Notes on dg (co)algebras and their (co)modules

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The aim of these notes to provide a simple exposition on the basic constructions appearing in the theory of augmented dg algebras and coaugmented dg coalgebras, as well as A_∞-algebras and A_∞-coalgebras. In particular we explain the (reduced and nonreduced) bar and cobar constructions associated to them, the theory of twisting of dg and A_∞-algebras, presenting our sign conventions in a completely explicit manner, and the connection to Hochschild (co)homology of aumented dg algebras, which is of interest to us.

1 Preliminaries on basic algebraic structures

We recall the following basic facts, which will also establish the notation. From now on, k will denote a commutative ring with unit (which we also consider as a unitary graded ring concentrated in degree zero). By *module* over k we will always mean a symmetric bimodule over k (although several constructions can be clearly performed without this symmetry assumption, we shall suppose it in order to simplify the exposition). We fix an abelian group G of the form $\mathbb{Z} \times G'$ (or also $\mathbb{Z}/2.\mathbb{Z} \times G'$ to which all these construction can be adapted straightforward), which we write additively. The character map used in the Koszul sign rule will be just given by the projection on the first component of G. A typical element of G will be denoted by g, h, etc, and the corresponding first component i_g, i_h, etc. For an object M, we will denote by id_M the identity endomorphism of M. We also remark that the expression *map* between to graded or dg modules over k (or maybe provided with further structure) will always mean the mapping between (say) the underlying modules or even the underlying sets, which comes from forgetting all the extra structure. This might be sometimes useful if we want to stress just the values of morphisms at elements of a graded or dg module over k.

1.1 Graded and differential graded modules over a fixed commutative ring k

A *(cohomological) graded module* over k is a module over k provided with a decomposition of k-modules of the form $M = \oplus_{g\in G} M^g$. If m is a nonzero homogeneous element of a graded module M over k we define the *degree* $\deg m \in \mathbb{Z}$ of m and the *weight (internal degree, or Adams degree)* $w(m) \in G'$ of m by $m \in M^{(\deg m, w(m))}$. We say in this case that the *complete degree* of m is $(\deg m, w(m)) \in G$. The commutative ring k will be considered as a graded module with the grading given by $k^g = 0$ if $g \neq 0_G$, and $k^{0_G} = k$. If M is a graded module and $g \in G$, define $M[g]$ to be the graded module over k with the same underlying structure of k-module but with a complete degree shift given by $M[g]^{g'} = M^{g+g'}$, for all $g' \in G$. We shall usually write $M[i]$, for $i \in \mathbb{Z}$, instead of the more correct $M[(i, 0_{G'})]$, for it causes

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no confusion. For any two graded modules M and N over k, $\hom_k(M, N)$ is the space of k-linear maps of complete degree zero, *i.e.* $f(M^g) \subseteq N^g$ for all $g \in G$. The internal space of morphisms if given by $\mathcal{H}om_k(M, N) = \oplus_{g\in G} \hom_k(M, N[g])$ and it is obviously a graded k-module. The graded module $\mathcal{H}om_k(M, k)$ will be also denoted by $M^\#$. Note that in this case the g-th graded component of $M^\#$ is given by $(M^{-g})^*$, where $(-)^*$ denotes the usual dual for modules over k, and by the previous comments we have that $(M[g])^\# = (M^\#)[-g]$, for $g \in G$.

We remark that by very definition the graded module $\mathcal{H}om_k(M[g], N[g'])$ exactly coincides with $\mathcal{H}om_k(M, N)[g' - g]$ for $g, g' \in G$. In the same manner, the graded modules $M[g] \otimes N[g']$ and $(M \otimes N)[g + g']$, for $g, g' \in G$, are also exactly coincident. These "identities" are however misleading since they do not (in general) respect the Koszul sign rule, and -in some sense more fundamentally- the mentioned phenomenon for the homomorphisms spaces is not in accordance with the axioms of category theory. We will regard such coincidences only as a consequence of the usual abuse of notation in the definitions of tensor product and morphisms spaces: since we are interested in considering the Koszul sign rule, we should in fact force them to be noncoincident. There is however an identification (and in fact many of them, but in general different from the identity) between the corresponding previous graded modules, which is compatible with the Koszul sign rule, and that will be explained in the penultimate paragraph of this subsection.

