1, \dots, x_N\}$ is a finite set of N points, so $A = C^\infty(M)$ is just a finite product k^N of copies of k (see Example 5.6.16). Hence, $\tilde{\Sigma}^m A \simeq k^N$ as well, for all $m \in \mathbb{N}$, which in turn implies that $\tilde{\Sigma}A \simeq k^N$. By standard arguments, one can in the end only deal with tensor products over the base field k. Even though this is far from being possible in our more general situation, we will see that the algebraic picture remains similar, despite being technically more involved.

3.8 The symmetric bialgebra of X over A

3.8.1. For the rest of the chapter we assume that A is a commutative unitary locally m-convex algebra which is further supposed to be Fréchet, and that X is a convenient locally convex module over the convenient algebra structure induced on A that is Fréchet and finitely generated projective (as an A-module). By Lemma 1.6.10, X is topologically projective, so bornologically projective as a bornological locally convex module over A, by Fact 1.7.21. To emphasize the fact that we are going to consider convenient tensor products $\tilde{\otimes}_\beta$ of LCS, we shall denote the space of invariants $(X^{\tilde{\otimes}_\beta m})^{\mathbb{S}_m}$ of a convenient LCS X simply by $\tilde{\Sigma}^m X$, whereas the corresponding space of coinvariants $(X^{\tilde{\otimes}_\beta m})/\mathbb{S}_m$ will be denoted by $\tilde{S}^m X$. The corresponding symmetric construction will be thus denoted by $\tilde{S}X = \oplus_{m \in \mathbb{N}_0} \tilde{S}^m X$ and $\tilde{S}^+ X = \oplus_{m \in \mathbb{N}} \tilde{S}^m X$, and the tensor constructions by $\tilde{T}X = \oplus_{m \in \mathbb{N}_0} X^{\tilde{\otimes}_\beta m}$ and $\tilde{S}^+ X = \oplus_{m \in \mathbb{N}} X^{\tilde{\otimes}_\beta m}$. On other hand, in order to distinguish the symmetric and tensor construction in the category of A-modules, we shall denote $(X^{\otimes_A m})/\mathbb{S}_m$ by $S_A^m X$, for $m \in \mathbb{N}_0$, and their direct sum by $S_A X$, whereas the corresponding direct sum of $X^{\otimes_A m}$ for $m \in \mathbb{N}_0$ will be denoted by $T_A X$. Since X is a finitely generated projective module over a Fréchet algebra, the results in Section 2.3 tells us that the previous symmetric construction $S_A X$ coincides with the one given by taking convenient tensor products of bornological locally convex A-modules (or completed tensor products of locally convex A-modules), as we now show.

3.8.2 Fact. *The canonical unitary and counitary bialgebra structure in the symmetric monoidal category* ${}_A\mathrm{Mod}$ *of the underlying object of* $S_A X$ *(see Section 3.1) is also a unitary and counitary bialgebra in the symmetric monoidal category* ${}_A\mathfrak{CMod}$ *for the canonical topologies constructed in* BLCS_{HD}*. As a consequence,* $S_A X$ *is a unitary and noncounitary bialgebra in* ${}_{\tilde{\Sigma}A}\mathfrak{BMod}$ *and in* ${}_{\underset{\sim}{\Sigma}A}\mathfrak{BMod}$*. The same statement holds if we consider the previous statements in the categories* ${}_A\mathfrak{CMod}$, CLCS_{HD}, ${}_{\tilde{\Sigma}A}\mathfrak{CMod}$ *and* ${}_{\underset{\sim}{\Sigma}A}\mathfrak{CMod}$*. Furthermore, we may also replace* $\tilde{\Sigma}A$ *by* $\tilde{\Phi}A$ *and* $\underset{\sim}{\Sigma}A$ *by* $\underset{\sim}{T}A$*.* New

Proof. We recall that the last phrase concerning the topologies means that $S_A X$ has the coproduct topology in the category BLCS_{HD}, where each of the homogeneous components $S_A^m X$ for $m \in \mathbb{N}_0$ has the quotient topology of the bornological tensor product $X^{\otimes_A m}$. Since X is bornologically projective and finitely generated, the same is true of $X^{\otimes_A m}$ and $S_A^m X$, so they are complete and thus Fréchet, so bornological as well. By Proposition 1.6.7 this is the unique Fréchet module structure over A on them. As a consequence, $S_A X$ is complete as well, and *a fortiori* convenient. By the definition of the bornological tensor product and the coproduct topologies, it is clear that $S_A X$ is a unitary bornological algebra. The latter mentioned proposition also implies that the corresponding coproduct is continuous. The counit and coaugmentation are trivially continuous. The last statement is immediate, for A is a quotient of $\tilde{\Sigma}A$, $\tilde{\Phi}A$, $\underset{\sim}{\Sigma}A$ and $\underset{\sim}{T}A$. □

3.8.3. Since the coalgebra $S_A X$ is cocommutative, we may turn any right comodule over $S_A X$ in ${}_A\mathfrak{BMod}$

into a left one, and conversely, so it becomes a cosymmetric bicomodule. In order to shorten our terminology we shall only call them *comodules* over $S_A X$. On the other hand, taking into account that $S_A X$ is a commutative unitary and counitary bialgebra, the bornological tensor product $\otimes_A$ endows the category of $S_A X$-comodules in ${}_A\mathfrak{BMod}$ with a structure of symmetric monoidal category (see [75], Def. 1.8.2). We shall denote this category by ${}^{S_A X}_{A}\mathfrak{coBMod}$. The analogous version using conveniently completed tensor products $\tilde{\otimes}_A$ defines the symmetric monoidal category ${}^{S_A X}_{A}\mathfrak{coCMod}$. Since A is a quotient of $\tilde{\Sigma} A$ (resp., $\underset{\sim}{\Sigma} A$), the previous statements trivially apply to $\tilde{\Sigma} A$ (resp., $\underset{\sim}{\Sigma} A$) instead of A, but where we obtain only a nonunitary symmetric monoidal category structures on ${}^{S_A X}_{\tilde{\Sigma} A}\mathfrak{coBMod}$ (resp., ${}^{S_A X}_{\underset{\sim}{\Sigma} A}\mathfrak{coBMod}$) and on ${}^{S_A X}_{\tilde{\Sigma} A}\mathfrak{coCMod}$ (resp., ${}^{S_A X}_{\underset{\sim}{\Sigma} A}\mathfrak{coCMod}$). We may also replace $\tilde{\Sigma} A$ by $\tilde{\Phi} A$ and $\underset{\sim}{\Sigma} A$ by $\underset{\sim}{T} A$.

3.9 Applications to the convenient tensor-symmetric coalgebra of X over k

New **3.9.1 Corollary.** *Given $m \in \mathbb{N}$, $(S_A X)^{\tilde{\otimes}_\beta m}$ is a cocommutative coaugmented coalgebra in the symmetric monoidal category of convenient locally convex modules over $A^{\tilde{\otimes}_\beta m}$ for the coproduct and the counits defined in Proposition 3.4.6 and the obvious coaugmentation. The previous statement also holds for the case $m = 0$, where $(S_A X)^{\tilde{\otimes}_\beta 0} = k$, $A^{\tilde{\otimes}_\beta 0} = k$, and the coproduct, counit and coaugmentation are the usual one of k.*

The statement is a straightforward consequence of Proposition 3.4.6 and the remarks after diagram (3.7.1).

New **3.9.2 Corollary.** *Given $m \in \mathbb{N}_0$, $(S_A X)^{\tilde{\otimes}_\beta m}$ is a cocommutative counitary coalgebra in the symmetric monoidal category ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ for the tensor product $\otimes_{\underset{\sim}{T} A}$, whose counit is the composition of the counit of $(S_A X)^{\tilde{\otimes}_\beta m}$ together with the map $\hbar_m$ defined in 3.3.2. Moreover, the direct sum of the previous coalgebras $\{(S_A X)^{\tilde{\otimes}_\beta m}\}_{m \in \mathbb{N}_0}$ gives us a structure of cocommutative counitary coalgebra on $\tilde{T} S_A X$ in the symmetric monoidal category ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$.*

This is a direct consequence of Corollary 3.9.1 and the remarks after diagram (3.7.1).

New **3.9.3 Corollary.** *$\tilde{T} S_A X$ is a unitary algebra in the symmetric monoidal category ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ for the tensor product $\boxtimes$ and the usual product. The unit is given by the canonical inclusion of $k = (S_A X)^{\tilde{\otimes}_\beta 0}$ inside of $\tilde{T} S_A X$. Furthermore, this unitary algebra structure together with the cocommutative counitary coalgebra structure described in Corollary 3.9.2 give us a structure of cocommutative unitary and counitary bialgebra on $\tilde{T} S_A X$ relative to the symmetric 2-monoidal category ${}_{\underset{\sim}{\Sigma} A}\mathfrak{CMod}$ described in Proposition 3.3.9.*

This is a direct consequence of Proposition 3.4.6.

New **3.9.4 Corollary.** *Given $m \in \mathbb{N}$, $(S_A X)^{\tilde{\otimes}_\beta m}$ is a cocommutative noncounitary coalgebra in the symmetric monoidal category ${}_{\tilde{\Phi} A}\mathfrak{CMod}$, and the product of $S_A X$ induces a morphism $(S_A X)^{\tilde{\otimes}_\beta m} \to S_A X$ of noncounitary coalgebras in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$. Moreover, the direct sum of the previous coalgebras $\{(S_A X)^{\tilde{\otimes}_\beta m}\}_{m \in \mathbb{N}}$ gives us a structure of cocommutative noncounitary coalgebra on $\tilde{T}^+ S_A X$ in the symmetric monoidal category of convenient locally convex $\tilde{\Phi} A$-modules, and the sum of the previous morphisms $(S_A X)^{\tilde{\otimes}_\beta m} \to S_A X$ is a morphism $\tilde{T}^+ S_A X \to S_A X$ of noncounitary coalgebras in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$. As a consequence, $\tilde{T}^+ S_A X$ is also a (cosymmetric bi)comodule over $S_A X$ in the symmetric monoidal category ${}_{\tilde{\Phi} A}\mathfrak{CMod}$.*

This is a direct consequence of Corollary 3.9.1 and (3.4.3).

3.9.5.*Remark.* We also remark that $\tilde{T}^+ S_A X$ is a nonunitary algebra in the symmetric monoidal category CLCS_{HD} for the usual product recalled in Section 3.1, and the latter is compatible with the coaction stated in the previous corollary, in the sense that the diagram

$$\begin{array}{ccc} \tilde{T}^+S_AX\tilde{\otimes}_\beta\tilde{T}^+S_AX & \xrightarrow{\mu_{\tilde{T}^+S_AX}} & \tilde{T}^+S_AX \\ \downarrow & & \downarrow \\ \tilde{T}^+S_AX\tilde{\otimes}_\beta\tilde{T}^+S_AX\otimes_{\tilde{\Phi}A}S_AX & \xrightarrow{\mu_{\tilde{T}^+S_AX}\otimes_{\tilde{\Phi}A}\mathrm{id}_{S_AX}} & \tilde{T}^+S_AX\otimes_{\tilde{\Phi}A}S_AX \end{array}$$

commutes, where the vertical maps are the coactions, and $\mu_{\tilde{T}^+S_AX}$ is the product of $\tilde{T}^+S_AX$. The coaction of $\tilde{T}^+S_AX\tilde{\otimes}_\beta\tilde{T}^+S_AX$ exists due to Proposition 3.2.12, since the category ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$ has a natural structure of symmetric 2-monoidal category, and to the comments after (3.7.1).

3.9.6 Fact. *Note that $\tilde{S}^+S_AX$ is also an object the symmetric monoidal category ${}_{\underset{\sim}{\Sigma}A}\mathfrak{CMod}$.* Trivial

3.10 The double tensor-symmetric coalgebra over A

3.10.1. For the rest of this chapter $Y = V\otimes_A S_AX$ will denote the cofree right comodule over S_AX in ${}_A\mathfrak{CMod}$ cogenerated by a bornological locally convex A-module V that is bornologically projective.

3.10.2.*Example.* If $A = C^\infty(M)$ and $X = \Gamma(J^iE)$, the space $\mathscr{L}_{i,c}(M,E)$ introduced in Definition 5.1.13 is naturally endowed with a right comodule structure over S_AX in the symmetric monoidal category of $C^\infty(M)$-modules, cogenerated by $V = \Gamma_c(\mathrm{Vol}(M))$ (see Corollary 2.2.4). The continuity of the coaction follows from Corollary 2.3.10, as well as the bornological projectiveness of V.

3.10.3. Let us denote the coaction of Y by δ^r. Since S_AX is cocommutative, then we may turn the right coaction into a left one by applying the usual twist, and we thus obtain a structure of a cosymmetric bicomodule over S_AX on Y. As indicated in Section 3.8, we shall call it simply *comodule* over S_AX in ${}_A\mathfrak{BMod}$. Denote the left comodule structure on Y by δ^l.

3.10.4 Lemma. *Let A be a unitary Fréchet algebra, X a Fréchet A-module that is finitely generated and projective, and $Y = V\otimes_A S_AX$ the cofree comodule over S_AX in ${}_A\mathfrak{BMod}$ cogenerated by a bornologically projective bornological locally convex A-module V. Then S_A^+Y and T_A^+Y are complete and bornological, so convenient, and comodules over S_AX in ${}_A\mathfrak{BMod}$ (and also in ${}_{\tilde{\Sigma}A}\mathfrak{BMod}$ and ${}_{\tilde{\Phi}A}\mathfrak{BMod}$, resp.). Furthermore, the conilpotent (resp., cocommutative) cofree noncounitary coalgebra (resp., conilpotent cocommutative cofree noncounitary coalgebra) T_A^+Y (resp., S_A^+Y) in ${}_A\mathfrak{BMod}$ is a S_AX-comodule noncounitary coalgebra (resp., S_AX-comodule cocommutative noncounitary coalgebra) in ${}_A\mathfrak{BMod}$ (and hence in ${}_{\tilde{\Phi}A}\mathfrak{BMod}$ (resp., ${}_{\tilde{\Sigma}A}\mathfrak{BMod}$) as well), i.e. a noncounitary coalgebra (resp., cocommutative noncounitary coalgebra) in the symmetric monoidal category ${}^{S_AX}_{A}\mathfrak{coBMod}$ (and ${}^{S_AX}_{\tilde{\Phi}A}\mathfrak{coBMod}$ (resp., ${}^{S_AX}_{\tilde{\Sigma}A}\mathfrak{coBMod}$)).* New

Proof. We will prove everything for the symmetric construction S_A^+Y, as the case of the tensor construction T_A^+Y follows analogously. We first show that S_A^+Y is complete and bornological, so convenient. In order to prove it, it suffices to show that $Y^{\otimes_A m}$ is complete and bornological for all $m\in\mathbb{N}$, for this implies that S_A^mY is also complete (by Proposition 1.1.4) and bornological, and thus S_A^+Y is also. Since X is topologically projective (by Lemma 1.6.10), $X^{\otimes_A m}$ is also by Fact 1.6.9. Hence, Proposition 1.1.4 tells us that S_A^mX is bornologically projective for all $m\in\mathbb{N}$, and we conclude thus S_AX is so. As V is bornologically projective, $Y = V\otimes_A S_AX$ is also, by 1.7.22. The same result tells us that $Y^{\otimes_A m}$ is bornologically projective for all $m\in\mathbb{N}$, and the same holds for S_A^mY by Proposition 1.1.4. By 1.7.20, S_A^mY is complete and bornological for all $m\in\mathbb{N}$, so S_A^+Y is complete and bornological as well.

To prove the second part note that, since S_AX is a bialgebra in ${}_A\mathfrak{BMod}$, S_A^mY is a comodule over S_AX in ${}_A\mathfrak{BMod}$ by the usual formula, and the same holds for S_A^+Y.

For the last part of the lemma, notice that the conilpotent cofree noncounitary coalgebra on the object Y of the symmetric monoidal category ${}^{S_AX}_{A}\mathfrak{coBMod}$ coincides precisely with the conilpotent cofree noncounitary coalgebra on the underlying object of Y in ${}_A\mathfrak{BMod}$, because the forgetful functor ${}^{S_AX}_{A}\mathfrak{coBMod} \to {}_A\mathfrak{BMod}$ is strong symmetric monoidal.

The results concerning $\tilde{\Sigma}A$ follow directly from Fact 1.5.2 and Porism 1.7.14, because A is a quotient of $\tilde{\Sigma}A$. □

3.11 More applications to the twisted tensor-symmetric convenient coalgebra over k

New **3.11.1 Corollary.** *Assume the same hypotheses of Lemma 3.10.4. Given $m \in \mathbb{N}$, $Y^{\tilde{\otimes}_\beta m}$ is a comodule over the counitary coalgebra $(S_AX)^{\tilde{\otimes}_\beta m}$ in the symmetric monoidal category ${}_{A^{\tilde{\otimes}_\beta m}}\mathfrak{CMod}$ for the coaction defined in Proposition 3.4.7. The previous statement also holds for the case $m = 0$, where $Y^{\tilde{\otimes}_\beta 0} = k$, $A^{\tilde{\otimes}_\beta 0} = k$, and the coaction is the usual one of k.*

This is a direct consequence of Proposition 3.4.7 and the comments after (3.7.1).

New **3.11.2 Corollary.** *Assume the same hypotheses of Lemma 3.10.4. Given $m \in \mathbb{N}_0$, $Y^{\tilde{\otimes}_\beta m}$ is a comodule over the counitary coalgebra $(S_AX)^{\tilde{\otimes}_\beta m}$ in the symmetric monoidal category ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$ for the tensor product $\otimes_{\underset{\sim}{T}A}$, that is described in Corollary 3.9.2. Moreover, the direct sum of the previous comodules $\{Y^{\tilde{\otimes}_\beta m}\}_{m\in\mathbb{N}_0}$ gives us a structure of comodule on $\tilde{T}Y$ over the counitary coalgebra $\tilde{T}(S_AX)$ in the symmetric monoidal category ${}_{\underset{\sim}{T}A}\mathfrak{CMod}$.*

This follows easily from Corollary 3.11.1 and the remarks after diagram (3.7.1).

New **3.11.3 Corollary.** *Assume the same hypotheses of Lemma 3.10.4. Given $m \in \mathbb{N}$, $Y^{\tilde{\otimes}_\beta m}$ is a comodule over the noncounitary coalgebra $\tilde{T}^+S_AX$ in ${}_{\tilde{\Phi}A}\mathfrak{CMod}$, and thus the same holds for $\tilde{T}^+Y$, whose comodule structure is that of a direct sum of the comodules $Y^{\tilde{\otimes}_\beta m}$ for $m \in \mathbb{N}$. As a consequence, both $Y^{\tilde{\otimes}_\beta m}$ for $m \in \mathbb{N}$ and $\tilde{T}^+Y$ are also comodules over the noncounitary coalgebra S_AX in ${}_{\tilde{\Phi}A}\mathfrak{CMod}$.*

This is just a consequence of Corollary 3.11.1, the remarks after diagram (3.7.1), and Corollary 3.9.4.

3.12 Comparing the tensor-symmetric coalgebras

3.12.1. For completeness, we will now compare the several tensor constructions on Y. Assume the same hypotheses of Lemma 3.10.4. We first remark that, given $m \in \mathbb{N}$, there is canonical mapping

$$S^mY \to S^m_AY \tag{3.12.1}$$

given as the obvious quotient. Analogously, for $m \in \mathbb{N}$, we have the canonical surjection

$$Y^{\tilde{\otimes}_\beta m} \to Y^{\otimes_A m}. \tag{3.12.2}$$

Since S^+_AY is convenient by Lemma 3.10.4, this defines a morphism of bornological locally convex ${}_{\tilde{\Sigma}A}\mathfrak{CMod}$-modules

$$\tilde{S}^+Y \to S^+_AY. \tag{3.12.3}$$

By the same reasons, we have the morphism of bornological locally convex ${}_{\tilde{\Phi}A}\mathfrak{CMod}$-modules

$$\tilde{T}^+Y \to T^+_AY. \tag{3.12.4}$$

.

New **3.12.2 Proposition.** *Assume the same hypotheses of Lemma 3.10.4. The map* (3.12.4) *is a morphism of comodules over the noncounitary coalgebra S_AX in ${}_{\tilde{\Phi}A}\mathfrak{CMod}$.*

Proof. Since the coproduct and the coaction of $\tilde{T}^+Y$ are the convenient completion of those of T^+Y (as an object of BLCS_{HD}), it will suffice to prove the proposition for the latter. The statement is now immediate from the explicit expression of the coaction of T^+Y, and that of T^+_AY, which is the same with the exception that all tensor products are over A (or $\tilde{\Phi}A$). □

3.12.3. We summarize the relevant information that we ascertained in the previous sections on the specific spaces we are going to use in the sequel. Recall that A is a unitary Fréchet algebra, X is a finitely generated projective Fréchet A-module, V is a bornologically projective bornological locally convex A-module, $Y = V \otimes_A S_A X$, $\tilde{\Phi} A = \lim_{m\in\mathbb{N}} A^{\tilde{\otimes}_\beta m}$ and $\tilde{\Sigma} A = \lim_{m\in\mathbb{N}} \tilde{\Sigma}^m A$ are inverse limits in BLCS_{HD}, and $\underset{\sim}{T} A = \oplus_{m\in\mathbb{N}_0} A^{\tilde{\otimes}_\beta m}$ and $\underset{\sim}{\Sigma} A = \oplus_{m\in\mathbb{N}_0} \tilde{\Sigma}^m A$ are locally convex direct sum in BLCS_{HD}. Then,

(a) $S_A X$ is a canonical cocommutative unitary and noncounitary bialgebra in ${}_{\tilde{\Sigma} A}\mathfrak{BMod}$ and in ${}_{\underset{\sim}{\Sigma} A}\mathfrak{BMod}$ (see Fact 3.8.2);

(b) Y is the cofree comodule over $S_A X$ in ${}_A\mathfrak{BMod}$, so Y is *a fortiori* a comodule over $S_A X$ in ${}_{\tilde{\Sigma} A}\mathfrak{BMod}$ (see Porism 1.7.14);

(c) $\tilde{T}^+ S_A X$ has the induced structure of cocommutative noncounitary coalgebra in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$, and there is a morphism of noncounitary coalgebras from $\tilde{T}^+ S_A X$ to $S_A X$ in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$ (see Corollary 3.9.4);

(d) $\tilde{T} S_A X$ has the induced structure of cocommutative counitary coalgebra in ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ for the tensor product $\otimes_{\underset{\sim}{T} A}$ (see Corollary 3.9.2) and that structure forms also a cocommutative unitary and counitary bialgebra on $\tilde{T} S_A X$ in the symmetric 2-monoidal category ${}_{\underset{\sim}{T} A}\mathfrak{CMod}$ (see Corollary 3.9.3);

(e) $S_A^+ Y$ has a canonical structure of cocommutative noncounitary coalgebra in ${}^{S_A X}_{\tilde{\Sigma} A}\mathfrak{coBMod}$ (see Lemma 3.10.4);

(f) $\tilde{T}^+ Y$ is a conilpotent cofree noncounitary coalgebra in CLCS_{HD} with comodule structure over $\tilde{T}^+ S_A X$, so *a fortiori* over $S_A X$, in ${}_{\tilde{\Phi} A}\mathfrak{CMod}$ (see Corollary 3.11.3);

(g) the canonical map $\tilde{T}^+ Y \to T_A^+ Y$ given by (3.12.4) is a morphism of comodules over $S_A X$ in ${}_{\tilde{\Phi} A}\mathfrak{coCMod}$ (see Proposition 3.12.2).

Chapter 4

Preliminaries on distributions on manifolds

4.1 Basic results

Mostly well-known

4.1.1. Given a smooth manifold M of dimension n, we will denote by $\mathrm{Vol}(M)$ the *density bundle* (also called the 1-*volume bundle*), which is the line bundle defined as the tensor product $\Lambda^n T^*M \otimes \mathfrak{o}_M$, where $\mathfrak{o}_M$ indicates the *orientation bundle* of M (see [11], p. 84). A direct definition can be found in [46], Def. 3.1.1. The global sections of $\mathrm{Vol}(M)$ are called *densities*. Note that $\mathrm{Vol}(M \times N) \simeq \mathrm{Vol}(M) \boxtimes \mathrm{Vol}(N)$, for any pair of manifolds M and N. Recall also that, if L is a line bundle over M, then $L \otimes L^*$ is isomorphic to the trivial bundle (see *e.g.* [77], 11.38), so $\mathrm{Vol}(M) \otimes \mathrm{Vol}(M)^*$ is isomorphic to the trivial line bundle as well. Moreover, since any manifold has a nonvanishing global density θ (by a standard argument using a partition of unity), the map $C^\infty(M) \to \Gamma(\mathrm{Vol}(M))$ sending f to $f\theta$ is an isomorphism of $C^\infty(M)$-modules. This isomorphism is however noncanonical.

4.1.2. From now on, given a LCS X, X' denotes the vector space of (complex-valued) continuous linear functionals on X.

4.1.3 Definition. *Given a vector bundle E over M, the space $\mathscr{D}'(M,E)$ of* distributions with values on E *(or* E-valued distributions*) is the LCS $\Gamma_c(E^* \otimes \mathrm{Vol}(M))'$ of continuous linear functionals provided with the bornologically strong topology (see 1.4.25). They were defined by L. Hörmander in* [55], *Def. 6.3.3 and the paragraph after Eq. (6.4.2). See also* [46], *Def. 3.1.4, where they are called* E-valued distributions of density character 0, *(or* section distributions of E*). If E is the trivial line bundle we write $\mathscr{D}'(M)$ instead of $\mathscr{D}'(M,E)$, and call their elements* distributions.[1]

Well-known

4.1.4. *Remark.* Recall that $\Gamma_c(E^* \otimes \mathrm{Vol}(M))$ is an (LF)-space, so it is bornological (see [95], II.8.2, Cor. 2). Moreover, each of the Fréchet spaces $\Gamma_K(E^* \otimes \mathrm{Vol}(M))$ forming the system defining $\Gamma_c(E^* \otimes \mathrm{Vol}(M))$ is reflexive, as noted in 2.3.9, so it is distinguished (see [95], IV.6.6). By Lemma 1.4.27, $\mathscr{D}'(M,E)$ is barreled and bornological. It is also complete (so convenient) for the strong topology (see [95], IV.6.1), which in turn coincides with bornologically strong topology. Since $\Gamma_c(E^* \otimes \mathrm{Vol}(M))$ is reflexive (see 2.3.9), by [95], IV.5.6, Cor. 1, $\mathscr{D}'(M,E)$ is also reflexive. Moreover, since $\mathscr{D}'(M,E)$ is the strong dual of $\Gamma_c(E^* \otimes \mathrm{Vol}(M))$, which is a strict inductive limit of a countable system, then we have the canonical isomorphism

$$\mathscr{D}'(M,E) \simeq \varprojlim_K \Big(\Gamma_K\big(E^* \otimes \mathrm{Vol}(M)\big)\Big)'_b \tag{4.1.1}$$

of LCS, where K runs over an exhausting sequence of M (see [8], Prop. 2.1). Since $\Gamma_K(E^* \otimes \mathrm{Vol}(M))$ is a nuclear and Fréchet, its strong dual is also nuclear (see [95], IV.9.6, Thm.), so the inverse limit $\mathscr{D}'(M,E)$ is nuclear as well (see [95], III.7.4, Cor.). Moreover, since $\Gamma_K(E^* \otimes \mathrm{Vol}(M))$ is Fréchet, so barreled, and Schwartz,

[1] We remark that other authors define (E-valued) distributions as elements of the space of continuous linear functionals $\Gamma_c(E^*)'$ (*e.g.* [29], XVII.3.1). We also stress that we consider *a priori* the bornologically strong topology on the spaces of distributions, but it coincides with the usual strong topology by Lemma 1.4.27. They are in principle different from the weak* topology, which is more common in the theory of PDE.

its strong dual satisfies the M.c.c. Indeed, any barreled space is infrabarreled (see [57], Def. 3.6.2), any Schwartz space is quasinormable (see [57], Exercise 3.15.6.(d)), and the strong dual of any infrabarreled Schwartz satisfies the s.M.c (see [57], Exercise 3.15.6.(c)), and in particular the M.c.c. (see 1.4.11). Finally, since the inverse limit of spaces satisfying the M.c.c. satisfies it as well (see 1.4.11), then $\mathscr{D}'(M,E)$ satisfies the M.c.c.

4.1.5. By Remark 1.6.4, $\mathscr{D}'(M,E)$ is naturally a $C^\infty(M)$-module. Furthermore, it is a bornological locally convex $C^\infty(M)$-module by Corollary 1.7.8. Note that usual definition of support $\operatorname{supp}(u)$ of a distribution u also makes sense for a distribution on a manifold with values in a vector bundle.

4.1.6. Given an isomorphism $t : E \to F$ of vector bundles over an isomorphism $f : M \to N$ of smooth manifolds, we will denote by

$$t^\wedge : \Gamma_c(M,E)' \to \Gamma_c(N,F)' \tag{4.1.2}$$

the map sending $u \in \Gamma_c(M,E)'$ to the distribution $t^\wedge(u)(\sigma) = u((t^{-1})_*(\sigma))$, for all $\sigma \in \Gamma_c(N,F)$, where we are using the morphism given in (2.2.6).

Well-known **4.1.7 Proposition** (see [46], Thm. 3.1.9)**.** *Let E be a vector bundle of rank r over a smooth manifold M, and let $\{(U_a,\tau_a)\}_{a\in A}$ be a trivialization of E whose underlying covering forms an atlas (U_a,ϕ_a) of M. Then, any element $u \in \mathscr{D}'(M,E)$ is equivalent to a collection $(u_a)_{a\in A}$, where $u_a \in \mathscr{D}'(\phi_a(U_a),k^r)$ is a (usual) distribution with values in k^r and*

$$u_a|_{\phi_a(U_a\cap U_{a'})} = (\tau_a \circ \tau_{a'}^{-1})^\wedge(u_{a'}|_{\phi_a(U_a\cap U_{a'})}),$$

for all $a,a' \in A$. The distribution u_a coincides with $\tau_a^\wedge(u|_{U_a})$.

This gives us an equivalent definition of distribution on a manifold as a collection of *usual distributions* (also called *Schwartz distributions*) satisfying a compatibility condition. The latter definition is in fact the original one introduced by Hörmander.

4.1.8. We recall that an E-valued distribution $u \in \mathscr{D}'(M,E)$ is called *smooth* if there is a (necessarily unique) section $\sigma \in \Gamma(E)$ such that $u(\lambda) = \int_M \langle\lambda,\sigma\rangle$, for all $\lambda \in \Gamma_c(E^*\otimes \mathrm{Vol}(M))$, where $\langle\lambda,\sigma\rangle \in \Gamma_c(\mathrm{Vol}(M))$ is induced by the usual evaluation map $E^*\otimes E \to M\times k$ of vector bundles. We recall that the *singular support* $\operatorname{singsupp}(u)$ of u is defined as the complement of the maximal open set U of M such that the restriction of u to U is smooth. More generally, an E-valued distribution $u \in \mathscr{D}'(M,E)$ is called *regular* if there exists a (necessarily unique) continuous section $\sigma \in \Gamma^0(E)$ such that $u(\lambda) = \int_M \langle\lambda,\sigma\rangle$, for all $\lambda \in \Gamma_c(E^*\otimes\mathrm{Vol}(M))$, where we have now $\langle\lambda,\sigma\rangle \in \Gamma_c^0(\mathrm{Vol}(M))$. The subspace of $\mathscr{D}'(M,E)$ formed by the regular distributions will be denoted by $\mathscr{D}'_{\mathrm{reg}}(M,E)$. We will incidentally need to consider the space $\mathscr{D}'_{\mathrm{lbc}}(M,E)$ of distributions given by sections $\sigma : M \to E$ of the projection $E \to M$ that are locally bounded and continuous except for a set of measure zero sections. [2]

Easy **4.1.9 Fact.** *Let $t : E \to F$ be an isomorphism of vector bundles over an isomorphism $f : M \to N$ of smooth manifolds, and let $u \in \mathscr{D}'_{\mathrm{reg}}(M,E)$ be induced by the continuous section $\sigma \in \Gamma^0(E)$. Then $t^\wedge(u)$ defined in* (4.1.2) *is the regular distribution induced by the continuous section $t\circ\sigma\circ f^{-1} \in \Gamma^0(F)$.*

This statement is a direct consequence of the definitions.

Well-known **4.1.10 Proposition.** *Given a vector bundle E over a smooth manifold M, consider the canonical map*

$$\mathscr{D}'(M)\times\Gamma(E) \to \mathscr{D}'(M,E) \tag{4.1.3}$$

sending (u,σ) to the linear functional $\lambda\otimes_{C^\infty(M)}\omega \mapsto u(\lambda(\sigma)\cdot\omega)$, where $u \in \mathscr{D}'(M)$, $\sigma \in \Gamma(E)$, $\lambda \in \Gamma(E)^\circledast \simeq \Gamma(E^*)$ and $\omega \in \Gamma_c(\mathrm{Vol}(M))$, and $\lambda(\sigma)\cdot\omega$ denotes the action of an element of $C^\infty(M)$ on an element $\Gamma_c(\mathrm{Vol}(M))$. Then, the induced map of $C^\infty(M)$-modules

$$\beta_E : \mathscr{D}'(M)\otimes_{C^\infty(M)}\Gamma(E) \to \mathscr{D}'(M,E) \tag{4.1.4}$$

[2] We recall that a set $S \subseteq M$ of a smooth manifold M is called of *measure zero* if for all charts (U,ϕ) of M, $\phi(S\cap U)$ has zero Lebesgue measure.

is an isomorphism, and its domain is canonically isomorphic to $\mathrm{Hom}_{C^\infty(M)}(\Gamma(E^*), \mathscr{D}'(M))$*, which coincides with* $\mathfrak{Hom}_{C^\infty(M)}(\Gamma(E^*), \mathscr{D}'(M))$*. Furthermore, the map* (4.1.4) *is also an isomorphism of bornological locally convex* $C^\infty(M)$*-modules if* $\mathscr{D}'(M) \otimes_{C^\infty(M)} \Gamma(E)$ *has the topology given by the bornological tensor product, and* $\mathscr{D}'(M,E)$ *has the (bornologically) strong topology.*

Proof. The bijectivity of β_E is proved in [46], Thm. 3.1.12. The second statement follows immediately from [3], Prop. 20.10, and the fact that $\Gamma(E)^{\circledast} = \Gamma(E^*)$ and that $\Gamma(E)$ is a finitely generated and projective $C^\infty(M)$-module. The penultimate statement is a direct consequence of Proposition 1.6.7. It also follows from the announced results in [45] (see Thm. 4.1 there), which unfortunately remain without proof. Finally, to prove the fact that (4.1.4) is also an isomorphism of bornological locally convex $C^\infty(M)$-modules we proceed as follows. For another proof, see [78], Thm. 15. Consider the functor G (resp., F) defined on the category of vector bundles over M to the category of bornological locally convex $C^\infty(M)$-modules sending E to the right (resp., left) member of (4.1.4), provided with the strong topology (resp., topology of the bornological tensor product). Recall that the strong topology coincides with the bornologically strong topology, by Lemma 1.4.27. By the definition of the involved topologies, it is clear that (4.1.3) is bounded (see [78], Lemma 16), so it defines a natural transformation from F to G. Moreover, by definition, (4.1.4) induces an isomorphism $F(C^\infty(M)) \simeq G(C^\infty(M))$ of bornological locally convex $C^\infty(M)$-modules. The fact that the bornologically strong dual of a locally convex direct sum of bornological LCS is the product of the corresponding bornologically strong duals (see (1.4.8)), that finite products and finite coproducts in all the involved categories coincide because they are k-linear, bornological tensor products commute with coproducts on each side, and that the functor Γ preserves finite coproducts imply that $F(E) \simeq G(E)$ (as bornological locally convex $C^\infty(M)$-modules), for all trivial vector bundles E over M. Since any vector bundle is a direct summand of a trivial vector bundle, the claim follows. □

4.1.11. By the previous result we will usually identify any E-valued distribution $u \in \mathscr{D}'(M,E)$ with the tensor product $\beta_E^{-1}(u) \in \mathscr{D}'(M) \otimes_{C^\infty(M)} \Gamma(E)$, or with the corresponding element in $\mathfrak{Hom}_{C^\infty(M)}(\Gamma(E^*), \mathscr{D}'(M))$. In particular, for any $e \in \Gamma(E^*)$, we shall denote by $\langle u, e\rangle \in \mathscr{D}'(M)$ the distribution given by evaluating the map in $\mathfrak{Hom}_{C^\infty(M)}(\Gamma(E^*), \mathscr{D}'(M))$ corresponding to u at e.

4.1.12 Lemma. *Let E be a vector bundle of rank r over a manifold M of dimension n, $u \in \mathscr{D}'(M,E)$ be an E-valued distribution with support $K \subseteq M$, and $\sigma \in \Gamma_c(E^* \otimes \mathrm{Vol}(M))$ be a section of compact support. Consider a trivializing covering $(U_a, \tau_a)_{a\in A}$ of E satisfying the assumptions of Proposition 4.1.7, and follow the notation there. If for any $a \in A$ the morphism $\sigma_a = \pi_2 \circ \tau_a \circ \sigma \circ \phi_a^{-1} : \phi_a(U_a) \to k^r$ satisfies that all the partial derivatives $\partial^\alpha \sigma_a$ vanish on $\phi_a(U_a \cap K)$, for all $\alpha \in \mathbb{N}_0^n$ then $u(\sigma)$ vanishes as well.* *New?*

Let us stress that we are not assuming that K is compact. We are almost convinced that this result should be well-known among the experts, but we were not able to find a reference.

Proof. By the local descriptions of u given in Proposition 4.1.7, it suffices to show that the associated distributions u_a vanish, for all $a \in A$. In order to do so, we will prove the same statement for usual distributions, *i.e.* for those distributions defined on open subsets of the Euclidean space. Let us first note that the support of u_a is clearly included in $\phi_a(U_a \cap K)$, for all $a \in A$. We will need thus to prove that if $u \in \mathscr{D}'(U)$ is a usual distribution on an open set $U \subseteq \mathbb{R}^n$ having support $K' \subseteq U$ and $\sigma \in C_c^\infty(U)$ satisfies that all the partial derivatives $\partial^\alpha \sigma$ vanish on K', then $u(\sigma) = 0$. We remark that K' is not assumed to be compact. Consider an increasing set $\{K_j\}_{j\in\mathbb{N}}$ of compact subsets of U such that $K_j \subseteq K_{j+1}^\circ$ and the union of all of them is U, and a collection of functions $\{f_j\}_{j\in\mathbb{N}} \in C_c^\infty(U)^{\mathbb{N}}$ such that $f_j|_{K_j} \equiv 1$, and $\mathrm{supp}(f_j) \subseteq K_{j+1}^\circ$ (see [33], Cor. 2.16). The inclusion $\mathrm{supp}(f_j u) \subseteq \mathrm{supp}(f_j) \cap \mathrm{supp}(u)$ tells us that the distribution $f_j u$ has compact support included in K'. It is easy to verify that $(f_j u)(\sigma)$ converges to $u(\sigma)$, since the sequence $(f_j u)(\sigma) = u(f_j \sigma)$ is eventually constant, because there exists $j_0 \in \mathbb{N}$ such that $\mathrm{supp}(\sigma) \subseteq K_{j_0}$. Finally, as the partial derivatives of all orders of σ vanish on the compact support of the distribution $f_j u$, [55], Thm. 2.3.3, tells us that $(f_j u)(\sigma) = 0$ for all $j \in \mathbb{N}$, and the result follows. □

Some new results

4.2 External product and symmetric distributions

4.2.1. Let M and N be smooth manifolds, and E and F be two vector bundles over M and N, respectively. Let $u \in \mathscr{D}'(M,E)$ and $v \in \mathscr{D}'(N,F)$. Recall that $\Gamma_c(\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*) \simeq \Gamma_c(\mathrm{Vol}(M)\otimes E^*)\tilde{\otimes}_\beta\Gamma_c(\mathrm{Vol}(N)\otimes F^*)$, by Proposition 2.3.13. Then, the canonical continuous map

$$\Gamma_c\Big(\mathrm{Vol}(M)\otimes E^*\Big)'\otimes_\beta \Gamma_c\Big(\mathrm{Vol}(N)\otimes F^*\Big)' \to \Big(\Gamma_c\Big(\mathrm{Vol}(M)\otimes E^*\Big)\otimes_\beta \Gamma_c\Big(\mathrm{Vol}(N)\otimes F^*\Big)\Big)'$$

sends the tensor product $u\otimes v$ to a continuous linear functional on $\Gamma_c(\mathrm{Vol}(M)\otimes E^*)\otimes_\beta\Gamma_c(\mathrm{Vol}(N)\otimes F^*)$. By the universal property of the convenient completion, the latter functional defines a unique continuous linear map $u\boxtimes v \in \Gamma_c(\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*)' = \mathscr{D}'(M\times N, E\boxtimes F)$ called the *external tensor product* $u\boxtimes v \in$ of u and v. This gives a continuous linear map

$$\mathscr{D}'(M,E)\otimes_\beta \mathscr{D}'(N,F) \to \mathscr{D}'(M\times N, E\boxtimes F) \tag{4.2.1}$$

sending $u\otimes v$ to $u\boxtimes v$. It is easy to verify that this product is associative and distributive.

Unknown? New? **4.2.2 Proposition.** *Let M and N be smooth manifolds, and E and F be two vector bundles over M and N, respectively. Then, the map* (4.2.1) *induces an isomorphism of bornological locally convex $C^\infty(M\times N)$-modules*

$$\iota : \mathscr{D}'(M,E)\tilde{\otimes}_\beta\mathscr{D}'(N,F) \longrightarrow \mathscr{D}'(M\times N, E\boxtimes F). \tag{4.2.2}$$

Moreover, the previous tensor product $\tilde{\otimes}_\beta$ on the domain of ι may be equivalently replaced by the completed projective tensor product or the completed injective tensor product. [3]

Proof. Since the map (4.2.1) is continuous, its convenient completion is the morphism (4.2.2) of bornological LCS, for its codomain is convenient. Furthermore, taking into account that the mapping (4.2.1) is a morphism of bornological locally convex $C^\infty(M)\otimes_\beta C^\infty(N)$, its convenient completion (4.2.2) is a morphism of bornological locally convex modules over $C^\infty(M)\tilde{\otimes}_\beta C^\infty(N) \simeq C^\infty(M\times N)$, by (1.7.12) and Lemma 2.3.11.

On the other hand, the map (4.2.1) clearly factorizes as the composition of (1.4.4) and

$$\mathscr{D}'(M,E)\otimes_\pi \mathscr{D}'(N,F) \longrightarrow \mathscr{D}'(M\times N, E\boxtimes F), \tag{4.2.3}$$

so in particular it factors through (1.4.16) and

$$\iota : \mathscr{D}'(M,E)\hat{\otimes}_\pi\mathscr{D}'(N,F) \longrightarrow \mathscr{D}'(M\times N, E\boxtimes F). \tag{4.2.4}$$

Furthermore, (4.2.4) is an isomorphism of LCS by exactly the same steps as in the proof given in [104], Thm. 51.7, for the local case.

We will now prove that $\mathscr{D}'(M,E)\hat{\otimes}_\pi\mathscr{D}'(N,F)$ coincides with $\mathscr{D}'(M,E)\hat{\otimes}_i\mathscr{D}'(N,F)$. In order to do so, it suffices to show that, given any complete LCS Z, the space $\mathscr{B}(\mathscr{D}'(M,E),\mathscr{D}'(N,F);Z)$ of (jointly) continuous bilinear maps coincides with the space $\mathfrak{B}(\mathscr{D}'(M,E),\mathscr{D}'(N,F);Z)$ of separately continuous bilinear maps. Since $\mathscr{D}'(M,E)$ and $\mathscr{D}'(N,F)$ are barreled and bornological, Lemma 1.4.38 tells us that the space $\mathfrak{B}(\mathscr{D}'(M,E),\mathscr{D}'(N,F);Z)$ coincides with the space $\mathfrak{B}_b(\mathscr{D}'(M,E),\mathscr{D}'(N,F);Z)$ of bounded bilinear maps.

Let us write $X = \Gamma_c(\mathrm{Vol}(M)\otimes E^*)$, $Y = \Gamma_c(\mathrm{Vol}(N)\otimes F^*)$ and $W = \Gamma_c(\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*)$. Recall that X, Y and W as well as their strong duals are bornological, barreled, complete, nuclear, and reflexive (see 2.3.9 and Remark 4.1.4). In particular, $X'_b = X'$ and $X \simeq (X'_b)'_b = (X'_b)' = X'' = (X')'_b$, and the same holds for Y and W. Let Z be any complete LCS, and consider the continuous linear map Λ_Z given as the composition of

$$\begin{aligned}\mathfrak{Hom}_b\Big(W',Z\Big) &\xrightarrow{\sim} W''\hat{\otimes}_\pi Z \xrightarrow{\sim} W\hat{\otimes}_\pi Z \xrightarrow{\sim} (X\tilde{\otimes}_\beta Y)\hat{\otimes}_\pi Z \to X\hat{\otimes}_\pi Y\hat{\otimes}_\pi Z \xrightarrow{\sim} X''\hat{\otimes}_\pi Y''\hat{\otimes}_\pi Z \\ &\xrightarrow{\sim} \mathfrak{Hom}_b\Big(X', Y''\hat{\otimes}_\pi Z\Big) \xrightarrow{\sim} \mathfrak{Hom}_b\Big(X', \mathfrak{Hom}_b\Big(Y',Z\Big)\Big) \xrightarrow{\sim} \mathfrak{B}_h\Big(X',Y';Z\Big) = \mathfrak{B}_b\Big(X',Y';Z\Big),\end{aligned} \tag{4.2.5}$$

[3] The only reason we believe this result is new is because we specifically asked several experts about it and they replied that they were not aware of it.

where the first map is the inverse of the isomorphism $\mathfrak{d}_{W',Z}$ of LCS defined in (1.4.10), the second one is $\iota_W^{-1}\hat{\otimes}_\pi \mathrm{id}_Z$, with $\iota_W : W \to W''$ the canonical isomorphism, the third is $\bar{\kappa}\hat{\otimes}_\pi \mathrm{id}_Z$ where $\bar{\kappa}$ is the isomorphism (2.3.6) of LCS given in Proposition 2.3.13, the fourth is continuous linear map $\tilde{\mathfrak{e}}_{X,Y}\hat{\otimes}_\pi \mathrm{id}_Z$ where $\tilde{\mathfrak{e}}_{X,Y}$ is the bijective continuous linear map (1.4.16) (see Remark 2.3.14), the fifth one is $\iota_X\hat{\otimes}_\pi\iota_Y\hat{\otimes}_\pi \mathrm{id}_Z$, with $\iota_X : X \to X''$ and $\iota_Y : Y \to Y''$ the canonical isomorphisms, the antepenultimate is the isomorphism $\mathfrak{d}_{X',Y''\hat{\otimes}_\pi Z}$ of LCS defined in (1.4.10) whereas the penultimate maps is the isomorphism $\mathfrak{Hom}(X', \mathfrak{d}_{Y',Z})$ of LCS also obtained from (1.4.10), the last isomorphism is the inverse of the one given in Fact 1.4.37, and the last equality follows from Lemma 1.4.38. We remark that we have also used the associativity of the completed projective tensor product.

We will need the following result.

4.2.3 Lemma. *Let Z be any complete LCS. Then the continuous linear map Λ_Z defined in* (4.2.5) *is bijective.* *Unknown? New?*

Proof. The proof is just an adaptation of that given in [47], II.3.3, Lemme 8 and Thm. 13. By definition of Λ_Z, it is bijective if and only if the continuous linear map $\tilde{\mathfrak{e}}_{X,Y}\hat{\otimes}_\pi \mathrm{id}_Z$ from $(X\tilde{\otimes}_\beta Y)\hat{\otimes}_\pi Z \to (X\hat{\otimes}_\pi Y)\hat{\otimes}_\pi Z$ is so. Since $\tilde{\mathfrak{e}}_{X,Y} : X\tilde{\otimes}_\beta Y \to X\hat{\otimes}_\pi Y$ is a bijective continuous linear map (1.4.16), we can identify $X\hat{\otimes}_\pi Y$ with a LCS (U,τ_1) and $X\tilde{\otimes}_\beta Y$ with a LCS (U,τ_0) on the same vector space U, where τ_1 is weaker than τ_0. Due to the isomorphism $\bar{\kappa} : X\tilde{\otimes}_\beta Y \to W$ of LCS, we can take U as the space of sections of compact support of the vector bundle $\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*$ over the space $M\times N$, and τ_0 is its usual topology (see 2.3.9). To abbreviate, will write U_j instead of (U,τ_j), for $j = 0,1$. We remark that they are both nuclear and complete (U_0 is clearly so, for it is isomorphic to $\Gamma_c(\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*)$, whereas U_1 satisfies both properties because it is the completed projective tensor products of the two nuclear spaces $X = \Gamma_c(\mathrm{Vol}(M)\otimes E^*)$ and $Y = \Gamma_c(\mathrm{Vol}(N)\otimes F^*)$), so semi-reflexive (see [95], III.7.2, Cor. 2 and IV.5.5). [4] Moreover, note that U_0 is an (LF)-space, whereas U_1 has a weaker topology than that of an (LF)-space (*e.g.* that of U_0).

Choose an inclusion i of the bundle $\mathrm{Vol}(M\times N)\otimes(E\boxtimes F)^*$ inside of a trivial bundle of the form $(M\times N)\times k^R$. This induces inclusions of vector spaces of U_0 and U_1 inside of the vector space of vector functions on $M\times N$ (with values on k^R) of compact support. This even respects the topologies in the case of U_0, if we regard $C_c^\infty(M\times N, k^R)$ with its usual topology (see 2.3.9), whereas U_1 has a weaker topology than the usual one for $C_c^\infty(M\times N, k^R)$.

From now, we will regard each U_j, with $j = 0,1$, as vector spaces of vector functions on $M\times N$ (with values on k^R), with their respective LCS topologies. As stated before, the underlying vector space of both U_0 and U_1 will be denoted by U, and the underlying vector space mapping of the continuous linear map $\tilde{\mathfrak{e}}_{X,Y} : U_0 \to U_1$ is just the identity of U. By definition, it is clear that U_0 has a finer topology than that of pointwise convergence of vector functions on $M\times N$ (with values on k^R). Furthermore, the same is true for U_1, as it follows from the explicit description of the corresponding topology (see [96], p. 116, or [34], (3)).

Let us define $\mathcal{F}_U(M\times N, Z)$ as the vector space of maps $f = (f_1,\dots,f_R)$ from $M\times N$ to Z^R, where $f_j : M\times N \to Z$, satisfying that the associated map $(p,p') \mapsto (\lambda(f_1(p,p')),\dots,\lambda(f_R(p,p')))$ from $M\times N$ to k^R belongs to U, for all $\lambda \in Z'$. Note that $\mathcal{F}_U(M\times N, Z)$ depends on the structure of LCS of Z, but it does not depend on any locally convex topology of the vector space U, and we have omitted the index R for simplicity.

We recall that, if X is a LCS, then X_σ denotes the LCS given by the same underlying vector space provided with the *weak topology*, *i.e.* the topology of uniform convergence on finite subsets of X', whereas X'_σ denotes the *weak** *topology*, *i.e.* the topology of pointwise convergence. For every $j = 0,1$, consider the linear map

$$\mathfrak{Hom}(Z'_\sigma, (U_j)_\sigma) \to \mathcal{F}_U(M\times N, Z) \tag{4.2.6}$$

that sends $u \in \mathfrak{Hom}(Z'_\sigma, (U_j)_\sigma)$ to the map F_u given as follows. For every $(p,p') \in M\times N$, define $F_u(p,p')$ as the unique element $(z_1,\dots,z_R)$ of Z^R such that $(\lambda(z_1),\dots,\lambda(z_R)) = u(\lambda)(p,p') \in k^R$, for all $\lambda \in Z'$. We remark that the map from Z'_σ to k^R given by $\lambda \mapsto u(\lambda)(p,p')$ is continuous, since it is the composition of the continuous map u and the evaluation map $(U_j)_\sigma \to k^R$ at a fixed point (p,p'), which is continuous since $((U_j)_\sigma)' = U'_j$ and the topology on U_j is finer than that of pointwise convergence. Hence, $(z_1,\dots,z_R)$ in the definition of

[4] We remark that a semi-reflexive space is called reflexive in [47] (see Introduction, III, p. 6)

F_u exists because the canonical map from Z into the continuous dual of Z'_σ is an isomorphism of vector spaces (see [104], Prop. 35.1), and it is unique by the Hahn-Banach theorem. By [47], II.3.3, Lemme 8, the linear map (4.2.6) is bijective, since the spaces U_0 and U_1 are semi-reflexive and have topologies weaker than that of an (LF)-space but stronger than the that of pointwise convergence. [5] The inverse map is given by sending $f = (f_1, \dots, f_R) \in \mathcal{F}_U(M \times N, Z)$ to the continuous linear map $G_f \in \mathfrak{Hom}(Z'_\sigma, (U_j)_\sigma)$ satisfying that $G_f(\lambda)(p,p') = (\lambda(f_1(p,p')), \dots, \lambda(f_R(p,p')))$, for all $\lambda \in Z'$ and $(p,p') \in M \times N$. Moreover, by [104], Prop. 42.2, (1), the vector spaces $\mathfrak{Hom}(Z'_\sigma, (U_j)_\sigma)$ and $\mathfrak{Hom}(Z'_\tau, U_j)$ coincide, so (4.2.6) gives a linear bijection between $\mathfrak{Hom}(Z'_\tau, U_0)$ and $\mathfrak{Hom}(Z'_\tau, U_1)$. By (1.4.9), we have the commutative diagram

$$\begin{array}{ccc}
\mathfrak{Hom}_e(Z'_\tau, U_0) & \xrightarrow{\mathfrak{Hom}_e(Z'_\tau, \tilde{\mathfrak{c}}_{X,Y})} & \mathfrak{Hom}_e(Z'_\tau, U_1) \\
{\scriptstyle \mathfrak{d}'_{U_0,Z}} \uparrow \wr & & {\scriptstyle \mathfrak{d}'_{U_1,Z}} \uparrow \wr \\
U_0 \hat{\otimes}_\pi Z & \xrightarrow{\tilde{\mathfrak{c}}_{X,Y} \hat{\otimes}_\pi \mathrm{id}_Z} & U_1 \hat{\otimes}_\pi Z
\end{array}$$

where the vertical maps are isomorphisms of LCS, and the upper horizontal map is a linear bijection by the previous comments. As a consequence, $\tilde{\mathfrak{c}}_{X,Y} \hat{\otimes}_\pi \mathrm{id}_Z$ is a linear bijection. The lemma is thus proved. □

Putting all the previous information together, we obtain thus a chain of isomorphisms of k-linear vector spaces

$$\begin{aligned}
\mathfrak{Hom}\Big(\mathscr{D}'(M,E) \hat{\otimes}_i \mathscr{D}'(N,F), Z\Big) &\simeq \mathfrak{B}\Big(\mathscr{D}'(M,E), \mathscr{D}'(N,F); Z\Big) = \mathfrak{B}_b\Big(\mathscr{D}'(M,E), \mathscr{D}'(N,F); Z\Big) \\
&\simeq \mathfrak{Hom}\Big(\mathscr{D}'(M \times N, E \boxtimes F), Z\Big) \simeq \mathfrak{Hom}\Big(\mathscr{D}'(M,E) \hat{\otimes}_\pi \mathscr{D}'(M,F), Z\Big),
\end{aligned} \tag{4.2.7}$$

for all complete LCS Z, where we have used Lemma 4.2.3 in the penultimate isomorphism, and isomorphism (4.2.4) in the last step. The equality follows from Lemma 1.4.38. This implies thus that the canonical map from $\mathscr{D}'(M,E) \hat{\otimes}_i \mathscr{D}'(N,F)$ to $\mathscr{D}'(M,E) \hat{\otimes}_\pi \mathscr{D}'(N,F)$ is an isomorphism of LCS.

Finally, since $\mathscr{D}'(M,E) \hat{\otimes}_\pi \mathscr{D}'(N,F) \simeq \mathscr{D}'(M \times N, E \boxtimes F)$, and any space of distributions satisfies the M.c.c. (see Remark 4.1.4), we see that $\mathscr{D}'(M,E) \hat{\otimes}_i \mathscr{D}'(N,F)$ satisfies the M.c.c. Furthermore, since $\mathscr{D}'(M,E)$ and $\mathscr{D}'(N,F)$ are barreled and bornological, so $\mathscr{D}'(M,E) \otimes_i \mathscr{D}'(N,F) \simeq \mathscr{D}'(M,E) \otimes_\beta \mathscr{D}'(N,F)$, the completion of $\mathscr{D}'(M,E) \otimes_\beta \mathscr{D}'(N,F)$ satisfies the M.c.c. Since $\mathscr{D}'(M,E) \otimes \mathscr{D}'(N,F)$ is sequentially dense in $\mathscr{D}'(M \times N, E \boxtimes F)$, because even $\Gamma_c(E) \otimes \Gamma_c(F)$ is sequentially dense in $\mathscr{D}'(M \times N, E \boxtimes F)$ (see [104], Thm. 28.2 and its Cor., and Thm. 39.2, for the local case, where the global case follows from an argument using partitions of unity), Fact 1.4.12 tells us that the convenient completion $\mathscr{D}'(M,E) \tilde{\otimes}_\beta \mathscr{D}'(N,F)$ of $\mathscr{D}'(M,E) \otimes_\beta \mathscr{D}'(N,F) \simeq \mathscr{D}'(M,E) \otimes_i \mathscr{D}'(N,F)$ coincides with its completion $\mathscr{D}'(M,E) \hat{\otimes}_i \mathscr{D}'(N,F)$. The proposition is thus proved. □

4.2.4. *Remark.* Note that Proposition 4.2.2 gives an example of a class of LCS which are not Fréchet and for which the completed projective tensor product coincides with the convenient tensor product, and also with the usual completion of the bornological tensor product. Furthermore, this implies that

$$\mathfrak{c}_{\mathscr{D}'(M,E), \mathscr{D}'(N,F)} : \mathscr{D}'(M,E) \otimes_\beta \mathscr{D}'(N,F) \to \mathscr{D}'(M,E) \otimes_\pi \mathscr{D}'(N,F), \tag{4.2.8}$$

whose underlying set-theoretic map is the identity, is an isomorphism of LCS. Indeed, the fact that the completion of (4.2.8) is an isomorphism tells us that $\mathfrak{B}_b(X,Y;Z') = \mathscr{B}(X,Y;Z')$, for all complete LCS Z', where $X = \mathscr{D}'(M,E)$ and $Y = \mathscr{D}'(N,F)$. To prove that (4.2.8) is an isomorphism it suffices to show that $\mathscr{B}(X,Y;Z) \subseteq \mathfrak{B}_b(X,Y;Z)$ is in fact an equality, for all Hausdorff LCS Z. Given $\phi \in \mathfrak{B}_b(X,Y;Z)$, we consider $i_Z \circ \phi \in \mathfrak{B}_b(X,Y;\hat{Z}) = \mathscr{B}(X,Y;\hat{Z})$, where $i_Z : Z \to \hat{Z}$ is the canonical inclusion. Since the image of the jointly continuous bilinear map $i_Z \circ \phi$ is included in Z, and Z has the subspace topology of $\hat{Z}$ under $i_Z : Z \to \hat{Z}$ (see [95], II.4.1), then ϕ is jointly continuous, *i.e.* $\phi \in \mathscr{B}(X,Y;Z)$, as was to be proved. This provides examples of spaces, beyond the class of Fréchet spaces, for which the bornological tensor product, or equivalently, the inductive tensor product (by Lemma 1.4.38), coincides with the projective tensor product, answering a question posed at the end of Section I.5.8 of [64].

[5] In [47], II.3.3, Lemme 8, the author deals with the case $R = 1$, but the case for any $R \in \mathbb{N}$ follows *verbatim*.

4.2.5. If the distributions u and v are regular, their external tensor product $u \boxtimes v$ is also regular, and if u is induced by $f \in \Gamma^0(E)$ and v is induced by $g \in \Gamma^0(F)$, then $u \boxtimes v$ is induced by the external tensor product $f \boxtimes g$, defined as in the case of smooth functions, *i.e.* the map $(p, p') \mapsto f(p) \otimes g(p')$. Given a sequence $u_1 \in \mathscr{D}'(M_1, E_1), \ldots, u_m \in \mathscr{D}'(M_m, E_m)$, we will also denote their ordered external product $u_1 \boxtimes \cdots \boxtimes u_m$ by

$$\prod_{j=1}^{m}{}^{\boxtimes} u_j. \tag{4.2.9}$$

4.2.6. Since $\mathscr{D}'(M^m, E^{\boxtimes m})$ is the continuous dual of $\Gamma_c((\mathrm{Vol}(M) \otimes E^*)^{\boxtimes m})$, which has an obvious action of the group of permutations $\mathbb{S}_m$, the former is also acted on by $\mathbb{S}_m$. To simplify, we shall usually denote $\mathscr{D}'(M^m, E^{\boxtimes m})$ simply by $\mathscr{D}'_m(M, E)$, and if the vector bundle E is trivial by $\mathscr{D}'_m(M)$. On the other hand, the surjective morphism $\Gamma_c(\mathrm{Vol}(M) \otimes E^*)^{\tilde{\otimes}_\beta m} \to \Gamma_c(\mathrm{Vol}(M) \otimes E^*)^{\tilde{\otimes}_\beta m}/\mathbb{S}_m$ of bornological locally convex $\tilde{\Sigma}^m C^\infty(M)$-modules induces in turn an injective morphism of bornological locally convex $\tilde{\Sigma}^m C^\infty(M)$-modules

$$\left(\Gamma_c(\mathrm{Vol}(M) \otimes E^*)^{\tilde{\otimes}_\beta m}/\mathbb{S}_m\right)' \to \mathscr{D}'(M^m, E^{\boxtimes m}). \tag{4.2.10}$$

4.2.7 Definition. *We call the domain of this map the* space of symmetric E-valued distributions of rank m, *and denote it by $\mathscr{D}'_{\mathrm{sym},m}(M, E)$. If E is the trivial bundle, we will just write $\mathscr{D}'_{\mathrm{sym},m}(M)$.* New?

4.2.8. By Proposition 1.1.4, $\mathscr{D}'_{\mathrm{sym},m}(M, E)$ is canonically identified with the continuous dual of the space of invariants of $\Gamma_c(\mathrm{Vol}(M) \otimes E^*)^{\tilde{\otimes}_\beta m}$ under the action of $\mathbb{S}_m$ (as bornological locally convex $\tilde{\Sigma}^m C^\infty(M)$-modules). Given a distribution $u \in \mathscr{D}'(M^m, E^{\boxtimes m})$, we define the *symmetrization* $\mathrm{sym}_m(u)$ of u as the dual of the action of the idempotent e recalled in the proof of Proposition 1.1.4, *i.e.* $\mathrm{sym}_m(u)([\sigma]) = u(\bar{e}.[\sigma])$, where $\sigma \in \Gamma_c(\mathrm{Vol}(M) \otimes E^*)^{\tilde{\otimes}_\beta m}$ and the brackets denote its class in the associated space of invariants under $\mathbb{S}_m$. It is a just a continuous section of (4.2.10).

4.2.9. Moreover, given $u \in \mathscr{D}'(M^m, E^{\boxtimes m})$ and $v \in \mathscr{D}'(M^{m'}, E^{\boxtimes m'})$, we define the *symmetrized external tensor product* $u \boxtimes_S v \in \mathscr{D}'_{\mathrm{sym},m+m'}(M, E)$ as the symmetrization of $u \boxtimes v$. The associativity of the usual external product of distributions implies that $\boxtimes_S$ is associative as well, and it is clearly seen to be commutative. Given a sequence $u_1 \in \mathscr{D}'(M, E), \ldots, u_m \in \mathscr{D}'(M, E)$, we will also denote their symmetrized external product $u_1 \boxtimes_S \cdots \boxtimes_S u_m$ by

$$\prod_{j=1}^{m}{}^{\boxtimes_S} u_j. \tag{4.2.11}$$

4.2.10. We also note that the support of a symmetric E-valued distributions of rank m, regarded as an $E^{\boxtimes m}$-valued distribution of M^m via (4.2.10), is a *symmetric* subset $S \subseteq M^m$, *i.e.* $\varsigma(S) \subseteq S$, for all $\varsigma \in \mathbb{S}_m$.

4.3 Pull-backs of distributions

Well-known

4.3.1. For the definition of *wave front set* $\mathrm{WF}(u) \subseteq T^*M$ of an E-valued distribution u we refer the reader to [55], Def. 8.1.2 and the two paragraphs before Example 8.2.5. A very nice coordinate-independent but equivalent definition is given in [32], Def. 1.3.1 (see also [102], §1). We will write $\mathrm{WF}_p(u) = T^*_pM \cap \mathrm{WF}(u)$ and recall that $\mathrm{WF}_p(u) \subseteq T^*_pM \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the origin of the vector space T^*_pM. We remark that the image of $\mathrm{WF}(u)$ under the canonical projection $T^*M \to M$ coincides with the singular support of u (see [55], (8.1.5) and Def. 8.1.2), and that

$$\mathrm{WF}(Du) \subseteq \mathrm{WF}(u), \tag{4.3.1}$$

for any E-valued distribution u, and any differential operator D acting on the sections of E (see [55], (8.1.11)).

Well-known **4.3.2 Fact.** *Let M be a manifold, and E a vector bundle over M. Given any E-valued distribution $u \in \mathscr{D}'(M,E)$, if we write $\beta_E^{-1}(u) = \sum_{j\in J} u_j \otimes_{C^\infty(M)} \sigma_j$, with $u_j \in \mathscr{D}'(M)$ and $\sigma_j \in \Gamma(E)$, where β_E is defined in* (4.1.4), *and where we recall that the sum is finite, then*

$$\mathrm{WF}(u) \subseteq \bigcup_{j\in J} \mathrm{WF}(u_j).$$

As a consequence, if F is another vector bundle over M and $t : E \to F$ is a morphism of vector bundles over M, we have

$$\mathrm{WF}\Big(\beta_F \circ \Big(\mathrm{id}_{\mathscr{D}'(M)} \otimes_{C^\infty(M)} \Gamma(t)\Big) \circ \beta_E^{-1}(u)\Big) \subseteq \mathrm{WF}(u),$$

where $\Gamma(t) : \Gamma(E) \to \Gamma(F)$ denotes the morphism induced by t.

This result is just a direct consequence of the definition, but we stated it to emphasize the properties of the wave front set.

4.3.3. We recall that a *cone* (resp., *convex cone*) $\mathscr{P}$ in a finite dimensional real (Euclidean) vector space V is a subset satisfying that $\lambda v \in \mathscr{P}$ (resp., $\lambda v + \lambda' v' \in \mathscr{P}$) for all $\lambda \in \mathbb{R}_{>0}$ and $v \in \mathscr{P}$ (resp., $\lambda, \lambda \in \mathbb{R}_{>0}$ and $v, v' \in \mathscr{P}$). A cone $\mathscr{P}$ is called *blunt* (resp., *blunt convex*) if the origin $\mathbf{0}$ of the vector space V does not belong to $\mathscr{P}$ (resp., and $\mathscr{P} \cup \{\mathbf{0}\}$ is a convex cone). We say that a cone $\mathscr{P}$ is *proper* if $\mathscr{P} \cap (-\mathscr{P}) \subseteq \{\mathbf{0}\}$. Since all the cones we will consider are going to be blunt, we shall usually omit to mention it. Moreover, we say that $\mathscr{P}$ is *closed* (resp., *open*, a *neighborhood*) if the intersection $\mathscr{P} \cap \mathbb{S}(V)$ is a closed subset (resp., an open subset, a neighborhood) in the unit sphere $\mathbb{S}(V)$ of V, where $\mathbb{S}(V)$ is endowed of the subspace topology. Finally, we say that a subset $\mathcal{P}$ of a vector bundle E over a smooth manifold M is *conic* if $\mathcal{P} \cap E_p$ is a cone of E_p for all $p \in M$. The other properties of a cone also apply to a conic subspace, and they mean that the involved condition holds over all the points $p \in M$.

4.3.4. Let $f : M \to N$ be a C^∞ map between smooth manifolds, and F a vector bundle over N. Define the *normal bundle*

$$N_f = \Big\{(q,\lambda) \in T^*N : \text{such that } q = f(p) \text{ and } \lambda\Big(\mathrm{Im}(df_p)\Big) = 0\Big\}.$$

Given a closed conic subspace $\mathcal{P} \subseteq T^*N$ satisfying that $\mathcal{P} \cap N_f = \emptyset$, define the vector space $\mathscr{D}'(N,F)$ formed by the distributions $u \in \mathscr{D}'(N,F)$ whose wave front set $\mathrm{WF}(u) \subseteq \mathcal{P}$. This is indeed a vector space (even a $C^\infty(N)$-module), because $\mathrm{WF}(gu+v) \subseteq \mathrm{WF}(u) \cup \mathrm{WF}(v)$ for any pair of F-valued distributions u and v on N and any $g \in C^\infty(N)$ (see [107], Thm. 2.31). If (V,ψ) is a chart of N, we denote by τ_ψ^* the corresponding trivialization of T^*N. Then, given (V,ψ) a chart of N, $m \in \mathbb{N}_0$, $g \in C_c^\infty(\mathbb{R}^{\dim N})$ whose support is included in $\psi(V)$, and C a closed cone in $\mathbb{R}^{\dim N}$ such that $\tau_\psi^*(\mathcal{P}) \cap (\mathrm{supp}(g) \times C) = \emptyset$, set

$$p^m_{(V,\psi),g,C}(u) = \sup_{y\in C} |y|^m \Big|\widehat{g(\tau_\psi^*)^\wedge(u|_V)}(y)\Big|,$$

where the hat indicates the Fourier transform, and we are using the notation of Proposition 4.1.7 (see [55], Def. 8.2.2, or [107], (2.41)). This defines a family of seminorms $\{p^m_{(V,\psi),g,C}\}$ in $\mathscr{D}'_{\mathcal{P}}(N,F)$, which, together with the usual seminorms given as the restriction to the latter space of the family of seminorms of $\mathscr{D}'(N,F)$ defining the strong topology, endow $\mathscr{D}'_{\mathcal{P}}(N,F)$ with a locally convex structure that is finer than the usual subspace topology of its inclusion inside of $\mathscr{D}'(N,F)$. This is called the *normal topology* of $\mathscr{D}'_{\mathcal{P}}(N,F)$ in [15], in contrast with the *Hörmander topology* of $\mathscr{D}'_{\mathcal{P}}(N,F)$, which is given by replacing the family of seminorms of $\mathscr{D}'(N,F)$ defining the strong topology by those of the weak* topology. We recall that $\mathscr{D}'_{\mathcal{P}}(N,F)$ is sequentially complete for the Hörmander topology (see [107], Thm. 2.51, for a proof where the manifold N is an open subset of the Euclidean space. The general case follows in the same fashion), and the subspace formed by F-valued smooth distributions is sequentially dense in $\mathscr{D}'_{\mathcal{P}}(N,F)$ (see [55], Thm. 8.2.3, for the case where N is an open subset of the Euclidean space. The general case is a direct consequence). It is even nuclear and quasi-complete (see [15], Prop. 14 and Prop. 29). On the other hand, $\mathscr{D}'_{\mathcal{P}}(N,F)$ is nuclear and complete for the normal topology (see [15], Prop. 12 and Cor. 25).

4.3.5 Theorem. *Let $f : M \to N$ be a C^∞ map between smooth manifolds, F a vector bundle over N, and $\mathcal{P} \subseteq T^*N$ be a closed conic subset satisfying that $\mathcal{P} \cap N_f = \emptyset$. Then, there exists a unique sequentially continuous linear map*

$$f^* : \mathscr{D}'_{\mathcal{P}}(N,F) \to \mathscr{D}'_{df^*(\mathcal{P})}(M, f^*F) \tag{4.3.2}$$

for the Hörmander topologies, called the pull-back, *extending the usual pull-back of smooth sections given by $\sigma \mapsto \sigma \circ f$, and we remark that $df^*(\mathcal{P})$ is the closed conic subset of T^*M defined as the direct image of $\mathcal{P}$ under $df^* : T^*N \to T^*M$, the dual of the differential $df : TM \to TN$. The previous pull-back is even continuous for the normal topologies.*

For a proof of the first part in the case of distributions on the Euclidean space see [55], Thm. 8.2.4. The construction of this map in the case of general manifolds is explained in detail in [107], Thm. 2.150, to which we refer. For the proof of the last statement in the case of distributions on the Euclidean space see [16], Prop. 5.1, whereas the case for general manifolds follows from a standard argument. We remark that, if f is an isomorphism of smooth manifolds, and we also denote by f the morphism $f^*F \to F$ of vector bundles, f^* coincides with $f^\wedge$ given in (4.1.2).

4.3.6. More generally, define the subspace $\mathscr{D}'(N,F)^{\perp f}$ of $\mathscr{D}'(N,F)$ formed by the distributions u satisfying that $\mathrm{WF}(u) \cap N_f = \emptyset$. It is a $C^\infty(N)$-submodule of $\mathscr{D}'(N,F)$ by the same arguments as those given in 4.3.4 for $\mathscr{D}'_{\mathcal{P}}(N,F)$. Then, Theorem 4.3.5 tells us that there exists a unique linear map

$$f^* : \mathscr{D}'(N,F)^{\perp f} \to \mathscr{D}'(M, f^*F) \tag{4.3.3}$$

extending the pull-back of smooth sections and satisfying a particular continuity condition. Furthermore, Theorem 4.3.5 also says that

$$\mathrm{WF}(f^*u) \subseteq df^*\mathrm{WF}(u). \tag{4.3.4}$$

The very definition of the pull-back of F-valued distributions on manifolds and the uniqueness of the pull-back of distributions on the Euclidean space under the sequential continuity condition imply that $(f \circ g)^*(u) = g^* \circ f^*(u)$, for any C^∞ maps $f : M \to N$ and $g : N' \to M$, and any $u \in (f^*)^{-1}(\mathscr{D}'(M, f^*F)^{\perp g}) \cap \mathscr{D}'(N,F)^{\perp f \circ g}$.

4.3.7 Fact. *Let u be an E-valued distribution on a manifold M and $t : E \to E$ an isomorphism of the bundle E over the smooth isomorphism $f : M \to M$ such that $t^\wedge(u) = u$. Then,* $\mathrm{WF}(u)$ *is invariant under f as well, i.e.* $df^*\mathrm{WF}(u) = \mathrm{WF}(u)$. *In particular, if $u \in \mathscr{D}'_{\mathrm{sym},m}(M,E)$ is a symmetric E-valued distribution on M, then* $\mathrm{WF}(u)$ *is invariant under the action of $\mathbb{S}_m$.* *Easy*

This is an immediate consequence of (4.3.4).

4.3.8. We will also consider another definition of pull-back on the space of regular distributions. More precisely, let $f : M \to N$ be a C^∞ map between smooth manifolds, and F a vector bundle over N, as before. Then, there exists a linear map

$$f^* : \mathscr{D}'_{\mathrm{reg}}(N,F) \to \mathscr{D}'_{\mathrm{reg}}(M, f^*F) \tag{4.3.5}$$

sending a regular distribution induced by the continuous section $\sigma' \in \Gamma^0(F)$ to the regular distribution induced by the continuous section $\sigma' \circ f \in \Gamma^0(f^*F)$. This is the usual pull-back of continuous maps, and, in particular, no wave front set condition is involved.

4.3.9. We claim that the maps (4.3.3) and (4.3.5) coincide on the intersection of their domains, as stated in the next result. This seems not to have been observed before, but we surmise that the experts should be fully aware of it. The proof is essentially an adaptation of that of [55], Thm. 8.2.4. An interesting consequence of the theorem is that the notation f^* for the pull-back of distributions satisfying the Hörmander wave front set condition and that for the usual pull-back of regular distributions induced by that of functions are compatible, which is an essential ingredient for the construction of a Feynman measure associated with a continuous local propagator of cut type in Section 7.2.

New? **4.3.10 Theorem.** *Let $f : M \to N$ be a C^∞ map between smooth manifolds of dimension n and m, respectively, and F a vector bundle over N. Given $u \in \mathscr{D}'(N,F)^{\perp f} \cap \mathscr{D}'_{\text{reg}}(N,F)$, then the image of u under* (4.3.3) *coincides with the image of u under* (4.3.5).

Proof. In order to distinguish (4.3.3) and (4.3.5) in this proof, we shall still denote the former by f^*, whereas the second will be denoted by $f^\star$. We first note that, by construction of f^*, $f^*(u) = f^\star(u)$ if u is smooth.

Suppose that u is the regular distribution induced by the continuous section $\sigma' \in \Gamma^0(F)$, and define the closed conic subset $\mathcal{P} = \mathrm{WF}(u)$ of T^*N. Consider an open covering $\{V_b\}_{b\in B}$ of N by charts (V_b, ψ_b) that also trivialize the vector bundle F by means of the trivialization τ_b, and an open covering $\{U_a\}_{a\in A}$ of M by charts (U_a, ϕ_a) satisfying that for any $a \in A$ there is an index $b_a \in B$ such that $f(U_a) \subseteq V_{b_a}$. We shall denote the trivialization of f^*F over U_a by ζ_a. By [107], Thm. 2.144, we have that

$$\zeta_a^\wedge(f^*u|_{U_a}) = (\psi_{b_a} \circ f \circ \phi_a^{-1})^*\Big(\tau_{b_a}^\wedge(u|_{V_{b_a}})\Big),$$

for all $a \in A$, whereas Fact 4.1.9 yields that

$$\zeta_a^\wedge(f^\star u|_{U_a}) = (\psi_{b_a} \circ f \circ \phi_a^{-1})^\star\Big(\tau_{b_a}^\wedge(u|_{V_{b_a}})\Big),$$

for all $a \in A$. Since any distribution –in particular f^*u or $f^\star u$– is determined by its restrictions to any open covering by Proposition 4.1.7, it suffices to prove the theorem for the case that M and N are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We suppose that the rank of F is $r \in \mathbb{N}$. We may thus assume that we are in the same situation as [55], Thm. 8.2.4. In particular, we will consider that $\mathcal{P} = \mathrm{WF}(u)$ is a closed conic subspace $N \times (\mathbb{R}^m \setminus \{0\})$. We recall $\mathscr{D}'_{\mathcal{P}}(N,F)$ is the subspace of $\mathscr{D}'(N,F)$ formed by the distributions v satisfying that $\mathrm{WF}(v) \subseteq \mathcal{P}$, and it has a natural locally convex topology given by a family of seminorms in 4.3.4. By [55], Thm. 8.2.3, there is a sequence $\{u_j\}_{j\in\mathbb{N}} \in \Gamma_c(N,F) = C_c^\infty(N)^r$ such that $u_j \to u$ in $\mathscr{D}'_{\mathcal{P}}(N,F)$. We now follow and tailor the proof of [55], Thm. 8.2.4 (see also [107], Thm. 2.61) to suit our requirements.

Consider any point $p_0 \in M$, and set $q_0 = f(p_0) \in N$. Choose a closed conic neighborhood C of $\mathcal{P}_{q_0}$ in $\mathbb{R}^m \setminus \{0\}$ such that $f'(p_0)(\eta) \neq 0$ for all $\eta \in C$; a compact neighborhood F of q_0 such that $\mathcal{P}_q \subseteq C$ for all $q \in F$; and a compact neighborhood G of p_0 such that $f(G)$ is included in the interior F° of F and $f'(p)(\eta) \neq 0$ for all $p \in G$ and $\eta \in C$. Take any function $\phi \in C_c^\infty(N)$ such that $\phi|_{f(G)} \equiv 1$ and $\mathrm{supp}(\phi) \subseteq F$. In the proof of the mentioned theorem, Hörmander shows that

$$f^*(u_j)(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \widehat{\phi u_j}(\eta) I_\xi(\eta)\, d\eta \tag{4.3.6}$$

converges for all $\xi \in C_c^\infty(G^\circ)$, where

$$I_\xi(\eta) = \int_{\mathbb{R}^n} \xi(x) e^{i\langle f(x),\eta\rangle} dx, \text{ and } \widehat{\phi u_j}(\eta) = \int_{\mathbb{R}^m} \phi(y) u_j(y) e^{-i\langle y,\eta\rangle} dy,$$

and $\langle , \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^m$. As a consequence, the Banach-Steinhaus theorem implies that the limit of (4.3.6) defines a distribution in $\mathscr{D}'(G^\circ, f^*F|_{G^\circ})$. We recall that the distributions (4.3.6) coincide on each of the intersections $G_\ell^\circ \cap G_{\ell'}^\circ$, where $\{G_\ell\}_{\ell\in\mathbb{N}}$ is a covering M by such sets, because given any two such sets G_ℓ and $G_{\ell'}$ the sequence $\{f^*(u_j)(\xi)\}_{j\in\mathbb{N}}$ in (4.3.6) is independent of G_ℓ or $G_{\ell'}$, for all $\xi \in C_c^\infty(G_\ell^\circ) \cap C_c^\infty(G_{\ell'}^\circ)$. The element in $\mathscr{D}'(M, f^*F)$ obtained from them is by definition f^*u.

It thus suffices to prove that (4.3.6) also converges to $f^\star u|_{G^\circ}$ in $\mathscr{D}'(G^\circ, f^*F|_{G^\circ})$. We first claim that

$$f^\star(u)(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \widehat{\phi u}(\eta) I_\xi(\eta)\, d\eta, \tag{4.3.7}$$

for all $\xi \in C_c^\infty(G^\circ)$, and we remark that we are assuming the same notation and hypotheses as for (4.3.6).

Indeed,

$$\begin{aligned}\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\widehat{\phi u}(\eta)I_\xi(\eta)d\eta &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\phi(y)u(y)e^{-i\langle y,\eta\rangle}\xi(x)e^{i\langle f(x),\eta\rangle}dxdyd\eta\\ &= \int_{\mathbb{R}^n}\xi(x)\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\phi(y)u(y)e^{-i\langle(y-f(x)),\eta\rangle}d\eta dydx\\ &= \int_{\mathbb{R}^n}\xi(x)\phi(f(x))u(f(x))dx = \int_{\mathbb{R}^n}\xi(x)u(f(x))dx = f^\star(u)(\xi),\end{aligned}$$

where we have used Fubini's theorem, the reciprocity relation

$$\phi(y')u(y') = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\phi(y)u(y)e^{-i\langle(y-y'),\eta\rangle}d\eta dy$$

for the Fourier transform in the space of tempered distributions (because ϕu is a distribution of compact support), and the fact that $\phi(f(x)) = 1$ for all $x \in G$. This proves (4.3.7).

We will now prove that the sequence (4.3.6) converges to (4.3.7). As noted in [55], Eq. (8.2.6), we have

$$\left|I_\xi(\eta)\right| \le C_{h,\xi}\left(1+|\eta|\right)^{-h}, \tag{4.3.8}$$

for all $h \in \mathbb{N}$ and $\eta \in C$, where $C_{h,\xi} > 0$ is a constant depending on h and ξ, but not on $\eta \in C$.

On the other hand, the definition of the convergence $u_j \to u$ in $\mathscr{D}'_{\mathcal{P}}(N,F)$ implies that

$$\left|\widehat{\phi u_j}(\eta) - \widehat{\phi u}(\eta)\right| \le C'_h\left(1+|\eta|\right)^{-h}, \tag{4.3.9}$$

for all $h \in \mathbb{N}$ and $\eta \notin C$, where $C'_h > 0$ is a constant depending on h, but not on $\eta \notin C$.

By the Banach-Steinhaus theorem, there is a constant $C > 0$ and a positive integer $h' \in \mathbb{N}$ such that

$$\left|\widehat{\phi u_j}(\eta)\right| = \left|u_j(\phi e^{-i\langle\eta,(-)\rangle})\right| \le C \sup_{\substack{|\alpha|\le h'\\ y\in\operatorname{supp}(\phi)}}\left|\partial^\alpha\phi(y)e^{-i\langle\eta,y\rangle}\right| \le C'\left(1+|\eta|\right)^{h'}, \tag{4.3.10}$$

for all $\eta \in \mathbb{R}^m$ and all $j \in \mathbb{N}$, where $C' > 0$ is a new constant, also independent of η and j. By definition of u, we have that

$$\left|\widehat{\phi u}(\eta)\right| \le C'',$$

for all $\eta \in \mathbb{R}^m$ and where C'' is a constant independent of η, since it is an integral of a uniformly bounded (with respect to η) continuous function over a compact set. We combine the two previous inequalities to get

$$\left|\widehat{\phi u_j}(\eta) - \widehat{\phi u}(\eta)\right| \le \bar{C}\left(1+|\eta|\right)^{h'}, \tag{4.3.11}$$

for all $\eta \in \mathbb{R}^m$, where $\bar{C} > 0$ is a constant independent of η, and $h' \in \mathbb{N}$ was given in (4.3.10).

By (4.3.7), we have

$$f^\star(u)(\xi) = \frac{1}{(2\pi)^m}\int_C\widehat{\phi u}(\eta)I_\xi(\eta)d\eta + \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m\setminus C}\widehat{\phi u}(\eta)I_\xi(\eta)d\eta. \tag{4.3.12}$$

The dominated convergence theorem together with (4.3.8), (4.3.9), (4.3.10) and (4.3.11) imply that the sequence (4.3.6) converges to (4.3.12), and the proposition follows. □

4.3.11 Theorem. *Let M and N be two manifolds, E and F be two vector bundles over M and N, respectively, and $\mathcal{P} \subseteq T^*M$ and $\mathcal{P}' \subseteq T^*N$ be two closed conic subspaces satisfying that $0 \notin \mathcal{P}_p$ for all $p \in M$ and $0 \notin \mathcal{P}'_{p'}$ for all $p' \in N$. Define $\mathcal{P}'' \subseteq T^*(M\times N) \simeq T^*M \times T^*N$ by*

$$\mathcal{P}'' = (\mathcal{P}\times\mathcal{P}') \cup \left(\mathcal{P}\times(N\times\{0\})\right) \cup \left((M\times\{0\})\times\mathcal{P}'\right),$$

where we regard $(M \times \{0\})$ *inside of* T^*M, *and the same for* $(N \times \{0\})$. *Then,* $\mathcal{P}''$ *is a closed conic subspace of* $T^*(M \times N)$ *and the external tensor product map* (4.2.1) *restricts to a sequentially continuous bilinear map*

$$\mathscr{D}'_{\mathcal{P}}(M,E) \times \mathscr{D}'_{\mathcal{P}'}(N,F) \to \mathscr{D}'_{\mathcal{P}''}(M \times N, E \boxtimes F) \tag{4.3.13}$$

for the Hörmander topologies. Moreover, the previous bilinear map is hypocontinuous for the normal topologies.

For a proof of the first part, see [24], Thm. 8.2.4 and Rk. 8.2.5. For a proof of the last part in the case of open sets of Euclidean spaces, see [16], Thm. 4.6. The general case follows from the same argument.

Well-known

4.4 Push-forwards of $\mathrm{Vol}(M)$-valued distributions

4.4.1. Let $f : M \to N$ be a C^∞ map between smooth manifolds. Assume further that f is *proper*, *i.e.* $f^{-1}(K)$ is compact for all compact subsets $K \subseteq N$. That is equivalent to the fact that the pull-back $f^* : C^\infty(N) \to C^\infty(M)$ restricts to a map $C_c^\infty(N) \to C_c^\infty(M)$ (see [55], Thm. 6.1.1). One defines the *push-forward* map as the continuous dual $f_* : C_c^\infty(M)' \to C_c^\infty(N)'$ of the previous mapping. In terms of distributions, this is precisely a continuous map $f_* : \mathscr{D}'(M, \mathrm{Vol}(M)) \to \mathscr{D}'(N, \mathrm{Vol}(N))$.

4.4.2 Theorem. *Let* M *and* N *be two manifolds and* $f : M \to N$ *be a proper smooth map. Given any closed conic subspace* $\mathcal{P} \subseteq T^*M$ *satisfying that* $0 \notin \mathcal{P}_p$ *for all* $p \in M$, *define* $\mathcal{P}'$ *as*

$$\begin{aligned}\mathcal{P}' &= \Big(T^*N \setminus \big\{(p',0) : p' \in N\big\}\Big) \cap (df^*)^{-1}\Big(\mathcal{P} \cup \big\{(p,0) : p \in M\big\}\Big)\\ &= \Big\{(p',w) \in T^*N : \text{ such that there exists } p \in M \text{ satisfying that } p' = f(p) \text{ and } (p, w \circ df_p) \in \mathcal{P}_p \cup \{0\}\Big\}.\end{aligned}$$

Then, $\mathcal{P}'$ *is a closed conic subspace of* T^*N, *and the push-forward* $f_* : \mathscr{D}'(M, \mathrm{Vol}(M)) \to \mathscr{D}'(N, \mathrm{Vol}(N))$ *restricts to sequentially continuous linear map from* $\mathscr{D}'_{\mathcal{P}}(M, \mathrm{Vol}(M))$ *to* $\mathscr{D}'_{\mathcal{P}'}(N, \mathrm{Vol}(N))$.

For a proof of the first statement, see [24], Thm. 8.3.5.

4.4.3. As a consequence, if $u \in \mathscr{D}'(M, \mathrm{Vol}(M))$ is a $\mathrm{Vol}(M)$-valued distribution, then

$$\mathrm{WF}(f_*(u)) \subseteq \Big\{(p',w) : \exists p \in \mathrm{supp}(u) \text{ such that } f(p) = p' \text{ and either } df_p^*(w) = 0 \text{ or } (p, df_p^*(w)) \in \mathrm{WF}(u)\Big\}. \tag{4.4.1}$$

For a direct proof of the previous inclusion, see [40], Prop. 11.3.3.