Given M and N two graded modules over k, *a morphism of graded modules of complete degree* $g \in G$ is an element $f \in \mathcal{H}om_k(M, N)$ of complete degree g. We remark that the morphism $s_{M,g} : M \to M[g]$ whose underlying map is given by the identity is a morphism of graded modules of degree $-g$, and we shall typically denote $s_{M,(1,0_{G'})}$ by s_M, or simply s, if M is clear from the context. All along this article, if we do not indicate the complete degree of a morphism (between graded modules, or later on dg modules, etc), it means that it is of complete degree zero. Also, for M and N two graded modules over k, the (usual) tensor product $M \otimes_k N$ has the structure of graded module over k with $(M \otimes_k N)^g = \oplus_{g'\in G} M^{g'} \otimes_k N^{g-g'}$. From now on, all unadorned tensor products $\otimes$ will mean $\otimes_k$. For $f : M \to N$ and $f' : M' \to N'$ two morphisms of graded modules over k of complete degrees g and g', respectively, the map $f \otimes f' : M \otimes M' \to N \otimes N'$ given by $(f \otimes f')(m \otimes m') = (-1)^{\deg f' \deg m} f(m) \otimes f'(m')$, for $m \in M$ and $m' \in M'$ homogeneous, is a morphism of graded modules over k of complete degree $g + g'$. Analogously, if $f : M \to N$ and $f' : N' \to M'$ are two morphisms of graded modules over k of complete degrees g and g', respectively, the map $\mathcal{H}om_k(f, f') : \mathcal{H}om_k(N', N) \to \mathcal{H}om_k(M', M)$ given by $\phi \mapsto (-1)^{\deg f(\deg \phi + \deg f')} f' \circ \phi \circ f$, for ϕ homogeneous, is a morphism of complete degree $g + g'$. We shall also denote $\mathcal{H}om_k(f, N) = \mathcal{H}om_k(f, \mathrm{id}_N)$ and $\mathcal{H}om_k(N', f') = \mathcal{H}om_k(\mathrm{id}_{N'}, f')$. Furthermore, we will usually denote $\mathcal{H}om_k(f, k)$ by $f^\#$, which is of course has the same complete degree as the one of f. As for the case of the tensor product we shall usually omit the commutative ring k in the notation of the homomorphism groups introduced before, and proceed to write $\mathcal{H}om$ instead of $\mathcal{H}om_k$. The canonical map $\iota_M : M \to (M^\#)^\#$ defined as $\iota_M(m)(f) = (-1)^{\deg m \deg f} f(m)$, for $m \in M$ and $f \in M^\#$ homogeneous, is a morphism of graded modules. Given M and N two graded modules over k, we will occasionally consider the morphism $\iota_{M,N} : M^\# \otimes N^\# \to (M \otimes N)^\#$ of graded modules defined as $\iota_{M,N}(\phi \otimes \psi)(m \otimes n) = (-1)^{\deg \psi \deg m} \phi(m)\psi(n)$. We also have the *flip* $\tau_{M,N} : M \otimes N \to N \otimes M$, which is the morphism of graded modules defined as $\tau_{M,N}(m \otimes n) = (-1)^{\deg m \deg n} n \otimes m$, for all $m \in M$ and $n \in N$ homogeneous elements.

A *differential graded module (or dg module)* over k is a graded k-module $M = \oplus_{g\in G} M^g$ together with a homogeneous k-linear map $d_M : M \to M$ of degree $+1$ and zero weight, *i.e.* $d_M(M^g) \subseteq M^{g+(1,0_{G'})}$ for all $g \in G$, such that it is a differential, *i.e.* $d_M^2 = 0$. The graded module structure on k explained before can be extended to a dg module by defining the differential $d_k = 0$, and more generally, any graded k-module M may be regarded as dg module with vanishing differential. For a dg module M over k, the *cohomology* $H^\bullet(M)$ of M, given by the quotient $\mathrm{Ker}(d_M)/\mathrm{Im}(d_M)$, is in fact a graded module over k. A dg module M is called *acyclic* if $H^\bullet(M)$ vanishes. If M is a dg module and $g \in G$, $M[g]$ is the dg module over k with the same graded module structure as before and differential $d_{M[g]} = (-1)^{i_g} d_M$. For M and N two dg modules over k, the tensor product $M \otimes N$ has the structure of dg module over k with the same underlying graded structure as before and with differential $d_{M\otimes N} = d_M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes d_N$. We endow the graded k-module $\mathcal{H}om(M,N)$ with the differential $d_{\mathcal{H}om(M,N)}(f) = d_N \circ f - (-1)^{\deg f} f \circ d_M$, so it becomes a dg module over k. In this case, for M a dg module we will still denote by $M^\#$ the dg module $\mathcal{H}om(M,k)$. Note that $d_{M^\#} = -d_M^\#$.

Given M and N two dg modules over k, $f : M \to N$ is a *morphism of differential graded modules over k of complete degree g* if it is a morphism between the underlying graded modules of complete degree g and satisfies that $d_N \circ f = (-1)^{i_g} f \circ d_M$, *i.e.* it is cocycle of complete degree g of the dg module $\mathcal{H}om(M,N)$. We stress that, as before, if we do not specify the complete degree of a morphism, it will be assumed to be zero. Note that $s_{M,g} : M \to M[g]$ introduced previously is in fact a morphism of dg modules of complete degree $-g$. This is in fact tantamount to the definition of dg module structure over k on $M[g]$. We stress that if $f : M \to N$ and $f' : M' \to N'$ are two morphisms of dg modules over k of complete degree g and g', respectively, then $f \otimes f' : M \otimes M' \to N \otimes N'$ is a morphism of dg modules over k of complete degree $g + g'$. Analogously, given $f : M \to N$ and $f' : N' \to M'$ two morphisms of dg modules over k of complete degrees g and g', respectively, the map $\mathcal{H}om(f, f') : \mathcal{H}om(N', N) \to \mathcal{H}om(M', M)$ defined for graded modules is moreover a morphism of dg modules over k of complete degree $g + g'$. Note that given two dg k-modules M and N, the maps ι_M, $\iota_{M,N}$ and $\tau_{M,N}$ defined in the third paragraph of this section are further morphisms of dg modules over k.