Well-known with a novelty

4.5 Internal product

4.5.1. We refer the reader to [80] for a nice discussion of the different definitions of (internal) products of distributions and their relations. Given distributions $u_1, \dots, u_m$ with values in vector bundles $E_1, \dots, E_m$ over M, resp., we say that they satisfy the *Hörmander wave front set condition* if for every point $p \in M$ and any subset $\{j_1 < \dots < j_\ell\} \subseteq \{1, \dots, m\}$, the space $\mathrm{WF}_p(u_{j_1}) + \dots + \mathrm{WF}_p(u_{j_\ell}) \subseteq T_p^*M$ does not include the null vector. In this case, we will also say that the *product* $u_1 \dots u_m$ is *defined in the sense of Hörmander*.[6] It is given by the pull-back (4.3.3) of $u_1 \boxtimes \dots \boxtimes u_m$ under the diagonal map $\mathrm{diag}_m : M \to M^m$. The Hörmander wave front set condition implies that $u_1 \boxtimes \dots \boxtimes u_m$ is in the domain of diag_m^*, so the distribution $u_1 \dots u_m = \mathrm{diag}_m^*(u_1 \boxtimes \dots \boxtimes u_m) \in \mathscr{D}'(M, E_1 \otimes \dots \otimes E_m)$ exists. This product is clearly distributive and commutative, where the last means that if $\varsigma \in \mathbb{S}_m$ and we denote by $\bar{\varsigma} : E_1 \otimes \dots \otimes E_m \to E_{\varsigma^{-1}(1)} \otimes \dots \otimes E_{\varsigma^{-1}(m)}$ the naturally associated morphism of

[6] We remark that we want our definition of product of distributions to imply associativity. This is a condition usually discarded in the theory of distributions, for already some basic examples are not associative, *e.g.* $0 = (\delta.x).\mathrm{vp}(1/x) \neq \delta.(x.\mathrm{vp}(1/x)) = \delta$ for distributions on $\mathbb{R}$ (see [46], 1.1.1, (i)). This also shows that unfortunately the associativity property stated in [89], Thm. IX.43, (c), cannot hold.

vector bundles, then $\bar{\varsigma}^*(u_{\varsigma^{-1}(1)} \dots u_{\varsigma^{-1}(m)}) = u_1 \dots u_m$. Moreover, using the functoriality of the pull-back and the fact that it commutes with direct products, we conclude that, if the product $u_1 \dots u_m$ is defined, then the product of the distributions of any subset of $\{u_1, \dots, u_m\}$ is also defined, and they are all associative: the C_{m-1} possible ways of fully parenthesizing the string $u_1 \dots u_m$ of m symbols via $m-1$ applications of the binary product are defined and they coincide. We recall that these ways of parenthesizing are parametrized by binary rooted trees with m leaves, and that C_m indicates the m-th Catalan number.

4.5.2. By, Theorem 4.3.11 (see also [55], Thm. 8.2.9), given $u \in \mathscr{D}'(M,E)$ and $v \in \mathscr{D}'(M,F)$, the wave front set of their external product $u \boxtimes v$ satisfies that

$$\mathrm{WF}(u \boxtimes v) \subseteq \Big(\Big(\operatorname{supp}(u) \times \{0\}\Big) \times \mathrm{WF}(v)\Big) \cup \Big(\mathrm{WF}(u) \times \mathrm{WF}(v)\Big) \cup \Big(\mathrm{WF}(u) \times \Big(\operatorname{supp}(v) \times \{0\}\Big)\Big). \tag{4.5.1}$$

This last inclusion together with $\mathrm{WF}(f^*u) \subseteq df^*\mathrm{WF}(u)$ applied to the diagonal map imply that, if the product of u and v is defined, then

$$\mathrm{WF}_p(uv) \subseteq \mathrm{WF}_p(u) \cup \Big(\mathrm{WF}_p(u) + \mathrm{WF}_p(v)\Big) \cup \mathrm{WF}_p(v), \tag{4.5.2}$$

for all $p \in M$ (see [55], Thm. 8.2.10). In particular, if $\mathcal{P} \subseteq T^*M$ is a closed conic subspace satisfying that $(\mathcal{P} + \mathcal{P}) \cap (M \times \{0\}) = \emptyset$, then Theorems 4.3.5 and 4.3.11 tells us that the internal product map

$$\mathscr{D}'_{\mathcal{P}}(M,E) \times \mathscr{D}'_{\mathcal{P}}(M,F) \to \mathscr{D}'_{\mathcal{P}}(M, E \otimes F) \tag{4.5.3}$$

sending (u,v) to uv is a hypocontinuous bilinear map for the normal topologies, as the composition of a hypocontinuous bilinear map and a continuous map is hypocontinuous (see for instance [16], Lemma 5.6).

4.5.3. We will also consider the product of regular distributions $u_1, \dots, u_m$ with values in vector bundles E_1, $\dots$, E_m over M, resp. It is also defined as the pull-back (4.3.5) of external tensor product $u_1 \boxtimes \cdots \boxtimes u_m$ under the diagonal map $\mathrm{diag}_m : M \to M^m$. Since $u_1 \boxtimes \cdots \boxtimes u_m$ is a regular distribution with values in the vector bundle $E_1 \boxtimes \cdots \boxtimes E_m$, the pull-back (4.3.5) exists. The product will also be denoted by $u_1 \dots u_m$. Taking into account that the external tensor products of sections of vector bundles is distributive and associative, the same holds for the product of regular distributions. It is also commutative in the sense explained in the two previous paragraphs. The compatibility of this definition of product of distributions with the previous one in the case that a tuple of regular distributions $u_1, \dots, u_m$ also satisfies the Hörmander wave front set condition is a consequence of Theorem 4.3.10.

4.5.4. We will also need the following result concerning the wave front of the usual product of regular distributions. It is similar to the results obtained in the work of D. Iagolnitzer for analytic wave front sets (see [59]). In order to do so, given two closed conic sets $\mathcal{P}$ and $\mathcal{P}'$ of a vector bundle E of rank m over M, define $\mathcal{P} \hat{+}_i \mathcal{P}'$ as the conic set of E such that, for any $p \in M$, $(\mathcal{P} \hat{+}_i \mathcal{P}') \cap E_p$ is

$$\begin{aligned}\Big\{ v \in E_p \setminus \{0\} : &\text{ there exist a trivialization } (U,\tau) \text{ of } E \text{ with } (U,\phi) \text{ a chart of } M \\ &\text{around } p \text{ and sequences } (p_j, v_j) \in (E_U \cap \mathcal{P})^{\mathbb{N}}, (q_j, w_j) \in (E_U \cap \mathcal{P}')^{\mathbb{N}} \\ &\text{such that } (p_j, q_j) \to (p,p) \text{ and } (\pi_2 \circ \tau)(v_j) + (\pi_2 \circ \tau)(w_j) \to v \Big\},\end{aligned} \tag{4.5.4}$$

where $E_U = \sqcup_{q \in U} E_q$, and $\pi_2 : U \times k^m \to k^m$ is the canonical projection. We note that by definition $\mathcal{P} \hat{+}_i \mathcal{P}'$ is a closed conic subset of E, and $\mathrm{WF}(u) \subseteq \mathrm{WF}(u) \hat{+}_i \mathrm{WF}(v)$ (and the same for v).

4.5.5 Fact. *The previous definition of $\mathcal{P} \hat{+}_i \mathcal{P}'$ depends on $\mathcal{P}$ and $\mathcal{P}'$ only locally, i.e. for any $p \in M$, any open set V including p, and any pair closed conic subsets $\mathcal{Q}$ and $\mathcal{Q}'$ such that $\mathcal{P}_q = \mathcal{Q}_q$ and $\mathcal{P}'_q = \mathcal{Q}'_q$ for all $q \in V$, then $(\mathcal{P} \hat{+}_i \mathcal{P}')_p = (\mathcal{Q} \hat{+}_i \mathcal{Q}')_p$.*

This follows immediately from the definition.

Follows the proof of Eskin/Dang

4.5.6 Theorem. *Let $u_1, u_2 \in \mathscr{D}'_{\mathrm{loc}}(M, \mathrm{Vol}(M))$ be two $\mathrm{Vol}(M)$-valued distributions over a manifold of dimension n, that are represented by functions $f_1, f_2 : M \to k$ that are locally bounded and continuous except for a set of measure zero, respectively. Then, the wave front set of the $\mathrm{Vol}(M)$-valued distribution $u_1 u_2$ induced by the function $f_1.f_2 : M \to k$ satisfies that*

$$\mathrm{WF}(u_1 u_2) \subseteq \mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2). \tag{4.5.5}$$

This result (and the proof) is essentially an adaptation of the one given by N. V. Dang in [27], Thm. 3.1, which is in turn based on a proof of G. Eskin in [36], Thm. 14.3 (see also [26], 4.2.2).

Proof. It suffices to prove (4.5.5) at every point $p \in M$. Since the wave front set of u_1, u_2 and $u_1 u_2$ is determined locally, without loss of generality and by multiplying by a smooth function $\phi \in C_c^\infty(M)$ such that $\phi(p) \neq 0$, we may assume that u_1 and u_2 are Schwartz distributions on $\mathbb{R}^n$ represented by continuous functions f_1 and f_2 on $M = \mathbb{R}^n$ of compact support, respectively. From now on, we assume that $M = \mathbb{R}^n$, and we identify the distributions of compact support u_1 and u_2 with f_1 and f_2, resp., to reduce our notation.

Consider $(p, w) \in (\{p\} \times (\mathbb{R}^n)^*) \setminus (\mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2))$. We recall that $\mathrm{WF}(u_1)$ and $\mathrm{WF}(u_2)$ are now regarded as conic subsets of $\mathbb{R}^n \times (\mathbb{R}^n)^*$ (with respect to the second variable). We have to prove that $(p, w) \notin \mathrm{WF}_p(u_1 u_2)$. By definition of wave front set, this is tantamount to prove that there exists an open cone $C \subseteq (\mathbb{R}^n)^*$ and a function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi(p) \neq 0$, $w \in C$, and for all $j \in \mathbb{N}$ we have that

$$\sup_{w' \in C} \left|\widehat{\psi u_1 u_2}(w')\right| \Big(1 + |w'|\Big)^j < \infty, \tag{4.5.6}$$

where $|w'|$ denotes the Euclidean norm of w'. The proof of this theorem is separated in several steps. Notice first that we may further assume without loss of generality that $w \in \mathbb{S}((\mathbb{R}^n)^*)$, where $\mathbb{S}((\mathbb{R}^n)^*)$ denotes the unit sphere of $(\mathbb{R}^n)^*$ for the Euclidean norm. We will now prove the next result.

Follows the proof of Dang

4.5.7 Lemma. *Let u_1, u_2 be two Schwartz distributions of compact support defined on $\mathbb{R}^n$, $p \in \mathbb{R}^n$ a fixed point, and $(p, w) \in \{p\} \times \mathbb{S}((\mathbb{R}^n)^*) \setminus (\mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2))$, where $\mathbb{S}((\mathbb{R}^n)^*)$ denotes the unit sphere of $(\mathbb{R}^n)^*$ for the Euclidean norm. Then, there exists a closed conic neighborhood $W \subseteq (\mathbb{R}^n)^*$ of w and an open set $V \subseteq \mathbb{R}^n$ with $p \in V$ such that for all $h \in C_c^\infty(V)$ we have that*

$$\Big(\Big(\Sigma(hu_1) \cup \{0\}\Big) + \Big(\Sigma(hu_2) \cup \{0\}\Big)\Big) \cap W = \emptyset, \tag{4.5.7}$$

where $\Sigma(u) \subseteq (\mathbb{R}^n)^*$ denotes the frequency set *of a distribution u (see* [40], *Def. 11.1.2).*

Proof. Suppose (4.5.7) is not true. We will show it produces an absurd. Indeed, if (4.5.7) does not hold, then, for any closed conic neighborhoods $W \subseteq (\mathbb{R}^n)^*$ of w and any open neighborhood $V \subseteq \mathbb{R}^n$ of p, there is $h \in C_c^\infty(V)$ and there are elements $w' \in \Sigma(hu_1) \cup \{0\}$ and $w'' \in \Sigma(hu_2) \cup \{0\}$ such that $w' + w'' \in W$. Since $\mathcal{P} \hat{+}_i \mathcal{P}'$ is a closed conic set and $w \notin \pi_2((\mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)) \cap (\{p\} \times (\mathbb{R}^n)^*))$, there is a closed conic neighborhood $W_0 \subseteq (\mathbb{R}^n)^*$ of w such that $W_0 \cap \pi_2((\mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)) \cap (\{p\} \times (\mathbb{R}^n)^*)) = \emptyset$, where $\pi_2 : \mathbb{R}^n \times (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$ is the canonical projection. For every $j \in \mathbb{N}$, take $W_j \subseteq (\mathbb{R}^n)^*$ a closed conic neighborhood of w included in W_0 such that $W_j \cap \mathbb{S}((\mathbb{R}^n)^*)$ has diameter less than or equal to $1/j$. Furthermore, take V_j equal to the ball of $\mathbb{R}^n$ centered at p of radius $1/j$. By hypothesis, there exist $h_j \in C_c^\infty(V_j)$, $w'_j \in \Sigma(h_j u_1) \cup \{0\}$ and $w''_j \in \Sigma(h_j u_2) \cup \{0\}$ such that $w'_j + w''_j \in W_j \subseteq W_0$. As W_j does not include the zero element, w'_j and w''_j are not both zero.

If there exists a subsequence of (w'_j, w''_j) such that one of the components always vanishes (say the second one), then we may assume we have a subsequence of the form $(w'_j, 0)$, where $w'_j \neq 0$ for all $j \in J \subseteq \mathbb{N}$, where J is an infinite subset of $\mathbb{N}$. By [55], Prop. 8.1.3 and (8.1.9),

$$\Sigma(h_j u_1) = \pi_2\Big(\mathrm{WF}(h_j u_1)\Big) \subseteq \pi_2\Big(\mathrm{WF}(u_1)\Big), \tag{4.5.8}$$

where $\pi_2 : \mathbb{R}^n \times (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$ denotes the canonical projection. As a consequence, there is $p_j \in V_j$ such that $(p_j, w'_j) \in \mathrm{WF}(u_1) \cap (V_j \times W_j)$ for all $j \in J \subseteq \mathbb{N}$. Then, the sequence $(p_j, w'_j/|w'_j|)_{j \in J}$ satisfies that $w'_j/|w'_j| \in$

$W_j \cap \mathbb{S}((\mathbb{R}^n)^*)$, so the compactness of $\mathbb{S}((\mathbb{R}^n)^*)$ tells us that by taking a subsequent $(p_j, w'_j/|w'_j|)_{j\in J'}$, where $J' \subseteq J$ is an infinite subset, $(p_j, w'_j/|w'_j|)_{j\in J'}$ is convergent. By construction $(p_j, w'_j/|w'_j|)_{j\in J'} \in \mathrm{WF}(u_1)^{J'}$ and it converges to (p, w), which implies that $(p, w) \in \mathrm{WF}(u_1)$, for the latter is a closed conic set. However, since $\mathrm{WF}(u_1) \subseteq \mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)$, we conclude that $(p, w) \in \mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)$, which is absurd.

It remains to consider the case that for both w'_j and w''_j are different from zero for all but a finite number of indices $j \in \mathbb{N}$. There exists thus an infinite subset $J \subseteq \mathbb{N}$ such that w'_j and w''_j are different zero for all $j \in J$. Using again (4.5.8), there exist $p'_j, p''_j \in V_j$ such that $(p'_j, w'_j) \in \mathrm{WF}(u_1) \cap (V_j \times W_j)$ and $(p''_j, w''_j) \in \mathrm{WF}(u_2) \cap (V_j \times W_j)$ for all $j \in J$. Consider the sequences $(p'_j, w'_j/|w'_j + w''_j|) \in \mathrm{WF}(u_1) \cap (V_j \times W_j)$ and $(p''_j, w''_j/|w'_j + w''_j|) \in \mathrm{WF}(u_2) \cap (V_j \times W_j)$ for all $j \in J$. Since $\tilde{w}_j = w'_j/|w'_j + w''_j| + w''_j/|w'_j + w''_j| \in \mathbb{S}((\mathbb{R}^n)^*)$, there is an infinite subset $J' \subseteq J$ such that $\tilde{w}_j$ converges. By construction $\tilde{w}_j$ converges to w, and p'_j and p''_j converge to p. Moreover, by definition of $\mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)$ we conclude that $(p, w) \in \mathrm{WF}(u_1) \hat{+}_i \mathrm{WF}(u_2)$, which contradicts the hypothesis of the lemma. □

By Lemma 4.5.7, there is a closed conic neighborhood W of w and a smooth function h of compact support such that $h(p) \neq 0$ satisfying that

$$\Big(\big(\Sigma(hu_1) + \Sigma(hu_2)\big) \cup \Sigma(hu_1) \cup \Sigma(hu_2)\Big) \cap W = \Big(\big(\Sigma(hu_1) \cup \{0\}\big) + \big(\Sigma(hu_2) \cup \{0\}\big)\Big) \cap W = \emptyset. \tag{4.5.9}$$

Note that the set intersecting W in the first member of the previous expression is a closed cone, for it is the union of three closed conic sets. Indeed, $\Sigma(hu_1)$ and $\Sigma(hu_2)$ are closed conic sets by definition, whereas $\Sigma(hu_1)+\Sigma(hu_2)$ is a closed conic set, because its intersection with $\mathbb{S}((\mathbb{R}^n)^*)$ is the image under the continuous map $\mathbb{S}((\mathbb{R}^n)^*) \times \mathbb{S}((\mathbb{R}^n)^*) \to \mathbb{S}((\mathbb{R}^n)^*)$ defined as $(w', w'') \mapsto (w' + w'')/|w' + w''|$ of the compact set $(\Sigma(hu_1) \cap \mathbb{S}((\mathbb{R}^n)^*)) \times (\Sigma(hu_2) \cap \mathbb{S}((\mathbb{R}^n)^*))$. Pick nonzero homogeneous functions h_1, h_2 on $(\mathbb{R}^n)^*$ of degree zero satisfying that $h_1(w), h_2(w) \in [0,1]$ for all $w \in (\mathbb{R}^n)^*$, $h_1|_{(\mathbb{R}^n)^* \setminus \{0\}}, h_2|_{(\mathbb{R}^n)^* \setminus \{0\}} \in C^\infty((\mathbb{R}^n)^* \setminus \{0\})$, $h_1(w) = 1$ for all w in a conic neighborhood of $\Sigma(hu_1)$, $h_2(w) = 1$ for all w in a conic neighborhood of $\Sigma(hu_2)$, and $(\mathrm{supp}(h_1) + \mathrm{supp}(h_2)) \cap W = \emptyset$. Since $0 \in \mathrm{supp}(h_j)$ for $j = 1, 2$, we also see that $\mathrm{supp}(h_j) \cap W = \emptyset$, for $j = 1, 2$.

Hence, the absolute convergent integral defining the Fourier transform of $h^2 u_1 u_2$ can be decomposed as

$$\widehat{h^2 u_1 u_2}(w') = (\widehat{hu}_1 * \widehat{hu}_2)(w') = I_1(w') + I_2(w') + I_3(w') + I_4(w'),$$

where $w' \in (\mathbb{R}^n)^*$, and

$$\begin{aligned}
I_1(w') &= \int_{(\mathbb{R}^n)^*} h_1(w' - w'')\widehat{hu}_1(w' - w'')h_2(w'')\widehat{hu}_2(w'')dw'', \\
I_2(w') &= \int_{(\mathbb{R}^n)^*} \big(1 - h_1(w' - w'')\big)\widehat{hu}_1(w' - w'')h_2(w'')\widehat{hu}_2(w'')dw'', \\
I_3(w') &= \int_{(\mathbb{R}^n)^*} \big(1 - h_2(w' - w'')\big)\widehat{hu}_2(w' - w'')h_1(w'')\widehat{hu}_1(w'')dw'', \\
I_4(w') &= \int_{(\mathbb{R}^n)^*} \big(1 - h_1(w' - w'')\big)\widehat{hu}_1(w' - w'')\big(1 - h_2(w'')\big)\widehat{hu}_2(w'')dw''.
\end{aligned}$$

The condition $(\mathrm{supp}(h_1) + \mathrm{supp}(h_2)) \cap W = \emptyset$ implies that $I_1(w') = 0$ for all $w' \in W$. Furthermore, since hu_1 and hu_2 are bounded of compact support, their Fourier transforms $\widehat{hu}_1$ and $\widehat{hu}_2$ satisfy that

$$\sup_{w' \in (\mathbb{R}^n)^*} \big|\widehat{hu}_j(w')\big| < C_j, \tag{4.5.10}$$

for $j = 1, 2$ and some $C_1, C_2 > 0$. The fact that $\mathrm{supp}(1 - h_j) \cap \Sigma(hu_j) = \emptyset$ for all $j = 1, 2$, tells us that

$$\sup_{w' \in (\mathbb{R}^n)^*} \big(1 - h_j(w')\big)\big|\widehat{hu}_j(w')\big|\big(1 + |w'|\big)^m < C'_{m,j}, \tag{4.5.11}$$

for all $m \in \mathbb{N}_0$ and $j = 1,2$, and some $C'_{m,1}, C'_{m,2} > 0$.

Let g_1 be the nowhere vanishing continuous function $(w',w'') \mapsto |w'-w''|$, where $w' \in W \cap \mathbb{S}((\mathbb{R}^n)^*)$ and $w'' \in \operatorname{supp}(h_1) \cup \operatorname{supp}(h_2)$; and let g_2 be the nowhere vanishing continuous function $(w',w'') \mapsto |w'-w''|$, where $w' \in W$ and $w'' \in (\operatorname{supp}(h_1) \cup \operatorname{supp}(h_2)) \cap \mathbb{S}((\mathbb{R}^n)^*)$. An easy argument taking a compact neighborhood of $(W \cap \mathbb{S}((\mathbb{R}^n)^*)) \times ((\operatorname{supp}(h_1) \cup \operatorname{supp}(h_2)) \cap \mathbb{S}((\mathbb{R}^n)^*))$ implies that each function g_j has a minimum $d_j > 0$ for $j = 1,2$. Set $d > 0$ as the minimum of d_1 and d_2. This implies that

$$|w'-w''| \geq d|w'| \quad \text{and} \quad |w'-w''| \geq d|w''|, \tag{4.5.12}$$

for all $w' \in W$ and $w'' \in \operatorname{supp}(h_1) \cup \operatorname{supp}(h_2)$. Hence, for $j = 1,2$ and $w' \in W$ we have

$$\begin{aligned}\left(1+|w'|\right)^m \left|I_{1+j}(w')\right| &\leq C'_{2m,j} C_{3-j} \int_{(\mathbb{R}^n)^*} \frac{\left(1+|w'|\right)^m}{\left(1+|w'-w''|\right)^{2m}} dw'' \leq C'_{2m,j} C_{3-j} \int_{(\mathbb{R}^n)^*} \frac{\left(1+|w'|\right)^m}{\left(1+d|w'|\right)^m \left(1+d|w''|\right)^m} dw'' \\ &\leq C'_{2m,j} C_{3-j} C \int_{(\mathbb{R}^n)^*} \frac{1}{\left(1+d|w''|\right)^m} dw'',\end{aligned} \tag{4.5.13}$$

where $m \in \mathbb{N}_0$, we have used (4.5.12) in the second inequality, and that $w'' \mapsto (1+|w''|)/(1+d|w''|)$ is a bounded function on $(\mathbb{R}^n)^*$ by a constant $C > 0$. The last integral in (4.5.13) converges if $m > n/2$. As a consequence,

$$\sup_{w' \in W} \left(1+|w'|\right)^m \left|I_j(w')\right| < \infty,$$

for $j = 2,3$ and all $m \in \mathbb{N}_0$ such that $m > n/2$. Since the function $w' \mapsto (1+|w'|)^m/(1+|w'|)^{1+n/2}$ on $(\mathbb{R}^n)^*$ is bounded for $m \leq n/2$, we conclude that

$$\sup_{w' \in W} \left(1+|w'|\right)^m \left|I_j(w')\right| < \infty,$$

for $j = 2,3$ and all $m \in \mathbb{N}_0$.

Finally,

$$\begin{aligned}\left(1+|w'|\right)^m \left|I_4(w')\right| &\leq C'_{2m,1} C'_{m,2} \int_{(\mathbb{R}^n)^*} \frac{\left(1+|w'|\right)^m}{\left(1+|w'-w''|\right)^{2m} \left(1+|w''|\right)^m} dw'' \\ &\leq C'_{2m,1} C'_{m,2} \int_{(\mathbb{R}^n)^*} \frac{1}{\left(1+|w'-w''|\right)^m} dw'',\end{aligned} \tag{4.5.14}$$

where we have used the inequality

$$\frac{1+(a+b)}{(1+a)(1+b)} \leq 1$$

for all $a, b \geq 0$ in the last step. The last integral in (4.5.14) converges for $m > n/2$. Hence,

$$\sup_{w' \in W} \left(1+|w'|\right)^m \left|I_4(w')\right| < \infty,$$

for all $m > n/2$. Taking into account that the function $w' \mapsto (1+|w'|)^m/(1+|w'|)^{1+n/2}$ on $(\mathbb{R}^n)^*$ is bounded for $m \leq n/2$, we conclude that

$$\sup_{w' \in W} \left(1+|w'|\right)^m \left|I_4(w')\right| < \infty,$$

for all $m \in \mathbb{N}_0$. As a consequence, for all $m \in \mathbb{N}_0$ we have that

$$\sup_{w' \in W} \left(1+|w'|\right)^m \left|\widehat{h^2 u_1 u_2}(w')\right| < \infty,$$

which proves (4.5.6) for the open conic set C given by the interior of W and $\psi = h^2$. The theorem is proved. □

4.5.8. Let M and N be two manifolds and $f: M \to N$ be a proper smooth map that is an embedding, *i.e.* it is an immersion that is homeomorphic to the subspace of N given by its image. Consider $u \in \mathscr{D}'(N)$ and $v \in \mathscr{D}'(M, \mathrm{Vol}(M))$ such that $\mathrm{WF}(u) \cap N_f = \emptyset$, and the product of $f^*(u)$ and v is defined in the sense of Hörmander (see 4.5.1). Then, an easy application of (4.4.1) tells us that the product of u and $f_*(v)$ is defined in the sense of Hörmander as well. Moreover, we have the identity

$$u.f_*(v) = f_*\big(f^*(u).v\big). \tag{4.5.15}$$

Indeed, the previous identity trivially holds if u and v are smooth, since in that case the left member of (4.5.15) sends $g \in C^\infty(N)$ to $(u.f_*(v))(g) = f_*(v)(u.g) = v((u \circ f).(g \circ f))$, and the right member sends $g \in C^\infty(N)$ to $f_*(f^*(u).v)(g) = (f^*(u).v)(g \circ f) = ((u \circ f).v)(g \circ f)$, which trivially coincides with the previous computation. Finally, the identity (4.5.15) follows from the sequential continuity of the pull-back (see Theorem 4.3.5), of the push-forward (see Theorem 4.4.2) and of the external product (see Theorem 4.3.11).

4.6 Boundary values of complex holomorphic functions as distributions

Well-known

4.6.1. Let $U \subseteq \mathbb{R}^N$ be an open set. Let $\bar{C} \subseteq \mathbb{S}(\mathbb{R}^N) \subseteq \mathbb{R}^N$ be an open subset of the $(N-1)$-dimensional sphere, and let $C \subseteq \mathbb{R}^N$ be an open convex cone given by

$$C = \{y = (y_1, \dots, y_n) \in \mathbb{R}^N : y = c\hat{y}, 0 < c < r, \bar{y} \in \bar{C}\}, \tag{4.6.1}$$

for some $r > 0$. Consider the open subset

$$W = U + C\sqrt{-1} \subseteq \mathbb{C}^N,$$

where $\sqrt{-1}$ denotes the imaginary unit. It is called the *local tube of base U and profile C*. Given $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, we shall denote by $\mathrm{Im}(z)$ the vector $(\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_N))$. Given a complex holomorphic function $f: W \to \mathbb{C}^r$, it defines a family of smooth functions on U of the form $\{f_y : y \in C\}$, where $f_y(x) = f(x + y\sqrt{-1})$ for all $x \in U$. We will regard f_y as a regular distribution on U with values in $\mathbb{C}^r$. Following [71], we recall that f has a *distributional boundary value* on U from W if, for any closed cone $C' \subseteq C$, the limit of the net of distributions

$$\lim_{\substack{y \to 0 \\ y \in C'}} f_y \tag{4.6.2}$$

exists (for the strong topology). The limit distribution f_0 will be called the *distributional boundary value* of f, and it will be also denoted by $\mathrm{bv}(f)$.

4.6.2 Theorem. *Let U, C, W and f be as before. Then, given any bounded open subset U' such that $\bar{U}' \subseteq U$ and any closed cone $C' \subseteq C$, the following conditions are equivalent:* *Well-known*

- *(i) the net of distributions $f_y|_{U'}$ in the space $C_c^\infty(U', \mathbb{C}^r)'$, endowed with the weak* topology, converges to a distribution $f_0 \in C_c^\infty(U', \mathbb{C}^r)'$, for $y \in C'$ and $y \to 0$;*
- *(ii) there is $\epsilon > 0$ such that the set of distributions $\{f_y|_{U'}\}_{\|y\|_\infty < \epsilon}$ is a bounded set in the space $C_c^\infty(U', \mathbb{C}^r)'$, endowed with the weak* topology, where $\|y\|_\infty$ denotes the sup norm of y;*
- *(iii)* f is of moderate growth near the reals, *i.e. there exist positive numbers $D, \epsilon > 0$ and a nonnegative integer $m \in \mathbb{N}_0$ such that*

$$\big\| f(x + y\sqrt{-1}) \big\|_\infty \le D\|y\|_\infty^{-m}, \tag{4.6.3}$$

for all $x \in U'$ and for all $y \in C'$ satisfying that $\|y\|_\infty < \epsilon$.

Moreover, if $m \in \mathbb{N}_0$ is the nonnegative integer in condition (iii), then the limit distribution f_0 has order less than or equal to $m+1$.

This result is due to Martineau in [71], V.2.

Proof. For the equivalence between (i), (ii) and (iii), see [79], Thm. 2.2. The last statement can be found in [55], Thm. 3.1.15. □

4.6.3.*Remark.* We want to stress first that, for any closed cone $C' \subseteq C$, the net given in (4.6.2) converges for the weak* topology $C_c^\infty(U,\mathbb{C}^r)'$ if and only if for any open bounded subset U' whose closure is included in U the net $f_y|_{U'}$ weakly* converges in $C_c^\infty(U',\mathbb{C}^r)'$ for $y \in C'$ and $y \to 0$. This follows directly from the definition of the weak* topology.

On the other hand, we remark that in the definition of boundary value of a holomorphic function we may have equivalently imposed that the limit (4.6.2) is considered with respect to the weak* topology of the space of distributions $C_c^\infty(U,\mathbb{C}^r)'$. Indeed, let C' be a closed cone $C' \subseteq C$ and suppose that the net f_y weakly* converges for $y \in C'$ and $y \to 0$. By the previous theorem, there is some $\epsilon > 0$ such that the set $\{f_y\}_{y\in C,\|y\|_\infty<s}$ is bounded for the weak* topology of $C_c^\infty(U,\mathbb{C}^r)'$. Since $C_c^\infty(U,\mathbb{C}^r)$ is barreled, any bounded set for the weak* topology is also bounded for the strong topology of $C_c^\infty(U,\mathbb{C}^r)'$ (see [95], IV.5.2). Furthermore, as $C_c^\infty(U,\mathbb{C}^r)$ is a Montel space (see [104], Prop. 34.4), [104], Prop. 34.6, tells us that any bounded net converges for the weak* topology if and only if it converges for the strong topology. Hence, the net f_y strongly converges for $y \in C'$ and $y \to 0$, as claimed.

Well-known **4.6.4 Proposition.** *Let U, C, W and f be as in the beginning of this section, and let*

$$C^\vee = \Big\{y' \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\} : y'(y) \geq 0 \text{ for all } y \in C\Big\}$$

be the (blunt) dual cone *of C. Then $C^\vee$ is a proper closed convex cone in $(\mathbb{R}^n)^*$ and* $\mathrm{WF}(\mathrm{bv}(f)) \subseteq U \times C^\vee$.

For a proof, see [55], Thm. 8.1.6, and the comments after it.

4.6.5.*Remark.* The previous result characterizes the wave front set of a boundary value of a holomorphic function (see also [55], Thm. 8.4.15, for the converse).

4.6.6. We shall now recall how to adapt the previous results to smooth manifolds, essentially following [13], Appendix A. We note however that we are not going to make use of the following results in this book. We first remark that given any smooth manifold M, it has a compatible structure of real analytic manifold (see [111], Thm. 1). Furthermore, given two structures of real analytic manifolds on M that are compatible with the smooth structure, then there is an analytic isomorphism between the two. Indeed, as indicated by M. Hirsch in [48], Thm. 2.5.1, it follows from [44], Prop 8, that the space of analytic maps from a real analytic manifold M to another real analytic manifold N is dense in the space of smooth maps from the underlying smooth structure of M to that of N for the strong topology defined in [48], Chapter 2, Section 1. Moreover, the space of smooth isomorphisms from the underlying smooth structure of M to that of N is also an open subset on the space of all smooth maps for the strong topology (see [48], Thm. 2.1.7). Hence, if M is a smooth manifold with two compatible real analytic structures that we denote by M' and M'', then, by the previous results there is an analytic morphism $\phi : M' \to M''$ such that the underlying map between the smooth structures is an isomorphism. In particular, ϕ is bijective and regular (*i.e.* the differential is everywhere invertible). Since a bijective regular analytic map between real analytic manifolds of the same dimension is an analytic isomorphism by the Inverse mapping theorem for analytic functions, we conclude that ϕ is an analytic isomorphism. This implies in particular that any smooth manifold has a compatible analytic structure, that is unique up to analytic isomorphism.

4.6.7. Let M be any real N-dimensional analytic manifold, and let $M_{\mathbb{C}}$ be a *complexification* of M, *i.e.* $M_{\mathbb{C}}$ is a complex N-dimensional manifold provided with an isomorphism ϕ of real analytic manifolds from M onto a real analytic submanifold of the underlying real analytic manifold of $M_{\mathbb{C}}$ such that for every point $p \in M_{\mathbb{C}}$ there is an open set $U \subseteq M_{\mathbb{C}}$ containing p and an biholomorphic isomorphism ψ from U to an open subset V of $\mathbb{C}^N$ such that $\psi(\phi(M) \cap U) = V \cap \mathbb{R}^N$. It is known that any real analytic manifold M admits a complexification, and even though there is in general no uniqueness of complexication, there is a local uniqueness result: given any two complexifications $(M_{\mathbb{C}}, \phi)$ and $(M'_{\mathbb{C}}, \phi')$ of M, there are open neighborhoods U and U' of $\phi(M)$ and of $\phi'(M)$ in $(M_{\mathbb{C}}, \phi)$ and $(M'_{\mathbb{C}}, \phi')$, respectively, and a biholomorphic isomorphism $\phi'' : U \to U'$ such that $\phi'' \circ \phi = \phi'$ (see [112], Prop. 1). From now on we fix a complexification $M_{\mathbb{C}}$ of M and we identify M with the image of M under the isomorphism ϕ of real analytic manifolds from M onto a real analytic submanifold of the underlying real analytic manifold of $M_{\mathbb{C}}$. We also recall that TM stands for the tangent bundle of M. We will also use the following notation. If (U, ϕ) is an analytic chart of M around $p \in M$, we pick $(U_{\mathbb{C}}, \phi_{\mathbb{C}})$ a holomorphic chart of $M_{\mathbb{C}}$ around p such that $U_{\mathbb{C}} \cap M = U$ and $\phi_{\mathbb{C}}|_U = \phi$. Moreover, if $(TU, T\phi)$ is the canonical chart of TM induced by (U, ϕ), we will denote by $(T_{\mathrm{loc}}U, T_{\mathrm{loc}}\phi)$ the subchart where $T_{\mathrm{loc}}U = T\phi^{-1}(\phi(U) \times V)$, for some open set $V \subseteq \mathbb{R}^N$ containing the origin, and $T_{\mathrm{loc}}\phi = T\phi|_{T_{\mathrm{loc}}U}$.

4.6.8 Definition. *Let M be a real N-dimensional analytic manifold and let $M_{\mathbb{C}}$ be a complexification of M as before. An* admissible local diffeomorphism *around $p \in M$ is a smooth isomorphism $\psi : T_{\mathrm{loc}}U \to U_{\mathbb{C}}$, where (U, ϕ) is an analytic chart around p and we regard the underlying structures of smooth manifolds of $T_{\mathrm{loc}}U$ and $U_{\mathbb{C}}$, satisfying that*

(i) for all $q \in U$, $\psi(q, \mathbf{0}) = q$, where $\mathbf{0}$ denotes the origin of T_qM;

(ii) for all $(q, v) \in T_qM$ such that $(q, tv) \in T_{\mathrm{loc}}U$ for all $0 < t \leq 1$, the map $t \mapsto \psi(q, tv)$ is such that

$$\frac{d}{dt}\psi(q, tv)|_{t=0} = \sqrt{-1}cv, \tag{4.6.4}$$

for some $c > 0$.

4.6.9. It is clear that the previous definition is invariant under any change of analytic charts $(T_{\mathrm{loc}}U, T_{\mathrm{loc}}\phi) \to (T_{\mathrm{loc}}U', T_{\mathrm{loc}}\phi')$ of TM induced by a change of analytic charts $(U, \phi) \to (U', \phi')$ of M, and and any change of biholomorphic charts $f : (U_{\mathbb{C}}, \phi_{\mathbb{C}}) \to (U'_{\mathbb{C}}, \phi'_{\mathbb{C}})$ restricting to a change of analytic charts $(U, \phi) \to (U', \phi')$. On the other hand, a *generalized local tube* in $\mathbb{C}^N$ is a subset of the form $\psi(W)$, where $W \subseteq \mathbb{C}^N \simeq T\mathbb{R}^N$ is a local tube and ψ is an admissible local diffeomorphism at the origin of $\mathbb{R}^N$.

4.6.10 Lemma. *Let $W' = \psi(U \times C)$ be a generalized local tube in $\mathbb{C}^N$, where C is the open convex cone determined by $\bar{C}$ according to* (4.6.1)*, and let $\bar{y_0} \in \bar{C}$. Then,* Well-known

(i) there exist a neighborhood $U_0 \times \bar{C}_0$ of $(0, \bar{y_0})$ in $\mathbb{R}^N \times \mathbb{S}(\mathbb{R}^N)$ and $r_0 > 0$ such that $x + c\bar{y}\sqrt{-1} \in W'$, for all $x \in U_0$, $0 < c < r_0$ and $\bar{y} \in \bar{C}_0$;

(ii) for every admissible local diffeomorphism ψ', there exist a neighborhood $U'_0 \times \bar{C}'_0$ of $(0, \bar{y_0})$ in $\mathbb{R}^N \times \mathbb{S}(\mathbb{R}^N)$ such that $\psi'(U'_0 \times \bar{C}'_0) \subseteq W'$.

For a proof, see [13], Lemma A.1.

4.6.11 Definition. *Assume the same hypotheses as in Definition 4.6.8. A* profile *above M is an open subset $\mathcal{C} \subseteq TM$ of the form $\mathcal{C} = \sqcup_{p \in M}\mathcal{C}_p$, where $\mathcal{C}_p \subseteq T_pM$ is a nonempty open (blunt) convex cone.*

4.6.12. Given the tangent bundle TM over M, define $\mathbb{S}_pM = (T_pM \setminus \{0\})/\mathbb{R}_{>0}$, for the obvious action of $\mathbb{R}_{>0}$, and the *sphere bundle* $\mathbb{S}M = \sqcup_{p \in M}\mathbb{S}_pM$. There is a canonical map $\pi_S : TM \setminus 0_M \to \mathbb{S}M$, where 0_M is the image of the zero section $M \to TM$, given by the canonical projection $(T_pM \setminus \{0\}) \to (T_pM \setminus \{0\})/\mathbb{R}_{>0}$, for all $p \in M$. It can be equivalently defined using a Riemannian metric on TM, and it has the structure of a fiber bundle over M (see [11], Section 11). Given a profile $\mathcal{C}$ above M, let $\mathbb{S}(\mathcal{C})$ be the open subset of $\mathbb{S}M$ given by $\pi_S(\mathcal{C})$. Let $\mathbb{S}(\mathcal{C}^c)$ be the complement in $\mathbb{S}M$ of the closure of $\mathbb{S}(\mathcal{C})$. Note that $\mathbb{S}(\mathcal{C}^c) \cap \mathbb{S}_pM$ is included in the complement in in $\mathbb{S}_pM$ of the closure of $\mathbb{S}(\mathcal{C}) \cap \mathbb{S}_pM$, for all $p \in M$.

4.6.13 Definition. *Assume the same hypotheses as in Definition 4.6.8. A subset $\mathcal{T} \subseteq M_{\mathbb{C}}$ is called a* tuboid *with profile $\mathscr{C}$ above M if for every $p \in M$ there is an admissible local diffeomorphism $\psi : T_{\mathrm{loc}}U \to U_{\mathbb{C}}$ around p satisfying that*

(i) for any $(q, \bar{v}) \in \mathbb{S}(\mathcal{C})$ there is a compact neighborhood $K \subseteq \mathbb{S}(\mathcal{C})$ such that $\psi(\pi_S^{-1}(K) \cap T'_{\mathrm{loc}}U) \subseteq \mathcal{T}$, where $T'_{\mathrm{loc}}U \subseteq T_{\mathrm{loc}}U$ is a sufficiently small neighborhood;

(ii) for any $(q,\bar{v}) \in \mathbb{S}(\mathcal{C}^c)$ there is a compact neighborhood $K' \subseteq \mathbb{S}(\mathcal{C}^c)$ such that $\psi(\pi_S^{-1}(K') \cap T''_{\mathrm{loc}}U) \cap \mathcal{T} = \emptyset$, where $T''_{\mathrm{loc}}U \subseteq T_{\mathrm{loc}}U$ is a sufficiently small neighborhood.

4.6.14. Since the previous definition only relies on that of admissible local diffeomorphism, the notion of tuboid is independent of the changes of charts as those explained in 4.6.9. Moreover, in the previous definition one may equivalently state properties (i) and (ii) with respect to any admissible local diffeomorphism (see [13], Prop. A.1). The importance of this result is to fix the possible admissible local diffeomorphisms we are going to handle, as follows. Let $U \subseteq \mathbb{R}^N$ be an open set and let $U_{\mathbb{C}} = U + V\sqrt{-1} \subseteq \mathbb{C}^N$ be a complexification of U, where $V \subseteq \mathbb{R}^N$ is an open subset containing the origin of $\mathbb{R}^N$. We say that a tuboid $\mathcal{T}$ in $U_{\mathbb{C}}$ with profile $\mathcal{C} \subseteq TU$ above U is *standardly described* if for all $p \in U$, the conditions (i) and (ii) in Definition 4.6.13 are fulfilled for the admissible local diffeomorphism given by the restriction of $\psi_0 : T_{\mathrm{loc}}U \to U_{\mathbb{C}}$ sending (p',v) to $p' + \sqrt{-1}v$. Let $U, U' \subseteq \mathbb{R}^N$ be an open sets as before together with their corresponding complexifications $U_{\mathbb{C}}$ and $U'_{\mathbb{C}}$, and let $\mathcal{T}$ and $\mathcal{T}'$ be two tuboids in $U_{\mathbb{C}}$ and $U'_{\mathbb{C}}$ with profiles $\mathcal{C} \subseteq TU$ and $\mathcal{C}' \subseteq TU'$ above U and U', respectively. Given a biholomorphic isomorphism $f : U_{\mathbb{C}} \to U'_{\mathbb{C}}$ restricting to an analytic isomorphism from U onto U', we say it induces an *isomorphism* of tuboids if $f(\mathcal{T}) = \mathcal{T}'$ and $Df|_U(\mathcal{C}) = \mathcal{C}'$.

4.6.15. Assume the same hypotheses as in Definition 4.6.8. Let $\mathcal{T} \subseteq M_{\mathbb{C}}$ be a tuboid with profile $\mathscr{C}$ above M, and let $f : \mathcal{T} \to \mathbb{C}$ be a holomorphic function. Given any point $p \in M$, choose an analytic chart (U,ϕ) around p, together with the complex chart $(U_{\mathbb{C}}, \phi_{\mathbb{C}})$. Consider the tuboid $\mathcal{T}_{(U,\phi)} = \mathcal{T} \cap U_{\mathbb{C}}$ in $U_{\mathbb{C}}$ with profile $(\mathcal{U},\phi) = \mathcal{C} \cap TU$ above U. Using the chart $\phi_{\mathbb{C}}$ we can transfer this structure to a tuboid $\mathcal{T}_{\phi(U)}$ in $\phi_{\mathbb{C}}(U_{\mathbb{C}})$ with profile $\mathcal{C}_{\phi(U)} = D\phi(\mathcal{C}) \cap T\phi(U)$ above $\phi(U)$. By Definition 4.6.13 and the comments on 4.6.14, $\mathcal{T}_{\phi(U)}$ can be standardly described as

$$\mathcal{T}_{\phi(U)} \supseteq \bigcup_{(x,\bar{y}) \in \mathbb{S}(\mathcal{C}_{\phi(U)})} \mathcal{T}_{\phi(U)}(x,\bar{y}), \tag{4.6.5}$$

for some local tube $\mathcal{T}(x,\bar{y}) = U_{(x,\bar{y})} + \sqrt{-1}C_{(x,\bar{y})}$, where $U_{(x,\bar{y})}$ is an open neighborhood of x, $C_{(x,\bar{y})}$ is a cone induced by the open set $\bar{C}_{(x,\bar{y})} \subseteq \mathbb{S}(\mathbb{R}^N)$ according to (4.6.1) and $\bar{C}_{(x,\bar{y})}$ is a neighborhood of $\bar{y}$.