If $f : M \to N$ is a morphism of dg modules over k of complete degree g, then, for each $g', g'' \in G$, the may consider the map

$$\mathcal{H}om(s_{M,g'}^{-1}, s_{N,g''})(f) : M[g'] \to N[g''],$$

which will be denoted by $f_{[g']}^{[g'']}$. It is trivial to see that $f_{[g']}^{[g'']}$ a morphism of dg modules over k of complete degree $g + g' - g''$, *i.e.*

$$d_{N[g']} \circ f = (-1)^{i_{g+g'-g''}} f \circ d_{M[g']}.$$

If f is a morphism of complete degree 0_G, it is rather usual to allow the abuse of notation given by denoting the map $(-1)^{\deg g' \deg g''} f_{[g']}^{[g'']}$ also by f. We shall only follow this convention when we consider it is unambiguous. We also remark that (unlike the case for graded modules) the dg modules $\mathcal{H}om(M[g], N[g'])$ and $\mathcal{H}om(M,N)[g' - g]$, for $g, g' \in G$, are not the same, for the identity map between the underlying graded modules is not a morphism of dg modules over k. Indeed, the corresponding isomorphism of dg modules from $\mathcal{H}om(M,N)[g'-g]$ to $\mathcal{H}om(M[g], N[g'])$, which we denote by $H_{M,N,g,g'}$, is given by $s_{\mathcal{H}om(M,N),g'-g}(f) \mapsto \mathcal{H}om(s_{M,g}^{-1}, s_{N,g'})(f)$, for $f \in \mathcal{H}om(M,N)$. The underlying map is thus the identity times a $(-1)^{(\deg f + i_{g'}) i_g}$ sign. In the same manner, the dg module structure on the tensor product and $M[g'] \otimes N[g']$ and $(M \otimes N)[g + g']$, for $g, g' \in G$, are not same. There is though a (not completely canonical) isomorphism of dg modules $M[g'] \otimes N[g'] \to (M \otimes N)[g + g']$ over k, denoted by $T_{M,N,g,g'}$, defined as $s_{M,g}(m) \otimes s_{N,g'}(n) \mapsto (-1)^{i_{g'} \deg m} s_{M\otimes N, g+g'}(m \otimes n)$, for $m \in M$ and $n \in N$ homogeneous elements.

If $f : M \to N$ is a morphism of dg modules, the *cone* $\operatorname{cone}(f)$ is the dg module whose underlying graded module is $M[1] \oplus N$ and whose differential is given by $d_{\operatorname{cone}(f)}(m, n) = (-d_M(m) + f(m), d_N(n))$. Given a morphism of dg modules $f : M \to N$ of complete degree g, it directly induces a morphism of graded modules $H^\bullet(M) \to H^\bullet(N)$ of the same complete degree, which we will denote by $H^\bullet(f)$. It is clear that $H^\bullet(\mathrm{id}_M) = \mathrm{id}_{H^\bullet(M)}$ and that $H^\bullet(f \circ f') = H^\bullet(f) \circ H^\bullet(f')$, for any two composable morphisms f and f' of dg modules of complete degrees g and g', resp. Furthermore, a morphism of dg modules $f : M \to N$ of complete degree 0_G is said to be a *quasi-isomorphism* if $H^\bullet(f)$ is an isomorphism of graded modules. It is well-known that f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is acyclic (see [23], Cor. 1.5.4).

1.2 Graded and differential graded algebras and coalgebras, and modules over the former

A *(nonunitary) graded algebra* over k is just a (nonunitary) algebra over k together with a decomposition of k-modules $A = \oplus_{g \in G} A^g$ satisfying that $A^g A^{g'} \subseteq A^{g+g'}$, for all $g, g' \in G$. We will also sometimes denote the product of A by (the morphism of graded modules) $\mu_A : A \otimes A \to A$. A *morphism of graded algebras* from a graded algebra A to a graded algebra B is a morphism of graded modules $f : A \to B$ such that $f(aa') = f(a)f(a')$ for all $a, a' \in A$. A *unitary graded algebra* over k is a nonunitary one together with an element $1_A \in A^{0_G}$, called the *unit* of A, satisfying the usual axiom $1_A a = a 1_A = a$ for all $a \in A$. We may also consider the unit of A as a morphism of graded modules $\eta_A : k \to A$ which satisfies that $\mu_A \circ (\mathrm{id}_A \otimes \eta_A)$ and $\mu_A \circ (\eta_A \otimes \mathrm{id}_A)$ coincide with the canonical isomorphisms $A \otimes k \to A$ and $k \otimes A \to A$, resp. Given two unitary graded algebras A and B, a *morphism of unitary graded algebras* is a morphism of the underlying nonunitary graded algebras $f : A \to B$ such that $f(1_A) = 1_B$. The *opposite graded algebra* A^{op} of a nonunitary graded algebra A is given by the same graded module over k but with the product $a \cdot_{\mathrm{op}} b = (-1)^{\deg a \deg b} ba$, for all $a, b \in A$ homogeneous. In case A is unitary, A^{op} also, with the same unit of A. If A and B are two nonunitary graded algebras, the graded module structure over k of the tensor product $A \otimes B$ is also a graded algebra with the product $(a \otimes a')(b \otimes b') = (-1)^{\deg a' \deg b} ab \otimes a'b'$. If A and B are unitary with units 1_A and 1_B, resp., then $A \otimes B$ is also unitary with unit $1_A \otimes 1_B$. We consider the graded algebra $A^e = A \otimes A^{\mathrm{op}}$, which is called the *enveloping algebra* of A.

We also have the dual definitions. A *(noncounitary) graded coalgebra* over k is a graded module $C = \oplus_{g \in G} C^g$ together with a morphism of graded modules $\Delta_C : C \to C \otimes C$ satisfying the coassociativity axiom $(\Delta_C \otimes \mathrm{id}_C) \circ \Delta_C = (\mathrm{id}_C \otimes \Delta_C) \circ \Delta_C$. We define $\Delta_C^{(n)} : C \to C^{\otimes n}$ as the composition $(\Delta_C^{(n-1)} \otimes \mathrm{id}_C) \circ \Delta_C$, for $n \in \mathbb{N}_{\geq 3}$, and $\Delta_C^{(2)} = \Delta_C$. As usual, we may use the *Sweedler notation* $\Delta_C^{(n)}(c) = c_{(1)} \otimes \cdots \otimes c_{(n)}$ for the iterated coproduct of an element $c \in C$ (by the coassociativity axiom this notation is consistent). A *morphism of graded coalgebras* from a graded coalgebra C to a graded algebra D is a morphism of graded modules $f : C \to D$ such that $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$. A graded coalgebra C is called *counitary* if there is a morphism of graded modules $\epsilon_C : C \to k$, called the *counit* of C, satisfying that $(\epsilon_C \otimes \mathrm{id}_C) \circ \Delta_C$ and $(\mathrm{id}_C \otimes \epsilon_C) \circ \Delta_C$ coincide with the canonical isomorphisms $C \simeq k \otimes C$ and $C \simeq C \otimes k$, resp. Given two counitary graded coalgebras C and D, a *morphism of counitary graded coalgebras* is a morphism of the underlying noncounitary graded coalgebras $f : C \to D$ such that $\epsilon_D \circ f = \epsilon_C$. The *coopposite graded coalgebra* C^{coop} of a noncounitary graded coalgebra C is given by the same graded module over k but with coproduct $\Delta_{C^{\mathrm{coop}}} = \tau_{C,C} \circ \Delta_C$. If C is counitary, C^{coop} is also, with the same counit as the one of C. If C and D are two noncounitary graded coalgebras, the graded module structure over k of the tensor product $C \otimes D$ is also a graded coalgebra with the coproduct $\Delta_{C\otimes D} = (\mathrm{id}_C \otimes \tau_{C,D} \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D)$. If C and D are counitary with counits ϵ_C and ϵ_D, resp., then $C \otimes D$ is also counitary with counit $\epsilon_C \otimes \epsilon_D$. The graded coalgebra $C \otimes C^{\mathrm{coop}}$ is called the *enveloping coalgebra* of C, and is denoted by C^{e}.

A left (resp., right) *graded module* over a nonunitary graded algebra A is just a left (resp., right) module over A such that it is a graded module over k for the action of k given by restriction (*i.e.* provided with a decomposition of k-modules of the form $M = \oplus_{g\in G} M^g$) satisfying that $A^{g'} M^g \subseteq M^{g'+g}$ (resp., $M^g A^{g'} \subseteq M^{g+g'}$), for all $g, g' \in G$. If A is unitary, we say that M is a left (resp., right) graded module if we further assume that $1_A m = m$ (resp., $m 1_A = m$) for all $m \in M$. A *graded bimodule* over A will be just a left graded module over the enveloping algebra A^{e}.

In the rest of this subsection, unless further explanation is required, we shall usually refer to the term graded algebra (resp., graded coalgebra), differential graded algebra (resp., differential graded coalgebra), module over a graded algebra (resp., comodule over a graded coalgebra), etc. without explicitly indicating whether there is a unit (resp., counit) or not to indicate that the definitions and constructions apply to each possibility in the sense that either the adjective nonunitary should be applied to them altogether, or else the adjective unitary.

If M is a left (resp., right) graded module and $g \in G$, define $M[g]$ to be the left (resp., right) graded module over A with the new action of A given by $a \cdot m = (-1)^{i_g \deg a} am$ (resp., with the same action) such that the complete degree shifts as $M[g]^{g'} = M^{g+g'}$. Note that, if M is a left (resp., right) graded module over A, then it is also a right (resp., left) graded module over A^{op} with the same structure of graded module over k and right (resp., left) action $ma = (-1)^{\deg a \deg m} am$ (resp., $am = (-1)^{\deg a \deg m} ma$) over A^{op}. For any two left (resp., right) graded modules M and N over A, $\hom_A(M,N)$ is the space of A-linear maps of complete degree zero, and $\mathcal{H}om_A(M,N) = \oplus_{g\in G} \hom_A(M, N[g])$, which is obviously a graded k-module. Note that, if M and N are left (resp., right) graded modules over A, this implies that $\mathcal{H}om_A(M,N)$ is the subspace of $\mathcal{H}om(M,N)$ given by sums of homogeneous maps satisfying that $f(am) = (-1)^{\deg f \deg a} af(m)$ (resp., $f(ma) = f(m)a$), for $a \in A$ and $m \in M$ homogeneous elements. These latter are called *morphisms of graded left (resp., right) A-modules* (of some complete degree). Notice that the graded left (resp., right) A-module structure on $M[g]$ is tantamount to requiring that the map $s_{M,g} : M \to M[g]$ is a morphism graded left (resp., right) A-modules. We may point out that there are similar definitions of graded comodules over graded coalgebras, to which the previous constructions also apply *mutatis mutandi*. Since we will not need these, we do not provide such definitions, but we let the interested reader to elaborate on them.