4.6.16 Definition. *Assume the same hypotheses as in the previous paragraph. We say that f admits a* distributional boundary value *on M from $\mathcal{T}$ if $f \circ \phi_{\mathbb{C}}$ has a distributional boundary value* $\mathrm{bv}(f \circ \phi_{\mathbb{C}}) \in \mathscr{D}'(U_{(x,\bar{y})})$ *on $U_{(x,\bar{y})}$ from $U_{(x,\bar{y})} + C_{(x,\bar{y})}\sqrt{-1}$ in the sense of Theorem 4.6.2, for all $(x,\bar{y}) \in \mathbb{S}(\mathcal{C}_{\phi(U)})$. We also say that f has* moderate growth near *M if the map $f \circ \phi_{\mathbb{C}}$ defined on $U_{(x,\bar{y})} + C_{(x,\bar{y})}\sqrt{-1}$ has moderate growth near the reals, for all $(x,\bar{y}) \in \mathbb{S}(\mathcal{C}_{\phi(U)})$.*

4.6.17. It can be shown that the previous definitions of distributional boundary value and of moderate growth are independent of the choice of local tubes made in (4.6.5) and of the holomorphic charts on the complexification $M_{\mathbb{C}}$ of M, so the previous notions are well-defined, and in fact equivalent (see [13], Thm. A.2). Moreover, it is not complicated to verify that, given any chart (U,ϕ), the locally defined distributional boundary values $\mathrm{bv}(f \circ \phi_{\mathbb{C}}) \in \mathscr{D}'(U_{(x,\bar{y})})$ for every $(x,\bar{y}) \in \mathbb{S}(\mathcal{C}_{\phi(U)})$ coincide on the intersections of their domains, so they define a unique distribution $\mathrm{bv}_{(U,\phi)}(f) \in \mathscr{D}'(U)$. Furthermore, the distributions $\mathrm{bv}_{(U,\phi)}(f) \in \mathscr{D}'(U)$ satisfy the coherence condition given in Proposition 4.1.7 when we change analytic charts, so they define a distribution on M, called the *boundary value* of f from the tuboid $\mathcal{T}$, and it denoted by $\mathrm{bv}(f)$ (see [13], Appendix A.2). Moreover, since the notion of moderate growth, or equivalently of distributional boundary value, only depend on a neighborhood of M inside of $M_{\mathbb{C}}$, we see that the previous definitions are also independent of the chosen complexification $M_{\mathbb{C}}$ of M.

Probably well-known

4.6.18 Proposition. *Assume the same hypotheses as in Definition 4.6.8. Let $\mathcal{T} \subseteq M_{\mathbb{C}}$ be a tuboid with profile $\mathscr{C}$ above M, and let $f : \mathcal{T} \to \mathbb{C}^r$ be a holomorphic function. Define the subset $\mathcal{C}^\vee \subseteq T^*M$ given by*

$$\mathcal{C}^\vee \cap T_p^*M = \left\{ y' \in T_p^*M \setminus \{0\} : y'(y) \geq 0 \text{ for all } y \in \mathcal{C}_p \right\},$$

for all $p \in M$. It is called the (blunt) dual conic subset *of $\mathcal{C}$. Then $\mathcal{C}^\vee$ is a proper closed convex conic set and* $\mathrm{WF}(\mathrm{bv}(f)) \subseteq \mathcal{C}^\vee$.

The proof follows from that of Proposition 4.6.4.

Chapter 5

Quantum field theory (after Borcherds)

5.1 The set-up

Based on Borcherds

5.1.1. Following [10], Def. 1, a *spacetime* will be a smooth manifold M provided with a (partial) order $\preceq \subseteq M \times M$ (*i.e.* a reflexive, antisymmetric and transitive relation) that is *closed* (*i.e.* $\preceq$ is a closed subset of $M \times M$). The fact $\preceq$ is closed means precisely that if $p, p' \in M$ satisfy that $p \npreceq p'$, then there exist open subsets $U, U' \subseteq M$ such that $p \in U$, $p' \in U'$, and $(U \times U') \cap \preceq = \emptyset$. In contrast to [10], Def. 1, we have assumed that $\preceq$ is also antisymmetric. The order $\preceq$ is called the *causal relation* of the spacetime, and two subsets $Z, Z' \subseteq M$ are called *spacelike-separated* if $((Z \times Z') \cup (Z' \times Z)) \cap \preceq = \emptyset$. As usual, $\prec = \preceq \setminus \mathrm{Diag}_2$, where $\mathrm{Diag}_m \subseteq M^m$ denotes the diagonal of M^m for all $m \in \mathbb{N}_{\geq 2}$.

5.1.2.*Example.* This definition of spacetime given previously is weaker than the notion of causally simple classical spacetime considered in physics, *i.e.* a connected time-oriented four-dimensional Lorentzian manifold (see [81], p. 163). More generally, a *classical spacetime* (M, g) is any connected Lorentzian manifold, that is time-oriented (see [81], p. 145, for the definition). Define the *classical causal order* $p \preceq p'$ if either $p = p'$ or there is a causal (*i.e.* non-spacelike) and future-pointing continuous and piecewise smooth curve from p to p' (see [81], pp. 146 and 163). It is clear to see that $\preceq$ is a preorder. Recall that $J^+(p) \subseteq M$ is defined by $\{p\} \times J^+(p) = (\{p\} \times M) \preceq$. The classical spacetime (M, g) is called *causal* if the previous preorder satisfies the antisymmetry condition (see [73], Def. 3.7), and *causally simple* if moreover the sets $J^+(p) \subseteq M$ are closed for all $p \in M$ (see [73], Def. 3.63). Hence, if M is a causally simple classical spacetime, then $\preceq \subseteq M \times M$ is a partial order that is closed (see [73], Prop. 3.68). We remark that any globally hyperbolic classical spacetime is causally simple (see [73], Prop. 3.71). There are however interesting causally simple classical spacetimes that are not globally hyperbolic, such as the (universal cover of the) anti-de Sitter spacetime (see [81], Def. 4.23, Examples 8.27 and 14.41).

5.1.3. The following definitions are somehow implicit in [10], even though they do not appear explicitly there. Given $m \in \mathbb{N}_{\geq 2}$, recall that $\mathrm{Par}(m, 2)$ is the set of ordered pairs (J', J'') of nonempty disjoint subsets $J', J'' \subseteq \{1, \dots, m\}$ such that $J' \cup J'' = \{1, \dots, m\}$. For any $(J', J'') \in \mathrm{Par}(m, 2)$, set

$$U_{J',J''} = \Big\{ \bar{p} = (p_1, \dots, p_m) \in M^m : p_{j'} \npreceq p_{j''}, \text{ for all } j' \in J' \text{ and } j'' \in J'' \Big\}. \tag{5.1.1}$$

It is clearly an open subset of M^m, for $\preceq$ is closed. We also have that

$$M^m \setminus \mathrm{Diag}_m = \bigcup_{(J',J'') \in \mathrm{Par}(m,2)} U_{J',J''}. \tag{5.1.2}$$

Indeed, if $\bar{p} \notin \mathrm{Diag}_m$, then there are two different indices j'_0, j''_0 such that $p_{j'_0} \neq p_{j''_0}$. Since $\preceq$ is antisymmetric, we may assume without loss of generality that $p_{j'_0} \npreceq p_{j''_0}$. Define J' as the subset of $\{1, \dots, m\}$ formed by the elements j' such that $p_{j'} \not\preceq p_{j_0''}$, and let $J'' = \{1,\dots,m\} \setminus J'$. Note that $j_0' \in J'$ and $j_0'' \in J''$, so J' and J'' are nonempty. Moreover, $p_{j''} \preceq p_{j_0''}$, for all $j'' \in J''$, and so $\bar{p} \in U_{J',J''}$, as was to be shown.[1] The identity (5.1.2) already appears in the Diplomarbeit of C. Bergbauer (see [6]), but it is usually attributed to C. Popineau and R. Stora (see [83], Section 3, which is the published version of a very old preprint of them, but also [19], Lemma 4.1, for the case of a globally hyperbolic spacetime).

Given any $m' \in \mathbb{N}$ strictly less than m, define

$$U_{m'<m} = \bigcup U_{J',J''}, \tag{5.1.3}$$

where the union is indexed over all $(J',J'') \in \mathrm{Par}(m,2)$ satisfying that $\#(J') = m'$. It is clearly a symmetric open subset of M^m.

New Expected **5.1.4 Lemma.** *Let M be a spacetime of dimension $n \geq 2$. Then $U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}} \neq \emptyset$. Furthermore,* $\mathrm{Diag}_2 \subseteq \overline{U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}}$.

Proof. Let us prove the first statement. By considering any connected component M' of M and $M' \cap U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$ instead of $U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$ it suffices to assume that M is connected. This implies that $M \times M$ is connected, or equivalently, arcwise connected. We will prove that $(M \times M) \setminus \mathrm{Diag}_2$ is arcwise connected. In order to do so, consider a tubular neighborhood $\zeta : E \to M \times M$ of the inclusion $\mathrm{diag}_2 : M \to M \times M$ sending p to (p,p), *i.e.* ζ is a smooth isomorphism from a vector bundle E of rank n over M to an open subset of $M \times M$ including the image of diag_2 such that $\zeta \circ \zeta_0 = \mathrm{diag}_2$, where $\zeta_0 : M \to E$ is the zero section. A simple argument shows that $(M \times M) \setminus \mathrm{Diag}_2$ is connected if $E \setminus \mathrm{Im}(\zeta_0) \simeq \mathrm{Im}(\zeta) \setminus \mathrm{Diag}_2$ is so. Indeed, from the Mayer-Vietoris sequence for the covering $\mathrm{Im}(\zeta) \cup ((M \times M) \setminus \mathrm{Diag}_2)$ of $M \times M$, we get the exact sequence

$$H_1(M \times M, k) \to H_0\Big(\mathrm{Im}(\zeta) \setminus \mathrm{Diag}_2, k\Big) \to H_0\Big(\mathrm{Im}(\zeta), k\Big) \oplus H_0\Big((M \times M) \setminus \mathrm{Diag}_2, k\Big) \to H_0(M \times M, k) \to 0.$$

Using that $H_0(M \times M, k) \simeq k \simeq H_0(\mathrm{Im}(\zeta), k)$, we see that $(M \times M) \setminus \mathrm{Diag}_2$ is connected if $\mathrm{Im}(\zeta) \setminus \mathrm{Diag}_2$ is so, proving the claim. It suffices to show now that $E \setminus \mathrm{Im}(\zeta_0)$ is connected. We first note that the rank of E being strictly greater than 1 tells us that given any point $p \in M$ and nonzero $v, w \in E_p$, there is a continuous path $\gamma : [0,1] \to E_p$ such that $\gamma(0) = v$, $\gamma(1) = w$ with $w \neq 0$ and $\gamma(t)$ never vanishes. It thus suffices to show that given $p, q \in M$ and any nonzero $v \in E_p$, there is a continuous path $\gamma : [0,1] \to E$ such that $\gamma(0) = (p,v)$ and $\gamma(1) = (q,w)$ with $w \neq 0$. In order to do so just pick a Riemannian metric on E, the Levi-Civita connection associated to it, any continuous path $\alpha : [0,1] \to M$ such that $\alpha(0) = p$ and $\alpha(1) = q$, and consider the parallel transport γ of α with respect to the Levi-Civita connection. Since the parallel transport preserves the metric we see that the vector component of γ never vanishes, as was to be shown. Finally, since $(M \times M) \setminus \mathrm{Diag}_2 = U_{\{1\},\{2\}} \cup U_{\{2\},\{1\}}$, the connectedness property of $(M \times M) \setminus \mathrm{Diag}_2$ implies that $U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}} \neq \emptyset$.

It remains to prove that $\mathrm{Diag}_2 \subseteq \overline{U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}}$, *i.e.* for every $p \in M$ and every open set $V \subseteq M$ such that $p \in V$, $(V \times V) \cap U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}} \neq \emptyset$. This last inequality follows from the first part of the lemma by replacing M by V, $U_{\{1\},\{2\}}$ by $U_{\{1\},\{2\}} \cap (V \times V)$ and $U_{\{2\},\{1\}}$ by $U_{\{2\},\{1\}} \cap (V \times V)$. The lemma is thus proved. □

New **5.1.5 Definition.** *We will say that the spacetime M of dimension n, provided with the causal order $\preceq$, is* admissible *if for every $p \in M$, there is an open chart (U, ϕ) of M around p and an hyperplane $H \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the image of the diagonal map $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ such that $(\phi \times \phi)(\preceq \cap (U \times U))$ is included in one of the closed half-spaces determined by H.*

5.1.6.*Remark.* It is easy to construct nonadmissible causal orders on a manifold M by choosing $\preceq$ to be the union of Diag_2 and a reasonable set of finite points.

5.1.7.*Example.* Let (M,g) be a causally simple classical spacetime provided with the usual causal order (see Example 5.1.2). Then, M satisfies the admissibility condition in Definition 5.1.5. Indeed, for any classical

[1] The antisymmetry condition is not imposed in [10], producing a minor gap in that exposition. Indeed, if $\preceq$ is not antisymmetric the set of elements $\bar{p} \in M^m \setminus \mathrm{Diag}_m$ satisfying that $p_{j'} \preceq p_{j''}$ for all $j', j'' = 1, \dots, m$ is nonempty, *i.e.* (5.1.2) does not hold. As a consequence, the so-called "Gaussian condition" in Def. 9 of that article does not recursively determine the Feynman measure outside of the diagonal, contrary to what is stated in the last paragraph of the proof of Thm. 15 there.

spacetime (M,g) and any point $p \in M$, consider a (geodesically) convex open neighborhood $U \subseteq M$ of p (see [81], Ch. 5, Def. 5) and the normal chart ϕ induced by the exponential map. Then, $(\phi\times\phi)(\preceq)$ is the same as the restriction to $(\phi\times\phi)(U\times U)$ of the classical causal order of the Minkowski space $\mathbb{R}^{1,n-1}$ (see [81], Ch. 14, Lemma 2), which is simply given by $p \preceq q$ if and only if $q-p \in J^+(\mathbf{0})$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^{1,n-1}$. Take $v_0 \in \mathbb{R}^{1,n-1}$ such that $\langle v_0, v\rangle \geq 0$ for all $v \in J^+(\mathbf{0})$ (*e.g.* $v_0 = (1,0,\dots,0)$). Define now $L \subseteq \mathbb{R}^{1,n-1}\times\mathbb{R}^{1,n-1}$ to be the hyperplane given by the vectors $(v,w) \in \mathbb{R}^{1,n-1}\times\mathbb{R}^{1,n-1}$ such that $\langle v, v_0\rangle = \langle w, v_0\rangle$, where $\langle\,,\rangle$ is the usual nondegenerate bilinear form of signature $(1,n-1)$. It is clear that H contains the image of the diagonal map $\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$ and that $(\phi\times\phi)(\preceq)$ is included in the closed half-space determined by H formed by the vectors $(v,w)\in\mathbb{R}^{1,n-1}\times\mathbb{R}^{1,n-1}$ such that $\langle v, v_0\rangle \leq \langle w, v_0\rangle$.

5.1.8. We will also need the following notation. Given positive integers $m \geq m'$, define $\mathrm{Par}(m,m')$ as the set of all tuples $T = (T_1,\dots,T_{m'})$, where $\emptyset \neq T_j \subseteq \{1,\dots,m\}$ for all $j = 1,\dots,m'$, $T_j \cap T_{j'} = \emptyset$ for all $j \neq j'$, and $\cup_{j=1}^{m'} T_j = \{1,\dots,m\}$. For any convenient LCS X and $T = (T_1,\dots,T_{m'}) \in \mathrm{Par}(m,m')$, define

$$\mu_X^T : X^{\otimes_\beta m} \to S^{|T_1|}X\tilde{\otimes}_\beta\dots\tilde{\otimes}_\beta S^{|T_{m'}|}X \tag{5.1.4}$$

as the unique continuous linear map satisfying

$$x_1\otimes\cdots\otimes x_m \mapsto x_{T_1}\otimes\cdots\otimes x_{T_{m'}},$$

for all $x_1,\dots,x_m \in X$, where $x_J = x_{j_1}ⓢ\dots ⓢ x_{j_{m''}}$, for $J = \{j_1 < \cdots < j_{m''}\} \subseteq \{1,\dots,m\}$. Its convenient completion will be denoted by $\tilde{\mu}_X^T$.

5.1.9 Lemma. *Let $m \geq m'$ be two positive integers, and $T = (T_1,\dots,T_{m'}) \in \mathrm{Par}(m,m')$. Let A be a commutative bornological locally m-convex algebra that is Fréchet, and X a bornological locally convex A-module. Then $\tilde{\mu}_X^T$ is a morphism of bornological locally convex $\tilde{\Sigma}^m A$-modules, where $\tilde{\Sigma}^m A$ acts by the tensor-wise action on the domain and the codomain of $\tilde{\mu}_X^T$.* *New Expected*

Proof. By continuity of $\tilde{\mu}_X^T$, it is enough to prove that

$$\tilde{\mu}_X^T(xa) = \tilde{\mu}_X^T(x)a$$

for $x = x_1\otimes\cdots\otimes x_m$ and $a = \sum_{\varsigma\in\mathbb{S}_m} a_{\varsigma(1)}\otimes\cdots\otimes a_{\varsigma(m)}$. We suppose even that the tuple $\bar{a} = (a_1,\dots,a_m) \in A^m$ is fixed. Set $y = xa = \sum_{\varsigma\in\mathbb{S}_m} x_1 a_{\varsigma(1)}\otimes\cdots\otimes x_m a_{\varsigma(m)}$. Let us denote $x_j a_{\varsigma(j)}$ by $y_{j,\varsigma}$, for $j = 1,\dots,m$, and $y_\varsigma = x_1 a_{\varsigma(1)}\otimes \cdots\otimes x_m a_{\varsigma(m)}$. Moreover, given any tuple $\bar{b} = (b_1,\dots,b_m) \in A^m$ and any set $J = \{j_1 < \cdots < j_{m''}\} \subseteq \{1,\dots,m\}$, define $b' = b_1\otimes\cdots\otimes b_m$ and

$$J \rightharpoonup b' = \frac{1}{m''!}\sum_{\varsigma\in\mathbb{S}_{m''}} b_{j_{\varsigma(1)}}\otimes\cdots\otimes b_{j_{\varsigma(m'')}} \in \tilde{\Sigma}^{m''}A.$$

Then,

$$\begin{aligned}
\tilde{\mu}_X^T(xa) &= \sum_{\varsigma\in\mathbb{S}_m}\tilde{\mu}_X^T(y_{1,\varsigma}\otimes\cdots\otimes y_{m,\varsigma}) = \sum_{\varsigma\in\mathbb{S}_m}(y_\varsigma)_{T_1}\otimes\cdots\otimes(y_\varsigma)_{T_{m'}} \\
&= \sum_{\varsigma\in\mathbb{S}_m}(x_{T_1}\otimes\cdots\otimes x_{T_{m'}})\Big((T_1 \rightharpoonup a_\varsigma)\otimes\cdots\otimes(T_{m'} \rightharpoonup a_\varsigma)\Big) \\
&= \tilde{\mu}_X^T(x)\sum_{\varsigma\in\mathbb{S}_m}\Big((T_1 \rightharpoonup a_\varsigma)\otimes\cdots\otimes(T_{m'} \rightharpoonup a_\varsigma)\Big) = \tilde{\mu}_X^T(x)a,
\end{aligned}$$

where $a_\varsigma = a_{\varsigma(1)}\otimes\cdots\otimes a_{\varsigma(m)}$. □

5.1.10. Let A be a commutative bornological locally m-convex algebra that is Fréchet. Define

$$\tilde{\mu}_{A,\mathrm{pro}}^T : A^{\tilde{\otimes}_\beta m} \to A^{\tilde{\otimes}_\beta m'} \tag{5.1.5}$$

by $(\mu_A^{(|T_1|)}\tilde{\otimes}_\beta\dots\tilde{\otimes}_\beta\mu_A^{(|T_m|)})\circ\tilde{\mu}_A^T$, where $\mu_A^{(m')}$ was defined in (1.5.1). It is clearly well-defined, because the product of A is commutative, and continuous, so a morphism of LCS.

5.1.11. We will now consider a geometric construction dual to the previous algebraic one. Let $m' \le m$ be two positive integers. For any $T \in \mathrm{Par}(m,m')$, define the map $t_T : \{1,\dots,m\} \to \{1,\dots,m'\}$ sending $j \in \{1,\dots,m\}$ to the unique index $j' \in \{1,\dots,m'\}$ such that $j \in T_{j'}$. Consider now the smooth map

$$\mathrm{diag}_T : M^{m'} \to M^m \tag{5.1.6}$$

given by sending $\bar{p} = (p_1,\dots,p_{m'})$ to the element $\mathrm{diag}_T(\bar{p})$ whose j-th component is $p_{t_T(j)}$, for $j \in \{1,\dots,m\}$. It is fairly straightforward to show that diag_T is a proper map for all $T \in \mathrm{Par}(m,m')$ and all positive integers $m \ge m'$. Note that the diagonal map $\mathrm{diag}_m : M \to M^m$ coincides with diag_T, for the partition T consisting of the only one entry $\{1,\dots,m\}$.

New Easy **5.1.12 Lemma.** *Let $m \ge m'$ be two positive integers, and $T = (T_1,\dots,T_{m'}) \in \mathrm{Par}(m,m')$. Then the pull-back $(\mathrm{diag}_T)^* : C^\infty(M^m) \to C^\infty(M^{m'})$ of the map diag_T given in (5.1.6) is $\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}$ under the identifications of Lemma 2.3.11.*

Proof. It suffices to prove the lemma for a function $f \in C^\infty(M^m)$ of the form $f_1 \otimes \cdots \otimes f_m$, where $f_j \in C^\infty(M)$. Then, given $\bar{p} \in M^{m'}$ we have

$$(\mathrm{diag}_T)^*(f)(\bar{p}) = (f \circ \mathrm{diag}_T)(\bar{p}) = f_1(p_{t_T(1)})\dots f_m(p_{t_T(m)}) = \prod_{j=1}^{m'}\Big(\prod_{j\in T_{j'}} f_j(p_{j'})\Big).$$

On the other hand,

$$\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}(f)(\bar{p}) = \Big(\Big(\prod_{j\in T_1} f_j\Big)\otimes\cdots\otimes\Big(\prod_{j\in T_{m'}} f_j\Big)\Big)(\bar{p}) = \Big(\prod_{j\in T_1} f_j(p_1)\Big)\dots\Big(\prod_{j\in T_{m'}} f_j(p_{m'})\Big) = \prod_{j=1}^{m'}\Big(\prod_{j\in T_{j'}} f_j(p_{j'})\Big),$$

so both expressions agree and the lemma follows. □

Based on Borcherds **5.1.13 Definition.** *A* quantum field theory (QFT) background of order $i \in \mathbb{N}_0$ *is the data of a spacetime M and a vector bundle E over M. We will denote by J^iE the* bundle of jets of order i of E *(see* [77], *11.59). From the QFT background one defines the* space of classical fields $\Gamma(E)$, *the* space of derivatives of classical fields (of order i) $\Gamma(J^iE)$, *the* space of Lagrangians (of order i) $S_{C^\infty(M)}\Gamma(J^iE)$ *and the* space of Lagrangian densities (of order i) $\Gamma(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S_{C^\infty(M)}\Gamma(J^iE)$. *We remark that, given any $C^\infty(M)$-module X, $S_{C^\infty(M)}X$ denotes the symmetric algebra of X in the symmetric monoidal category of $C^\infty(M)$-modules. We shall denote $\Gamma(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S_{C^\infty(M)}\Gamma(J^iE)$ by $\mathscr{L}_i(M,E)$, and $\Gamma_c(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S_{C^\infty(M)}\Gamma(J^iE)$ by $\mathscr{L}_{i,c}(M,E)$.*

Finally, the space of (nonlocal) actions *is the symmetric algebra $\tilde{S}\mathscr{L}_i(M,E)$ of the underlying convenient space of $\mathscr{L}_i(M,E)$ in the symmetric monoidal category* CLCS_{HD} *for the convenient tensor product $\tilde{\otimes}_\beta$. Its topology is the that of the coproduct in the category* BLCS_{HD} *of the homogeneous components $\Gamma(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S^m_{C^\infty(M)}\Gamma(J^iE)$, whose topologies were described in Section 2.3. The same comments apply to the* space of (nonlocal) actions of compact support *$\tilde{S}\mathscr{L}_{i,c}(M,E)$.*

The previous definitions are based on [10], Def. 2-6 (see however Section 5.2 for a discussion of the differences between these definitions and those of Borcherds). Notice that $\tilde{S}\mathscr{L}_{i,c}(M,E)$ is a convenient LCS, and it will be regarded as a quotient of the convenient locally convex $\tilde{\Sigma}C^\infty(M)$-module $\tilde{T}\mathscr{L}_{i,c}(M,E)$ (see Corollary 3.11.3).

5.1.14. *Remark.* Taking the symmetric algebra in the symmetric monoidal category CLCS_{HD} is equivalent to Borcherds' definition, because we want to study the continuous linear functionals on $\tilde{S}\mathscr{L}_{i,c}(M,E)$. Indeed, as the direct sum of convenient LCS is convenient, the only difference is in the tensor product, which has now been conveniently completed. However, the Hahn-Banach theorem and (1.7.12) tell us that the continuous linear functionals on the homogeneous piece $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ coincide with the continuous functionals on the usual symmetric product $S^m\mathscr{L}_{i,c}(M,E)$ in the category BLCS_{HD}.

5.1.15. *Notation.* To avoid confusion we will denote the unit element of $\tilde{S}\mathscr{L}_i(M,E)$ (or $S\mathscr{L}_i(M,E)$) by 1_S, whereas the unit element of $\tilde{T}\mathscr{L}_i(M,E)$ (or $T\mathscr{L}_i(M,E)$) will be denoted by by 1_T. Given $m \in \mathbb{N}$, an elementary symmetric tensor of $S^m\mathscr{L}_{i,c}(M,E)$ (resp., $S^m\mathscr{L}_i(M,E)$) will be typically denoted by $\ell_1 \circledS \dots \circledS \ell_m$, where $\ell_1,\dots,\ell_m \in \mathscr{L}_{i,c}(M,E)$ (resp., $\ell_1,\dots,\ell_m \in \mathscr{L}_i(M,E)$). Analogously, an elementary tensor of $\mathscr{L}_{i,c}(M,E)^{\otimes_\beta m}$ (resp., $\mathscr{L}_i(M,E)^{\otimes_\beta m}$) shall be denoted by $\ell_1|\dots|\ell_m$, for the same choice of elements $\ell_1,\dots,\ell_m$, and we are replacing the symmetric bornological tensor product $\otimes_\beta$ by a bar. The elements of the previous tensor algebras will be usually denoted by the capital letter L, whereas their class in the corresponding symmetric algebra will be denoted by $[L]$. For $m \in \mathbb{N}$, any elementary symmetric tensor element of $S^m_{C^\infty(M)}\Gamma(J^iE)$ will be denoted by $\bar{\sigma} = \sigma_1 \dots \sigma_m$, where $\sigma_1,\dots,\sigma_m \in \Gamma(J^iE)$, where we are omitting the tensor products $\otimes_{C^\infty(M)}$. Hence, any element of $\mathscr{L}_{i,c}(M,E)$ (resp., $\mathscr{L}_i(M,E)$) can be written as a finite sum of terms of the form $\theta\bar{\sigma} = \theta\sigma_1 \dots \sigma_m$, where $\theta \in \Gamma_c(\mathrm{Vol}(M))$ (resp., $\theta \in \Gamma(\mathrm{Vol}(M))$) and we are also omitting the tensor product $\otimes_{C^\infty(M)}$.

5.2 Some technical differences with the article of Borcherds

5.2.1. In his article [10], R. Borcherds works in the general category of sheaves of $\mathscr{O}_M$-modules, where $\mathscr{O}_M$ is the sheaf of rings of the manifold M, without any topology, whereas we work within the category of bornological locally convex $C^\infty(M)$-modules. Moreover, the direct sums and tensor products he considers for the symmetric algebras in his Definitions 4 and 6 are not in the completed sense, as he remarks in the paragraphs following each of the mentioned definitions. The only topology he considers are on the spaces of global sections (of compact support or not). As a consequence, the topology on the symmetric algebra he considers in his Definition 6 is that of a coproduct in the category of LCS (see the second paragraph of p. 631), which in this case is never metrizable (see [95], Ch. II, Ex. 7). In particular, he does not work with the category of sheaves recalled in Theorem 2.3.8.

We also want to observe that, if the underlying manifold M is noncompact, the space of global sections of the symmetric algebra construction $S_{\mathscr{O}_M}\mathscr{E}$ of a sheaf of $\mathscr{O}_M$-modules $\mathscr{E}$ does not coincide with the symmetric algebra $S_{C^\infty(M)}\Gamma(\mathscr{E})$ in the category of $C^\infty(M)$-modules, where we remark that the direct sums and tensor products in the definition of $S_{\mathscr{O}_M}\mathscr{E}$ are considered in the category of sheaves of $\mathscr{O}_M$-modules. This is a straightforward consequence of the fact that the global section functor does not preserve nontrivial infinite direct sums if the base space is noncompact. Indeed, consider a family of pairwise disjoint open sets $\{U_n\}_{n\in\mathbb{N}}$, and let $\{\sigma_n\}_{n\in\mathbb{N}}$ be such that σ_n is a section of $S^n_{\mathscr{O}_M}\mathscr{E}$ having support on a compact set included in U_n. Then, the sheaf property on $S_{\mathscr{O}_M}\mathscr{E}$ tells us that $\{\sigma_n\}_{n\in\mathbb{N}}$ defines a unique section of the former, whereas it does not give any element of $S_{C^\infty(M)}\Gamma(\mathscr{E})$. We believe in particular that the statement before Lemma 14 in [10] is somehow misleading.

5.3 The notion of support

New

5.3.1. The following definitions and discussions seem to be somehow implicit in [10].

Given $m \in \mathbb{N}_0$, define $\mathscr{L}^m_{i,c}(M,E)$ as $\Gamma_c(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S^m_{C^\infty(M)}\Gamma(J^iE)$. Then,

$$\mathscr{L}_{i,c}(M,E) = \Gamma_c\Big(\mathrm{Vol}(M)\Big) \otimes_{C^\infty(M)} S_{C^\infty(M)}\Gamma(J^iE) = \bigoplus_{m\in\mathbb{N}_0} \mathscr{L}^m_{i,c}(M,E). \tag{5.3.1}$$

Note that $\mathscr{L}^m_{i,c}(M,E)$ is the space of sections of compact support of the vector bundle $\mathrm{Vol}(M) \otimes S^mJ^iE$, so it has a natural bornological locally convex topology, which is complete, nuclear and reflexive by the comments in 2.3.9. The space $\mathscr{L}_{i,c}(M,E)$ is endowed thus with the coproduct topology in the category of LCS. By [95], II.6.6, and IV.5.8, we see that $\mathscr{L}_{i,c}(M,E)$ is also complete and reflexive. Furthermore, since the direct sum is countable, by [95], III.7.4, Thm., $\mathscr{L}_{i,c}(M,E)$ is a nuclear space. We consider the convenient tensor product topology on each $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ and so $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ has the corresponding convenient locally convex quotient topology. By (1.4.14), (5.3.1), and isomorphism (2.3.6) in Proposition 2.3.13 we conclude that $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ is also a countable direct sum of spaces of sections of compact support of vector bundles, so by the previous argument $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ is bornological, complete (so convenient), nuclear and reflexive. Taking into account that $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ is a direct summand of the bornological complete nuclear space $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ in the category BLCS_{HD}, the former is also bornological, complete and nuclear (see [95], II.6.2, II.8.2, and III.7.4, Thm.). Since $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ is reflexive, it is barreled (see [95], IV.5.6, Thm.), so its quotient $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ is also barreled (see [95], II.7.2, Cor. 1). Taking into account that a barreled complete nuclear space is always reflexive (see [95], Ch. IV, Exercise 19, (b)), we see that $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ is reflexive. Finally, $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ is endowed with the coproduct topology in the category BLCS_{HD}. We remark that it is bornological, complete, nuclear and reflexive by the same arguments stated above. They also show that $\tilde{T}^+\mathscr{L}_{i,c}(M,E)$ is bornological, complete, nuclear and reflexive.

5.3.2. We note that $\tilde{S}^+\mathscr{L}_i(M,E)$ (resp., $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$) can be decomposed as a direct sum of subspaces

$$\mathscr{L}_i^{m_1}(M,E)\tilde{\circledS}_\beta\ldots\tilde{\circledS}_\beta\mathscr{L}_i^{m_q}(M,E)\ \Big(\text{resp., } \mathscr{L}_{i,c}^{m_1}(M,E)\tilde{\circledS}_\beta\ldots\tilde{\circledS}_\beta\mathscr{L}_{i,c}^{m_q}(M,E)\Big), \tag{5.3.2}$$

indexed by $q\in\mathbb{N}$ and $\bar{m}=(m_1,\ldots,m_q)\in\mathbb{N}_0^q$. According to 3.6.7, any element $[L]$ in (5.3.2) can be canonically regarded as a section (resp., of compact support) of the vector bundle $F_{\bar{m}}^{\boxtimes q}$ over M^q, where

$$F_{\bar{m}} = \bigoplus_{m\in\{m_1,\ldots,m_q\}} S^m J^i E.$$

In particular, its support is defined. Moreover, given $[L]$ in $\tilde{S}^+\mathscr{L}_i(M,E)$ (resp., $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$), we define its *total support* $\mathrm{Supp}([L])$ as the sequence $(\mathrm{supp}_q([L]))_{q\in\mathbb{N}}$ where $\mathrm{supp}_q([L])$ is the union of the supports of all the components of $[L]$ that are sections of vector bundles over M^q. Since $[L]$ has only a finite number of nonzero such components, the union is also finite, and thus a closed subset of M^q. It is a compact set if $[L]\in\tilde{S}^+\mathscr{L}_{i,c}(M,E)$. Furthermore, notice that the sequence $\mathrm{Supp}([L])$ vanishes almost always.

5.3.3. Given a compact $K\subseteq M$, define $\mathscr{L}_{i,K}^m(M,E)$ as the subspace of elements $[L]$ of $\mathscr{L}_{i,c}^m(M,E)$ such that $\mathrm{supp}_1([L])\subseteq K$. We recall that, by the comments in 2.3.9, the subspace of sections of a vector bundle with support contained in a fixed compact set is a closed subset (so complete) of the space of all sections (of compact support of that vector bundle, and it is bornological, because it is Fréchet. In particular, $\mathscr{L}_{i,K}^m(M,E)$ is bornological, complete and a closed subspace of $\mathscr{L}_i^m(M,E)$. We note that the tensor product

$$\mathscr{L}_{i,K}^{m_1}(M,E)\circledS_\beta\ldots\circledS_\beta\mathscr{L}_{i,K}^{m_q}(M,E) \tag{5.3.3}$$

has the initial topology of its inclusion inside of (5.3.2) for any $q\in\mathbb{N}$ and $\bar{m}=(m_1,\ldots,m_q)\in\mathbb{N}_0^q$. Indeed, by the same arguments as those in Lemma 1.4.38, we may equivalently replace the convenient completions of the tensor products in (5.3.2) by usual completions, and the result follows from the fact a Hausdorff LCS has the initial topology of its inclusion inside of its completion. This also implies that the usual completion of the inclusion map of (5.3.3) inside of (5.3.2) is an injection (see [104], Exercise 5.3). Corollary 1.4.21 and the comments in 1.4.23 tell us that we may equivalently replace the usual completion of (5.3.3) by its convenient completion in the previous statement. By Proposition 2.3.13 and the previous arguments, the image of the previous injection is included in the subset of elements of

$$\mathscr{L}_i^{m_1}(M,E)\tilde{\circledS}_\beta\ldots\tilde{\circledS}_\beta\mathscr{L}_i^{m_q}(M,E) \tag{5.3.4}$$

whose support is contained in K^q. This implies that there is a continuous canonical injective map of the symmetric construction $\tilde{S}^+\mathscr{L}_{i,K}(M,E)$ of $\mathscr{L}_{i,K}(M,E)$ into $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$, and the image is included in the subspace formed by the elements $[L]\in\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ satisfying that $\mathrm{supp}_m([L])\subseteq K^m$ for all $m\in\mathbb{N}$. By construction, $\tilde{S}^+\mathscr{L}_{i,K}(M,E)$ is complete and $S^+\mathscr{L}_{i,K}(M,E)=\tilde{S}^+\mathscr{L}_{i,K}(M,E)\cap S^+\mathscr{L}_i(M,E)$ is dense in it, where we recall that $S^+\mathscr{L}_i(M,E)$ is the symmetric construction in the category BLCS_{HD} (*i.e.* where the tensor products are not conveniently completed).

5.3.4 Fact. *Let $K \subseteq M$ be a compact subset and let $[L] \in \tilde{S}^{+}\mathscr{L}_{i,c}(M,E)$ be an element such that $\pi_j(\mathrm{supp}_m([L])) \subseteq K$ for all $m \in \mathbb{N}$ and $j \in \{1,\dots,m\}$. Then $[L] \in \tilde{S}\mathscr{L}_{i,K}(M,E)$.* *Easy*

This result is a direct consequence of the previous comments.

5.4 Generalities on propagators

5.4.1. In this section we will convey the basic definitions of propagators. They will essentially include both Feynman propagators and two-point functions. From now on, we fix a QFT background of order i given by a spacetime M and a vector bundle E.

5.4.2 Definition. *Following* [10], *Def. 7, a* propagator *associated with that QFT background of order i is a separately continuous bilinear map* *Based on Borcherds but new explanations*

$$\Delta : \Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\times\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\to\mathbb{C}.$$

By the very definition of the inductive tensor product $\otimes_i$ (see 1.4.35) and the Hahn-Banach theorem, the previous morphism is equivalent to a continuous linear map

$$\bar{\Delta} : \Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\hat{\otimes}_i\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\to\mathbb{C}.$$

Lemma 1.4.38 together with the fact that the LCS appearing in the previous map are (LF)-spaces (see 2.3.9) tell us that $\bar{\Delta}$ is can be regarded as a continuous linear map

$$\bar{\Delta} : \Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\tilde{\otimes}_\beta\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^i E\Big)\to\mathbb{C}.$$

We denote by $\mathrm{Prop}_i(M,E)$ the $C^\infty(M\times M)$-module of propagators. A propagator Δ is said to be local *if $\Delta(\sigma,\sigma') = \Delta(\sigma',\sigma)$, for all $\sigma,\sigma' \in \Gamma_c(\mathrm{Vol}(M)\otimes J^i E)$ whose supports are spacelike-separated. It is said to be* Feynman *if $\Delta(\sigma,\sigma') = \Delta(\sigma',\sigma)$, for all $\sigma,\sigma' \in \Gamma_c(\mathrm{Vol}(M)\otimes J^i E)$.*

5.4.3 Lemma. *There is a canonical bijective $C^\infty(M\times M)$-linear map* *New Implicit*

$$\bar{\iota} : \mathscr{D}'\Big(M\times M,(J^iE\boxtimes J^iE)^*\Big)\to\mathrm{Prop}_i(M,E). \tag{5.4.1}$$

Moreover, there is a canonical isomorphism of $C^\infty(M\times M)$-modules

$$\mathscr{D}'\Big(M\times M,(J^iE\boxtimes J^iE)^*\Big)\simeq\mathrm{Hom}_{C^\infty(M)\otimes C^\infty(M)}\Big(\Gamma(J^iE)\otimes\Gamma(J^iE),\mathscr{D}'(M\times M)\Big), \tag{5.4.2}$$

where the codomain also coincides with the space

$$\mathfrak{Hom}_{C^\infty(M)\otimes_\beta C^\infty(M)}\Big(\Gamma(J^iE)\otimes_\beta\Gamma(J^iE),\mathscr{D}'(M\times M)\Big).$$

This result is implicit in the exposition of Borcherds, and it gives different incarnations of what a propagator is.

Proof. The first isomorphism is a particular case of Proposition 4.2.2. Finally, the isomorphism (5.4.2) is a direct consequence of Corollary 2.1.7 and Proposition 4.1.10. The last statement is proved as follows. Let F be in the codomain of (5.4.2). By Proposition 1.6.7 it is a separately continuous bilinear map, so it is jointly continuous, for $\Gamma(J^iE)$ is Fréchet and then barreled (see [95], II.7.2, Cor. 2, and III.5.1, Cor. 1)). Since the bornological (resp., convenient) tensor product coincides with the projective tensor product (resp., completed projective tensor product) of metrizable LCS coincide (by 1.4.23 and 1.4.34), the result follows. The lemma is thus proved. □

From now on we will use the identifications of the previous lemma without further explanation, and so we will usually say that Δ is a $(J^iE\boxtimes J^iE)^*$-valued distribution.

5.4.4. *Remark.* Note that all the isomorphisms of Lemma 5.4.3 preserve regular distributions, *i.e.* they restrict to isomorphisms of the same form if we replace $\mathscr{D}'$ by $\mathscr{D}'_{\text{reg}}$ in all the occurrences there.

5.4.5 Definition. *Based on* [10], *Def. 7, we will say that a propagator* $\Delta \in \text{Prop}_i(M,E)$ *is* precut *with family of proper closed convex cones* $\{\mathscr{P}_p\}_{p\in M}$, *where* $\mathscr{P}_p \subseteq T^*_pM$, *if*

(i) if $(v,w) \in \text{WF}_{(p,q)}(\Delta)$, *for any point* $(p,q) \in M \times M$, *then* $-v \in \mathscr{P}_p$ *and* $w \in \mathscr{P}_q$;

(ii) if $(v,w) \in \text{WF}_{(p,p)}(\Delta)$, *for any point* $p \in M$, *then* $w = -v$.

This definition essentially tries to capture the notion of two-point function in the physics literature.

5.4.6. *Example.* Let (M,g) be a classical spacetime and consider the family of proper closed convex cones $\{\mathscr{P}_p\}_{p\in M}$, where $\mathscr{P}_p$ is the image of the forward closed light cone $V^+_p \subseteq T_pM$ at p, *i.e.* the one formed by all future-pointing causal vectors, under the isomorphism $T_pM \simeq T^*_pM$ induced by g_p (see [88], Sections 2 and 4). Then, condition (i) in Definition 5.4.5 is precisely the one given in [88], Def. 4.1. Moreover, consider the usual construction of scalar field theory on a globally hyperbolic classical spacetime (M,g) (see *e.g.* [86]), and assume that the Wightman propagator Δ satisfies the usual global Hadamard assumption, as well as the so-called (KG) and (Com) mod C^∞ assumptions, in pp. 532–533 of [87]. Then, Δ satisfies conditions (i) and (ii) in Definition 5.4.5, *i.e.* Δ is precut. This is a direct consequence of [87], Thm. 5.1, 3 (see also [100], 4.3.4 and 4.5). We also refer the reader to the nice exposition [100], where the wave front sets of the most typical examples of propagators are presented. We also note that the Wightman function of the Klein-Gordon scalar field in the anti-de Sitter space is also precut, as one can easily deduce from [12], Section 6, together with Prop. 4.6.4.

5.4.7. Another definition we will utilize in the sequel is the following. Assume that the $(J^iE \boxtimes J^iE)^*$-valued distribution associated to a propagator $\Delta \in \text{Prop}_i(M,E)$ by means of (5.4.1) is regular, *i.e.* it is induced by a continuous section of $(J^iE \boxtimes J^iE)^*$. We will say in this case that Δ is *continuous*.

New Implicit **5.4.8 Lemma.** *Let* $\Delta \in \text{Prop}_i(M,E)$ *be a local precut propagator. Then the restriction of* Δ *to the open set* $U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}} \subseteq M \times M$ *defined in* (5.1.1) *is smooth, i.e. the intersection of the latter set and the singular support of* Δ *is empty.*

This result is not mentioned in [10], but it is somehow implicit.

Proof. Consider the map fl from $U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$ to itself given by $(p,p') \mapsto (p',p)$. Since Δ is local, the pull-back $\text{fl}^*(\Delta|_{U_{\{1\},\{2\}}\cap U_{\{2\},\{1\}}})$ coincides with itself. We first note that the definition of (4.3.3) and the fact that fl is an isomorphism imply that for all $(p,p') \in U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$, $(v,v') \in \text{WF}_{(p,p')}(\Delta)$ if and only if $(v',v) \in \text{WF}_{(p',p)}(\Delta)$. Assume there is an element $(v,v') \in \text{WF}_{(p,p')}(\Delta)$, where $(p,p') \in U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$. Condition (ii) of the definition of precut propagator now tells us that $(v,v') \in (-\mathscr{P}_p) \times \mathscr{P}_{p'}$ and $(v',v) \in (-\mathscr{P}_{p'}) \times \mathscr{P}_p$, which gives $v \in (-\mathscr{P}_p) \cap \mathscr{P}_p$ and $v' \in (-\mathscr{P}_{p'}) \cap \mathscr{P}_{p'}$, both of which are empty conditions. Taking into account that the image of $\text{WF}(\Delta)$ under the canonical projection $T^*(M\times M) \to M \times M$ is the singular support of Δ, the lemma follows. □

Easy Implicit **5.4.9 Fact.** *Let* $\Delta_1,\dots,\Delta_m \in \text{Prop}_i(M,E)$ *be any family of precut propagators satisfying conditions (i) and (ii) in the previous definition for the same collection of cones* $\{\mathscr{P}_p\}_{p\in M}$. *Then the product* $\Delta_1 \dots \Delta_m \in \mathscr{D}'\big(M \times M, ((J^iE \boxtimes J^iE)^*)^{\otimes m}\big)$ *is defined.*

This is a direct consequence of the Hörmander wave front set condition recalled in Section 4.5, (4.5.1) and (4.5.2).

5.5 Propagators of cut type

5.5.1. Based on [10], Def. 7, we will now present the definition of propagator of cut type. Since the condition stated in [10] is somehow ambiguous, because the author does not state how the limit of the boundary value is precisely defined, we prefer to use a slightly changed terminology, for we are not sure it completely coincides with the idea of Borcherds. We also remark that this condition is only needed to prove the existence of Feynman measures (see Chapter 7).

5.5.2 Definition. *Let Δ be a precut propagator with family of proper closed convex cones $\{\mathscr{P}_p\}_{p\in M}$. It is called* of cut type *with respect to that family of cones if there a collection $\mathcal{U} = \{(U_a, \tau_a, \phi_a)\}_{a\in A}$ satisfying that*

(a) (U_a, τ_a) *is a trivialization of* $(J^iE \boxtimes J^iE)^*$ *and* (U_a, ϕ_a) *is a chart of* $M\times M$*, for all* $a \in A$*, and* $\{U_a : a \in A\}$ *is a locally finite covering of* $M \times M$*;*

(b) there is an involution $\operatorname{invd} : A \to A$ *such that* $\operatorname{invd}(a) = a$ *if and only if* $U_a \cap \operatorname{Diag}_2 \neq \emptyset$*, and it satisfies that* $(U_{\operatorname{invd}(a)}, \tau_{\operatorname{invd}(a)}, \phi_{\operatorname{invd}(a)}) = (\operatorname{fl}(U_a), \operatorname{fl}\circ\tau_a\circ\operatorname{fl}, \operatorname{fl}\circ\phi_a\circ\operatorname{fl})$*, for all* $a\in A$*, where* fl *stands for the usual flip* $(p,q)\mapsto(q,p)$*, for* p,q *in* M *or in* $\mathbb{R}^n$*, or for the flip* $v\otimes w\mapsto w\otimes v$*, for* $v\in(J^iE)^*_p$ *and* $w\in(J^iE)^*_q$*;*

(c) there is a finite set J *such that, for every* $a\in A$*,*

$$\tau_a^{\wedge}(\Delta|_{U_a}) = \sum_{j\in J}\sigma_{a,j}\,\operatorname{bv}\Big(G_{a,j}\Big), \tag{5.5.1}$$

where $\sigma_{a,j}$ *is a smooth vector function on* $\phi_a(U_a)$*, and* $\operatorname{bv}(-)$ *indicates the boundary value of a holomorphic function* $G_{a,j} : W_{a,j} \to \mathbb{C}$ *with*

$$W_{a,j} = \Big\{ z = (z_1,\dots,z_{2n}) \in \phi_a(U_a) + C_{a,j}\sqrt{-1} : \big|\operatorname{Im}(z_j)\big| < r \text{ for all } j = 1,\dots,n \Big\} \subseteq \mathbb{C}^{2n}, \tag{5.5.2}$$

where $\sqrt{-1}$ *denotes the imaginary unit,* $C_{a,j}\subseteq\mathbb{R}^{2n}$ *is an open convex cone, and*

$$G_{a,j} = \Big(\log(p_{j,0})\prod_{\ell\in L_j} p_\ell^{s_\ell}\Big)\Big|_{W_{a,j}}, \tag{5.5.3}$$

where L_j *is a finite set,* $p_{j,0}, p_\ell \in \mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n]$ *are real polynomial which are (uniquely) regarded as holomorphic functions, and in* (5.5.3) *we are considering the usual definition of the logarithm on* $\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0}$*;*

(d) for all $j\in J$*, the polynomials* $p_{j,0}, p_\ell \in \mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n]$ *in* (5.5.3) *are symmetric under the interchange between* $(x_1,\dots,x_n)$ *and* $(y_1,\dots,y_n)$*;*

(e) for all $j\in J$ *and all* $a\in A$*,* $\sigma_{\operatorname{invd}(a),j} = \sigma_{a,j}\circ\operatorname{fl}$*, where* fl *denotes the usual flip* $(p,q)\mapsto(q,p)$*, for* p,q *in* M*;*

(f) for every $a\in A$*,* $j\in J$ *and* $(p,p')\in U_a$*,* $(d\phi_a)^*_{(p,p')}(C^\vee_{a,j})$ *is included in the cone* $(-\mathscr{P}_p)\times\mathscr{P}_{p'}$*, where we identify canonically* TU_a *and* $U_a+\sqrt{-1}\mathbb{R}^{2n}$*, and if we further assume that* $a = \operatorname{invd}(a)$*, then it is is included in the cone* $\{(v,-v) : v\in T^*_pM\}\cap((-\mathscr{P}_p)\times\mathscr{P}_p)$*.*

We remark that the polynomials $\{p_{j,0}, p_\ell : \ell\in L_j\}$ given in (5.5.3) are in principle different for every $j\in J$. We also note that the conditions (a) and (b) for the collection $\mathcal{U}$ are of topological nature and can always be satisfied.

5.5.3.*Remark.* The condition given by 5.5.1 and 5.5.3 in the definition of propagator of cut type is very similar to the Hadamard parametrix expansion proposed by B. Kay and R. Wald in [62], Section 3.3, for the case of the scalar fields satisfying the Klein-Gordon on a (four dimensional) globally hyperbolic classical spacetime (M,g), called the *global Hadamard condition*, which was in turn a more precise version of the *old Hadamard condition* (see [86], Def. 1.18), that was prevalent in the literature. However, the condition of being of cut type is *a priori* stronger, because the expression (5.5.1) is valid on the whole $M\times M$, whereas the global Hadamard condition only expresses the propagator as a boundary value over a set of the form $N\times N$, where N is a causal normal neighborhood of a Cauchy hypersurface of M (see [86], Def. 1.12, for the definition). Hence, it would remain to know if the condition of being of cut type is implied by the global Hadamard condition for the case of scalar fields satisfying the Klein-Gordon on a (four dimensional) globally hyperbolic classical spacetime (M,g). Borcherds claims that the cut condition is "satisfied by almost any reasonable example" (see [10], p. 630, l. 8), however it seems that the precise expression of the two-point functions is known in only a few examples of general globally hyperbolic spacetimes.

For further details on the scalar theory on globally hyperbolic classical spacetimes, we refer the reader to the article [87] by M. Radzikowski, who proved that a state satisfies the global Hadamard condition, the Klein-Gordon equation and a commutator condition modulo C^∞ if and only if the associated Feynman propagator is a distinguished parametrix of the Klein-Gordon operator modulo C^∞, and if and only if the former state satisfies the so-called wave front set spectrum condition and the commutator condition modulo C^∞ (see [87], Thm. 5.1). For a nice exposition on the Hadamard parametrix expansion we refer the reader to [56], Section 17.4, or [94], Section 4.

We finally remark that item (f) provide only a compatibility between the condition on the wave front sets of the propagator coming from the precut assumption and the wave front set that is obtained from the boundary value operation.

5.5.4. *Example.* Assume the same hypotheses of Example 5.4.6. We consider the case where (M,g) is the Minkowski spacetime, and let Δ be the usual Wightman 2-point function of the scalar field theory. Then Δ is of cut type. Indeed, it is clearly of precut by Example 5.4.6. Moreover, by taking a global chart in that case, conditions 5.5.1 and 5.5.3 follow from the expression of the Wightman 2-point function given in [9], Ch. 8, App. F, (F.8b), (see also [87], (12)). We recall that in this case the tuboid is given by $M^2 + C\sqrt{-1}$, where $C \subseteq M^2$ is the set formed by the pairs (p,p') such that $p'-p$ is in the forward light cone V^+ recalled in Example 5.4.6 (see [99], (3.4.5)). It is easy to verify that $C^\vee \subseteq ((-V^+)\times V^+)\cap\{(-v,v) : v \in M\}$, so item (f) holds. Moreover, using the explicit description of the wave front set of Δ given in [100], 4.3.4, we see that it is in fact included in $C^\vee$. Conditions (d) and (e) follow from the symmetry property of the analytic extension of the Wightman functions (see *e.g.* [99], (3.4.11)). A nice and detailed explanation of the holomorphic extension of the Wightman functions on Minkowski space can be found in [98] (see Sections 2-4, and Thms. 3-5 and 3-6), and [99], Section 3.4. Moreover, since the components of the propagators for the Dirac theory on the Minkowski spacetime are given by the corresponding propagators of the scalar theory on the Minkowski spacetime (see [9], Ch. 8, App. F, (F.21)), we also see that propagators for the Dirac theory on the Minkowski spacetime are of cut type. On the other hand, let (M,g) be the de Sitter spacetime and Δ be the usual 2-point function of the Klein-Gordon scalar field theory, as explained in [13]. By using the usual embedding of the de Sitter spacetime into the Minkowski spacetime we can see that Δ is also of cut type. Indeed, conditions 5.5.1 and 5.5.3 follow from [13], Cor. 4.1, specially eq. (4.21) and (4.22), whereas conditions (d) and (e) follows from the symmetry property of the Wightman function (see the comments before Section 3.1 in [13] and those in Section 3.4). Item (f) also holds in this case by a simple computation similar to that of the Minkowski case. Analogously, the two-point Wightman function Δ of the Klein-Gordon scalar theory on the anti-de Sitter spacetime is also of cut type (see [12], Section 6).

5.6 Extensions of propagators

5.6.1. In this section we shall construct a *Laplace pairing* from a given propagator Δ that is either continuous or precut. This follows the general philosophy introduced by C. Brouder, B. Fauser, A. Frabetti and R. Oeckl in [17], to which we refer for a very nice and detailed discussion about the use of Laplace pairings in QFT. The notion of Laplace pairing was introduced by P. Doubilet, G.-C. Rota and J. Stein in [31], Section 7, (1) (see also [91]). Our definition is however an extension of the one considered by them, for they have (essentially) dealt with symmetric monoidal categories, whereas we work with more general ones.

New **5.6.2 Definition.** *Let* $(\mathscr{C},\otimes_{\mathscr{C}},I_\otimes,\tau,\boxtimes_{\mathscr{C}},I_\boxtimes,\mathrm{sh})$ *be a k-linear framed symmetric 2-monoidal category inside of* $(\mathscr{C}',\boxtimes_{\mathscr{C}'},I'_\boxtimes,\tau')$ *via the faithful functor F (see Definitions 3.2.2 and 3.2.6). Given a unitary and counitary bialgebra* $(C,\mu_C,\eta_C,\Delta_C,\epsilon_C)$ *relative to the previous k-linear symmetric 2-monoidal category and a unitary algebra* $(A,\mu_{A,\ell},\eta_{A,\ell})$ *in* $\mathscr{C}$ *with respect to* $\otimes_{\mathscr{C}}$ *and* $I_\otimes$*, a* left Laplace pairing *on C relative to the previous k-linear* *symmetric 2-monoidal category and with values on A is a morphism $\langle\,,\rangle : C \boxtimes_{\mathscr{C}} C \to A$ of $\mathscr{C}$ satisfying that the diagrams*

$$
\begin{array}{ccccccc}
F(C)\boxtimes_{\mathscr{C}'} F(C)\boxtimes_{\mathscr{C}'} F(C) & \xrightarrow[\sim]{\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}\psi_2(C,C)} & F(C)\boxtimes_{\mathscr{C}'} F(C\boxtimes_{\mathscr{C}} C) & \xrightarrow{\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}F(\mu_C)} & F(C)\boxtimes_{\mathscr{C}'} F(C) & \xrightarrow[\sim]{\psi_2(C,C)} & F(C\boxtimes_{\mathscr{C}} C) \\
\downarrow{\scriptstyle F(\Delta_C)\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}} & & & & & & \downarrow{\scriptstyle F(\langle,\rangle)} \\
F(C\otimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}'} F(C)\boxtimes_{\mathscr{C}'} F(C) & & & & & & F(A) \\
\downarrow{\scriptstyle \mathrm{id}_{F(C\otimes_{\mathscr{C}} C)}\boxtimes_{\mathscr{C}'}\varphi_2(C,C)} & & & & & & \uparrow{\scriptstyle F(\mu_{A,\ell})} \\
F(C\otimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}'} F(C\otimes_{\mathscr{C}} C) & \xrightarrow[\sim]{\psi_2(C\otimes_{\mathscr{C}} C,\,C\otimes_{\mathscr{C}} C)} & F\big((C\otimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}}(C\otimes_{\mathscr{C}} C)\big) & \xrightarrow{F(\mathrm{sh}_{C,C,C,C})} & F\big((C\boxtimes_{\mathscr{C}} C)\otimes_{\mathscr{C}}(C\boxtimes_{\mathscr{C}} C)\big) & \xrightarrow{F(\langle,\rangle\otimes_{\mathscr{C}}\langle,\rangle)} & F(A\otimes_{\mathscr{C}} A)
\end{array}
\tag{5.6.1}
$$

commutes in $\mathscr{C}'$ and

$$
\begin{array}{ccccc}
C\boxtimes_{\mathscr{C}} I_{\boxtimes} & & \xrightarrow{\mathrm{id}_C\boxtimes_{\mathscr{C}}\eta_C} & & C\boxtimes_{\mathscr{C}} C \\
\downarrow{\scriptstyle r^{\boxtimes}_C} & & & & \downarrow{\scriptstyle \langle,\rangle} \\
C & \xrightarrow{\epsilon_C} & I_{\otimes} & \xrightarrow{\eta_{A,\ell}} & A
\end{array}
\tag{5.6.2}
$$

commutes in $\mathscr{C}$. Analogously, given the same unitary and counitary bialgebra as before and a unitary algebra $(A,\mu_{A,r},\eta_{A,r})$ in $\mathscr{C}$ with respect to $\otimes_{\mathscr{C}}$ and $I_{\otimes}$, a right Laplace pairing *on C relative to the previous k-linear symmetric 2-monoidal category and with values on A is a morphism $\langle\,,\rangle : C \boxtimes_{\mathscr{C}} C \to A$ of $\mathscr{C}$ satisfying that the diagrams*

$$
\begin{array}{ccccccc}
F(C)\boxtimes_{\mathscr{C}'} F(C)\boxtimes_{\mathscr{C}'} F(C) & \xrightarrow[\sim]{\psi_2(C,C)\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}} & F(C\boxtimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}'} F(C) & \xrightarrow{F(\mu_C)\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}} & F(C)\boxtimes_{\mathscr{C}'} F(C) & \xrightarrow[\sim]{\psi_2(C,C)} & F(C\boxtimes_{\mathscr{C}} C) \\
\downarrow{\scriptstyle \mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C)}\boxtimes_{\mathscr{C}'}F(\Delta_C)} & & & & & & \downarrow{\scriptstyle F(\langle,\rangle)} \\
F(C)\boxtimes_{\mathscr{C}'} F(C)\boxtimes_{\mathscr{C}'} F(C\otimes_{\mathscr{C}} C) & & & & & & F(A) \\
\downarrow{\scriptstyle \varphi_2(C,C)\boxtimes_{\mathscr{C}'}\mathrm{id}_{F(C\otimes_{\mathscr{C}} C)}} & & & & & & \uparrow{\scriptstyle F(\mu_{A,r})} \\
F(C\otimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}'} F(C\otimes_{\mathscr{C}} C) & \xrightarrow[\sim]{\psi_2(C\otimes_{\mathscr{C}} C,\,C\otimes_{\mathscr{C}} C)} & F\big((C\otimes_{\mathscr{C}} C)\boxtimes_{\mathscr{C}}(C\otimes_{\mathscr{C}} C)\big) & \xrightarrow{F(\mathrm{sh}_{C,C,C,C})} & F\big((C\boxtimes_{\mathscr{C}} C)\otimes_{\mathscr{C}}(C\boxtimes_{\mathscr{C}} C)\big) & \xrightarrow{F(\langle,\rangle\otimes_{\mathscr{C}}\langle,\rangle)} & F(A\otimes_{\mathscr{C}} A)
\end{array}
\tag{5.6.3}
$$

commutes in $\mathscr{C}'$ and

$$
\begin{array}{ccccc}
I_{\boxtimes}\boxtimes_{\mathscr{C}} C & & \xrightarrow{\eta_C\boxtimes_{\mathscr{C}}\mathrm{id}_C} & & C\boxtimes_{\mathscr{C}} C \\
\downarrow{\scriptstyle \ell^{\boxtimes}_C} & & & & \downarrow{\scriptstyle \langle,\rangle} \\
C & \xrightarrow{\epsilon_C} & I_{\otimes} & \xrightarrow{\eta_{A,r}} & A
\end{array}
\tag{5.6.4}
$$

commutes in $\mathscr{C}$.

5.6.3. *Remark.* Note that we need the symmetric 2-monoidal category to be framed in order to be able to define a left or right Laplace pairing. By abuse of notation concerning the two diagrams (5.6.1) and (5.6.3), we will write the previous identities (5.6.3) and (5.6.4) (resp., (5.6.1) and (5.6.2)) in the following form

$$
\begin{aligned}
&\langle cc', d\rangle = \langle c, d_{(1)}\rangle \cdot_r \langle c', d_{(2)}\rangle \ \big(\text{resp., } \langle c, dd'\rangle = \langle c_{(1)}, d\rangle \cdot_\ell \langle c_{(2)}, d'\rangle\big),\\
&\langle\,,\rangle \circ (\eta_C \boxtimes_{\mathscr{C}} \mathrm{id}_C) = \eta_{A,r} \circ \epsilon_C \ \big(\text{resp., } \langle\,,\rangle \circ (\mathrm{id}_C \boxtimes_{\mathscr{C}} \eta_C) = \eta_{A,\ell} \circ \epsilon_C\big),
\end{aligned}
\tag{5.6.5}
$$

for all $c, c', d, d' \in C$, where we write a comma instead of a tensor product symbol, $\cdot_\ell$ and $\cdot_r$ instead of μ_ℓ and μ_r, respectively, and where we use the usual Sweedler convention $\Delta_C(c) = c_{(1)} \otimes_{\mathscr{C}} c_{(2)}$ and $\Delta_C(d) = d_{(1)} \otimes_{\mathscr{C}} d_{(2)}$ for the coproduct of C.

5.6.4. Consider a propagator $\Delta \in \mathrm{Prop}_i(M,E)$ that is either precut or continuous. By (5.4.1) and (5.4.2), it has a unique element $\bar{\Delta} \in \mathrm{Hom}_{C^\infty(M)\otimes C^\infty(M)}(\Gamma(J^iE)\otimes\Gamma(J^iE),\mathscr{D}'(M\times M))$ associated to it. Following [10] (see the paragraph before Def. 9), $\bar{\Delta}$ induces a unique map

$$\tilde{\Delta} \in \mathrm{Hom}_{C^\infty(M)\otimes C^\infty(M)}\Big(S_{C^\infty(M)}\Gamma(J^iE)\otimes S_{C^\infty(M)}\Gamma(J^iE),\mathscr{D}'(M\times M)\Big)$$

given by $\tilde{\Delta}(1_M,1_M)=1_{M^2}$, where $1_M \in C^\infty(M)$ is the constant unit function on M and analogously for 1_{M^2}, $\tilde{\Delta}(1_M,\sigma_1\dots\sigma_n)=0$ for all $\sigma_1,\dots,\sigma_n\in\Gamma(J^iE)$ and $n\in\mathbb{N}$, and

$$\tilde{\Delta}(\sigma_1\dots\sigma_n,\tau_1\dots\tau_m)=\begin{cases}\sum\limits_{\varsigma\in\mathbb{S}_n}\bar{\Delta}(\sigma_1,\tau_{\varsigma(1)})\dots\bar{\Delta}(\sigma_n,\tau_{\varsigma(n)}), & \text{if } n=m\in\mathbb{N},\\ 0, & \text{else,}\end{cases} \tag{5.6.6}$$

for all $\sigma_1,\dots,\sigma_n,\tau_1,\dots,\tau_m\in\Gamma(J^iE)$ and $n,m\in\mathbb{N}$, where we use commas instead of tensor products over k. Note that Fact 5.4.9 implies that the products appearing in (5.6.6) are defined if Δ is precut. They are also clearly defined in the case Δ is continuous. Furthermore, Theorem 4.3.10 implies that both definitions agree if Δ is precut and continuous.

The argument in the proof of Lemma 5.4.3 says that the restriction of $\tilde{\Delta}$ to $S^m_{C^\infty(M)}\Gamma(J^iE)\otimes_\pi S^{m'}_{C^\infty(M)}\Gamma(J^iE)$ is continuous for all $m,m'\in\mathbb{N}_0$. Since it is a finitely generated locally convex module over $C^\infty(M)\otimes_\pi C^\infty(M)$ for all $m,m'\in\mathbb{N}_0$, its completion $S^m_{C^\infty(M)}\Gamma(J^iE)\hat{\otimes}_\pi S^{m'}_{C^\infty(M)}\Gamma(J^iE)$ is a finitely generated locally convex module over the Fréchet algebra $C^\infty(M)\hat{\otimes}_\pi C^\infty(M)\simeq C^\infty(M\times M)$. Moreover, the fact that $\mathscr{D}'(M\times M)$ is complete (see Remark 4.1.4) implies that $\tilde{\Delta}$ uniquely extends to an element

$$\check{\Delta}\in\mathfrak{Hom}_{C^\infty(M)\hat{\otimes}_\pi C^\infty(M)}\Big(S_{C^\infty(M)}\Gamma(J^iE)\hat{\otimes}_\pi S_{C^\infty(M)}\Gamma(J^iE),\mathscr{D}'(M\times M)\Big).$$

As the LCS involving tensor products are metrizable, the previous mapping may be regarded as a morphism

$$\check{\Delta}\in\mathfrak{Hom}_{C^\infty(M)\tilde{\otimes}_\beta C^\infty(M)}\Big(S_{C^\infty(M)}\Gamma(J^iE)\tilde{\otimes}_\beta S_{C^\infty(M)}\Gamma(J^iE),\mathscr{D}'(M\times M)\Big).$$

5.6.5. Inspired by the work of Borcherds, we consider a map $\hat{\Delta}=\sum_{m,m'\in\mathbb{N}_0}\hat{\Delta}^{m,m'}$, where $\hat{\Delta}^{m,m'}$ is a morphism of bornological locally convex $C^\infty(M)^{\tilde{\otimes}_\beta m}\tilde{\otimes}_\beta C^\infty(M)^{\tilde{\otimes}_\beta m'}$-modules

$$\hat{\Delta}^{m,m'}:\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta m}\tilde{\otimes}_\beta\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta m'}\longrightarrow\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_{m'}(M) \tag{5.6.7}$$

satisfying the next conditions. Note that the codomain of $\hat{\Delta}^{m,m'}$ is isomorphic to $\mathscr{D}'(M)^{\tilde{\otimes}_\beta m}\tilde{\otimes}_\beta\mathscr{D}'(M)^{\tilde{\otimes}_\beta m'}$ by Proposition 4.2.2.

Recall that $C^\infty(M)^{\hat{\otimes}_\pi 0}=k$, so we may identify the Fréchet algebras $C^\infty(M)^{\hat{\otimes}_\pi m}\hat{\otimes}_\pi C^\infty(M)^{\hat{\otimes}_\pi 0}$, $C^\infty(M)^{\hat{\otimes}_\pi m}$ and $C^\infty(M)^{\hat{\otimes}_\pi 0}\hat{\otimes}_\pi C^\infty(M)^{\hat{\otimes}_\pi m}$. Since the involved spaces are metrizable, we may replace the completed projective tensor products by convenient tensor products, and the usual completions by convenient completions. Furthermore, setting $\mathscr{D}'_0(M)=k$, we have the isomorphisms of LCS between $\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_0(M)$, $\mathscr{D}'_{,m}(M)$ and $\mathscr{D}'_0(M)\tilde{\otimes}_\beta\mathscr{D}'_m(M)$. Recall also that $(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta 0}=k$. Let us define $\hat{\Delta}^{m,0}$ and $\hat{\Delta}^{0,m}$ as the morphisms of bornological locally convex $C^\infty(M)^{\tilde{\otimes}_\beta m}\tilde{\otimes}_\beta C^\infty(M)^{\tilde{\otimes}_\beta 0}$-modules and $C^\infty(M)^{\tilde{\otimes}_\beta 0}\tilde{\otimes}_\beta C^\infty(M)^{\tilde{\otimes}_\beta m}$-modules, respectively,

$$\begin{aligned}\hat{\Delta}^{m,0}&:\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta m}\tilde{\otimes}_\beta\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta 0}\longrightarrow\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_0(M)\\ \hat{\Delta}^{0,m}&:\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta 0}\tilde{\otimes}_\beta\Big(S_{C^\infty(M)}\Gamma(J^iE)\Big)^{\tilde{\otimes}_\beta m}\longrightarrow\mathscr{D}'_0(M)\tilde{\otimes}_\beta\mathscr{D}'_m(M)\end{aligned} \tag{5.6.8}$$

given as follows. Using the previous identifications, $\hat{\Delta}^{m,0}$ and $\hat{\Delta}^{0,m}$ are defined as the composition of the counit of $(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta m}$ stated in Corollary 3.9.1, and the canonical inclusion of $C^\infty(M)^{\tilde{\otimes}_\beta m}$ inside of

$\mathscr{D}'_m(M)$ as the smooth distributions of M^m. However, we will need later to keep track whether the zero index appears on the right or the left.

Let us now assume that $m, m' \in \mathbb{N}$, and let us define $\hat{\Delta}^{m,m'}$. We first note that (1.7.12) and (1.7.14) tell us that any morphism of the form (5.6.7) is equivalent to a morphism of $C^\infty(M)^{\bar{\otimes}_\beta m} \otimes C^\infty(M)^{\bar{\otimes}_\beta m}$-modules

$$\left(S_{C^\infty(M)}\Gamma(J^iE)\right)^{\bar{\otimes}_\beta m} \otimes_\beta \left(S_{C^\infty(M)}\Gamma(J^iE)\right)^{\bar{\otimes}_\beta m'} \longrightarrow \mathscr{D}'_m(M) \bar{\otimes}_\beta \mathscr{D}'_{m'}(M), \tag{5.6.9}$$

which is just the restriction of $\hat{\Delta}^{m,m'}$ to the domain of (5.6.9). We recall that any morphism of $C^\infty(M)^{\bar{\otimes}_\beta m} \otimes C^\infty(M)^{\bar{\otimes}_\beta m'}$-modules of the form (5.6.9) is automatically continuous by Proposition 1.6.7. Set now

$$\mathrm{ddiag}_{m,m'} : M^m \times M^{m'} \longrightarrow \prod_{(j,j') \in \mathbb{N}_{\leq m} \times \mathbb{N}_{\leq m'}} (M \times M)$$

to be the morphism of smooth manifolds given by $(\mathrm{ddiag}_{m,m'}(\bar{p}, \bar{p}'))_{(j,j')} = (p_j, p'_{j'})$, for all j, j', where $\bar{p} = (p_1, \dots, p_m) \in M^m$ and $\bar{p}' = (p'_1, \dots, p'_{m'}) \in M^{m'}$. Define then

$$\hat{\Delta}^{m,m'}(\bar{\sigma}^1|\dots|\bar{\sigma}^m, \bar{\tau}^1|\dots|\bar{\tau}^{m'}) = \mathrm{ddiag}^*_{m,m'}\Bigg(\prod_{j=1}^{m}{}^{\boxtimes} \prod_{j'=1}^{m'}{}^{\boxtimes} \tilde{\Delta}\big(\bar{\sigma}^j_{(j')}, \bar{\tau}^{j'}_{(j)}\big)\Bigg), \tag{5.6.10}$$

where $\bar{\sigma}^1, \dots, \bar{\sigma}^m, \bar{\tau}^1, \dots, \bar{\tau}^{m'} \in S_{C^\infty(M)}\Gamma(J^iE)$, we have replaced tensor products by bars as indicated in Notation 5.1.15, the products indexed by j and j' are external products of distributions, and we use the Sweedler notation $\Delta^{(m'')}_{S_{C^\infty(M)}\Gamma(J^iE)}(\bar{\sigma}) = \bar{\sigma}_{(1)} \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \bar{\sigma}_{(m'')}$ for the coproduct of any element of the form $\bar{\sigma} \in S_{C^\infty(M)}\Gamma(J^iE)$ (see Section 3.1). We will come to the question of the good definition of the previous expression in 5.6.8.

5.6.6 Fact. *Let $\Delta \in \mathrm{Prop}_i(M, E)$ be an either continuous or precut propagator. If $(v_1, \dots, v_m, v'_1, \dots, v'_{m'})$ is in the wave front set at $\bar{p}$ of the distribution $\hat{\Delta}(U)$ given in* (5.6.7) *for $U \in (S_{C^\infty(M)}\Gamma(J^iE))^{\bar{\otimes}_\beta m} \bar{\otimes}_\beta (S_{C^\infty(M)}\Gamma(J^iE))^{\bar{\otimes}_\beta m'}$ and $\bar{p} = (p_1, \dots, p_m, p'_1, \dots, p'_{m'}) \in M^{m+m'}$, then $-v_i \in \mathscr{P}_{p_i} \cup \{0\}$ and $v'_j \in \mathscr{P}_{p'_j} \cup \{0\}$, for all $i = 1, \dots, m$ and $j = 1, \dots, m'$; and there exists one pair (i, j) such that $-v_i \in \mathscr{P}_{p_i}$ and $v'_j \in \mathscr{P}_{p'_j}$.* *Easy Implicit*

The proof of this result is easy, which is somehow an extension of Fact 5.4.9.

5.6.7 Proposition. *Let $m, m' \in \mathbb{N}_0$. The map $\hat{\Delta}^{m,m'}$ of bornological locally convex $C^\infty(M)^{\bar{\otimes}_\beta m} \bar{\otimes}_\beta C^\infty(M)^{\bar{\otimes}_\beta m'}$-modules given in* (5.6.7) *is $\mathbb{S}_m \times \mathbb{S}_{m'}$-equivariant, where the domain and the codomain have the obvious action of $\mathbb{S}_m \times \mathbb{S}_{m'}$ by permuting the corresponding tensors.* *Easy Implicit*

The proof is a direct consequence of expressions (5.6.8) and (5.6.10), together with the cocommutativity of the coproduct of $(S_{C^\infty(M)}\Gamma(J^iE))^{\bar{\otimes}_\beta m} \bar{\otimes}_\beta (S_{C^\infty(M)}\Gamma(J^iE))^{\bar{\otimes}_\beta m'}$.

5.6.8. Note that the pull-back in (5.6.10) exists due to condition (ii) in the definition of precut propagator, if Δ is precut. The pull-back clearly exists if Δ is continuous, and both definitions agree if Δ is continuous and precut by Theorem 4.3.10. We leave to the reader the verification that (5.6.10) is well-defined, *i.e.* the expression in the right member of (5.6.10) is independent of the choice of representatives in $S\Gamma(J^iE)$ for $\bar{\sigma}_{(1)} \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \bar{\sigma}_{(m'')}$. We only indicate that this involves showing that (5.6.10) is $C^\infty(M)$-balanced with respect to the tensor appearing within the elements $\bar{\sigma}_{(1)} \otimes \cdots \otimes \bar{\sigma}_{(m'')} \in S\Gamma(J^iE)$ lifting $\Delta^{(m'')}_{S_{C^\infty(M)}\Gamma(J^iE)}(\bar{\sigma})$, and this latter fact follows from the pull-back by $\mathrm{ddiag}_{m,m'}$.

5.6.9.*Remark.* Notice that the right hand side of (5.6.10) is a form of "mixed" product: we are taking the internal product of distributions on some variables and the external product on others. For example,

$$\hat{\Delta}^{1,2}(\bar{\sigma}^1, \bar{\tau}^1|\bar{\tau}^2) = \mathrm{ddiag}^*_{1,2}\Big(\tilde{\Delta}(\bar{\sigma}^1_{(1)}, \bar{\tau}^1) \boxtimes \tilde{\Delta}(\bar{\sigma}^1_{(2)}, \bar{\tau}^2)\Big)$$

involves an internal product of the distribution tensor factors corresponding to the first argument of the propagators and a external product of the distribution tensor factors corresponding to the second argument.

5.6.10. Note that since each LCS $\mathscr{D}'_m(M)$ is a bornological locally convex module over $C^\infty(M)^{\tilde{\otimes}_\beta m}$, for all $m \in \mathbb{N}_0$, then it is *a fortiori* a bornological locally convex module over $\underset{\sim}{T}C^\infty(M)$, by means of the canonical projection $\mathfrak{s}_m$ considered in 3.3.2. As a consequence,

$$\bigoplus_{m,m'\in\mathbb{N}_0} \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'}(M) = \Big(\bigoplus_{m\in\mathbb{N}_0} \mathscr{D}'_m(M)\Big)^{\boxtimes 2} \tag{5.6.11}$$

is also a bornological locally convex module over $\underset{\sim}{T}C^\infty(M)$, by means of the second monoidal structure $\boxtimes$ defined in the symmetric 2-monoidal category of modules over $\underset{\sim}{T}C^\infty(M)$ (see Proposition 3.3.9). We remark that the action is given by the usual tensor-wise product.

5.6.11. In order to state the next results we introduce the following notation. We recall that $\delta_{m,m'}$ is the Kronecker delta of $m, m' \in \mathbb{N}_0$. Given $m_1, m_2, m' \in \mathbb{N}$, let

$$\mathrm{ddiag}_{m_1,m_2;m'} : M^{m_1+m_2} \times M^{m'} \to (M^{m_1} \times M^{m'}) \times (M^{m_2} \times M^{m'})$$

be the map $(\bar{p}, \bar{p}') \mapsto (\bar{p}_1, \bar{p}', \bar{p}_2, \bar{p}')$, where $\bar{p} = (\bar{p}_1, \bar{p}_2)$ with $\bar{p}_j \in M^{m_j}$ and $\bar{p}' \in M^{m'}$. The map $\mathrm{ddiag}_{m;m'_1,m_2}$: $M^m \times M^{m'_1+m'_2} \to (M^m \times M^{m'_1}) \times (M^m \times M^{m'_2})$ is defined analogously.

Using these two maps we introduce the following partially defined operations $\cdot_\ell$ and $\cdot_r$ on the object $\oplus_{m,m'\in\mathbb{N}_0} \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'}(M)$ regarded in the monoidal category BLCS_{HD} as follows:

(i) the element $1 \in k \simeq k\tilde{\otimes}_\beta k = \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)$ satisfies that

$$1 \cdot_\ell v = v \cdot_\ell 1 = \delta_{0,m} v, \quad \text{and} \quad 1 \cdot_r v = v \cdot_r 1 = \delta_{0,m'} v,$$

if $v \in \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'}(M)$, for $m, m' \in \mathbb{N}_0$;

(ii) given $u \in \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)$ and $v \in \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'}(M)$, with $m, m' \in \mathbb{N}$, we set $u \cdot_\ell v = v \cdot_\ell u = 0 = u \cdot_r v = v \cdot_r u$;

(iii) for elements $u \in \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_m(M)$ and $v \in \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'}(M)$ (resp., $u \in \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)$ and $v \in \mathscr{D}'_{m'}(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)$), with $m, m' \in \mathbb{N}$, then the partially defined product is given by the expression (whenever it is defined)

$$u \cdot_r v = \delta_{m,m'} uv \in \mathscr{D}'_m(M) \simeq \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_m(M) \quad \Big(\text{resp., } u \cdot_\ell v = \delta_{m,m'} uv \in \mathscr{D}'_m(M) \simeq \mathscr{D}'_m(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)\Big),$$

whereas $u \cdot_\ell v$ (resp., $u \cdot_r v$) is the external product

$$u \boxtimes v \in \mathscr{D}'_{m+m'}(M) \simeq \mathscr{D}'_0(M)\tilde{\otimes}_\beta \mathscr{D}'_{m+m'}(M) \quad \Big(\text{resp., } u \boxtimes v \in \mathscr{D}'_{m+m'}(M) \simeq \mathscr{D}'_{m+m'}(M)\tilde{\otimes}_\beta \mathscr{D}'_0(M)\Big);$$

(iv) for $m_1, m_2, m'_1, m'_2 \in \mathbb{N}$, $u \in \mathscr{D}'_{m_1}(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'_1}(M)$ and $v \in \mathscr{D}'_{m_2}(M)\tilde{\otimes}_\beta \mathscr{D}'_{m'_2}(M)$, set

$$u \cdot_\ell v = \delta_{m_1,m_2}\, \mathrm{ddiag}^*_{m_1;m'_1,m'_2}(u \boxtimes v), \tag{5.6.12}$$

and

$$u \cdot_r v = \delta_{m'_1,m'_2}\, \mathrm{ddiag}^*_{m_1,m_2;m'_1}(u \boxtimes v). \tag{5.6.13}$$

Note that (5.6.12) and (5.6.13) are well-defined whenever the corresponding pull-backs exist.

We remark that these products $\cdot_\ell$ and $\cdot_r$ are however not $\underset{\sim}{T}C^\infty(M)$-linear for the structure of convenient locally convex module over $\underset{\sim}{T}C^\infty(M)$ referred to in (5.6.11).

5.6.12 Definition. *Let $\Delta \in \mathrm{Prop}_i(M,E)$ be a precut or continuous propagator. Consider the convenient locally convex $\underset{\sim}{T}C^\infty(M)$-module $\oplus_{m,m'\in\mathbb{N}_0}\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_{m'}(M)$ mentioned in* (5.6.11)*. Define $\mathscr{D}\mathscr{A}(M)$ to be the convenient locally convex $\underset{\sim}{T}C^\infty(M)$-submodule of $\oplus_{m,m'\in\mathbb{N}_0}\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_{m'}(M)$ generated by a finite iteration of application of the products $\cdot_\ell$ and $\cdot_r$ to elements in the image of the maps $\{\hat{\Delta}^{m,m'}\}_{m,m'\in\mathbb{N}_0}$. These products exist by Fact 5.6.6.*

Set two structures of convenient locally convex $\underset{\sim}{T}C^\infty(M)$-module on $\oplus_{m,m'\in\mathbb{N}_0}\mathscr{D}'_m(M)\tilde{\otimes}_\beta\mathscr{D}'_{m'}(M)$, and thus on its subspace $\mathscr{D}\mathscr{A}(M)$, by means of the standard module structures of each of the two tensor factors of the former space. More precisely,

(i) the left structure *of convenient locally convex $\underset{\sim}{T}C^\infty(M)$-module is defined as the only one such that, given $v_1 \in \mathscr{D}'_{m_1}(M)$, $v_2 \in \mathscr{D}'_{m_2}(M)$, and $f = f_1 \otimes f_2 \in \underset{\sim}{T}C^\infty(M)$ with $f_1 \in C^\infty(M)^{\tilde{\otimes}_\beta m_1}$,*

$$(f_1 \otimes f_2)\cdot(v_1 \otimes v_2) = \big(f_1 v_1\big)\otimes\big(\mathfrak{s}_0(f_2)v_2\big);$$

(ii) the right structure *of convenient locally convex $\underset{\sim}{T}C^\infty(M)$-module is the unique one satisfying that, given $v_1 \in \mathscr{D}'_{m_1}(M)$, $v_2 \in \mathscr{D}'_{m_2}(M)$, and $f = f_1 \otimes f_2 \in \underset{\sim}{T}C^\infty(M)$ with $f_2 \in C^\infty(M)^{\tilde{\otimes}_\beta m_2}$,*

$$(f_1 \otimes f_2)\cdot(v_1 \otimes v_2) = \big(\mathfrak{s}_0(f_1)v_1\big)\otimes\big(f_2 v_2\big).$$

We recall that the morphism $\mathfrak{s}_0$ was introduced in 3.3.2.

This definition makes sense as a direct consequence of Fact 5.6.6.

5.6.13. Let $\jmath : \underset{\sim}{T}C^\infty(M) \to \oplus_{m\in\mathbb{N}_0}\mathscr{D}'_m(M)$ be the direct sum of the maps $\{\jmath_m\}_{m\in\mathbb{N}_0}$,

$$\jmath_m : C^\infty(M)^{\tilde{\otimes}_\beta m} \to \mathscr{D}'_m(M),$$

where $\jmath_0$ is the identity map of k, and, for $m \in \mathbb{N}$, $\jmath_m$ is the canonical inclusion of smooth functions inside the space of distributions on M^m. It is clear that each $\jmath_m$ is continuous (due to Proposition 1.6.7), so $\jmath$ is also continuous by the universal property of the coproduct, and a morphism of convenient LCS.

5.6.14 Lemma. *Let $\mathscr{D}\mathscr{A}(M)$ be the LCS introduced in Definition 5.6.12. We regard it with the right (resp., left) structure of bornological locally convex module over $\underset{\sim}{T}C^\infty(M)$, in which case we will denote it by $\mathscr{D}\mathscr{A}_r(M)$ (resp., $\mathscr{D}\mathscr{A}_\ell(M)$). Then, $\mathscr{D}\mathscr{A}_r(M)$ (resp., $\mathscr{D}\mathscr{A}_\ell(M)$) is a unitary algebra in the symmetric monoidal category of convenient locally convex $\underset{\sim}{T}C^\infty(M)$-modules for the tensor product $\otimes_{\underset{\sim}{T}C^\infty(M)}$, with the product given by $\cdot_r$ (resp., $\cdot_\ell$), and the unit by $\mathfrak{s}_0\tilde{\otimes}_\beta\jmath$ (resp., $\jmath\tilde{\otimes}_\beta\mathfrak{s}_0$), where $\mathfrak{s}_0$ is defined in 3.3.2.* *Easy*

This result is straightforward to check, and left to the reader

5.6.15 Proposition. *The map $\hat{\Delta}$ given in* (5.6.10) *is a right (resp., left) Laplace pairing on the cocommutative unitary and counitary bialgebra $\tilde{T}S_{C^\infty(M)}\Gamma(J^iE)$ defined in Proposition 3.4.6 relative to the framed symmetric 2-monoidal category ${}_{\underset{\sim}{T}C^\infty(M)}\mathfrak{CMod}$ (see Proposition 3.3.9) and with values on the unitary algebra $\mathscr{D}\mathscr{A}_r(M)$ (resp., $\mathscr{D}\mathscr{A}_\ell(M)$) introduced in Fact 5.6.14. In particular, we have* *Lengthy*

$$\hat{\Delta}^{m,m'_1+m'_2}(u,v_1v_2) = \sum \hat{\Delta}^{m,m'_1}(u_{(1)},v_1)\cdot_\ell \hat{\Delta}^{m,m'_2}(u_{(2)},v_2), \tag{5.6.14}$$

and

$$\hat{\Delta}^{m_1+m_2,m'}(u_1u_2,v) = \sum \hat{\Delta}^{m_1,m'}(u_1,v_{(1)})\cdot_r \hat{\Delta}^{m_2,m'}(u_2,v_{(2)}), \tag{5.6.15}$$

for all $m,m',m_1,m_2,m'_1,m'_2 \in \mathbb{N}_0$, where the sums are indexed as in the following coproducts of the counitary coalgebras $(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta m'}$ and $(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta m}$, respectively, given by

$$\Delta_{(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta m'}}(v) = \sum v_{(1)}\otimes v_{(2)} \quad \textit{and} \quad \Delta_{(S_{C^\infty(M)}\Gamma(J^iE))^{\tilde{\otimes}_\beta m}}(u) = \sum u_{(1)}\otimes u_{(2)}$$

(see Corollary 3.9.1).

The proof is lengthy but straightforward, and it is proved by induction.

5.6.16. *Example.* To illustrate the previous definition, consider the case referred to in Remark 3.7.6, where M is a finite set $\{x_1,\dots,x_N\}$ of N points, E is the trivial line bundle over M and $i = 0$. Then, $A = C^\infty(M) = X = \Gamma(J^0E) = k^N$ and $\mathscr{D}'(M) = A^*$. We denote by e_j the N-tuple having zeros except in position j, where it is 1. Moreover, we fix the linear isomorphism $C^\infty(M) = A \simeq A^* = \mathscr{D}'(M)$ sending e_j to the functional $e_{j'} \mapsto \delta_{j,j'}$, which identifies smooth maps with their corresponding smooth distributions (see 4.1.8). The previous isomorphism identifies the algebra structure on $\mathscr{D}'(M)$ (for the internal product) with A, which we shall use from now on. A propagator is just a bilinear map $\Delta : A \otimes A \to \mathbb{C}$ or, equivalently, $\Delta \in \mathrm{Hom}_{A\otimes A}(A \otimes A, A^* \otimes A^*) \simeq \mathrm{Hom}_{A\otimes A}(A\otimes A, A\otimes A)$. Then, $S_A\Gamma(J^0E) \simeq S_AA \simeq A\otimes k[\phi]$, where ϕ is an indeterminate and $k[\phi]$ is the polynomial algebra on ϕ. Note that S_AA has the basis $\{\phi^n(x_j) : n \in \mathbb{N}_0, j \in \{1,\dots,N\}\}$ over k, where we write $\phi^n(x_j)$ instead of $e_j \otimes \phi^n$. This has the counitary coalgebra structure over A for the coproduct $\Delta_{S_AA} : S_AA \to S_AA \otimes_A S_AA$ given by

$$\Delta_{S_AA}\Big(\phi^n(x_j)\Big) = \sum_{\ell=0}^{n} \binom{n}{\ell} \phi^\ell(x_j) \otimes_A \phi^{n-\ell}(x_j),$$

and the counit $\epsilon : S_AA \to A$ defined as $\epsilon(\phi^n(x_j)) = \delta_{n,0}e_j$ (*cf.* [14], Section 2.1, where S_AA is denoted by $\mathcal{C}$, but it is only regarded as a counitary coalgebra over k). It is *a fortiori* a counitary coalgebra in the symmetric monoidal category of modules over ${}^{\mu}TA$ in the symmetric monoidal category of vector spaces over k. As explained in the Proposition 3.4.6, the previous structure induces a unitary and counitary bialgebra on ${}_{\Delta_{S_AA}}T(S_AA)$ relative to the framed symmetric 2-monoidal category of modules over $T^{\mu}A$ (*cf.* [14], Section 2.1, where the author considers the symmetric construction $S(\mathcal{C})$, which is a subalgebra of $T(\mathcal{C})$). The expression (5.6.6) for $\tilde{\Delta} : S_AA \otimes S_AA \to A \otimes A$ gives precisely

$$\tilde{\Delta}\Big(\phi^n(x_j), \phi^{n'}(x_{j'})\Big) = \delta_{n,n'} n! \Delta\Big(\phi(x_j), \phi(x_{j'})\Big),$$

which coincides with the one given in [14], Lemma 2.4, and the expression (5.6.10) for $\hat{\Delta} : T(S_AA) \otimes T(S_AA) \to T(A) \otimes T(A)$ coincides essentially with the formula appearing in [14], Lemma 2.3. The difference between our expression and the latter formula is the meaning of the products: in [14], Lemma 2.3, the author uses the product of k, whereas we use the product of A for the first (resp., second) tensor factors of expressions of the form $\Delta(\phi(x_j), \phi(x_\ell))\Delta(\phi(x_j), \phi(x_m))$ (resp., $\Delta(\phi(x_\ell), \phi(x_j))\Delta(\phi(x_m), \phi(x_j))$), and the product of $T(A)$ for the second (resp., first) tensor factors. This is compatible with the points on the fields inside of the propagators.