A *nonunitary (resp., unitary) differential graded algebra (or dg algebra)* over k is a nonunitary (resp., unitary) graded algebra over k together with a homogeneous k-linear map $d_A : A \to A$ of complete degree $(1, 0_{G'})$ satisfying the Leibniz identity, *i.e.* $d_A(ab) = d_A(a)b + (-1)^{\deg a} a d_A(b)$, for all $a, b \in A$ homogeneous, and $d_A^2 = 0$ (resp., $d_A(1_A) = 0$ and $d_A^2 = 0$). As in the case of unitary graded algebras, we may also consider the unit of A as a morphism of dg modules $\eta_A : k \to A$ which satisfies the same axioms as before. Note that the dg module structure on k stated before is compatible with its structure of unitary algebra, turning k into a unitary dg algebra. The graded k-module given by the cohomology $H^\bullet(A)$ of A is in fact a nonunitary (resp., unitary) graded algebra with the product induced by that of A (resp., and the unit of $H^\bullet(A)$ is the cohomology class of the unit of A). Note that if A is a dg algebra over k, then the opposite graded algebra together with the same differential d_A is also a dg algebra over k. Analogously, for A and B two dg algebras, the dg module structure over k of the tensor product $A \otimes B$ with the product (and unit if A and B are unitary) described above for graded algebras is also a dg algebra. In this case, the enveloping algebra A^e of a dg algebra A is also a dg algebra.

A *noncounitary (resp., counitary) differential graded coalgebra (or dg coalgebra)* over k is a noncounitary (resp., counitary) graded coalgebra C over k provided with a morphism of graded k-modules $d_C : C \to C$ of complete degree $(1, 0_{G'})$ satisfying that $\Delta_C \circ d_C = (\mathrm{id}_C \otimes d_C + d_C \otimes \mathrm{id}_C) \circ \Delta_C$, and $d_C^2 = 0$ (resp., $\epsilon_C \circ d_C = 0$ and $d_C^2 = 0$). Note also that the canonical isomorphism $k \to k \otimes k$ turns the dg module k into a dg coalgebra, which is further counitary by setting $\epsilon_k = \mathrm{id}_k$. If k is Von Neumann regular, the graded k-module given by the cohomology $H^\bullet(C)$ of C has a coproduct (resp., and a counit) induced by that of C, by the Künneth formula, so it becomes a noncounitary (resp., counitary) graded coalgebra. If C is a dg coalgebra over k, then the coopposite graded coalgebra together with the same differential d_C is also a dg coalgebra over k. Analogously, for C and D two dg coalgebras, the dg module structure over k of the tensor product $C \otimes D$ with the coproduct (and counit if C and D are counitary) described above for graded coalgebras is also a dg coalgebra. As for the case of algebras, the enveloping coalgebra C^e of a dg coalgebra C is also a dg coalgebra.

A left (resp., right) *differential graded module (or dg module)* over a dg algebra A is a left (resp., right) graded A-module $M = \oplus_{g \in G} M^g$ such that it is also a dg module over k, for the action of k coming from restriction (*i.e.* together with a homogeneous k-linear map $d_M : M \to M$ of complete degree $(1, 0_{G'})$, such that $d_M^2 = 0$), which satisfies the Leibniz identity, *i.e.* $d_M(am) = d_A(a)m + (-1)^{\deg a} a d_M(m)$ (resp., $d_M(ma) = d_M(m)a + (-1)^{\deg m} m d_A(a)$), for all $a \in A$ and $m \in M$ homogeneous. If M is a left (resp., right) dg module over A and $g \in G$, $M[g]$ is the left (resp., right) dg module over A with the same graded module structure over A as defined previously and differential given by its structure of dg module over k, *i.e.* $d_{M[g]} = (-1)^{i_g} d_M$. Note that for any two left (resp., right) dg modules M and N over a dg algebra A the space $\mathcal{H}om_A(M, N)$ is obviously a dg k-module for $d_{\mathcal{H}om_A(M,N)}(f) = d_N \circ f - (-1)^{\deg f} f \circ d_M$. A *morphism of differential graded modules over A of complete degree g* is an element $f \in \mathcal{H}om_A(M, N)$ of degree d satisfying that $d_N \circ f = (-1)^{i_g} f \circ d_M$, *i.e.* it is cocycle of complete degree g of the dg k-module $\mathcal{H}om_A(M, N)$. Note that $s_{M,g}$ is a morphism of dg modules over A, for any dg A-module M and $g \in G$. As in the case of dg modules over k, if $f : M \to N$ is a morphism of dg modules over A of complete degree g, then, for each $g', g'' \in \mathbb{Z}$, we may consider

$$f_{[g']}^{[g'']} : M[g'] \to N[g''],$$

which is a morphism of dg modules over A of degree $g + g' - g''$. Also, notice that, if M is a left (resp., right) dg module over A, then it is also a right (resp., left) dg module over A^{op} with the same structure of dg module over k and right (resp., left) action as in the case of graded modules over A^{op}. Indeed, it is trivial to check that this satisfies the Leibniz identity, so it defines a structure of dg module over A. A *differential graded bimodule (or dg bimodule)* over A is defined as a left dg module over the enveloping algebra A^e. As before, we endow the graded k-module $\mathcal{H}om_A(M, N)$ with differential $d(f) = d_N \circ f - (-1)^{\deg f} f \circ d_M$, so it becomes a dg module over k. Again, we notice that there are similar definitions of differential graded comodules over dg coalgebras, to which the previous constructions also apply straightforward, but we will not give them for they are not going to be required.

Let M be a dg module over a dg algebra A. It is called *free* if it is isomorphic to a direct sum of dg modules over A of the form $A[g_i]$, for a family $\{g_i : i \in I\}$ of elements of G, where I is a set of indices. We say that M is *semi-free* if there is an increasing filtration $\{M_i\}_{i \in \mathbb{N}_0}$ of dg submodules of M over A such that that $M_0 = 0$

(*i.e.* the filtration is *Hausdorff*), $\cup_{i\in\mathbb{N}_0} M_i = M$ (*i.e.* the filtration is *exhaustive*) and M_{i+1}/M_i is a free dg module over A for all $i \in \mathbb{N}_0$ (see [1], Subsection 1.11, (4)). Equivalently, M is semi-free if there exists a set $\mathcal{B} \subseteq M$ of homogeneous elements which gives a basis of the underlying graded module of M over the underlying graded algebra of A with the following property. For any $S \subseteq \mathcal{B}$, let $\delta(S) \subseteq \mathcal{B}$ be the smallest subset among all of the subsets T of $\mathcal{B}$ such that $d(S)$ is included in the A-linear span of T. Then, the previously mentioned property is that for every $b \in \mathcal{B}$, there is $n \in \mathbb{N}$ such that $\delta^n(\{b\}) = \emptyset$ (see [2], Prop. 8.2.3). It is a very simple exercise to prove that if M is provided with an increasing Hausdorff and exhaustive filtration $\{M_i\}_{i\in\mathbb{N}_0}$ of dg submodules of M over A such that M_{i+1}/M_i is a semi-free dg module over A for all $i \in \mathbb{N}_0$, then M is also semi-free (see [2], Cor. 8.2.4).

We say that a dg module M over a dg algebra A is *homotopically projective* if given any acyclic dg module N over A (*i.e.* $H^\bullet(N) = 0$) and every morphism of dg modules $f : M \to N$, there is $h \in \hom_A(M, N[-1])$ (called a *homotopy* between f and 0) such that $d(h) = f$. As noticed by [2], any semi-free dg module is homotopically projective. This follows directly from the easy fact that any homotopy between $f|_{M_i}$ and the zero map can be extended to a homotopy between $f|_{M_{i+1}}$ and the corresponding zero map. Indeed, this can be easily proved by diagram chasing arguments applied to the following exact sequence of dg modules over k provided with morphisms of complete degree zero

$$\mathcal{H}om_A(M_{i+1}/M_i, N) \to \mathcal{H}om_A(M_{i+1}, N) \to \mathcal{H}om_A(M_i, N) \to 0.$$

A *semi-free resolution* of M is a dg module F over A together with a morphism of dg A-modules $f : F \to M$ of complete degree zero such that it is a quasi-isomorphism. As noted in [1], Subsection 1.11, (6), a semi-free resolution always exists, and the morphism f can be even choosen to be surjective (see [2], Thm. 8.3.2). The construction of the pair (F, f) is given by the direct limit of a recursive contruction of pairs $(F_i, f_i)_{i\in\mathbb{N}_0}$ satisfying that $\{F_i\}_{i\in\mathbb{N}_0}$ is an increasing sequence of dg A-modules with $F_0 = 0$ and F_{i+1}/F_i free, $f_i : F_i \to M$ is a morphism of dg A-modules of complete degree zero and $f_{i+1}|_{M_i} = f_i$ for all $i \in \mathbb{N}_0$. The inductive step is given as follows. Suppose we have constructed $(F_j, f_j)_{j=0,\dots,i}$ as before, for some $i \in \mathbb{N}_0$, then one takes a free dg A-module P together with a morphism $P \to \operatorname{cone}(f_i)[-1]$ which induces a surjective morphism between cohomology groups (this can be easily done by taking P the free dg A-module generated by a set of cocycles, whose cohomology class generate the cohomology of the cone $\operatorname{cone}(f_i)[-1]$). Set

$$F_{i+1} = \operatorname{cone}((p_1)^{[-1]}_{[-1]} \circ \pi),$$

where $p_1 : \operatorname{cone}(f_i) \to F_i[1]$ is the morphism of dg modules given by the canonical projection, and define $f_{i+1} : F_{i+1} \to M$ as

$$f_{i+1}(p, e) = (p_2 \circ \pi^{[1]}_{[1]})(p) + f_i(e),$$

where $(p, e) \in P[1] \oplus F_i$, $p_2 : \operatorname{cone}(f_i) \to M$ is the morphism of graded modules given by the canonical projection (it is not a morphism of dg modules!). It is easy to check that f_{i+1} is a morphism of dg A-modules, there is a canonical inclusion of dg A-modules $F_i \subseteq F_{i+1}$, $f_{i+1}|_{F_i} = f_i$ and $F_{i+1}/F_i \simeq P$ is a free dg A-module. It is clear that F is semi-free and f is surjective. Let us see that it is a quasi-isomorphism. It is easy to see that the inclusion $F_i \subseteq F_{i+1}$ of dg A-modules induce in turn an inclusion $\operatorname{cone}(f_i) \to \operatorname{cone}(f_{i+1})$ of dg A-modules, thanks to the property $f_{i+1}|_{F_i} = f_i$. We thus obtain an increasing Hausdorff and exhaustive filtration $\{\operatorname{cone}(f_i)\}_{i\in\mathbb{N}_0}$ of dg A-modules of $\operatorname{cone}(f)$. Since filtered colimits are exact (see [23], Thm. 2.6.15), they commute with taking cohomology, so the cohomology of

$\mathrm{cone}(f)$ is the direct limit of the system given by $\{H^{\bullet}(\mathrm{cone}(f_i))\}_{i\in\mathbb{N}_0}$ together with the cohomology classes of the maps $\mathrm{cone}(f_i) \to \mathrm{cone}(f_{i+1})$, for $i \in \mathbb{N}_0$. The latter morphisms $H^{\bullet}(\mathrm{cone}(f_i)) \to H^{\bullet}(\mathrm{cone}(f_{i+1}))$ vanish by construction, which implies thus that $H^{\bullet}(\mathrm{cone}(f)) = 0$, which in turn implies that f is a quasi-isomorphism.