New explanations

5.7 Feynman measures

Based on Borcherds

5.7.1 Definition. *Following* [10], *Def. 9, we define a* Feynman measure ω *as a continuous linear functional* $\omega : \tilde{S}\mathscr{L}_{i,c}(M,E) \to \mathbb{C}$ *satisfying that* $\omega(1_S) = 1$.

5.7.2. *Remark.* Note that the definition means that ω is a continuous linear functional on $\tilde{T}\mathscr{L}_{i,c}(M,E)$ sending the unit 1_T to 1 and whose restriction to $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ is invariant under the permutation group $\mathbb{S}_m$, for all $m \in \mathbb{N}$. In other words, we may equivalently regard ω as an element of

$$\prod_{m\in\mathbb{N}} \Big(\mathfrak{Hom}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}, \mathbb{C}\Big)\Big)^{\mathbb{S}_m},$$

where the homomorphism space is taken in the category CLCS_{HD}.

We also want to stress that, the space of invariants under the action of the group $\mathbb{S}_m$ on the space of morphisms from $X^{\otimes_{\mathscr{C}} m}$ to $Y^{\otimes_{\mathscr{C}} m}$, where X and Y are objects in a symmetric monoidal category $\mathscr{C}$, coincides precisely with the space of morphisms in $\mathscr{C}$ that are also $\mathbb{S}_m$-equivariant between the mentioned tensor products.

5.7.3 Definition. *Following* [10], *p. 633, the Feynman propagator* $\Delta_{\omega,F}$ *associated with a Feynman measure* ω *is given by* $\Delta_{\omega,F} = \omega|_{\tilde{S}^2\mathscr{L}_{i,c}(M,E)} \circ \mathrm{proj}$, *where* proj *is the composition of the canonical projection* *Based on Borcherds*

$$\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^iE\Big)\tilde{\otimes}_\beta\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^iE\Big) \to \tilde{S}^2\Gamma_c\Big(\mathrm{Vol}(M)\otimes J^iE\Big)$$

and the canonical inclusion of the codomain of the previous map inside of $\tilde{S}^2\mathscr{L}_{i,c}(M,E)$, *and where we are using the comments in Definition 5.4.2.*

5.7.4. Given a Feynman measure ω, its restriction to (5.3.2) is denoted by $\omega_{q,\bar{m}}$. By (4.2.10) $\omega_{q,\bar{m}}$ is a symmetric $F_{\bar{m}}$-valued distribution of rank q. More generally, for $q \in \mathbb{N}$ and a vector bundle F over M constructed as a finite direct sum of some of the bundles $\{S^mJ^iE\}_{m\in\mathbb{N}_0}$, ω induces an $F^{\boxtimes q}$-valued distribution $\omega_{q,F}$ given by applying ω to the sections of $F^{\boxtimes q}$. It is easy to see that $\omega_{q,F}$ is a sum of certain $\omega_{q,\bar{m}}$. Another way to regard these distributions $\omega_{q,\bar{m}}$ or $\omega_{q,F}$ associated to the Feynman measure ω is given in the following result.

5.7.5 Proposition. *The convenient vector space of Feynman measures is isomorphic to*

$$\begin{aligned}\mathfrak{Hom}\Big(\tilde{S}^+\mathscr{L}_{i,c}(M,E),\mathbb{C}\Big) &\simeq \mathfrak{Hom}_{\underset{\sim}{\Sigma} C^\infty(M)}\Big(\tilde{S}^+\mathscr{L}_{i,c}(M,E),\mathfrak{Hom}^\flat\Big(\underset{\sim}{\Sigma} C^\infty(M),\mathbb{C}\Big)\Big) \simeq \prod_{m\in\mathbb{N}}\Big(\mathfrak{Hom}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big)\Big)^{\mathbb{S}_m}\\ &\simeq \prod_{m\in\mathbb{N}}\Big[\mathfrak{Hom}_{C^\infty(M)^{\tilde{\otimes}_\beta m}}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathfrak{Hom}^\flat\Big(C^\infty(M)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big)\Big)\Big]^{\mathbb{S}_m}.\end{aligned} \tag{5.7.1}$$

Proof. Let $m \in \mathbb{N}$. Using that $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ is a unitary convenient locally convex $\tilde{\Sigma}^mC^\infty(M)$-module, we obtain the isomorphism

$$\begin{aligned}\mathfrak{Hom}\Big(\tilde{S}^m\mathscr{L}_{i,c}(M,E),\mathbb{C}\Big) &\simeq \mathfrak{Hom}\Big(\tilde{S}^m\mathscr{L}_{i,c}(M,E)\tilde{\otimes}_{\tilde{\Sigma}^mC^\infty(M)}\tilde{\Sigma}^mC^\infty(M),\mathbb{C}\Big)\\ &\simeq \mathfrak{Hom}_{\tilde{\Sigma}^mC^\infty(M)}\Big(\tilde{S}^m\mathscr{L}_{i,c}(M,E),\mathfrak{Hom}^\flat\Big(\tilde{\Sigma}^mC^\infty(M),\mathbb{C}\Big)\Big),\end{aligned} \tag{5.7.2}$$

where we have used the adjunction between the convenient tensor product over $\tilde{\Sigma}^mC^\infty(M)$ and the space of internal homomorphisms. This proves the first isomorphism in (5.7.1).

The second isomorphism was already observed in Remark 5.7.2. Finally, to prove the last isomorphism, note $\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ is a unitary convenient locally convex $C^\infty(M)^{\tilde{\otimes}_\beta m}$-module, so the same argument proving (5.7.2) shows that

$$\mathfrak{Hom}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big) \simeq \mathfrak{Hom}_{C^\infty(M)^{\tilde{\otimes}_\beta m}}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathfrak{Hom}^\flat\Big(C^\infty(M)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big)\Big), \tag{5.7.3}$$

for all $m \in \mathbb{N}$, which in turn implies

$$\Big(\mathfrak{Hom}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big)\Big)^{\mathbb{S}_m} \simeq \Big[\mathfrak{Hom}_{C^\infty(M)^{\tilde{\otimes}_\beta m}}\Big(\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m},\mathfrak{Hom}^\flat\Big(C^\infty(M)^{\tilde{\otimes}_\beta m},\mathbb{C}\Big)\Big)\Big]^{\mathbb{S}_m}. \tag{5.7.4}$$

This proves the proposition. □

5.7.6. We shall denote the $\mathbb{S}$-equivariant (this is usual terminology in the theory of operads) continuous linear map

$$\tilde{T}^+\mathscr{L}_{i,c}(M,E) \to \bigoplus_{m\in\mathbb{N}}\Big(C^\infty(M)^{\tilde{\otimes}_\beta m}\Big)' \tag{5.7.5}$$

corresponding to a Feynman measure ω under the isomorphism in (5.7.1) by ρ_ω. It is clear that, if $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$, then $\rho_\omega(L)(f) = \omega(fL)$, for $f \in C^\infty(M)^{\tilde{\otimes}_\beta m} \simeq C^\infty(M^m)$. Moreover, the continuous linear map

$$\tilde{S}^+\mathscr{L}_{i,c}(M,E) \to \bigoplus_{m\in\mathbb{N}}\Big(\tilde{\Sigma}^mC^\infty(M)\Big)' \tag{5.7.6}$$

equivalent to a Feynman measure ω under (5.7.1) will be denoted by $\bar{\rho}_\omega$. It is clear that $\bar{\rho}_\omega$ is the linear map induced by sending $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$ to the restriction of $\rho_\omega(L)$ to $\tilde{\Sigma}^m C^\infty(M)$. Conversely, $\omega([L]) = \rho_\omega(L)(1_{M^m})$ if $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$, where 1_{M^m} denotes the constant unit function on M^m.

5.7.7. Note that $\rho_\omega(L)(f)$ has compact support included in the support $\operatorname{supp}_m(L) \subseteq M^m$ of $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$. Recall that $C^\infty(M^m)'$ is identified (in the category ${}_{C^\infty(M^m)}\mathfrak{BMod}$) to the space $\mathscr{D}'(M^m, \operatorname{Vol}(M^m))$ of $\operatorname{Vol}(M)^{\boxtimes m}$-valued distributions on M^m, and furthermore, by the comments in 4.2.8, the image of the restriction of (5.7.5) to $\tilde{\Sigma}^m \mathscr{L}_{i,c}(M,E)$ is included in the space $\mathscr{D}'_{sym,m}(M, \operatorname{Vol}(M))$ of $\operatorname{Vol}(M)$-valued symmetric distributions on M of rank m. We also remark that, given $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}$, the support $\operatorname{supp}(\rho_\omega(L))$ of $\rho_\omega(L)$ is a symmetric subset of M^m.

5.7.8. *Remark.* We may canonically extend the map ρ_ω defined in (5.7.5) to a linear map

$$\tilde{T}\mathscr{L}_{i,c}(M,E) \to \bigoplus_{m \in \mathbb{N}_0} \Big(C^\infty(M^m)\Big)' \tag{5.7.7}$$

that we are also going to denote by ρ_ω, sending $1_T \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta 0}$ to the identity map in $k' = (\tilde{\Sigma}^0 C^\infty(M))'$, and analogously for $\bar{\rho}_\omega$. Note that ρ_ω does not factor through $\tilde{S}\mathscr{L}_{i,c}(M,E)$.

Based on Borcherds

5.7.9 Definition. *Following* [10], *Def. 9, a Feynman measure ω is said to be* associated with a precut propagator Δ *if it satisfies the following conditions:*

(1) given any $q \in \mathbb{N}$ and $\bar{m} = (m_1, \dots, m_q) \in \mathbb{N}_0^q$, if $(v_1, \dots, v_q) \in \mathrm{WF}_{(p,\dots,p)}(\iota(\omega_{q,\bar{m}}))$ for some $p \in M$, then $v_1 + \dots + v_q = 0$;

(2) there is a nowhere vanishing function $f \in C^\infty(M)$ satisfying that $\omega(\theta) = \int_M f\theta$, for all $\theta \in \mathscr{L}^0_{i,c}(M,E) = \Gamma_c(\operatorname{Vol}(M))$;

(3) given any $m', m'' \in \mathbb{N}$, $L' \in \mathscr{L}_{i,c}(M,E)^{\otimes_\beta m'}$ and $L'' \in \mathscr{L}_{i,c}(M,E)^{\otimes_\beta m''}$, whose quotients in $S^{m'}\mathscr{L}_{i,c}(M,E)$ and in $S^{m''}\mathscr{L}_{i,c}(M,E)$ are denoted by $[L']$ and $[L'']$ respectively, such that

$$\Big(\pi_{j'}\Big(\operatorname{supp}_{m'}([L'])\Big) \times \pi_{j''}\Big(\operatorname{supp}_{m''}([L''])\Big)\Big) \cap \leq \, = \emptyset$$

for all $j' \in \{1, \dots, m'\}$ and $j'' \in \{1, \dots, m''\}$, where $\pi_{j'} : M^{m'} \to M$ denotes the canonical projection and analogously for $\pi_{j''} : M^{m''} \to M$, then the products of distributions on $M^{m'+m''}$ appearing in the sum below are defined and they give

$$\rho_\omega(L'L'') = \sum \Big(\rho_\omega(L'_{(0)}) \boxtimes \rho_\omega(L''_{(0)})\Big)\hat{\Delta}^{m',m''}(L'_{(1)}, L''_{(1)}), \tag{5.7.8}$$

where $\hat{\Delta}$ is the extension of Δ in (5.6.7), $\tilde{T}^{m'}\delta^r(L') = L'_{(0)} \otimes_{C^\infty(M)^{\tilde{\otimes}_\beta m'}} L'_{(1)}$ and $\tilde{T}^{m''}\delta^r(L'') = L''_{(0)} \otimes_{C^\infty(M)^{\tilde{\otimes}_\beta m''}} L''_{(1)}$ are the expressions of the right coactions described in Proposition 3.4.7. The right member of (5.7.8) *is clearly independent of the choice of L' and L'', for fixed $[L']$ and $[L'']$ (see Remark 5.7.12). We note that $\hat{\Delta}^{m',m''}(L'_{(1)}, L''_{(1)})$ is in $\mathscr{D}'_m(M) \tilde{\otimes}_\beta \mathscr{D}'_{m'}(M)$, whereas $\rho_\omega(L'_{(0)}) \boxtimes \rho_\omega(L''_{(0)})$ is a* $\operatorname{Vol}(M)^{\boxtimes(m'+m'')}$*-valued distribution on $M^{m'+m''}$.*

We will denote by $\mathscr{F}_\Delta$ the set of Feynman measures associated with a precut propagator Δ. The good definition of the products of distribution appearing in (5.7.8) *is given by Lemma 5.7.14.*

5.7.10. *Remark.* By Proposition 4.1.10 and Fact 4.3.2, we see that condition (1) is tantamount to the fact that, if $(v_1, \dots, v_q) \in \mathrm{WF}_{(p,\dots,p)}(\rho_\omega(L)) = \mathrm{WF}_{(p,\dots,p)}(\bar{\rho}_\omega([L]))$ for some $p \in M$ and $L \in \mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta q}$, with $q \in \mathbb{N}$, then $v_1 + \dots + v_q = 0$.

5.7.11. *Remark.* Note that the use of Sweedler's notation on the right member of (5.7.8) makes sense because $\hat{\Delta}^{m',m''}$ is $C^\infty(M)^{\tilde{\otimes}_\beta m'} \otimes C^\infty(M)^{\tilde{\otimes}_\beta m''}$-linear (see (5.6.7)), $\rho_\omega|_{\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta m}}$ is $C^\infty(M)^{\tilde{\otimes}_\beta m}$-linear, and the product of distributions appearing in the right member of (5.7.8) is $C^\infty(M)^{\tilde{\otimes}_\beta m}$-balanced.

5.7.12. *Remark.* To prove the independence of the right member of (5.7.8) with respect to the choice of L' and L'', Remark 5.7.2 implies that it suffices to check that the right member of (5.7.8) is $\mathbb{S}_{m'} \times \mathbb{S}_{m''}$-equivariant. The claim follows from the explicit expression of the coaction of L' and L'' described in Proposition 3.4.7, which is equivariant with respect to permutations, the fact that $\rho_\omega|_{\mathscr{L}_{i,c}(M,E)^{\tilde{\otimes}_\beta q}}$ is $\mathbb{S}_q$-equivariant for all $q \in \mathbb{N}$ (see Proposition 5.7.5), and that $\hat{\Delta}^{m',m''}$ is $\mathbb{S}_{m'} \times \mathbb{S}_{m''}$-equivariant (see Proposition 5.6.7).

5.7.13. *Remark.* Recall the canonical extension of ρ_ω explained in Remark 5.7.8. We see that the condition given in item (3) of Definition 5.7.9 is equivalent to the same expression where we further allow $m', m'' \in \mathbb{N}_0$. Indeed, if either $m' = 0$ or $m'' = 0$, then any pair of elements $[L'] \in \tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ and $[L''] \in \tilde{S}^{m''}\mathscr{L}_{i,c}(M,E)$ satisfy the condition about the supports, and (5.7.8) is trivially verified, as we now show. Suppose $m' = 0$ and thus $[L'] = 1_S$ without loss of generality. Hence, the coaction on it is trivial, so $\tilde{\mathcal{T}}^{m'}\delta^r(L') = L'_{(0)} \otimes_{C^\infty(M^{m'})} L'_{(1)} = 1_S \otimes 1$, using that $C^\infty(M)^{\tilde{\otimes}_\beta 0} = k$ and $(S_{C^\infty(M)}\Gamma(J^i E))^{\tilde{\otimes}_\beta 0} = k$. Then,

$$\sum \Big(\rho_\omega(1_S) \boxtimes \rho_\omega(L''_{(0)})\Big)\hat{\Delta}^{0,m''}(1, L''_{(1)}) = \sum \rho_\omega(L''_{(0)})\epsilon_{\tilde{\Sigma}^{m''}C^\infty(M)}(L''_{(1)}) = \rho_\omega(L'') = \rho_\omega(L'L''), \tag{5.7.9}$$

where we have used the definition of $\hat{\Delta}^{0,m''}$ given in (5.6.8).

5.7.14 Lemma. *Let ω be a Feynman measure associated with a precut propagator Δ with family of proper closed convex cones $\{\mathscr{P}_p\}_{p\in M}$, $[L] \in S^m\mathscr{L}_{i,c}(M,E)$ with $m \in \mathbb{N}$, $\bar{p} = (p_1,\dots,p_m) \in M^m$ and $\bar{v} = (v_1,\dots,v_m) \in T^*_{\bar{p}}M^m$. If $\bar{v} \in \mathrm{WF}_{\bar{p}}(\rho_\omega(L))$, then* New Implicit

$$\bar{v} \notin \Big(\prod_{i=1}^m \big(-\mathscr{P}_{p_i} \cup \{0\}\big)\Big) \cup \Big(\prod_{i=1}^m \big(\mathscr{P}_{p_i} \cup \{0\}\big)\Big). \tag{5.7.10}$$

As a consequence, the products of distributions appearing in (5.7.8) *are defined.*

This result is claimed in the first paragraph of p. 633 in [10], without proof.

Proof. We prove the last part first. Assume thus that (5.7.10) holds. We will prove that the products appearing in (5.7.8) are defined, so assume the notation of that equation. We suppose first that $[L'] \in S^{m'}\mathscr{L}_{i,c}(M,E)$ and $[L''] \in S^{m''}\mathscr{L}_{i,c}(M,E)$. By (4.5.1), any vector $\bar{v} = (v_1,\dots,v_{m'+m''})$ of the wave front set of the symmetrized external tensor product $\rho_\omega(L'_{(0)}) \boxtimes_S \rho_\omega(L''_{(0)})$ in (5.7.8) satisfies also (5.7.10). Fact 5.6.6 together with (4.5.2) tell us in turn that the products of distributions in (5.7.8) are defined. Finally, since $S^m\mathscr{L}_{i,c}(M,E)$ is dense in $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$ for all $m \in \mathbb{N}$, the good definition of (5.7.8) follows directly from the obvious continuity of the left and right expressions there.

We will now prove the first statement of the lemma by induction on m. If $m = 1$ then the wave front set of $\rho_\omega(L)$ is empty (by Definition 5.7.9, (1)), and the claim is immediate. We suppose now that the lemma has been proved for all positive integers strictly less than $m \geq 2$. If $\bar{p}$ lies in the diagonal of M^m, then the statement follows from Definition 5.7.9, (1), applied to $\rho_\omega(L)$, taking into account that $\bar{v} \in \mathrm{WF}_{\bar{p}}(\rho_\omega(L))$ implies that $\bar{v} \neq 0$. If $\bar{p}$ does not lie in the diagonal of M^m, then (5.1.2) implies that there exists $(J', J'') \in \mathrm{Par}(m,2)$ such that $\bar{p} \in U_{J',J''}$. Let $m' = \#(J')$. We may assume without loss of generality that $J' = \{1,\dots,m'\}$ and $J'' = \{m'+1,\dots,m\}$. Since $U_{J',J''}$ is open, there exist open sets $U' \subseteq M^{m'}$ and $U'' \subseteq M^{m''}$ such that $\bar{p} \in U' \times U'' \subseteq U_{J',J''}$. By multiplying if necessary by a symmetric function $f \in C^\infty(M^m)$ such that $f(\bar{p}) \neq 0$ and whose support is included in $\cup_{\varsigma\in\mathbb{S}_m}\varsigma(U' \times U'')$ –because this does not change the wave front set of $\rho_\omega(L)$ at $\bar{p}$– we may assume that $[L]$ has support included in $\cup_{\varsigma\in\mathbb{S}_m}\varsigma(U' \times U'')$. As the canonical map $S^{m'}\mathscr{L}_{i,c}(M,E) \otimes S^{m''}\mathscr{L}_{i,c}(M,E) \to S^m\mathscr{L}_{i,c}(M,E)$ is surjective, we may even assume that L is a finite sum $\sum_{j\in J} L'_j L''_j$, with $[L'_j] \in S^{m'}\mathscr{L}_{i,c}(M,E)$ and $[L''_j] \in S^{m''}\mathscr{L}_{i,c}(M,E)$ such that the support of $[L'_j]$ is included in U' and that of $[L''_j]$ is included in U''. Hence, $[L'_j]$ and $[L''_j]$ satisfy the hypotheses of Definition 5.7.9, (3). By the inductive hypothesis applied to $\rho_\omega(L'_j)$ and $\rho_\omega(L''_j)$, any vector in the wave front set of any of them satisfies

(5.7.10). As explained at the beginning of the proof, this implies that the expression (5.7.8) is defined for $L_j = L'_j L''_j$. This latter expression of $\rho_\omega(L)$ in terms of $\rho_\omega(L'_j)$ and $\rho_\omega(L''_j)$ together with (4.5.2) and Fact 5.6.6 yield that the elements of the wave front set of $\rho_\omega(L_j)$ also satisfy (5.7.10), for all $j \in J$. Since the wave front set of a finite sum is included in the union of the wave front sets of the summands, the wave front set of $\rho_\omega(L)$ also satisfies (5.7.10). The lemma is thus proved. □

5.7.15. *Remark.* We note that the previous condition (5.7.8) is a form of Wick's theorem, where one computes the S-matrix in terms of Feynman diagrams. Indeed, consider the situation analyzed in Example 5.6.16. From [14], Lemma 2.11 and Prop. 2.13 and the nice explanations in Section 2.7 of that article, combined with our description of $\hat{\Delta}$ given in the previous example, we see that (5.7.8) is precisely the expansion of the S-matrix in terms of Feynman diagrams.

Based on Borcherds

5.8 Renormalization

5.8.1. We recall that $S_{C^\infty(M)}\Gamma(J^iE)$ is a canonical commutative and cocommutative unitary and counitary bialgebra in the symmetric monoidal category ${}_{C^\infty(M)}\mathfrak{BMod}$ (see Section 3.1). Given any bornological locally convex $C^\infty(M)$-module X, then $Y = X \otimes_{C^\infty(M)} S_{C^\infty(M)}\Gamma(J^iE)$ is a (cosymmetric bi)comodule over $S_{C^\infty(M)}\Gamma(J^iE)$ for the regular coaction(s), in the symmetric monoidal category of bornological locally convex $C^\infty(M)$-modules. Since $S_{C^\infty(M)}\Gamma(J^iE)$ is a bialgebra, the (symmetric or not) tensor products of Y over $C^\infty(M)$ are also (cosymmetric bi)comodules over $S_{C^\infty(M)}\Gamma(J^iE)$ and thus $S^+_{C^\infty(M)}Y$ (or $S_{C^\infty(M)}Y$) is a (cosymmetric bi)comodule as well. This applies in particular to X given by $\Gamma(\mathrm{Vol}(M))$ or $\Gamma_c(\mathrm{Vol}(M))$.

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5.8.2 Definition. *Following* [10], *Def. 10, we introduce the following notions. For a QFT background* (M,E) *of order* i, *the* group of renormalizations (resp., of compact support) $\mathscr{G}_i(M,E)$ *(resp.,* $\mathscr{G}_{i,c}(M,E)$*) is the group of automorphisms of the conilpotent cofree noncounitary coalgebra* $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$ *(resp.,* $S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$*) in the symmetric monoidal category* ${}_{C^\infty(M)}\mathfrak{BMod}$ *that are also morphisms of comodules over the coaugmented coalgebra* $S_{C^\infty(M)}\Gamma(J^iE)$ *in* ${}_{C^\infty(M)}\mathfrak{BMod}$.

5.8.3. *Remark.* Using the usual equivalence between coaugmented coalgebras and noncounitary coalgebras (see [66], 2.1.2), it is clear that $\mathscr{G}_i(M,E)$ (resp., $\mathscr{G}_{i,c}(M,E)$) is the group of automorphisms of the conilpotent cofree coaugmented coalgebra $S_{C^\infty(M)}\mathscr{L}_i(M,E)$ (resp., $S_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$) in the symmetric monoidal category ${}_{C^\infty(M)}\mathfrak{BMod}$ whose restriction to $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$ (resp., $S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$) are also morphisms of comodules over the coaugmented coalgebra $S_{C^\infty(M)}\Gamma(J^iE)$ in ${}_{C^\infty(M)}\mathfrak{BMod}$. We will use this identification without further explanation.

5.8.4. *Remark.* Note that what we have called group of renormalizations is what Borcherds calls the *ultraviolet group* in [10], Def. 10, even though he also refers to it there as "the group of renormalizations". As it is remarked in the third paragraph of p. 633 of [10], this group does not coincide with the usual definition of "renormalization group" in physics, even though it is (essentially) what other authors call the "Stückelberg-Petermann renormalization group" (see for instance [18], pp. 19–20). We also remark that the previous definition of group of renormalizations is of *local* nature, since it is used in the typical process in Quantum Field Theory of introducing local counterterms in the Lagrangian.

Well-known

5.8.5 Proposition. *The morphism*

$$\mathscr{G}_i(M,E) \to \mathrm{Hom}_{C^\infty(M)}\Big(S^+_{C^\infty(M)}\mathscr{L}_i(M,E), \Gamma\Big(\mathrm{Vol}(M)\Big)\Big) \tag{5.8.1}$$

sending $g \in \mathscr{G}_i(M,E)$ *to* $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}) \circ p_1 \circ g$, *where* $p_1 : S^+_{C^\infty(M)}\mathscr{L}_i(M,E) \to \mathscr{L}_i(M,E)$ *is the canonical projection, is a bijection onto the subset of the codomain of* (5.8.1) *formed by the elements* θ *satisfying that their restriction to* $\Gamma(\mathrm{Vol}(M)) \otimes_{C^\infty(M)} S^0_{C^\infty(M)}\Gamma(J^iE) \simeq \Gamma(\mathrm{Vol}(M))$ *is an isomorphism. The analogous result holds for* $\mathscr{G}_{i,c}(M,E)$, *where one should consider* $\mathscr{L}_{i,c}(M,E)$ *and* $\Gamma_c(\mathrm{Vol}(M))$ *instead.*

This result is a special case of the general theory of conilpotent cofree coaugmented coalgebras and cofree counitary comodules recalled in Section 3.1.

5.8.6. An easy and useful consequence of the previous result together with (3.1.2) is that any element $g \in \mathscr{G}_i(M,E)$ satisfies that, given any $m \in \mathbb{N}$,

$$g\Big(S^m_{C^\infty(M)}\mathscr{L}_i(M,E)\Big) \subseteq \bigoplus_{m'=1}^{m} S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E). \tag{5.8.2}$$

Note that $S_{C^\infty(M)}\mathscr{L}_i(M,E)$ (resp., $S_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$) can be decomposed as the direct sum of the spaces of sections (resp., of compact support) of the vector bundles over M given by

$$\Big(\mathrm{Vol}(M) \otimes S^{m_1} J^i E\Big) \circledS \dots \circledS \Big(\mathrm{Vol}(M) \otimes S^{m_q} J^i E\Big), \tag{5.8.3}$$

indexed by $q \in \mathbb{N}_0$ and $\bar{m} = (m_1, \dots, m_q) \in \mathbb{N}_0^q$. Define the *support* of $\sigma \in S_{C^\infty(M)}\mathscr{L}_i(M,E)$ (resp., $\sigma \in S_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$) to be the subset of M given as the union of the supports of its finite components in (5.8.3).

By 3.12.1, we have the following maps

$$\tilde{S}^+\mathscr{L}_i(M,E) \to S^+_{C^\infty(M)}\mathscr{L}_i(M,E) \tag{5.8.4}$$

and

$$\tilde{S}^+\mathscr{L}_{i,c}(M,E) \to S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E). \tag{5.8.5}$$

They are compatible in the sense that

$$\begin{array}{ccc}
\tilde{S}^+\mathscr{L}_{i,c}(M,E) & \longrightarrow & S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E) \\
\downarrow & & \downarrow \\
\tilde{S}^+\mathscr{L}_{i}(M,E) & \longrightarrow & S^+_{C^\infty(M)}\mathscr{L}_{i}(M,E)
\end{array}$$

is commutative, where the left vertical map is the convenient completion of the inclusion $S^+\mathscr{L}_{i,c}(M,E) \to S^+\mathscr{L}_i(M,E)$ and the right vertical map is the canonical inclusion. Notice that $\tilde{S}^+\mathscr{L}_i(M,E)$ and $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ are bornological locally convex $\tilde{\Sigma}C^\infty(M)$-modules (see Fact 3.9.6), and the definition of $\tilde{\Sigma}C^\infty(M)$ tells us that same holds for $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$ and $S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$. Moreover, the maps (5.8.4) and (5.8.5) are morphisms of bornological locally convex $\tilde{\Sigma}C^\infty(M)$-modules.

Also note that the restriction to $S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$ induces a canonical morphism of groups

$$\mathscr{G}_i(M,E) \to \mathscr{G}_{i,c}(M,E). \tag{5.8.6}$$

Indeed, if $A \in S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$, choose $f \in C_c^\infty(M)$ such that $f|_{\mathrm{supp}(A)} \equiv 1$. Hence, given any $g \in \mathscr{G}_i(M,E)$, we have that $g(A) = g(fA) = fg(A)$, which has compact support (included in the support of f). Using Proposition 2.2.5 and Proposition 5.8.5, we see that (5.8.6) is injective.

5.8.7 Lemma. *There is a morphism of groups*

Based on Borcherds with new stuff

$$\eta : \mathscr{G}_{i,c}(M,E) \to \mathrm{Aut}_{\mathrm{Cog}(\mathrm{CLCS}_{HD})}\Big(\tilde{S}^+\mathscr{L}_{i,c}(M,E)\Big), \tag{5.8.7}$$

where the codomain is the set of automorphisms of the conilpotent cofree noncounitary coalgebra $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ in the symmetric monoidal category CLCS_{HD}. Furthermore, $\eta(g)(\tilde{S}^+\mathscr{L}_{i,K}(M,E)) \subseteq \tilde{S}^+\mathscr{L}_{i,K}(M,E)$, for all compact subsets $K \subseteq M$, and $\eta(g)(L'L'') = \eta(g)(L')\eta(g)(L'')$, if $L' \in \tilde{S}^+\mathscr{L}_{i,K'}(M,E)$ and $L'' \in \tilde{S}^+\mathscr{L}_{i,K''}(M,E)$ with $K' \cap K'' = \emptyset$.

This result is stated in [10] as Lemma 14, but the proof there does not seem clear to us. We refer to Chapter 3 for the structures appearing in it. Moreover, the author of the mentioned reference states that $\eta(g)$ should be a morphism of comodules over $S_{C^\infty(M)}\Gamma(J^iE)$, but as explained in Section 3.5 we cannot understand this condition in any sensible way. On the other hand, we have also added the fact that the group of renormalizations "preserves the support", which was somehow implicit in [10].

Proof. The mapping (5.8.7) is defined as follows. Given $g \in \mathscr{G}_{i,c}(M,E)$, consider $p_1 \circ g : S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E) \to \mathscr{L}_{i,c}(M,E)$, where $p_1 : S^+_{C^\infty(M)}\mathscr{L}_{i,c}(M,E) \to \mathscr{L}_{i,c}(M,E)$ is the canonical projection. Note that $p_1 \circ g$ is a morphism of $C^\infty(M)$-modules, so in particular it is k-linear. The composition of (5.8.5) with $p_1 \circ g$ gives a k-linear map $\tilde{g} : \tilde{S}^+\mathscr{L}_{i,c}(M,E) \to \mathscr{L}_{i,c}(M,E)$, which is continuous by the same reason as the one for (5.8.5). Set $\eta(g)$ as the unique endomorphism of the conilpotent cofree noncounitary coalgebra $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ (in the symmetric monoidal category LCS^c_{HD}) such that $\tilde{p}_1 \circ \eta(g) = \tilde{g}$, where $\tilde{p}_1 : \tilde{S}^+\mathscr{L}_{i,c}(M,E) \to \mathscr{L}_{i,c}(M,E)$ is the canonical projection. Taking into account that $\tilde{g}|_{\mathscr{L}_{i,c}(M,E)} = (p_1 \circ g)|_{\mathscr{L}_{i,c}(M,E)}$ is an isomorphism of $C^\infty(M)$-modules, it is *a fortiori* one of vector spaces. Since it is continuous and $\mathscr{L}_{i,c}(M,E)$ is an (LF)-space, then it is an isomorphism of LCS spaces (see [95], III.2.2), which, by the properties of the conilpotent coalgebra recalled in Section 3.1, implies that $\eta(g)$ is an isomorphism as well. The fact that $\eta(g \circ g') = \eta(g) \circ \eta(g')$, for all $g, g' \in \mathscr{G}_{i,c}(M,E)$, follows directly from the (3.1.2).

The part concerning the support follows directly from the fact that map (5.8.5) is of $\tilde{\Sigma}C^\infty(M)$-modules and from (3.1.2). In order to prove the last statement it suffices to show it under the additional assumption that $[L'] \in S^+\mathscr{L}_{i,K'}(M,E)$ and $[L''] \in S^+\mathscr{L}_{i,K''}(M,E)$ with $K' \cap K'' = \emptyset$, because $\eta(g)$ is a continuous map and the product of $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$ is (jointly) continuous. We may further assume that $[L'] = \ell'_1 \circledS \dots \circledS \ell'_{q'}$ and $[L''] = \ell''_1 \circledS \dots \circledS \ell''_{q''}$ are elementary tensor expressions belonging to spaces of the form (5.3.3), where $q, q' \in \mathbb{N}$, $\mathrm{supp}_1(\ell'_{j'}) \subseteq K'$ for all $j' = 1, \dots, q'$ and $\mathrm{supp}_1(\ell''_{j''}) \subseteq K''$ for all $j'' = 1, \dots, q''$. By (3.1.2) and using the notation explained there, we can write

$$\eta(g)\big([L'][L'']\big) = \sum_{m\in\mathbb{N}} \frac{1}{m!} \underbrace{\tilde{g}^{\otimes m} \circ \Delta^{(m)}_{S^+\mathscr{L}_{i,c}(M,E)}\big([L'][L'']\big)}_{t_m}.$$

The hypothesis $K' \cap K'' = \emptyset$ implies that $\tilde{g}$ vanishes on any product containing factors $\ell'_{j'}$ and $\ell''_{j''}$, for some j' and j'', which in particular implies that $t_1 = 0$. Using this and the explicit expression of the coproduct of $S^+\mathscr{L}_{i,c}(M,E)$ we see that

$$t_m = \sum_{m'=1}^{m-1} \binom{m}{m'} t'_{m'} t''_{m-m'}, \tag{5.8.8}$$

where

$$t'_{m'} = \tilde{g}^{\otimes m'} \circ \Delta^{(m')}_{S^+\mathscr{L}_{i,c}(M,E)}\big([L']\big) \quad \text{and} \quad t''_{m''} = \tilde{g}^{\otimes m''} \circ \Delta^{(m'')}_{S^+\mathscr{L}_{i,c}(M,E)}\big([L'']\big), \tag{5.8.9}$$

and we recall that we are omitting in (5.8.8) and (5.8.9) the canonical projection of the tensor algebra onto the symmetric algebra, according to the conventions explained in 3.1.2. Hence,

$$\begin{aligned} \eta(g)\big([L'][L'']\big) &= \sum_{m\in\mathbb{N}} \frac{1}{m!} \sum_{m'=1}^{m-1} \binom{m}{m'} t'_{m'} t''_{m-m'} \\ &= \Big(\sum_{m'\in\mathbb{N}} \frac{1}{m'!} \tilde{g}^{\otimes m'} \circ \Delta^{(m')}_{S^+\mathscr{L}_{i,c}(M,E)}\big([L']\big) \Big)\Big(\sum_{m''\in\mathbb{N}} \frac{1}{m''!} \tilde{g}^{\otimes m''} \circ \Delta^{(m'')}_{S^+\mathscr{L}_{i,c}(M,E)}\big([L'']\big) \Big) \\ &= \eta(g)\big([L']\big)\eta(g)\big([L'']\big), \end{aligned} \tag{5.8.10}$$

as was to be shown. The proposition is thus proved. □

5.8.8. Following [10], Section 2, given $m \in \mathbb{N}_0$ define $\mathscr{G}_{i,\geq m}(M,E)$ as the subset formed by the elements $g \in \mathscr{G}_i(M,E)$ satisfying that $g|_{\tilde{S}^{m'}_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)} = \mathrm{id}_{\tilde{S}^{m'}_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)}$ for all $m' = 1,\dots,m$. It is clear that $\mathscr{G}_{i,\geq m+1}(M,E) \subseteq \mathscr{G}_{i,\geq m}(M,E)$ and that $\mathscr{G}_{i,\geq m}(M,E)$ is a subgroup of $\mathscr{G}_i(M,E)$ for all $m \in \mathbb{N}_0$. Notice that $\mathscr{G}_{i,\geq 0}(M,E) = \mathscr{G}_i(M,E)$. Using Proposition 5.8.5 it is fairly straightforward (but rather lengthy) to prove that $\mathscr{G}_{i,\geq m}(M,E)$ is a normal subgroup of $\mathscr{G}_i(M,E)$ and that the linear topology on it defined by $\{\mathscr{G}_{i,\geq m}(M,E)\}_{m\in\mathbb{N}_0}$ is complete, but we will not need it. Instead we will use the two following simpler results (*cf.* [10], Lemma 13).

5.8.9 Lemma. *Consider $(g_{-m})_{m\in\mathbb{N}_0}$, where $g_{-m} \in \mathscr{G}_{i,\geq m}(M,E)$. Then, for any element $\sigma \in S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$, the sequence $((g_{-m}\circ\cdots\circ g_0)(\sigma))_{m\in\mathbb{N}_0}$ is eventually constant and it defines an element of $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$, denoted by $g(\sigma)$. Moreover, the map $\sigma \mapsto g(\sigma)$ belongs to $\mathscr{G}_i(M,E)$.* Based on Borcherds

Proof. Property (5.8.2) tells us that $((g_{-m}\circ\cdots\circ g_0)(\sigma))_{m\in\mathbb{N}_0}$ is constant for all $m \geq m'$, if $\sigma \in S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E)$. Since $S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E)$ is a subcoalgebra of $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$ and $g_{-m}\circ\cdots\circ g_0$ is a morphism of coalgebras in ${}_{C^\infty(M)}\mathfrak{BMod}$, its restriction to $S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E)$ is a morphism of coalgebras in ${}_{C^\infty(M)}\mathfrak{BMod}$, which in turn implies the same holds for g. Similarly, using that $S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E)$ is a $S_{C^\infty(M)}\Gamma(J^iE)$-subcomodule of $S^+_{C^\infty(M)}\mathscr{L}_i(M,E)$ in the symmetric monoidal category ${}_{C^\infty(M)}\mathfrak{BMod}$ and that $g_{-m}\circ\cdots\circ g_0$ is a morphism of $S_{C^\infty(M)}\Gamma(J^iE)$-comodules in ${}_{C^\infty(M)}\mathfrak{BMod}$, we see that g is a morphism of $S_{C^\infty(M)}\Gamma(J^iE)$-comodules in ${}_{C^\infty(M)}\mathfrak{BMod}$ as well.

Finally, g is an element of $\mathscr{G}_i(M,E)$. Indeed, this follows from Proposition 5.8.5 taking into account that (5.8.2) tells us that $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))}\otimes_{C^\infty(M)}\epsilon_{S_{C^\infty(M)}\Gamma(J^iE)})\circ p_1\circ g|_{\Gamma(\mathrm{Vol}(M))}$ coincides with the isomorphism $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))}\otimes_{C^\infty(M)}\epsilon_{S_{C^\infty(M)}\Gamma(J^iE)})\circ p_1\circ g_0|_{\Gamma(\mathrm{Vol}(M))}$, where $p_1 : S^+_{C^\infty(M)}\mathscr{L}_i(M,E) \to \mathscr{L}_i(M,E)$ denotes the canonical projection. □

5.8.10. The element g constructed in the previous result is called the *infinite product* of $(g_{-m})_{m\in\mathbb{N}_0}$, and it is denoted by

$$\prod_{m=-\infty}^{0} g_m.$$

Given $m \in \mathbb{N}_0$, define $\mathscr{G}_{i,m}(M,E)$ as the subset formed by the elements g of $\mathscr{G}_{i,\geq m}(M,E)$ such that the restriction of $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))}\otimes_{C^\infty(M)}\epsilon_{S_{C^\infty(M)}\Gamma(J^iE)})\circ p_1\circ g$ to $S^{m'}_{C^\infty(M)}\mathscr{L}_i(M,E)$ vanishes if $m' \in \mathbb{N}_{\geq 2}\setminus\{m+1\}$, and its restriction to $\mathscr{L}_i(M,E)$ is $\mathrm{id}_{\Gamma(\mathrm{Vol}(M))}\otimes_{C^\infty(M)}\epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}$ if $m \in \mathbb{N}$. Using Proposition 5.8.5 we see easily that $\mathscr{G}_{i,0}(M,E)$ is the subgroup of $\mathscr{G}_i(M,E)$ formed by the elements g satisfying that $g(S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)) \subseteq S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$ for all $m \in \mathbb{N}$. However, $\mathscr{G}_{i,m}(M,E)$ is not a subgroup of $\mathscr{G}_i(M,E)$ for all $m \in \mathbb{N}$ (*cf.* paragraph before Lemma 13 in [10]), because even the composition of two elements of $\mathscr{G}_{i,m}(M,E)$ is not in $\mathscr{G}_{i,m}(M,E)$. Moreover, by Proposition 5.8.5 it is clear that, for all $m \in \mathbb{N}_0$, any element $g \in \mathscr{G}_{i,\geq m}(M,E)$ is uniquely written as $g'\circ g''$ with $g' \in \mathscr{G}_{i,\geq m+1}(M,E)$ and $g'' \in \mathscr{G}_{i,m}(M,E)$. A direct consequence of Lemma 5.8.9 and the previous statement is the following result.

5.8.11 Lemma. *Given $g \in \mathscr{G}_i(M,E)$, there exists a unique sequence $(g_{-m})_{m\in\mathbb{N}_0}$, with $g_{-m} \in \mathscr{G}_{i,m}(M,E)$, such that* Easy

$$g = \prod_{m=-\infty}^{0} g_m.$$

5.9 Meromorphic families of propagators, Feynman measures and renormalizations

New but expected

5.9.1. In this section we shall introduce the notion of meromorphic family of Feynman measures, and meromorphic family of renormalizations, that will be used in the sequel. They are implicit in [10].

5.9.2. We recall that, if X is a complex complete reflexive LCS and $\Omega \subseteq \mathbb{C}^m$ is an open subset, an X-valued function $f : \Omega \subseteq \mathbb{C} \to X$ is called *holomorphic* if for all $\lambda \in X'$ the induced map $\langle \lambda, f \rangle : \Omega \to \mathbb{C}$ defined as $z \mapsto \lambda(f(z))$ is holomorphic (see [58], 1.1). If $m = 1$, and given any point $z_0 \in \mathbb{C}$ such that $B(z_0, r) \setminus \{z_0\} \subseteq \Omega$ for some $r > 0$, we say that z_0 is a *pole* of f if there is $m \in \mathbb{N}_0$ such that $(z - z_0)^m \langle \lambda, f \rangle$ has a removable singularity at z_0 for all $\lambda \in X'$. Finally, f is called *meromorphic* if, for all $\lambda \in X'$, $\langle \lambda, f \rangle$ is meromorphic, and any pole z_0 of $\langle \lambda, f \rangle$ is also a pole of f (see [58], 1.4). In case $X = \Gamma_c(E^* \otimes \mathrm{Vol}(M))'$ we say that f is a *distribution valued meromorphic function* on Ω. If $X = (\Gamma_c(\mathrm{Vol}(M) \otimes J^i E) \tilde{\otimes}_\beta \Gamma_c(\mathrm{Vol}(M) \otimes J^i E))'$, we will say that we have a *meromorphic family of propagators* on Ω.

New but expected **5.9.3 Definition.** *Let $r \mapsto \omega^r$ be a map defined on an open set $\Omega \subseteq \mathbb{C}$, where $\omega^r : \tilde{S}\mathscr{L}_{i,c}(M,E) \to \mathbb{C}$ is a Feynman measure, for all $r \in \Omega$. We say that it is* holomorphic *(resp.,* meromorphic*) if, given any $m \in \mathbb{N}$, the map $r \mapsto \omega^r|_{\tilde{S}^m \mathscr{L}_{i,c}(M,E)}$ is holomorphic (resp., meromorphic) on Ω.*

We say that meromorphic family of Feynman measures $r \mapsto \omega^r$ defined on an open set $\Omega \subseteq \mathbb{C}$ is associated with a meromorphic family of propagators $r \mapsto \Delta^r$ also defined on Ω, if is ω^r associated with Δ^r for all $r \in \Omega$.