1.3 The bar resolution and Hochschild (co)homology of dg algebras

We recall that, for A and B two unitary dg algebras over k, the *free product* $A *_k B$ of A and B is given as a unitary graded algebra over k by

$$T_k(A \oplus B)/\langle 1_A - 1_B, a \otimes a' - aa', b \otimes b' - bb' : \text{ for all } a, a' \in A \text{ and } b, b' \in B\rangle,$$

where $T_k(V)$ is the tensor algebra on a graded module V over k. Note that the canonical inclusions $i_A : A \to A *_k B$ and $i_B : B \to A *_k B$ are morphisms of graded k-algebras. Then $A *_k B$ has a natural structure of graded A-bimodule via i_A and of graded B-bimodule via i_B. The differential d_A of A can be extended as the unique derivation $d_{A*_kB|A}$ of $A *_k B$ satisfying that $d_{A*_kB|A} \circ i_A = d_A$ and $d_{A*_kB|A} \circ i_B = 0$. The same applies to the differential d_B, providing a derivation $d_{A*_kB|B}$ on $A *_k B$. Note that $d_{A*_kB|A} d_{A*_kB|B} = -d_{A*_kB|B} d_{A*_kB|A}$. The differential d_{A*_kB} of $A *_k B$ is just the derivation $d_{A*_kB|A} + d_{A*_kB|B}$. Hence, we see that $A *_k B$ has in fact a natural structure of dg A-bimodule via i_A and of dg B-bimodule via i_B. By abuse of notation, we usually write d_A instead of $d_{A*_kB|A}$ and d_B instead of $d_{A*_kB|B}$. Note that $A *_k B$ is just the coproduct of A and B in the category of the dg algebras over k.

Let $k[\epsilon]$ be the differential graded algebra whose underlying k-module is the usual polynomial algebra on the indeterminate ϵ, where the degree of ϵ is -1 and the weight is zero, provided with the differential of complete degree $(1, 0_{G'})$ given by the derivation $\partial/\partial\epsilon$, *i.e.* the unique derivation satisfying that $\partial/\partial\epsilon(\epsilon) = 1$. Consider the differential graded algebra given by the free product $A *_k k[\epsilon]$, and the differential induced by d_A and $\partial/\partial\epsilon$. Following V. Drinfeld (*cf.* [12], Subsection 4.3), the *augmented (nonreduced or unnormalized) bar complex* of A is just another "presentation" of the differential graded algebra $A *_k k[\epsilon]$ with the differential given by $d_A + \partial/\partial\epsilon$. We will explain what this means. Consider the graded A-bimodule given by $\mathrm{Bar}(A) = \oplus_{n\in\mathbb{N}_0}(A \otimes A[1]^{\otimes n} \otimes A)$. If $n \in \mathbb{N}$ we will typically denote an element $a_0 \otimes s(a_1) \otimes \cdots \otimes s(a_n) \otimes a_{n+1} \in A \otimes A[1]^{\otimes n} \otimes A$ in the form $a_0[a_1|\dots|a_n]a_{n+1}$, where $a_0, \dots, a_{n+1} \in A$ and $s : A \to A[1]$ is the canonical morphism of degree -1 recalled in the third paragraph of Subsection 1.1. In the same manner, we may usually denote $a_0 \otimes a_1$ by $a_0[]a_1$. There is a canonical identification (as graded A-bimodules, so the morphism is of complete degree zero) of $\mathrm{Bar}(A)[1]$ inside $A *_k k[\epsilon]$ given by $s_{\mathrm{Bar}(A)}(a_0[a_1|\dots|a_n]a_{n+1}) \mapsto (-1)^{\deg a_0 + \dots + \deg a_n + n} a_0\epsilon \dots \epsilon a_{n+1}$, for $n \geq 0$, where we have replaced each occurrence of the tensor on the left member by ϵ and added a sign. Under this identification $\mathrm{Bar}(A)$ gets a differential b' of complete degree $(1, 0_{G'})$, such that $\mathrm{Bar}(A)$ is a differential graded A-bimodule (if we forget about the map $\partial/\partial\epsilon$ applied to elements $a_0\epsilon a_1$). Moreover, under the previous identifications, and seeing A inside $A *_k k[\epsilon]$ via i_A, the differential $d_{A*_kk[\epsilon]}$ of $A *_k k[\epsilon]$ induces the map of graded A-bimodules (of complete degree zero) from $\mathrm{Bar}(A)$ to A whose restriction to $A \otimes A$ is given by the product of A, and the restriction to $A \otimes A[1]^{\otimes n} \otimes A$, for $n \neq 0$, vanishes. In fact, it is clear that using the previous maps $A *_k k[\epsilon]$ is identified (as a graded A-bimodule) with the cone of the morphism of dg A-bimodules $\mathrm{Bar}(A) \to A$ of complete degree 0_G. That this map $\mathrm{Bar}(A) \to A$ is a quasi-isomorphism is tantamount to the fact that its cone is acyclic, or, under the previous identification, that $A *_k k[\epsilon]$ is acyclic. This last statement follows easily from the fact that the cohomology $H^{\bullet}(A *_k k[\epsilon])$ is a unitary algebra whose unit vanishes, since $1_{A*_kk[\epsilon]} = d_{A*_kk[\epsilon]}(\epsilon)$.