New but expected **5.9.4 Definition.** *Let $r \mapsto g^r$ be a map defined on an open set $\Omega \subseteq \mathbb{C}$, where $g^r \in \mathscr{G}_i(M,E)$ is a renormalization, for all $r \in \Omega$. We say that it is* holomorphic *(resp.,* meromorphic*) if, given any $m \in \mathbb{N}$, the induced map $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^i E)}) \circ p_1 \circ g^r|_{S^m_{C^\infty(M)}\mathscr{L}_i(M,E)}$ stated in Proposition 5.8.5, regarded as a map from Ω to $X = S^m_{C^\infty(M)} \mathscr{L}_i(M,E)^{\circledast} \otimes_{C^\infty(M)} \Gamma(\mathrm{Vol}(M))$, is holomorphic (resp., meromorphic) on Ω.*

Easy **5.9.5 Fact.** *Given an open set $\Omega \subseteq \mathbb{C}$, the set formed by the holomorphic (resp., meromorphic) families of renormalizations $r \mapsto g^r \in \mathscr{G}_i(M,E)$ defined on Ω forms a group for the pointwise composition $r \mapsto g_1^r \circ g_2^r$ of $r \mapsto g_1^r$ and $r \mapsto g_2^r$.*

This result is a direct consequence of the explicit expression of the product and the inverse of elements of $\mathscr{G}_i(M,E)$ deduced from the results recalled in Section 3.1.

Easy **5.9.6 Fact.** *Given a sequence $(g^r_{-m})_{m \in \mathbb{N}_0}$, where $r \mapsto g^r_{-m}$ is a meromorphic family of renormalizations defined on the open set Ω with $g^r_{-m} \in \mathscr{G}_{i,m}(M,E)$ for all $m \in \mathbb{N}_0$, then the element*

$$g^r = \prod_{m=-\infty}^{0} g^r_m$$

defined for every fixed $r \in \Omega$ is a meromorphic family of renormalizations on Ω.

Proof. The fact that g^r is uniquely defined for every $r \in \Omega$ follows from Lemma 5.8.9. To prove the remaining statements we will use the notation of Definition 5.9.4. Since, given any $m \in \mathbb{N}$, the induced map $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^i E)}) \circ p_1 \circ g^r|_{S^m_{C^\infty(M)}\mathscr{L}_i(M,E)}$ clearly coincides with the composition of the restriction to $S^m_{C^\infty(M)} \mathscr{L}_i(M,E)$ of the finite product $\prod_{m'=-m}^{0} g^r_{m'}$ and $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^i E)}) \circ p_1$, the meromorphic property follows from the previous Fact 5.9.5. □

Chapter 6

The first main result: The simply transitive action of the group of renormalizations

6.0.1. The composition of (5.8.6) and (5.8.7) gives an action (by coalgebra automorphisms in the category CLCS_{HD}) of the group of renormalizations $\mathscr{G}_i(M,E)$ on the space $\tilde{S}^+\mathscr{L}_{i,c}(M,E)$, which we denote by $g\cdot[L]$, where $g\in\mathscr{G}_i(M,E)$ and $[L]\in\tilde{S}^+\mathscr{L}_{i,c}(M,E)$. By Remark 5.8.3, this action of $\mathscr{G}_i(M,E)$ can be canonically extended to $\tilde{S}\mathscr{L}_{i,c}(M,E)$, and we will also denote it by $g\cdot[L]$. This induces the dual action on the space of its unitary continuous functionals, *i.e.* on the space of Feynman measures. We shall denote the action of $g\in\mathscr{G}_i(M,E)$ on $\omega\in(\tilde{S}\mathscr{L}_{i,c}(M,E))'$ simply by $g\cdot\omega$, *i.e.* $(g\cdot\omega)([L])=\omega(g^{-1}\cdot[L])$.

6.0.2 Lemma. *Let $[L]\in\tilde{S}^m\mathscr{L}_{i,c}(M,E)$, with $m\in\mathbb{N}$, $f\in\tilde{\Sigma}^m C^\infty(M)$, and $g\in\mathscr{G}_i(M,E)$. Assume the conventions and notation in the proof of Lemma 5.8.7, and to shorten the notation set $\underline{g}=g^{-1}$. We have the identity* New

$$
\begin{aligned}
\bar{\rho}_{g\cdot\omega}\Big([L]\Big)(f) &= \sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}\rho_\omega\Big[\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)\Big)\Big]\Big(\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}(f)\Big)\\
&= \sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}(\mathrm{diag}_T)_*\Big\{\rho_\omega\Big[\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)\Big)\Big]\Big\}(f),
\end{aligned}
\tag{6.0.1}
$$

where $\mathrm{diag}_T: M^{m'}\to M^m$ *is given by* (5.1.6), $\tilde{\underline{g}}_j$ *is the restriction of* $\tilde{\underline{g}}$ *to* $\tilde{S}^j\mathscr{L}_{i,c}(M,E)$, $\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}$ *is defined in 5.1.8, and the map* $\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}: C^\infty(M)^{\tilde{\otimes}_\beta m}\to C^\infty(M)^{\tilde{\otimes}_\beta m'}$ *is given in 5.1.10.*

Proof. By continuity of the expressions in (6.0.1), it suffices to prove them for L given by an elementary tensor $\ell_1|\dots|\ell_m$, where $\ell_1,\dots,\ell_m\in\mathscr{L}_{i,c}(M,E)$, and $f=\sum_{\varsigma\in\mathbb{S}_m}f_{\varsigma(1)}|\dots|f_{\varsigma(m)}$, with $f_1,\dots,f_m\in C^\infty(M)$. We have

$$
\begin{aligned}
\bar{\rho}_{g\cdot\omega}\Big([L]\Big)(f) &= (g\cdot\omega)\Big([L]f\Big)=\omega\Big(g^{-1}\cdot\Big([L]f\Big)\Big)=\omega\Big(\sum_{m'=1}^{m}\frac{1}{m'!}\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\circ\Delta^{(m')}_{\tilde{S}^+\mathscr{L}_{i,c}(M,E)}\Big([L]f\Big)\Big)\\
&=\omega\Big(\sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\circ\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]f\Big)\Big)=\omega\Big[\sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)f\Big)\Big]\\
&=\omega\Big[\sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)\Big)\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}(f)\Big]\\
&=\sum_{m'=1}^{m}\sum_{T\in\mathrm{Par}(m,m')}\rho_\omega\Big[\tilde{\underline{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)\Big)\Big]\Big(\tilde{\mu}^T_{C^\infty(M),\mathrm{pro}}(f)\Big),
\end{aligned}
\tag{6.0.2}
$$

where we have used Lemma 5.1.9 in the fifth identity, and the definition of $\tilde{\underline{g}}$ in the sixth equality. This proves the first identity of (6.0.1), whereas the second follow from the definition of push-forward and Lemma 5.1.12. The lemma is thus proved. □

6.0.3. *Remark.* We stress that the hypothesis in (6.0.1) stating that $f \in \tilde{\Sigma}^m C^\infty(M)$ is essential. Indeed, our inability to find a similar expression as in (6.0.1) for general $f \in C^\infty(M)^{\tilde{\otimes}_\beta m}$ is a shadow of the fact that we cannot express $[Lf]$ in terms of $[L]$ and f, and we cannot regard the action of the group of renormalizations at the level of the tensor algebra $\tilde{T}\mathscr{L}_{i,c}(M,E)$.

6.0.4. The following result is the first part of [10], Thm. 15, which we have decided to decouple into two for clarity. We give a complete proof of it, which seems very different from that of Borcherds.

New/Last part based on Borcherds

6.0.5 Proposition. *Let Δ be a local precut propagator. Then the action of the group $\mathscr{G}_i(M,E)$ on the set of Feynman measures restricts to an action on the set $\mathscr{F}_\Delta$ of Feynman measures associated with Δ.*

Proof. We have to show that the action of $\mathscr{G}_i(M,E)$ leaves the subset $\mathscr{F}_\Delta$ fixed inside of the set of all Feynman measures. Let $g \in \mathscr{G}_i(M,E)$ and $\omega \in \mathscr{F}_\Delta$. We will first prove that g respects condition (2) of Definition 5.7.9. Since the restriction of $(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}) \circ p_1 \circ g^{-1}$ to $\Gamma(\mathrm{Vol}(M))$ is an isomorphism of $C^\infty(M)$-modules (see Proposition 5.8.5) and $\Gamma(\mathrm{Vol}(M))$ is isomorphic to $C^\infty(M)$ as $C^\infty(M)$-modules (see 4.1.1), the former restriction is given by multiplying with a nonvanishing function $h \in C^\infty(M)$. Hence, the condition $\omega(\theta) = \int_M f\theta$, for $\theta \in \Gamma_c(\mathrm{Vol}(M))$ implies that $(g\cdot\omega)(\theta) = \int_M hf\theta$, so $(g\cdot\omega)$ satisfies condition (2) of Definition 5.7.9.

We will prove that $g\cdot\omega$ satisfies condition (1) of Definition 5.7.9. Assume the conventions and notation in the proof of Lemma 5.8.7. By Remark 5.7.10, it suffices to show that if $(v_1,\dots,v_m) \in \mathrm{WF}_{(p,\dots,p)}(\bar{\rho}_{g\cdot\omega}([L]))$ for some $p \in M$ and $[L] \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$, with $m \in \mathbb{N}$, then $v_1 + \cdots + v_m = 0$. Let $f \in \tilde{\Sigma}^m C^\infty(M)$. To shorten the notation set $\not{g} = g^{-1}$. By Lemma 6.0.2, we have

$$\bar{\rho}_{g\cdot\omega}\Big([L]\Big)(f) = \sum_{m'=1}^{m} \sum_{T\in\mathrm{Par}(m,m')} (\mathrm{diag}_T)_*\Big\{\rho_\omega\Big[\tilde{\not{g}}^{\tilde{\otimes}_\beta m'}\Big(\tilde{\mu}^T_{\mathscr{L}_{i,c}(M,E)}\Big([L]\Big)\Big)\Big]\Big\}(f). \tag{6.0.3}$$

Using this, the wave front set condition (1) of Definition 5.7.9 for ω and the inclusion (4.4.1), we conclude that the mentioned wave front set condition is also fulfilled by $g\cdot\omega$.

We will now show that $g\cdot\omega$ satisfies the condition (3) of Definition 5.7.9. Let L' and L'' be as in (3), and set $L = L'L''$. Recall the notation $L_J = \ell_{j_1} ⓢ \dots ⓢ \ell_{j_{m''}}$ for $J = \{j_1 < \cdots < j_{m''}\} \subseteq \{1,\dots,m\}$ (see 5.1.8). On the one hand, we have

$$\begin{aligned}\rho_{g\cdot\omega}(L'L'') &= \sum_{j=1}^{m'+m''} \frac{1}{j!} \sum_{T\in\mathrm{Par}(m'+m'',j)} (\mathrm{diag}_T)_*\Big(\rho_\omega\Big(\tilde{\not{g}}_{|T_1|}(L_{T_1})|\dots|\tilde{\not{g}}_{|T_j|}(L_{T_j})\Big)\Big)\\ &= \sum_{j'=1}^{m'}\sum_{j''=1}^{m''}\frac{1}{j'!}\frac{1}{j''!}\sum_{\substack{T'\in\mathrm{Par}(m',j')\\ T''\in\mathrm{Par}(m'',j'')}} (\mathrm{diag}_{(T',T'')})_*\underbrace{\Big(\rho_\omega\Big(\tilde{\not{g}}_{|T'_1|}(L'_{T'_1})|\dots|\tilde{\not{g}}_{|T'_{j'}|}(L'_{T'_{j'}})|\tilde{\not{g}}_{|T''_1|}(L''_{T''_1})|\dots|\tilde{\not{g}}_{|T''_{j''}|}(L''_{T''_{j''}})\Big)\Big)}_{w_{T',T''}},\end{aligned} \tag{6.0.4}$$

where we have used Lemma 6.0.2 in the first equality, (5.8.8) in the second one, and we denote by (T',T'') the element T of $\mathrm{Par}(m'+m'',j'+j'')$ satisfying that $T_j = T'_j$ for $j = 1,\dots,j'$, and $T_j = T''_{j-j'}$ for $j = j'+1,\dots,j'+j'$.

By the hypothesis (3) of Definition 5.7.9 for ω and the explicit expression of the coactions in that definition (described in Proposition 3.4.7) in terms of the coaction of $S_{C^\infty(M)}\Gamma(J^iE)$ on $\mathscr{L}_i(M,E)$ in the symmetric monoidal category of $C^\infty(M)$-modules, we may write $w_{T',T''}$ in the last member of (6.0.4) as

$$\begin{aligned}&\sum\Big(\rho_\omega\Big(\tilde{g}_{|T'_1|}(L'_{T'_1})_{(0)}|\dots|\tilde{g}_{|T'_{j'}|}(L'_{T'_{j'}})_{(0)}\Big)\boxtimes\rho_\omega\Big(\tilde{g}_{|T''_1|}(L''_{T''_1})_{(0)}|\dots|\tilde{g}_{|T''_{j''}|}(L''_{T''_{j''}})_{(0)}\Big)\Big)\\&\quad\hat{\Delta}^{j',j''}\Big(\tilde{g}_{|T'_1|}(L'_{T'_1})_{(1)}|\dots|\tilde{g}_{|T'_{j'}|}(L'_{T'_{j'}})_{(1)},\tilde{g}_{|T''_1|}(L''_{T''_1})_{(1)}|\dots|\tilde{g}_{|T''_{j''}|}(L''_{T''_{j''}})_{(1)}\Big)\\&=\sum\Big[\rho_\omega\Big(g_{|T'_1|}\big((\bar{L}'_{T'_1})_{(0)}\big)|\dots|g_{|T'_{j'}|}\big((\bar{L}'_{T'_{j'}})_{(0)}\big)\Big)\boxtimes\rho_\omega\Big(g_{|T''_1|}\big((\bar{L}''_{T''_1})_{(0)}\big)|\dots|g_{|T''_{j''}|}\big((\bar{L}''_{T''_{j''}})_{(0)}\big)\Big)\Big]\\&\quad\hat{\Delta}^{j',j''}\Big((\bar{L}'_{T'_1})_{(1)}|\dots|(\bar{L}'_{T'_{j'}})_{(1)},(\bar{L}''_{T''_1})_{(1)}|\dots|(\bar{L}''_{T''_{j''}})_{(1)}\Big),\end{aligned}\tag{6.0.5}$$

where all the subscripts of coactions are those of the coaction of $S_{C^\infty(M)}\Gamma(J^iE)$ on $S_{C^\infty(M)}\mathscr{L}_i(M,E)$ in the symmetric monoidal category of $C^\infty(M)$-modules, $\bar{L}'_J$ is the image under the map (5.8.5) of L'_J and the same for $\bar{L}''_J$, we have used that

$$\tilde{g}_{|J|}(L_J)=g|_{S^{|J|}_{C^\infty(M)}\mathscr{L}_i(M,E)}(\bar{L}_J),$$

and that g is a morphism of comodules over $S_{C^\infty(M)}\Gamma(J^iE)$ in the symmetric monoidal category of $C^\infty(M)$-modules.

On the other hand,

$$\Big(\rho_{g\cdot\omega}(L'_{(0)})\boxtimes\rho_{g\cdot\omega}(L''_{(0)})\Big)\hat{\Delta}^{m',m''}(L'_{(1)},L''_{(1)})=\sum_{j'=1}^{m'}\sum_{j''=1}^{m''}\frac{1}{j'!}\frac{1}{j''!}\sum_{\substack{T'\in\mathrm{Par}(m',j')\\T''\in\mathrm{Par}(m'',j'')}}\underbrace{w''_{T',T''}\hat{\Delta}^{m',m''}(L'_{(1)},L''_{(1)})}_{w'_{T',T''}},\tag{6.0.6}$$

where $w''_{T',T''}$ is given by

$$(\mathrm{diag}_{T'})_*\rho_\omega\Big(\tilde{g}_{|T'_1|}((L'_{(0)})_{T'_1})|\dots|\tilde{g}_{|T'_{j'}|}((L'_{(0)})_{T'_{j'}})\Big)\boxtimes(\mathrm{diag}_{T''})_*\rho_\omega\Big(\tilde{g}_{|T''_1|}((L''_{(0)})_{T''_1})|\dots|\tilde{g}_{|T''_{j''}|}((L''_{(0)})_{T''_{j''}})\Big).\tag{6.0.7}$$

Notice that $w'_{T',T''}$ is given by

$$(\mathrm{diag}_{(T',T'')})_*\Big[\rho_\omega\Big(\tilde{g}_{|T'_1|}\big((L'_{(0)})_{T'_1}\big)|\dots|\tilde{g}_{|T'_{j'}|}\big((L'_{(0)})_{T'_{j'}}\big)\Big)\boxtimes\rho_\omega\Big(\tilde{g}_{|T''_1|}\big((L''_{(0)})_{T''_1}\big)|\dots|\tilde{g}_{|T''_{j''}|}\big((L''_{(0)})_{T''_{j''}}\big)\Big)\Big]\hat{\Delta}^{m',m''}(L'_{(1)},L''_{(1)}).\tag{6.0.8}$$

Note also that for any $L\in\mathscr{L}_i(M,E)^{\bar{\otimes}_\beta m}$ and $m\in\mathbb{N}$,

$$\tilde{g}_{|J|}\Big((L_{(0)})_J\Big)=g|_{S^{|J|}_{C^\infty(M)}\mathscr{L}_i(M,E)}\Big((\bar{L}_J)_{(0)}\Big),\tag{6.0.9}$$

where the first 0 subscript comes from the coaction on an element of $\mathscr{L}_i(M,E)^{\bar{\otimes}_\beta m}$ as described in Definition 5.7.9, (3), whereas the second subscript 0 comes the coaction of the coalgebra $S_{C^\infty(M)}\Gamma(J^iE)$ on an element of $\mathscr{L}_i(M,E)^{\bar{\otimes}_{C^\infty(M)} m}$ in the symmetric monoidal category of $C^\infty(M)$-modules. Analogously, the reader can check that

$$\hat{\Delta}^{j',j''}\Big((\bar{L}_{T'_1})_{(1)}|\dots|(\bar{L}_{T'_{j'}})_{(1)},(\bar{L}_{T''_1})_{(1)}|\dots|(\bar{L}_{T''_{j''}})_{(1)}\Big)=(\mathrm{diag}_{(T',T'')})^*\Big(\hat{\Delta}^{m',m''}(L'_{(1)},L''_{(1)})\Big),\tag{6.0.10}$$

where similar comments concerning the subscripts indicating the coactions apply.

Hence, by (4.5.15), together with (6.0.9) and (6.0.10), we conclude that (6.0.8) coincides with (6.0.5), which in turn implies that the equality between (6.0.6) and (6.0.4), as was to be shown. The proposition is thus proved. □

6.0.6 Proposition. *Let $r\mapsto\Delta^r$ be a meromorphic family of local precut propagators on an open set $\Omega\subseteq\mathbb{C}$, $r\mapsto\omega^r$ a meromorphic family of Feynman measures associated with $r\mapsto\Delta^r$, and $r\mapsto g^r$ a meromorphic family of renormalizations also defined on Ω. Then, the map $r\mapsto g^r\cdot\omega^r$ is a meromorphic family of Feynman measures associated with $r\mapsto\Delta^r$.* New Implicit

Proof. By the previous proposition, it suffices to show that $r \mapsto g^r \cdot \omega^r$ is a meromorphic family of Feynman measures. Taking into account that the inverse operation of $\mathscr{G}_i(M,E)$ sends meromorphic family of renormalizations to meromorphic families of renormalization by Fact 5.9.6, and using the explicit expression of the composition of the maps (5.8.6) and (5.8.7), we clearly see that $r \mapsto g^r \cdot \omega^r$ is a meromorphic family of Feynman measures. The proposition is thus proved. □

6.0.7. We now prove the first main result of this manuscript, which is precisely (the second part of) [10], Thm. 15. The proof there is however not completely clear to us (in particular the construction of the element of the group of renormalizations done in the last paragraph), so we prove it completely. Even though our methods are inspired by the proof in [10], there are several steps that are rather different, or at least that have not been mentioned by Borcherds.

New **6.0.8 Theorem.** *Let Δ be a local precut propagator. The action of the group $\mathscr{G}_i(M,E)$ on the set $\mathscr{F}_\Delta$ is free and transitive.*

Proof. Let $\omega, \omega' \in \mathscr{F}_\Delta$. We will show that there exists a unique $g \in \mathscr{G}_i(M,E)$ such that $g^{-1} \cdot \omega = \omega'$. Since $\omega|_{\tilde{S}^0\mathscr{L}_{i,c}(M,E)} = \omega'|_{\tilde{S}^0\mathscr{L}_{i,c}(M,E)}$ by definition of Feynman measure, we assume that $\omega|_{\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)} = \omega'|_{\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)}$, for all $m' = 0,\dots,m-1$ and some $m \in \mathbb{N}$. We will thus prove that there exists a unique $g_{-m+1} \in \mathscr{G}_{i,m-1}(M,E)$ such that $g_{-m+1} \cdot \omega$ and ω' coincide on $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ for all $m' = 0,\dots,m$. The theorem clearly follows from this statement together with Lemmas 5.8.9 and 5.8.11. Define the map $\tilde{\omega} = (\omega - \omega')|_{\tilde{S}^m\mathscr{L}_{i,c}(M,E)}$ if $m > 1$, and $\tilde{\omega} = \omega'|_{\mathscr{L}_{i,c}(M,E)}$, if $m = 1$.

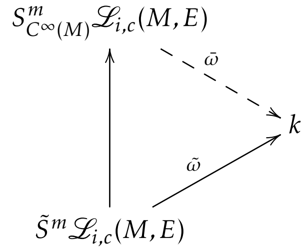
We claim that the hypothesis $\omega|_{\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)} = \omega'|_{\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)}$, for all $m' = 0,\dots,m-1$, tells us that $\rho_{\tilde{\omega}}(L) = \rho_\omega(L) - \rho_{\omega'}(L)$ vanishes for any $L \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$ whose support has trivial intersection with the diagonal of M^m if $m > 1$. We first note that it suffices to prove the previous statement for $L \in S^m\mathscr{L}_{i,c}(M,E)$, because $S^m\mathscr{L}_{i,c}(M,E)$ is dense in $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$. Now, given any point $\bar{p} = (p_1,\dots,p_m) \in \operatorname{supp}(L) \subseteq M^m \setminus \mathrm{Diag}_m$, (5.1.2) tells us that there exists $(J',J'') \in \mathrm{Par}(m,2)$ such that $\bar{p} \in U_{J',J''}$. We assume without loss of generality that $J' = \{1,\dots,m'\}$ and $J'' = \{m'+1,\dots,m\}$ for some $m' \in \mathbb{N}$ satisfying that $m' < m$. Let $m'' = m - m'$. By the definition of the product topology and the fact that $U_{J',J''}$ is open, there exist open sets $U' \subseteq M^{m'}$ and $U'' \subseteq M^{m''}$ such that $\bar{p} \in U' \times U'' \subseteq U_{J',J''}$. Furthermore, by choosing a refinement of the covering of $M^m \setminus \mathrm{Diag}_m$ given in (5.1.3) by open sets of the form $\cup_{\varsigma \in \mathbb{S}_m} \varsigma(U' \times U'')$ where U' and U'' are as before, and a partition of unity subordinate to it formed by smooth symmetric functions $f_{U',U''} \in \tilde{\Sigma}^m C^\infty(M)$, we may assume that the support of L is included in $\cup_{\varsigma \in \mathbb{S}_m} \varsigma(U' \times U'')$, for some open sets U' and U'' as before. Moreover, since the canonical map $S^{m'}\mathscr{L}_{i,c}(M,E) \otimes S^{m''}\mathscr{L}_{i,c}(M,E) \to S^m\mathscr{L}_{i,c}(M,E)$ is surjective, we may even assume that $L = \sum_{j \in J''} L'_j L''_j$, with J'' a finite set of indices, $L'_j \in \tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ and $L''_j \in \tilde{S}^{m''}\mathscr{L}_{i,c}(M,E)$ such that L'_j and L''_j have support included in U' and U'', respectively, and they satisfy the hypotheses of Definition 5.7.9, (3). By inductive hypothesis, $\rho_{\tilde{\omega}}(L'_j)$ and $\rho_{\tilde{\omega}}(L''_j)$ vanish for all $j \in J''$, so (5.7.8) tells us that $\rho_{\tilde{\omega}}(L) = 0$, as was to be shown. We have in particular thus proved that the support of the distribution $\rho_{\tilde{\omega}}(L)$ is included in the diagonal Diag_m of M^m for all $L \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$, if $m > 1$.

The theorem is now a consequence of the following lemma. Since we shall use the following result in the sequel, we will present it separately. □

New **6.0.9 Lemma.** *Let ω be a Feynman measure associated with the local precut propagator Δ. Let $m \in \mathbb{N}$ be a positive integer, and let $\tilde{\omega} : \tilde{S}^m\mathscr{L}_{i,c}(M,E) \to k$ be a continuous linear map such that the support of the associated distribution $\rho_{\tilde{\omega}}(L)$ is included in the diagonal Diag_m of M^m for all $L \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$ and it satisfies conditions (1) and (2) of Definition 5.7.9. Set $\omega' = \tilde{\omega}$ if $m = 1$, and $\omega' = \omega - \tilde{\omega}$ if $m \geq 2$. Then there is a unique element $g_{-m+1} \in \mathscr{G}_{i,m}(M,E)$ such that $g_{-m+1} \cdot \omega$ and ω' coincide on $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ for all $m' = 0,\dots,m$.*

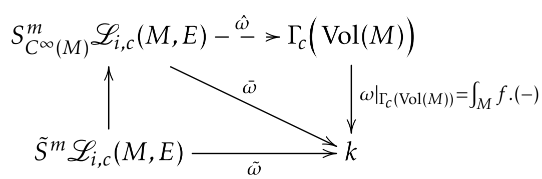
Proof. We first claim that $\tilde{\omega}$ factors through the canonical projection $\tilde{S}^m\mathscr{L}_{i,c}(M,E) \to S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$, *i.e.*

we have the commutative diagram



This is trivial if $m = 1$, so we will assume that $m > 1$. In order to prove so, it suffices to show that $\tilde{\omega}$ vanishes on any element of the form $L = \ell_1 g \circledS \ell_2 \circledS \dots \circledS \ell_m - \ell_1 \circledS g\ell_2 \circledS \dots \circledS \ell_m$, where $\ell_j \in \mathscr{L}_{i,c}(M,E)$ and $g \in C^\infty(M)$. We may assume that all the elements ℓ_j are in fact sections of the same vector bundle F over M, constructed as a finite direct sum of some of the bundles $\{S^\ell J^i E\}_{\ell\in\mathbb{N}_0}$. It is a straightforward but rather tedious computation that the restrictions to the diagonal Diag_m of all partial derivatives of L with respect to a trivializing covering of the bundle $F^{\boxtimes m}$ vanish. By Lemma 4.1.12, the $F^{\boxtimes m}$-valued distribution $\tilde{\omega}_{m,F}$ associated to $\tilde{\omega}$ and introduced in 5.7.4 vanish on L, so $\tilde{\omega}(L) = 0$ as was to be shown.

We now prove that there is a unique $C^\infty(M)$-linear map $\hat{\omega} : S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E) \to \Gamma_c(\mathrm{Vol}(M))$ making the diagram

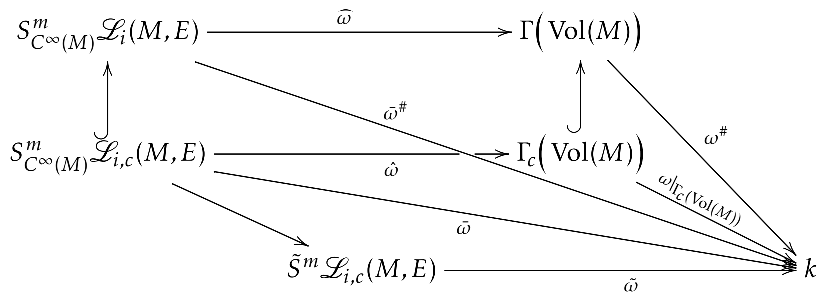


(6.0.11)

commute, where $f \in C^\infty(M)$ is the nonvanishing function of the condition (2) of Definition 5.7.9 for ω (or ω'). This is trivially achieved by taking into account that $S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$ is a projective $C^\infty(M)$-module (by Corollary 2.2.3), so a direct summand of a free $C^\infty(M)$-module X with basis $\{x_j\}_{j\in J}$. Indeed, extend $\bar{\omega}$ to a linear map $\tau : X \to k$, and then define a unique morphism $\tau' : X \to \Gamma_c(\mathrm{Vol}(M))$ of $C^\infty(M)$-modules sending x_j to an inverse image under the right vertical map of the previous diagram of $\tau(x_j)$. This is possible because the mentioned right vertical map is clearly surjective. The map $\hat{\omega}$ is defined as the restriction of τ' to $S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$. To prove the uniqueness of $\hat{\omega}$, suppose that two possible $\hat{\omega}'$ and $\hat{\omega}''$ make the previous diagram commute. The image of the difference $\hat{\omega}' - \hat{\omega}''$, which is a $C^\infty(M)$-submodule of $\Gamma_c(\mathrm{Vol}(M))$, is thus in the kernel of the integration map given by $\omega|_{\Gamma_c(\mathrm{Vol}(M))}$. If this image is nontrivial, there is a nonzero element $\theta \in \Gamma_c(\mathrm{Vol}(M))$ such that $\omega(\theta) = \int_M f\theta = 0$. Since there is $p \in M$ such that $\theta(p) \neq 0$, we may choose a small chart (U,ϕ) of M such that $p \in U$ and $\theta|_U$ is a nonvanishing function on U times the absolute value of the canonical top form associated to ϕ. Choose any nonnegative function $g \in C^\infty(M)$ such that $g(p) \neq 0$ and $\mathrm{supp}(g) \subseteq U$. Hence, $g\theta \neq 0$ is an element of the image of $\hat{\omega}' - \hat{\omega}''$, so we must have $\omega(g\theta) = 0$, but on the other hand $\omega(g\theta) = \int_M fg\theta \neq 0$, which is a contradiction. By Corollary 2.3.10, $\hat{\omega}$ is automatically a morphism of bornological locally convex $C^\infty(M)$-modules.

We will now show that $\hat{\omega}$ is the image under (5.8.6) of a $C^\infty(M)$-linear map $\widehat{\omega} : S^m_{C^\infty(M)}\mathscr{L}_i(M,E) \to \Gamma(\mathrm{Vol}(M))$. The injectivity of (5.8.6) (see Proposition 2.2.5) tells us that this element is unique. By the Hahn-Banach theorem, we can consider continuous linear functionals $\bar{\omega}^\# : S^m_{C^\infty(M)}\mathscr{L}_i(M,E) \to k$ and $\omega^\# : \Gamma(\mathrm{Vol}(M)) \to k$ extending $\bar{\omega}$ and $\omega|_{\Gamma_c(\mathrm{Vol}(M))}$, respectively. Since $S^m_{C^\infty(M)}\mathscr{L}_i(M,E)$ is also a projective $C^\infty(M)$-module (due to Theorem 2.1.3), by the same argument as the one given in the previous paragraph for $S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$ and $\Gamma_c(\mathrm{Vol}(M))$ we see that there is $C^\infty(M)$-linear map $\widehat{\omega} : S^m_{C^\infty(M)}\mathscr{L}_i(M,E) \to \Gamma(\mathrm{Vol}(M))$

such that $\omega^\# \circ \widehat{\omega} = \bar{\omega}^\#$. This implies that the diagram



(6.0.12)

commutes with the possible exception of the back vertical square, where the upward vertical maps are the canonical inclusions. Let us prove that this square also commutes. This is tantamount to the fact that the image of $\widehat{\omega}$ under (5.8.6) is $\hat{\omega}$. Consider the restriction $\hat{\omega}' = \widehat{\omega}|_{S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)}$. By the fact that (2.2.4) is an isomorphism, we see that the image of $\hat{\omega}'$ lies inside of $\Gamma_c(\mathrm{Vol}(M))$. Furthermore, if we replace $\hat{\omega}$ by $\hat{\omega}'$ in in (6.0.12), the back vertical square commutes by definition. Since the vertical triangles are commutative, we see also that $\omega|_{\Gamma_c(\mathrm{Vol}(M))} \circ \hat{\omega}' = \bar{\omega}$. Taking into account that $\hat{\omega}$ was the unique morphism of $C^\infty(M)$-modules satisfying this property, we conclude that $\hat{\omega} = \hat{\omega}'$, and the diagram (6.0.12) is commutative. By Corollary 2.3.6, $\widehat{\omega}$ is automatically a morphism of bornological locally convex $C^\infty(M)$-modules.

Using Proposition 5.8.5, set g_{-m+1} as the unique element of $\mathscr{G}_{i,m}(M,E)$ whose image under (5.8.1) is

$$(\mathrm{id}_{\Gamma(\mathrm{Vol}(M))} \otimes_{C^\infty(M)} \epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}) \circ \pi_1 + \widehat{\omega} \circ \pi_m,$$

if $m > 1$, and whose inverse is given by $\widehat{\omega} \circ \pi_1$, if $m = 1$, where $\pi_m : S_{C^\infty(M)}\mathscr{L}_{i,c}(M,E) \to S^m_{C^\infty(M)}\mathscr{L}_{i,c}(M,E)$ is the canonical projection. It is clear that $g_{-m+1} \cdot \omega$ and ω' coincide on $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ for all $m' = 0,\dots,m-1$. Finally, the commutativity of the diagram (6.0.11) is equivalent to the fact that $g_{-m+1} \cdot \omega$ and ω' coincide on $\tilde{S}^m\mathscr{L}_{i,c}(M,E)$. The lemma is thus proved. □

6.0.10.*Remark.* Note that the previous theorem does not need to assume any kind of homogeneity on the distributions involved nor any finiteness condition with respect to any scaling degree, in contrast to what is typically the case in pQFT (*cf.* [19, 97]).

New **6.0.11 Lemma.** *Let $r \mapsto \omega^r$ be a meromorphic family of Feynman measures associated with a meromorphic family of local precut propagator $r \mapsto \Delta^r$ defined on an open set $\Omega \subseteq \mathbb{C}$. Let $m \in \mathbb{N}$ be a positive integer, and let $r \mapsto \tilde{\omega}^r$ be a meromorphic function defined on Ω such that $\tilde{\omega}^r : \tilde{S}^m\mathscr{L}_{i,c}(M,E) \to k$, the associated distribution $\rho_{\tilde{\omega}^r}(L)$ is included in the diagonal Diag_m of M^m for all $L \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$ and it satisfies conditions (1) and (2) of Definition 5.7.9, for all $r \in \Omega$. Set $(\omega')^r = \tilde{\omega}^r$ if $m = 1$, and $(\omega')^r = \omega^r - \tilde{\omega}^r$ if $m \geq 2$. Then the map $r \mapsto g^r_{-m+1} \in \mathscr{G}_{i,m}(M,E)$, where g^r_{-m+1} is the unique element constructed in Lemma 6.0.9 such that $g^r_{-m+1} \cdot \omega^r$ and $(\omega')^r$ coincide on $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ for all $m' = 0,\dots,m$, is a meromorphic family of renormalizations.*

The proof follows by noting that each step in the proof of Lemma 6.0.9 clearly preserves the corresponding meromorphic property.

Chapter 7

The second main result: The existence of a Feynman measure associated with a manageable local propagator of cut type

7.1 The Bernstein-Sato polynomial and extensions of distributions

Well-known with new stuff

7.1.1. For this section we refer the reader to the nice exposition [43], and the references therein. We only recall that the N-th *Weyl algebra* $A_N(k)$ over the field k is the ring of polynomial differential operators with coefficients in $k[x_1,\dots,x_N]$, *i.e.* any element of $A_N(k)$ is a finite sum of the form $\sum_{\alpha\in\mathbb{N}_0^n} f_\alpha \partial^\alpha$, where $f_\alpha \in k[x_1,\dots,x_N]$ and we are using a similar notation to that in (2.3.1) for the partial derivatives. We refer the reader to [63], Ch. 8, for more details.

7.1.2 Theorem. *Let $f_1,\dots,f_m$ be $m \in \mathbb{N}$ elements in the ring $k[x_1,\dots,x_N]$ of polynomials in N indeterminates. Then, given any $j = 1,\dots,m$, there exist a nonzero polynomial $b_j(s_1,\dots,s_m) \in k[s_1,\dots,s_m]$ in m indeterminates and a polynomial $P_j(s_1,\dots,s_m) \in A_N(k)[s_1,\dots,s_m]$ such that*

$$b_j(s_1,\dots,s_m) f_1^{s_1}\dots f_m^{s_m} = P_j(s_1,\dots,s_m) f_j f_1^{s_1}\dots f_m^{s_m}. \tag{7.1.1}$$

Proof. The proof given by I. Bernšteĭn in [7], Ch. I, can be adapted to this cover case as well, as stated in [10], Lemma 16. More precisely, one first proves that the module $k(s_1,\dots,s_m)[x_1,\dots,x_m,f^{-1}]f_1^{s_1}\dots f_m^{s_m}$ over the Weyl algebra $A_N(k(s_1,\dots,s_m))$ is of finite length (and, even, holonomic), where $f = f_1\dots f_m$. The proof of [63], Lemma 8.10, applies almost *verbatim* to show this, using the same definition of filtration with f in that reference equal to our previously defined polynomial f. Finally, to prove (7.1.1), one considers the decreasing filtration $\{A_N(k(s_1,\dots,s_m))f_j^\ell f_1^{s_1}\dots f_m^{s_m}\}_{\ell\in\mathbb{N}_0}$ of submodules of $k(s_1,\dots,s_m)[x_1,\dots,x_m,f^{-1}]f_1^{s_1}\dots f_m^{s_m}$ over the Weyl algebra $A_N(k(s_1,\dots,s_m))$ and follows the argument given in the proof of [25], Thm. 10.3.3. □

7.1.3. Given any $j = 1,\dots,m$, the set of polynomials $b_j(s_1,\dots,s_m) \in k[s_1,\dots,s_m]$ satisfying that there exists another polynomial $P_j(s_1,\dots,s_m) \in A_N(k)[s_1,\dots,s_m]$ such that (7.1.1) holds is clearly an ideal of $k[s_1,\dots,s_m]$, called the *j-th Bernstein-Sato ideal* $\mathcal{B}^j_{f_1,\dots,f_m}$ of $f_1,\dots,f_m$. If $m = 1$ and we write $f = f_1$, the ideal $\mathcal{B}_f \subseteq k[s]$ has a unique monic generator (for $k[s]$ is a principal ideal domain), which is denoted by $b_f(s)$ and it is called the *Bernstein-Sato polynomial* of f.

7.1.4. More generally, let $\mathcal{O}$ be the ring of germs of analytic functions at the origin of k^N and let $f_1,\dots,f_m \in \mathcal{O}$. One may wonder if, given any $j = 1,\dots,m$, (7.1.1) is satisfied for a nonzero polynomial $b_j(s_1,\dots,s_m) \in k[s_1,\dots,s_m]$ and a polynomial $P_j(s_1,\dots,s_m) \in \mathcal{D}[s_1,\dots,s_m]$, where $\mathcal{D}$ is the ring of differential operators with coefficients in $\mathcal{O}$. This is also true (at least if $m = 1$), but the proof is highly nontrivial (see [43], Section 5). Given analytic functions $f_1,\dots,f_m$ defined on the same open set, we will say that they have *global Bernstein-Sato polynomials* if, for each $j = 1,\dots,m$, (7.1.1) is fulfilled by the same polynomials $b_j(s_1,\dots,s_m)$ and $P_j(s_1,\dots,s_m)$.

7.1.5. We will need the following variation of the well-known result by I. Bernšteĭn on the extension of distributions given by the existence of Bernstein-Sato polynomials (see for instance [43], Prop. 3.1 and Cor. 3.2).

New **7.1.6 Proposition.** *Let $g : W \to \mathbb{C}$ be a nonzero holomorphic function defined on $W = U + \sqrt{-1}C$, where $U \subseteq \mathbb{R}^N$ is an open set and $C \subseteq \mathbb{R}^N$ is an open convex cone, such that $g(W) \cap \sqrt{-1}\mathbb{R}_{\leq 0} = \emptyset$, and suppose that the distributional boundary value $g_0 = \mathrm{bv}(g)$ exists. Assume that g admits a global Bernstein-Sato polynomial $b_g(s)$, and let $P(s) = \sum_{\alpha \in \mathbb{N}_0^n} p_\alpha(s) f_\alpha \partial^\alpha$ be the differential operator fulfilling (7.1.1) together with $b_g(s)$. Consider here the unique logarithm defined on the set $\mathbb{C} \setminus \sqrt{-1}\mathbb{R}_{\leq 0}$. Then,*

(a) given any $s \in \mathbb{C}$, the map $g^s : W \to \mathbb{C}$ sending $p \in W$ to $\exp(s \log(g(p)))$ is well-defined and holomorphic, and its boundary value $g_0^s = \mathrm{bv}(g^s)$ also exists for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq 0$;

(b) the map $s \mapsto g_0^s$ is a holomorphic function from $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ to $C_c^\infty(U)'$, and the derivative of g_0^s at s is the boundary value of $g^s \log(g)$;

(c) the distribution $g_0^s = \mathrm{bv}(g^s)$ satisfies the functional equation

$$b_g(s) g_0^s = P(s)|_U g_0^{s+1} \tag{7.1.2}$$

for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$, where $P(s)|_U = \sum_{\alpha \in \mathbb{N}_0^n} p_\alpha(s) f_\alpha|_U \partial|_U^\alpha$, and $\partial|_U$ is the real part of ∂.

Hence, there is a unique distribution valued meromorphic function $\bar{g}^s : \mathbb{C} \setminus Z \to C_c^\infty(U)'$ that extends g^s, where Z is included in $A - \mathbb{N}_0$, and A is a finite subset of $\mathbb{Q}_{<0}$ given by the zeros of the Bernstein-Sato polynomial of g. Moreover, the order of each pole $s_0 \in Z$ of g^s is equal to the number $N(b_g, s_0)$ of roots $a \in A$ of $b_g(s)$ such that s_0 lies in the set $A - \mathbb{N}_0$, and it is thus less than or equal to the degree of b_g.

Proof. The fact that the map $g^s : W \to \mathbb{C}$ is holomorphic and well-defined is immediate. The existence of $g_0^s = \mathrm{bv}(g^s)$ is a direct consequence of Theorem 4.6.2. Indeed, by item (iii) of that result and the existence of $g_0 = \mathrm{bv}(g)$, there exist positive numbers $D, \epsilon > 0$ and a nonnegative integer $m \in \mathbb{N}_0$ such that

$$\left| g(x + y\sqrt{-1}) \right| \leq D \|y\|_\infty^{-m},$$

for all $x \in U'$ and for all $y \in C'$ satisfying that $\|y\|_\infty < \epsilon$, where U' is an open bounded set whose closure is included in U and $C' \subseteq C$ is a closed cone. Hence, given any fixed $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq 0$, we have that

$$\left| g^s(x + y\sqrt{-1}) \right| \leq D^{\mathrm{Re}(s)} e^{2\pi |\mathrm{Im}(s)|} \|y\|_\infty^{-m\mathrm{Re}(s)},$$

for all $x \in U'$ and for all $y \in C'$ satisfying that $\|y\|_\infty < \epsilon$, and the existence of $g_0^s = \mathrm{bv}(g^s)$ follows.

For the proof of item (b), note first that, given any $y \in C$, the derivative of the map $s \mapsto g^s((-) + \sqrt{-1}y)$ from $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ to $C_c^\infty(U)'$ is just $g^s((-) + \sqrt{-1}y)\log(g((-) + \sqrt{-1}y))$, which trivially satisfies the inequality of (iii) in Theorem 4.6.2. Indeed, for $s > 0$, take $0 < s' < s$, and re-write the previous map as $g^{s-s'}((-) + \sqrt{-1}y) g^{s'}((-) + \sqrt{-1}y)\log(g((-) + \sqrt{-1}y))$, for which inequality (4.6.3) clearly holds. Hence, its boundary value exists. Let $s_0 \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$ is a fixed complex. Since the map $g^s((-) + \sqrt{-1}y)$ is uniformly bounded on a neighborhood of s_0, the limit

$$\lim_{\substack{y \to 0 \\ y \in C'}} g^s\Big((-) + \sqrt{-1}y\Big)$$

exists uniformly (for the strong topology) on a neighborhood of s_0. By the Moore-Osgood theorem we have thus

$$\lim_{s\to s_0}\lim_{\substack{y\to 0\\ y\in C'}} \frac{g^s\Big((-)+\sqrt{-1}y\Big)-g^{s_0}\Big((-)+\sqrt{-1}y\Big)}{s-s_0}$$
$$=\lim_{\substack{y\to 0\\ y\in C'}}\lim_{s\to s_0} \frac{g^s\Big((-)+\sqrt{-1}y\Big)-g^{s_0}\Big((-)+\sqrt{-1}y\Big)}{s-s_0}=\mathrm{bv}\Big(g^s\log(g)\Big),$$

as was to be shown.

The proof of item (c), is immediate, for (7.1.2) is the boundary value of the corresponding functional equation (7.1.1) for the complex map g^s, and the corresponding derivatives commute with taking boundary value (see [33], Problem 12.14.(viii)). The last statement is a direct consequence of (7.1.2), as in the proof of the classical theorem of Bernstein explained in [43], Prop. 3.1 and Cor. 3.2. □

7.1.7. We will finally state the next result, whose proof is similar to that of the previous proposition (see else [27], Thm. 1.2, for a proof based on a clever procedure established by M. Atiyah for the analytic continuation of distributions using Hironaka's resolution of singularities), and that will be used in the construction of a Feynman measure. We will first explain the elements needed in the statement.

7.1.8. Let $g_1,\dots,g_m : U \to \mathbb{C}$ be nonzero real polynomial maps defined on an open set $U \subseteq \mathbb{R}^N$. Let $\tilde{g}_1,\dots,\tilde{g}_m : \mathbb{C}^N \to \mathbb{C}$ be the unique polynomials extending them, respectively. Given any $\epsilon > 0$, define the open subset

$$W_\epsilon = \bigcap_{j=1}^{m}(\tilde{g}_j+\sqrt{-1}\epsilon)^{-1}(\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0})$$

of $\mathbb{C}^N$. For $s_1,\dots,s_m \in \mathbb{C}$, the function $\prod_{j=1}^m (g_j+\sqrt{-1}\epsilon)^{s_j} : W_\epsilon \to \mathbb{C}$ is well-defined and holomorphic. Note that W_ϵ contains U and also the open subset

$$W = \bigcap_{j=1}^{m}\tilde{g}_j^{-1}(\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0})$$

of $\mathbb{C}^N$. Denote by

$$\prod_{j=1}^{m}(g_j+\sqrt{-1}0)^{s_j} \tag{7.1.3}$$

the distribution defined as the distributional limit of

$$\prod_{j=1}^{m}(g_j+\sqrt{-1}\epsilon)^{s_j} \tag{7.1.4}$$

as $\epsilon \to 0^+$, for $s_1,\dots,s_m \in \mathbb{C}$ such that $\mathrm{Re}(s_1),\dots,\mathrm{Re}(s_m) > 0$. Note that the previous limit exists, because (7.1.3) is the regular distribution given by the function $G(s_1,\dots,s_m)$ on U defined by

$$\prod_{j=1}^{m} g_j^{s_j} \tag{7.1.5}$$

on the open subset $\tilde{U}$ of U where all the factors are nonvanishing, and zero else (*cf.* [33], Problem 12.14.(x)). Let $b_j(s_1,\dots,s_m)$ be the j-th Bernstein-Sato polynomial of $g_1,\dots,g_m$, and $P_j(s_1,\dots,s_m)$ the differential operator $\sum_{\alpha\in\mathbb{N}_0^n} p_{j,\alpha}(s_1,\dots,s_m) f_{j,\alpha}\partial^\alpha$ fulfilling (7.1.1). This implies that, given $j = 1,\dots,m$, the function (7.1.5) satisfies the functional equation

$$b_j(s_1,\dots,s_m)\prod_{j'=1}^{m} g_{j'}^{s_{j'}} = P_j(s_1,\dots,s_m)\prod_{j'=1}^{m} g_{j'}^{s_{j'}+\delta_{j,j'}} \tag{7.1.6}$$

on $\tilde{U}$, for all $s_1,\dots,s_m \in \mathbb{C}$ such that $\mathrm{Re}(s_1),\dots,\mathrm{Re}(s_m) > \max\{\mathrm{ord}(P_j) : j = 1,\dots,m\}$. Since the subset of U where one of the factors of (7.1.5) vanishes has zero measure, for the set of zeros of any nonzero polynomial has zero measure (see [23]), (7.1.6) is valid for the corresponding regular distributions on U for all $s_1,\dots,s_m \in \mathbb{C}$ such that $\mathrm{Re}(s_1),\dots,\mathrm{Re}(s_m) > \max\{\mathrm{ord}(P_j) : j = 1,\dots,m\}$, *i.e.* it holds for (7.1.3). The case for all $s_1,\dots,s_m \in \mathbb{C}$ such that $\mathrm{Re}(s_1),\dots,\mathrm{Re}(s_m) > 0$ follows from analytic continuation. Incidentally, this proves item (b) of the following statement, provided item (a) holds.

New **7.1.9 Proposition.** *Let $g_1,\dots,g_m : U \to \mathbb{C}$ be nonzero real polynomials defined on an open set $U \subseteq \mathbb{R}^n$. Let $b_j(s_1,\dots,s_m)$ be the j-th Bernstein-Sato polynomial of $g_1,\dots,g_m$, and $P_j(s_1,\dots,s_m)$ be the differential operator $\sum_{\alpha\in\mathbb{N}_0^n} p_{j,\alpha}(s_1,\dots,s_m) f_{j,\alpha}\partial^\alpha$ fulfilling (7.1.1). Consider here the unique logarithm defined on the set $\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0}$. Then,*

(a) the map $(s_1,\dots,s_m) \mapsto \prod_{j=1}^m (g_j + \sqrt{-1}0)^{s_j}$ gives a holomorphic function from $\{(s_1,\dots,s_m) \in \mathbb{C}^m : \mathrm{Re}(s_j) > 0, \forall j\}$ to $C_c^\infty(U)'$, and

$$\frac{d^{k_1}}{ds_1^{k_1}}\cdots\frac{d^{k_m}}{ds_m^{k_m}}\Big(\prod_{j=1}^m (g_j + \sqrt{-1}0)^{s_j}\Big) = \prod_{j=1}^m \log^{k_j}(g_j + \sqrt{-1}0)(g_j + \sqrt{-1}0)^{s_j}, \tag{7.1.7}$$

where the right member denotes the limit of the sequence of distributions

$$\prod_{j=1}^m \log^{k_j}(g_j + \sqrt{-1}\epsilon)(g_j + \sqrt{-1}\epsilon)^{s_j}$$

as $\epsilon \to 0^+$;

(b) given $j = 1,\dots,m$, the distribution (7.1.3) satisfies the functional equation

$$b_j(s_1,\dots,s_m)\prod_{j'=1}^m (g_{j'} + \sqrt{-1}0)^{s_{j'}} = P_j(s_1,\dots,s_m)\prod_{j'=1}^m (g_{j'} + \sqrt{-1}0)^{s_{j'}+\delta_{j,j'}}, \tag{7.1.8}$$

for all $s_1,\dots,s_m \in \mathbb{C}$ such that $\mathrm{Re}(s_1),\dots,\mathrm{Re}(s_m) > 0$.

Hence, there is a unique distribution valued holomorphic function

$$\prod_{j=1}^m (g_j + \sqrt{-1}0)^{s_j} : \prod_{j=1}^m (\mathbb{C}\setminus Z_j) \to C_c^\infty(U)' \tag{7.1.9}$$

that extends $\prod_{j=1}^m (g_j + \sqrt{-1}0)^{s_j}$, where Z_j is included in $A_j - \mathbb{N}_0$, and A_j is a finite subset of $\mathbb{Q}_{<0}$ given by the zeros of the j-th Bernstein-Sato polynomial of $g_1,\dots,g_m$. The same holds for (7.1.7).

7.1.10. We will generically call the distribution valued holomorphic extension of (7.1.3) the *almost boundary value* of the product powers of $g_1,\dots,g_m$.

New **7.1.11 Corollary.** *Assume the same hypotheses of the previous proposition. Consider the extension of the distribution valued function (7.1.7), with the same domain as (7.1.9), and let $(s_1^0,\dots,s_m^0) \in Z_1\times\cdots\times Z_m$. Define the distribution valued function*

$$\prod_{j=1}^m \log^{k_j}(g_j + \sqrt{-1}0)(g_j + \sqrt{-1}0)^{s_j^0+s} : \mathbb{C}\setminus Z \to C_c^\infty(U)', \tag{7.1.10}$$

where Z is the subset of $\mathbb{C}$ formed by the elements s such that $(s_1^0+s,\dots,s_m^0+s) \notin Z_1\times\cdots\times Z_m$. Then (7.1.10) is holomorphic on $\mathbb{C}\setminus Z$, Z is discrete, and every element of Z is a pole. Moreover, the order of each pole s_0 of (7.1.10) is less than or equal to a fixed number that depends only on $(s_1^0,\dots,s_m^0)$, s_0 and each of the j-th Bernstein-Sato polynomial of $g_1,\dots,g_m$ for all $j = 1,\dots,m$.

This result is a clear consequence of Proposition 7.1.9 and in particular of the functional equations (7.1.8).

7.2 The existence of Feynman measures in the continuous case

7.2.1. The contents of this section are somehow implicit in the exposition [10], and we surmise that they are essentially well-known in the physics community, even though probably not in this form. We provide them for completeness.

7.2.2. We recall that a propagator $\Delta \in \mathrm{Prop}_i(M,E)$ is said to be *continuous* (resp., *smooth*) if the $(J^iE \boxtimes J^iE)^*$-valued distribution associated to it by means of (5.4.1) is regular (resp., smooth), *i.e.* it is induced by a continuous (resp., smooth) section of $J^iE \boxtimes J^iE$. Analogously, we say that a Feynman measure $\omega : \tilde{S}\mathscr{L}_{i,c}(M,E) \to k$ is *continuous* (resp., *smooth*) if for all $m \in \mathbb{N}$ and $[L] \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$, the $\mathrm{Vol}(M^m)$-valued distribution $\rho_\omega(L)$ is continuous (resp., smooth).

7.2.3. Given $m \in \mathbb{N}$, define the morphism of smooth manifolds

$$\mathrm{ddiag}_m : M^m \to \prod_{1\leq j<j'\leq m} (M\times M) \tag{7.2.1}$$

sending $\bar{p} = (p_1,\dots,p_m)$ to the tuple satisfying that $\mathrm{ddiag}_m(\bar{p})_{(j,j')} = (p_j, p_{j'})$ for all $1 \leq j < j' \leq m$. We will also use the terminology explained in Notation 5.1.15.

7.2.4 Definition. *Following* [10], *p. 633, we say that a continuous Feynman measure ω is said to be* constructed from a continuous Feynman propagator Δ_F *if the following conditions hold:* *Well-known?*

(1) there is a nowhere vanishing function $f \in C^\infty(M)$ such that the restriction of ω to $\mathscr{L}_{i,c}(M,E)$ is the linear mapping induced by

$$(\theta,\sigma) \mapsto \int_M \epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}(\sigma) f\theta, \tag{7.2.2}$$

where $\theta \in \mathrm{Vol}(M)$ and $\sigma \in S_{C^\infty(M)}\Gamma(J^iE)$;

(2) for all $m \in \mathbb{N}$, $\theta_1,\dots,\theta_m \in \Gamma_c(\mathrm{Vol}(M))$ and $\bar{\sigma}^1,\dots,\bar{\sigma}^m \in S_{C^\infty(M)}\Gamma(J^iE)$ we have

$$\rho_\omega(\theta_1\bar{\sigma}^1|\dots|\theta_m\bar{\sigma}^m) = \sum\Big(\prod_{j=1}^{m}{}^{\boxtimes}\rho_\omega(\theta_j\bar{\sigma}^j_{(1)})\Big)\mathrm{ddiag}_m^*\Big(\prod_{1\leq j<j'\leq m}{}^{\boxtimes}\tilde{\Delta}_F\big(\bar{\sigma}^j_{(j'-j+1)},\bar{\sigma}^{j'}_{(m-j+1)}\big)\Big), \tag{7.2.3}$$

where the sum is indexed over the different summands in the coproducts of $\bar{\sigma}^1,\dots,\bar{\sigma}^m \in S_{C^\infty(M)}\Gamma(J^iE)$, and the product between the two factors enclosed by parentheses is one of distributions on M^m.

7.2.5. Note that $\tilde{\Delta}_F$ (and also $\hat{\Delta}_F$) is a well-defined regular distribution, because Δ_F is, as well as the pull-backs appearing in (7.2.3). Furthermore, the product appearing in (7.2.3) is well-defined, for the left factor of the right member is a smooth distribution by (1), so in particular it is regular. Finally, the complete expression (7.2.3) is $\mathbb{S}_m$-equivariant, because Δ_F is symmetric and the coproduct of $S_{C^\infty(M)}\Gamma(J^iE)$ is cocommutative, so it defines a Feynman measure by Proposition 5.7.5.

7.2.6. *Remark.* It is easy to see that each of the expressions

$$\mathrm{ddiag}_m^*\Big(\prod_{1\leq j<j'\leq m}{}^{\boxtimes}\tilde{\Delta}_F\big(\bar{\sigma}^j_{(j'-j+1)},\bar{\sigma}^{j'}_{(m-j+1)}\big)\Big) \tag{7.2.4}$$

in (7.2.3) are in fact given by the internal product of $\mathrm{Vol}(M^m)$-valued distributions, each of whose factors is the pull-back of $\tilde{\Delta}_F(\bar{\sigma}^j_{(j'-j+1)},\bar{\sigma}^{j'}_{(m-j+1)})$ under the obvious projection $M^m \to M^2$ sending $(p_1,\dots,p_m) \in M^m$ to $(p_j,p_{j'})$.

New **7.2.7 Lemma.** *Let Δ_F be a continuous Feynman propagator satisfying that*

$$\mathrm{WF}_{(p,p)}(\Delta_F) \subseteq \Big\{(\lambda, -\lambda) : \lambda \in T_p^*M \setminus \{0\}\Big\}, \tag{7.2.5}$$

for all $p \in M$. Then the Feynman measure ω constructed from Δ_F satisfies condition (1) of Definition 5.7.9.

Proof. By Remark 7.2.6, the terms (7.2.4) in (7.2.3) are internal products of $\mathrm{Vol}(M^m)$-valued distributions, each of whose factors is the pull-back of $\tilde{\Delta}_F(\bar{\sigma}^j_{(j'-j+1)}, \bar{\sigma}^{j'}_{(m-j+1)})$ under the obvious projection $M^m \to M^2$. By (4.3.4), the wave front set at a point $(p,\dots,p) \in \mathrm{Diag}_m$ of each of these pull-backs is included in

$$\Big\{(\lambda_1, \dots, \lambda_m) : \lambda \in (T_p^*M \setminus \{0\})^m \text{ and } \lambda_1 + \cdots + \lambda_m = 0\Big\}. \tag{7.2.6}$$

Hence, by Theorem 4.5.6 we conclude that the wave front set at a point $(p,\dots,p) \in \mathrm{Diag}_m$ of (7.2.4) is included in (7.2.6), so the same holds for $\rho_\omega(\theta_1\bar{\sigma}^1|\dots|\theta_m\bar{\sigma}^m)$ given in (7.2.3). The lemma is thus proved. □

New **7.2.8 Lemma.** *Let Δ be a local propagator that is continuous, i.e. its associated $(J^iE \boxtimes J^iE)^*$-valued distribution by means of* (5.4.1) *is regular. Then, there is a unique continuous Feynman propagator Δ_F such that $\Delta_F|_{U_{\{1\},\{2\}}} = \Delta|_{U_{\{1\},\{2\}}}$.*

Proof. Since Δ is regular there is a continuous section $\Sigma \in \Gamma^0((J^iE \boxtimes J^iE)^*)$ such that $\Delta(\sigma,\sigma') = \int_{M^2} \langle \Sigma, \sigma \boxtimes \sigma' \rangle$, for all $\sigma, \sigma' \in \Gamma_c(\mathrm{Vol}(M) \otimes J^iE)$. As Δ is local, $\Sigma|_{U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}}$ is symmetric, *i.e.* $\Sigma(p,p') = \Sigma(p',p)$, for all $(p,p') \in U_{\{1\},\{2\}} \cap U_{\{2\},\{1\}}$. Let $\Sigma^F \in \Gamma^0((J^iE \boxtimes J^iE)^*)^{\mathbb{S}_2}$ be the continuous section given by

$$\Sigma^F(p,p') = \begin{cases} \Sigma(p,p'), & \text{if } (p,p') \in U_{\{1\},\{2\}} \cup \mathrm{Diag}_2, \\ \Sigma(p',p), & \text{if } (p,p') \in U_{\{2\},\{1\}} \cup \mathrm{Diag}_2. \end{cases} \tag{7.2.7}$$

This is clearly a well-defined symmetric continuous section of $(J^iE \boxtimes J^iE)^*$. Define the propagator Δ_F by $\Delta_F(\sigma,\sigma') = \int_{M^2} \langle \Sigma^F, \sigma \boxtimes \sigma' \rangle$, for all $\sigma, \sigma' \in \Gamma_c(\mathrm{Vol}(M) \otimes J^iE)$. It is direct that Δ_F is symmetric, so a Feynman propagator. Moreover, it is continuous, and by definition $\Delta_F(\sigma,\sigma') = \Delta(\sigma,\sigma')$ if $(\mathrm{supp}(\sigma) \times \mathrm{supp}(\sigma')) \subseteq U_{\{1\},\{2\}}$. The lemma is proved. □

7.2.9. Given a continuous local propagator Δ, the unique continuous Feynman propagator Δ_F satisfying that $\Delta_F|_{U_{\{1\},\{2\}}} = \Delta|_{U_{\{1\},\{2\}}}$ will be called the *continuous Feynman propagator associated with* Δ. It always exists by Lemma 7.2.8.

New **7.2.10 Definition.** *We say that a continuous local propagator Δ is* admissible *if the continuous Feynman propagator Δ_F associated with Δ satisfies that*

$$\mathrm{WF}_{(p,p)}(\Delta_F) \subseteq \Big\{(\lambda, -\lambda) : \lambda \in T_p^*M \setminus \{0\}\Big\}, \tag{7.2.8}$$

for all $p \in M$. This hypothesis does not appear in [10].

New **7.2.11 Lemma.** *Assume that the n-dimensional spacetime M is admissible (see Definition 5.1.5), and let Δ be a continuous local propagator with associated continuous Feynman propagator Δ_F. Then, Δ is admissible.*

Proof. Given any $p \in M$, consider a chart (U,ϕ) around p and multiply Δ (and thus Δ_F) by a smooth function f of compact support included in $U \times U$ such that $f(p,p) \neq 0$. Since this does not change the wave front of Δ_F, we may equivalently work with the regular distributions $(\phi^{-1} \times \phi^{-1})^\wedge(f\Delta)$ and $(\phi^{-1} \times \phi^{-1})^\wedge(f\Delta_F)$ on $\phi(U) \times \phi(U) \subseteq \mathbb{R}^{2n}$, that we will denote by Δ^U and Δ_F^U, respectively. Moreover, since M is admissible, we may even take the previous chart (U,ϕ) around p satisfying the conditions in Definition 5.1.5. Let $H \subseteq \mathbb{R}^{2n}$ be the hyperplane there and let H_+ be the closed half-space including $(\phi \times \phi)(\preceq \cap (U \times U))$. Let χ_+ be the characteristic function of H_+, and χ_- be the characteristic function of the complement of H_+. Then, $\Delta_F^U = \chi_+\Delta^U + \chi_-\Delta^U \circ \mathrm{fl}$, where $\mathrm{fl} : U \times U \to U \times U$ is the usual flip map. The result now follows from Theorem 4.5.6. □

7.2.12. *Remark.* The admissibility condition can also be immediately deduced from other assumptions in many situations of interest. For instance, in the case where M is the Minkowski space and the Feynman propagator Δ_F at a point $(p,q) \in M^2$ is assumed to depend only on the difference $p-q$, *i.e.* Δ_F is the pull-back of a (vector valued) distribution on M under the submersion $(p,q) \mapsto p-q$, then condition (7.2.8) is a direct consequence of (4.3.4).

7.2.13. We have the following statement, which is one of the main results of this section.

7.2.14 Theorem. *Consider a local precut propagator $\Delta \in \mathrm{Prop}_i(M,E)$ that is continuous, i.e. its associated $(J^iE \boxtimes J^iE)^*$-valued distribution by means of* (5.4.1) *is regular, and admissible, i.e. its associated Feynman propagator $\Delta_F \in \mathrm{Prop}_i(M,E)$ satisfies* (7.2.8)*. Then, there is a Feynman measure ω associated with Δ given explicitly by* (7.2.3). *New*

Proof. Taking into account that $\omega(1_S) = 1$, we have to define $\omega(L)$, for $[L] \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$ and $m \in \mathbb{N}$. Choose any nowhere vanishing function $f \in C^\infty(M)$, and define $\tau' : \mathrm{Vol}(M) \to k$ to be the map $\theta \mapsto \int_M \theta f$. Set $\tau : \mathscr{L}_{i,c}(M,E) \to k$ as the linear mapping induced by

$$(\theta,\sigma) \mapsto \int_M \epsilon_{S_{C^\infty(M)}\Gamma(J^iE)}(\sigma) f\theta, \tag{7.2.9}$$

where $\theta \in \mathrm{Vol}(M)$ and $\sigma \in S_{C^\infty(M)}\Gamma(J^iE)$. It is clearly continuous, and it satisfies conditions (1), (2) and (3) of Definition 5.7.9, for all $L \in k \oplus \mathscr{L}_{i,c}(M,E)$. Furthermore, (7.2.9) implies that the map $\rho_\tau(\ell)$ given by sending $f \in C^\infty(M)$ to $\tau(f\ell)$ is a smooth $\mathrm{Vol}(M)$-valued distribution for all $\ell \in \mathscr{L}_{i,c}(M,E)$.

We recall that Δ_F is the Feynman propagator defined in Lemma 7.2.8. Let $\omega : \tilde{S}\mathscr{L}_{i,c}(M,E) \to k$ be the unique continuous linear map satisfying that

$$\rho_\omega(\theta_1\bar{\sigma}^1|\dots|\theta_m\bar{\sigma}^m) = \sum\Big(\prod_{j=1}^{m}{}^{\boxtimes}\rho_\tau(\theta_j\bar{\sigma}^j_{(1)})\Big)\mathrm{ddiag}^*_m\Big(\prod_{1\le j<j'\le m}{}^{\boxtimes}\tilde{\Delta}_F\big(\bar{\sigma}^j_{(j'-j+1)},\bar{\sigma}^{j'}_{(m-j+1)}\big)\Big), \tag{7.2.10}$$

where we are using the same conventions as in (7.2.3). By definition, it is a Feynman measure constructed from the continuous Feynman propagator Δ_F. Since $\Delta_F(\sigma,\sigma') = \Delta(\sigma,\sigma')$ if $(\mathrm{supp}(\sigma)\times\mathrm{supp}(\sigma')) \subseteq U_{\{1\},\{2\}}$, an easy inductive argument tells us that condition (5.7.8) for ω and Δ holds. Hence, ω verifies condition (3) of Definition 5.7.9, and (7.2.9) implies that the same occurs with condition (2) of the mentioned definition. Condition (1) of Definition 5.7.9 is a direct consequence of Lemmas 7.2.7. The theorem is proved. □

7.3 The meromorphic family of Feynman measures

7.3.1. Let $\{(U_a,\tau_a,\phi_a)\}_{a\in A}$ be a family satisfying the conditions (a) and (b) in Definition 5.5.2. Since $\{U_a\}_{a\in A}$ is locally finite, let $\{f_a\}_{a\in A}$ be a partition of unity such that $f_a \in C^\infty(M\times M)$ whose support is included in U_a for all $a \in A$. Denote by Δ_a the restriction of Δ to $\Gamma_c(U_a,(\mathrm{Vol}(M)\otimes J^iE)\boxtimes(\mathrm{Vol}(M)\otimes J^iE))$. Hence,

$$\Delta = \sum_{a\in A} f_a\Delta = \sum_{a\in A} f_a\Delta_a. \tag{7.3.1}$$

7.3.2. By the hypothesis on Δ, $\tau_a^\wedge(\Delta_a)$ can be written as a finite sum

$$\sum_{j\in J}\sigma_{a,j}\underbrace{\mathrm{bv}\big(G_{a,j}\big)}_{G'_{a,j}}, \tag{7.3.2}$$

where $\sigma_{a,j}$ is a smooth vector function on U_a (see (5.5.1)). Define

$$H^{\bar{r}'}_{a,j} = p^{r_0}_{j,0}\prod_{\ell\in L_j}(p_\ell)^{r_\ell}, \tag{7.3.3}$$

for $\bar{r}'$ the complex vector composed of r_0 and $(r_\ell)_{\ell\in L_j}$, and where the polynomials $p_{j,0}$ and p_ℓ for $\ell \in L_j$ are those of (5.5.3). We should in principle write $(\bar{r}')^j$ instead of just $\bar{r}'$, to emphasize the dependence on the index j, but, as we will now see, this omission is justified. We may regard $H^{\bar{r}'}_{a,j}$ as a holomorphic function on an open subset $W_{a,j} \subseteq \mathbb{C}^{t_j+1}$. By choosing if necessary in the expression of $G_{a,j}$ powers of the identity polynomial 1 to new complex variables r, we may assume without loss of generality that the size t_j of the vector $\bar{s}$ indicated there does not depend on j (or on a), so we will denote it just by t. By Theorem 4.6.2 and the dominated convergence theorem, it is easy to see that the distributional boundary value of $H^{\bar{r}'}_{a,j}$ exists and it is a distribution valued holomorphic function of $\bar{r}'$. The derivative of $H^{\bar{r}'}_{a,j}$ with respect to r_0 at $\bar{r} = (0, \bar{s})$ is precisely $G_{a,j}$, and the same holds for the corresponding boundary values. It is clear that the distributional limit of $H^{\bar{r}'}_{a,j}$ is regular, *i.e.* it is given by a continuous function, for $\bar{r}'$ satisfying that $\mathrm{Re}(\bar{r}') \gg 0$.

7.3.3. On the other hand, by Proposition 7.1.6, each of the boundary values $\mathrm{bv}(H^{\bar{r}'}_{a,j})$ can be extended to a distribution valued holomorphic function defined for $\bar{r}' \in W' = (\mathbb{C}\setminus Z)^{t+1}$, where Z is included in $A - \mathbb{N}_0$ and A is a finite subset of $\mathbb{Q}_{<0}$. Set

$$\underline{\Delta}^{\bar{r}'} = \sum_{a\in A} f_a \sum_{j\in J} (\tau_a^{-1})^\wedge\Big(\underbrace{\sigma_{a,j}\mathrm{bv}\big(H^{\bar{r}'}_{a,j}\big)}_{(G'_{a,j})^{\bar{r}'}}\Big), \tag{7.3.4}$$

for all $\bar{r}' \in W' = (\mathbb{C}\setminus Z)^{t+1}$. It is clearly a distribution valued holomorphic function on $W' = (\mathbb{C}\setminus Z)^{t+1}$. Define $\Delta^{\bar{r}}$ as the derivative of $\underline{\Delta}^{\bar{r}'}$ with respect to r_0 at $\bar{r}' = (r_0, \bar{r})$ with $r_0 = 0$. We note that $\Delta^{\bar{r}}$ is thus defined for $\bar{r} \in W = (\mathbb{C}\setminus Z)^t$, for we have fixed the value of r_0 to be 0. Furthermore, since the domains of definition of the holomorphic functions $H^{\bar{r}'}_{a,j}$ do not change when $\bar{r}'$ varies, Proposition 4.6.4 implies that the wave front set of the corresponding boundary values satisfies the assumptions stated in item (f) of Definition 5.5.2, so both $\underline{\Delta}^{\bar{r}'}$ and $\Delta^{\bar{r}}$ fulfill all the conditions of a propagator of cut type with the same family of cones as Δ, for all $\bar{r}' \in W' = (\mathbb{C}\setminus Z)^{t+1}$ and $\bar{r} \in W = (\mathbb{C}\setminus Z)^t$, respectively, for the sum in (7.3.4) is locally finite. By the symmetry assumptions in Definition 5.5.2 (see (d)), both $\underline{\Delta}^{\bar{r}'}$ and $\Delta^{\bar{r}}$ are also local. We have thus obtained a holomorphic family of propagators on the open set $W' = (\mathbb{C}\setminus Z)^{t+1}$ and another on $W = (\mathbb{C}\setminus Z)^t$.

New **7.3.4 Definition.** *We will say that a local propagator of cut type Δ is* manageable *if the continuous propagator of cut type $\underline{\Delta}^{\bar{r}'}$ defined after* (7.3.4) *is admissible (see Definition 7.2.10) for all $\bar{r}'$ such that* $\mathrm{Re}(\bar{r}') \gg 0$, *and moreover for all $\bar{r}'$ such that* $\mathrm{Re}(\bar{r}') \gg 0$, *the associated Feynman propagator $\underline{\Delta}^{\bar{r}'}_F$ satisfies that*

$$\tau_a^\wedge\big(\underline{\Delta}^{\bar{r}'}_F\big) = \sum_{j\in J} \sigma_{a,j}\Big((p_{j,0} + \sqrt{-1}0)^{r_0} \prod_{\ell\in L_j} (p_\ell + \sqrt{-1}0)^{s_\ell + r_\ell}\Big), \tag{7.3.5}$$

for all $a \in A$, where $\sigma_{a,j}$ are the smooth functions in (7.3.2).

7.3.5.*Remark.* The manageability condition seems in our opinion to be a reasonable one to link the starting propagator of cut type Δ and its associated Feynman propagator. Even though it seems to be a strong restriction on the propagator Δ, it only involves a limit of continuous functions, which is in general easy to check, and that depend more on the tubes appearing in the expression (5.5.1) for Δ than its actual functional form. More precisely, the manageability condition is immediately satisfied for Δ if, for all $\bar{r}'$ such that $\mathrm{Re}(\bar{r}') \gg 0$, the limits of continuous functions

$$\lim_{\epsilon\to 0^+} \sum_{j\in J} \sigma_{a,j}\Big((p_{j,0}(\bar{x}_0) + \sqrt{-1}\epsilon)^{r_0} \prod_{\ell\in L_j} (p_\ell(\bar{x}) + \sqrt{-1}\epsilon)^{s_\ell + r_\ell}\Big)$$

and

$$\lim_{\epsilon\to 0^+} \sum_{j\in J} \sigma_{a,j}\Big((p_{j,0}(\bar{x} + \sqrt{-1}\bar{y}_j\epsilon))^{r_0} \prod_{\ell\in L_j} (p_\ell(\bar{x} + \sqrt{-1}\bar{y}_j\epsilon))^{s_\ell + r_\ell}\Big)$$

coincide, for all $a \in A$, $\bar{x} \in \phi_a(U_a \cap U_{\{1\},\{2\}})$ and $\bar{y}_j \in C_{a,j}$ (see Definition 5.5.2).

Let us remark that the manageability condition essentially expresses that the holomorphic family of continuous propagators extending the given local propagator of cut type and the associated Feynman propagators should be related in some manner. This relation is needed in order to control the forthcoming holomorphic extension. As for the admissibility condition, the hypothesis of manageability does not appear in [10]. We would also like to stress that if the spacetime M is admissible, then the admissibility condition of the continuous propagator of cut type $\underline{\Delta}^{\bar{r}'}$ appearing in Definition 7.3.4 is automatic (see Lemma 7.2.11).

7.3.6.*Example.* The manageability condition is satisfied in all the situations we have mentioned in Example 5.5.4. Indeed, it is satisfied by the Klein-Gordon scalar field theory on Minkowski spacetime. This follows from comparing the expression of the Wightman 2-point function given in [9], Ch. 8, App. F, (F.8b), (see also [87], (12)), and the expression of the Feynman propagator obtained by combining [39], 6.5, (6.48), and [28], (5.11) and (5.14). These formulas even show that the analytic continuation of the Feynman propagator $\underline{\Delta}_F^{\bar{r}'}$ in Definition 7.3.4 associated to the usual Wightman 2-point function is in fact the usual Feynman propagator considered in scalar field theory. Furthermore, since the components of the propagators for the Dirac theory on the Minkowski spacetime are given by the corresponding propagators of the scalar theory on the Minkowski spacetime (see [9], Ch. 8, App. F, (F.21)), we also see that propagators for the Dirac theory on the Minkowski spacetime are manageable. We also remark that the manageability condition is clearly fulfilled in the case of Klein Gordon scalar on de Sitter or anti-de Sitter spacetime mentioned in Example 5.5.4.

Concerning the scalar field theory on a globally hyperbolic classical spacetime, it seems that a weaker version of the manageability condition is assumed in several articles in the physical literature, were they suppose that (7.3.5) holds at least for open sets of the form $U \times U$, where U is a geodesically convex open set of M (see *e.g.* [42], item 1 on p. 14). It remains to be seen if the manageability condition is satisfied for other classical spacetimes.

7.3.7. Let us assume that the local propagator of cut type Δ is manageable, and denote by $\underline{\Delta}_F^{\bar{r}'}$ and $\Delta_F^{\bar{r}}$ the Feynman propagators associated with the continuous local propagator of cut type $\underline{\Delta}^{\bar{r}'}$ and $\Delta^{\bar{r}}$, respectively, for all $\bar{r}'$ and $\bar{r}$ such that $\mathrm{Re}(\bar{r}'), \mathrm{Re}(\bar{r}) \gg 0$. Since $\Delta^{\bar{r}}$ is the derivative of $\underline{\Delta}^{\bar{r}'}$ with respect to r_0 at $\bar{r}' = (r_0, \bar{r})$ with $r_0 = 0$, and the space of distributions with wave front set included in a fixed closed conic set is sequentially complete, we conclude that $\Delta^{\bar{r}}$ is also admissible for all $\bar{r}$ such that $\mathrm{Re}(\bar{r}) \gg 0$. By Theorem 7.2.14, there are Feynman measures $\underline{\omega}^{\bar{r}'}$ and $\omega^{\bar{r}}$ constructed from $\underline{\Delta}_F^{\bar{r}'}$ and $\Delta_F^{\bar{r}}$, respectively. Moreover, the same result tells us that $\underline{\omega}^{\bar{r}'}$ and $\omega^{\bar{r}}$ are associated with the local propagator of cut type $\underline{\Delta}^{\bar{r}'}$ and $\Delta^{\bar{r}}$, respectively, for all $\bar{r}'$ and $\bar{r}$ such that $\mathrm{Re}(\bar{r}'), \mathrm{Re}(\bar{r}) \gg 0$. It is also trivial to verify that, given any $[L] \in \tilde{S}\mathscr{L}_{i,c}(M,E)$, the map $\bar{r}' \mapsto \underline{\omega}^{\bar{r}'}([L])$ is a holomorphic function for all $\bar{r}'$ such that $\mathrm{Re}(\bar{r}') \gg 0$, and the same holds for $\omega^{\bar{r}}$.

7.3.8 Theorem. *Let Δ be a manageable local propagator of cut type and let $\bar{r} \mapsto \omega^{\bar{r}}$ be the holomorphic family of Feynman measures defined for all $\bar{r}$ such that* $\mathrm{Re}(\bar{r}) \gg 0$ *in the previous paragraph. It can be extended to a holomorphic family of Feynman measures for all $\bar{r} \in W = (\mathbb{C} \setminus Z)^t$, where Z is included in $A - \mathbb{N}_0$ and A is a finite subset of $\mathbb{Q}_{<0}$, such that, given any $[L] \in \tilde{S}\mathscr{L}_{i,c}(M,E)$, the map $\bar{r} \mapsto \omega^{\bar{r}}([L])$ is holomorphic on W. Furthermore, $\omega^{\bar{r}}$ is a Feynman measure associated with $\Delta^{\bar{r}}$ for all $\bar{r} \in W = (\mathbb{C} \setminus Z)^t$.* *New/Based on Borcherds*

This is the first main result of this section, which is parallel to [10], Thm. 18.

Proof. Define $\omega^{\bar{r}}(1_S) = 1$, for all $\bar{r} \in \mathbb{C}^t$. Moreover, by definition of the Feynman measure in Theorem 7.2.14, given $\ell \in \mathscr{L}_{i,c}(M,E)$, the value $\omega^{\bar{r}}(\ell)$ for all $\bar{r} \in \mathbb{C}^t$ such that $\mathrm{Re}(\bar{r}) \gg 0$, is independent of $\bar{r}$. This defines thus $\omega^{\bar{r}}(\ell)$ for all $\bar{r} \in \mathbb{C}^t$ and $\ell \in \mathscr{L}_{i,c}(M,E)$. Conditions (1) for $\omega^{\bar{r}}|_{\mathscr{L}_{i,c}(M,E)}$ and (2) of Definition 5.7.9 are satisfied by construction.

Let $[L] \in \tilde{S}^m\mathscr{L}_{i,c}(M,E)$ be a fixed element, for $m \in \mathbb{N}_{\geq 2}$. Since (7.2.1) is a morphism of smooth manifolds, there exists a collection of charts $\{(V_a^m, \tau_a^m)\}_{a \in A_m}$ of M^m such that $M^m = \cup_{a \in A_m} V_a^m$ is a locally finite covering, and that $\mathrm{ddiag}_m(V_a^m)$ is included in a chart of $\prod_{1 \leq j < j' \leq m}(M \times M)$ of the form $\prod_{1 \leq j < j' \leq m} U_{a_{j,j'}}$, for some $a_{j,j'} \in A$, for all $1 \leq j < j' \leq m$. By taking a partition of unity subordinated to $\{V_a^m\}_{a \in A_m}$, we can assume whitout loss of generality that the support of L is included in some V_a^m. Consider the distribution valued holomorphic function $\bar{r} \mapsto \omega^{\bar{r}}([L])$ for all $\bar{r}$ such that $\mathrm{Re}(\bar{r}) \gg 0$. Since Δ is manageable, the explicit expression of $\omega^{\bar{r}}([L])$ given in (7.2.3) is also a product of the almost boundary values of the logarithms and powers of the same real polynomials we considered for the propagator of cut type. By Proposition 7.1.9, the map $\bar{r} \mapsto \omega^{\bar{r}}([L])$ can be thus extended to a meromorphic map on $W = (\mathbb{C} \setminus Z)^t$, for some discrete set Z included in $A - \mathbb{N}_0$ and A is a finite subset of $\mathbb{Q}_{<0}$. Furthermore, Theorem 7.2.14 tells us that $\omega^{\bar{r}}$ satisfies the identity (5.7.8) given in Definition 5.7.9, (3), with respect to the local propagator of cut type $\Delta^{\bar{r}}$ for all $\bar{r}$ such that $\mathrm{Re}(\bar{r}) \gg 0$. By analytic continuation, it holds for all $\bar{r} \in W = (\mathbb{C} \setminus Z)^t$. On the other hand, for any $[L] \in \tilde{S}\mathscr{L}_{i,c}(M,E)$, Proposition 7.1.9 tells us that the distribution valued map $\bar{r} \mapsto \omega^{\bar{r}}([L])$ at a particular fixed value $\bar{r}_0 \in W$ is given as the successive application of a differential operator on $\omega^{\bar{r}}([L])$ for some $\bar{r}$ such that $\mathrm{Re}(\bar{r}) \gg 0$. The fact that the wave front set of the latter satisfies condition (1) of Definition 5.7.9 together with property (4.3.1) tell us that that $\omega^{\bar{r}}([L])$ also satisfies the mentioned condition. The theorem is thus proved. □

7.3.9. *Remark.* The proof of the previous theorem also extends to the more general situation in which one assumes that the functions $p_{j,0}$ and p_ℓ appearing in the local expression (5.5.3) of the manageable local propagator Δ of cut type (and also in (7.3.5)) are analytic, not only polynomials. In this case, the analytic continuation process is based on a more general version of the Bernstein-Sato theorem for analytic functions, due to C. Sabbah and R. Bahloul (see [92, 93] and [5]).

New/Based on Borcherds

7.3.10 Theorem. *We follow the same notations as in Theorem 7.3.8. Let Δ be a manageable local propagator of cut type, and $\bar{s}$ be the tuple formed by the exponents appearing in* (5.5.1). *Define the family of propagators $r \mapsto \Delta^r$ by $\Delta^r = \Delta^{(r,\dots,r)+\bar{s}}$, and the family of Feynman measures $r \mapsto \omega^r$ given by $\omega^r = \omega^{(r,\dots,r)+\bar{s}}$, where $r \in \mathbb{C} \setminus Z'$, with $Z' = \{r \in \mathbb{C} : r + s_\ell \notin Z, \forall \ell\}$. Then*

(1) Z' is discrete;

(2) $r \mapsto \Delta^r$ is a meromorphic family of local propagators of cut type that can extended to a holomorphic function on an open set containing 0;

(3) $r \mapsto \omega^r$ is a meromorphic family of Feynman measures associated with Δ^r for all $r \in \mathbb{C} \setminus Z'$.

This is a direct consequence of Theorem 7.3.8 and of Corollary 7.1.11.

7.4 The holomorphic family of Feynman measures

7.4.1. Consider the family of Feynman measures ω^r, where $r \in \mathbb{C} \setminus Z'$, and $\omega^r([L])$ is a holomorphic function on r, for all $[L] \in \tilde{S}\mathscr{L}_{i,c}(M,E)$. We have the main result of this section, which is parallel to [10], Thm. 20. The proof is essentially an expansion of the one sketched there.

New/Based on Borcherds

7.4.2 Theorem. *Let $r \mapsto \omega^r$ be the meromorphic family of Feynman measures associated with the family of propagators Δ^r given in Theorem 7.3.10 for all $r \in \mathbb{C} \setminus Z'$. Then, there is a meromorphic family of renormalizations $r \mapsto g^r$ for all $r \in \mathbb{C} \setminus Z'$, such that the meromorphic family of Feynman measures $r \mapsto (g^r \cdot \omega^r)$ associated with the family of propagators Δ^r and defined for $r \in \mathbb{C} \setminus Z'$ can be (uniquely) extended to a meromorphic family of Feynman measures $r \mapsto \tilde{\omega}^r$ associated with the family of propagators Δ^r but defined for all $r \in (\mathbb{C} \setminus Z') \cup \{0\}$.*

Proof. By the construction of ω^r, we see that $\rho_{\omega^r}|_{\mathscr{L}_{i,c}(M,E)}$ is a smooth $\mathrm{Vol}(M)$-valued distribution for all $r \in \mathbb{C}$. By Theorem 7.3.10, (2), the family of propagators $r \mapsto \Delta^r$ can be uniquely extended to a holomorphic function on an open set including 0 (and $\mathbb{C} \setminus Z'$).

Given $m \geq 2$, let us denote the restriction operation of a $\mathrm{Vol}(M^m)$-valued distribution u to $M^m \setminus \mathrm{Diag}_m$ by $\mathrm{res}_m(u)$. For $m \geq 2$, assume that we have constructed holomorphic families of renormalizations $r \mapsto g^r_{-2}$, $\dots$, $r \mapsto g^r_{-m+1}$, with $g^r_{-j} \in \mathscr{G}_{i,j}(M,E)$ for all $j = 2,\dots,m-1$, such that the restriction of $(g^r_{-m+1} \cdots g^r_{-2}) \cdot \omega^r$ to $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ can be (uniquely) extended to a holomorphic function on $r = 0$, for $m' = 0,\dots,m-1$. We shall prove it for m. Let us denote $(g^r_{-m+1} \cdots g^r_{-2}) \cdot \omega^r$ by ω^r_{m-1}. By condition (3) of Definition 5.7.9, we see that the restriction $\mathrm{res}_m(\rho_{\omega^r_{m-1}}|_{\tilde{S}^m\mathscr{L}_{i,c}(M,E)})$ of the $\mathrm{Vol}(M^m)$-valued distribution $\rho_{\omega^r_{m-1}}|_{\tilde{S}^m\mathscr{L}_{i,c}(M,E)}$ to

$M^m \setminus \mathrm{Diag}_m$ is uniquely determined by Δ^r and $\rho_{\omega^r_{m-1}}|_{\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)}$, for $m' = 0,\ldots,m-1$. As a consequence, $\mathrm{res}_m(\rho_{\omega^r_{m-1}}|_{\tilde{S}^m\mathscr{L}_{i,c}(M,E)})$ is holomorphic for all $r \in (\mathbb{C} \setminus Z') \cup \{0\}$. Indeed, the holomorphicity follows from [47], §2, Rk. 4 (a), together with the fact that the internal product map (4.5.3) is a hypocontinuous bilinear map defined between complete locally convex spaces, so *a fortiori* a separately continuous between spaces satisfying the condition (a) stated by the author (see [95], II.4.3, Cor.). Since the map $r \mapsto \rho_{\omega^r_{m-1}}|_{\tilde{S}^m\mathscr{L}_{i,c}(M,E)}$ is holomorphic on an open set of the form $B(0,\delta) \setminus \{0\}$, and 0 is a pole, it has a Laurent expansion of the form

$$\sum_{j=-j_m}^{\infty} u_j^m r^j, \tag{7.4.1}$$

where u_j^m is a $\mathrm{Vol}(M^m)$-valued distribution. Applying the restriction operator res_m to it we get

$$\sum_{j=-j_m}^{\infty} \mathrm{res}_m(u_j^m) r^j, \tag{7.4.2}$$

which by the comments above is a holomorphic function at 0. Hence, the $\mathrm{Vol}(M^m)$-valued distribution

$$U_m = \sum_{j=-j_m}^{-1} u_j^m r^j, \tag{7.4.3}$$

has support in the diagonal Diag_m. It clearly satisfies condition (1) of Definition 5.7.9, because ω^r_{m-1} does and the space of distributions with wave front set in a fixed closed conic set is sequentially complete. By Lemma 6.0.11, there is a meromorphic family of renormalizations $r \mapsto g^r_{-m}$, with $g^r_{-m} \in \mathscr{G}_{i,m}(M,E)$ such that $g^r_{-m} \cdot \omega^r_{m-1}$ and $\omega^r_{m-1} - U_m$ coincide on $\tilde{S}^{m'}\mathscr{L}_{i,c}(M,E)$ for $m' = 0,\ldots,m$. Since the expansion of $\omega^r_{m-1} - U_m$ around $r = 0$ has no singular terms, it is uniquely extended to a holomorphic map 0, so we obtain a holomorphic for all $r \in (\mathbb{C}\setminus Z')\cup\{0\}$. By Lemma 5.8.9, given any fixed $r \in \mathbb{C}\setminus Z'$, the infinite product element

$$g^r = \prod_{m=-\infty}^{-2} g^r_m$$

defines a unique element of $\mathscr{G}_i(M,E)$. By Fact 5.9.6, it is meromorphic on $\mathbb{C} \setminus Z'$. The theorem is thus proved. □

7.5 The existence of a Feynman measure for a manageable local propagator of cut type

7.5.1. We have now the main result of this chapter, which is parallel to [10], Thm. 21.

7.5.2 Theorem. *Let Δ be a manageable local propagator of cut type. Then there is a Feynman measure ω associated with Δ.* *New/Based on Borcherds*

Proof. Let $r \mapsto \tilde{\omega}^r$ be the meromorphic family of Feynman measures associated with the family of propagators Δ^r given in Theorem 7.4.2 for all $r \in (\mathbb{C} \setminus Z') \cup \{0\}$, and let ω be the value of $r \mapsto \tilde{\omega}^r$ at $r = 0$. Then ω satisfies the required properties. □

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