The *augmented reduced (or normalized) bar complex* is just what becomes identified when we consider $(A *_k k[\epsilon])/\langle \epsilon^2 \rangle$ instead of $A *_k k[\epsilon]$. In this case, the underlying graded A-bimodule will be given by $\overline{\mathrm{Bar}}(A) = \oplus_{n \in \mathbb{N}_0} A \otimes \bar{A}[1]^{\otimes n} \otimes A$, where $\bar{A} = A/k.1_A$ (as graded k-modules). We will still denote an element $a_0 \otimes s(\bar{a}_1) \otimes \cdots \otimes s(\bar{a}_n) \otimes a_{n+1} \in A \otimes \bar{A}[1]^{\otimes n} \otimes A$ in the form $a_0[a_1|\dots|a_n]a_{n+1}$, where $a_0, \dots, a_{n+1} \in A$, thus omitting the bars for simplicity. This will mean in particular that an element $a_0[a_1|\dots|a_n]a_{n+1}$ in the reduced bar complex of A vanishes if there is some index $i \in \{1, \dots, n\}$ such that a_i is scalar multiple of 1_A. The previous isomorphisms now induce an identification of $\overline{\mathrm{Bar}}(A)[1]$ inside $(A *_k k[\epsilon])/\langle \epsilon^2 \rangle$, and by the same arguments it becomes a dg A-bimodule with differential $\bar{b}'$, which is a resolution of A, also by the map whose restriction to $A \otimes A$ is given by the product of A, and whose restriction to $A \otimes \bar{A}[1]^{\otimes n} \otimes A$ vanishes for $n \neq 0$. As for the case of the bar complex, the dg A-bimodule $(A *_k k[\epsilon])/\langle \epsilon^2 \rangle$ is identified by the previous map with the cone of the morphism of dg A-bimodules $\overline{\mathrm{Bar}}(A) \to A$ of complete degree 0_G. This last map is proved to be a quasi-isomorphism using the same arguments as in the previous paragraph. This in turn implies that the morphism of differential graded A-bimodules $\mathrm{Bar}(A) \to \overline{\mathrm{Bar}}(A)$ given by taking quotients is a quasi-isomorphism, for the latter is induced (using the previous identifications) by the canonical quotient morphism $A *_k k[\epsilon] \to (A *_k k[\epsilon])/\langle \epsilon^2 \rangle$ of differential graded algebras. We remark that the previously defined differential $\bar{b}'$ coincides with the differential $d_0 + d_1$ defined in [9], Subsection 2.2, of the differential graded A-bimodule $\bar{B}(A; A; A)$ considered there (which, as a graded A-bimodule, coincides with $\overline{\mathrm{Bar}}(A)$). More explicitly, the previously referred maps are given by

$$\begin{aligned} d_0(a_0[a_1|\dots|a_n]a_{n+1}) =& d_A(a_0)[a_1|\dots|a_n]a_{n+1} \\ & - \sum_{i=1}^{n} (-1)^{\epsilon_i} a_0[a_1|\dots|d_A(a_i)|\dots|a_n]a_{n+1} \\ & + (-1)^{\epsilon_{n+1}} a_0[a_1|\dots|a_n]d_A(a_{n+1}), \end{aligned}$$

and

$$\begin{aligned} d_1(a_0[a_1|\dots|a_n]a_{n+1}) =& (-1)^{\deg a_0} a_0 a_1[a_2|\dots|a_n]a_{n+1} \\ & + \sum_{i=2}^{n} (-1)^{\epsilon_i} a_0[a_1|\dots|a_{i-1}a_i|\dots|a_n]a_{n+1} \\ & - (-1)^{\epsilon_n} a_0[a_1|\dots|a_{n-1}]a_n a_{n+1}, \end{aligned}$$

where $\epsilon_i = \deg a_0 + (\sum_{j=1}^{i-1} \deg a_j) - i + 1$, and where it is assumed that the expression $a_0[a_1|\dots|a_n]a_{n+1}$ vanishes if $a_i = \lambda 1_A$, for some $\lambda \in k$ and $i \in \{1, \dots, n\}$. The same expression of the differential hold for the nonreduced bar complex.

The following result justifies the relevance of the bar resolution. It was proved for augmented dg algebras and the reduced bar resolution in [6], Lemma 4.3, though exactly the same proof applies verbatim to this more general case. We will provide it for completeness.

Lemma 1.1. *Let A be a unitary differential graded algebra such that the underlying dg k-module of A is semi-free. Then the previously considered morphism of differential graded A-bimodules* $\mathrm{Bar}(A) \to A$ *(or* $\overline{\mathrm{Bar}}(A) \to A$*) is in fact a semi-free resolution of A.*

Proof. The fact that $\mathrm{Bar}(A) \to A$ (or $\overline{\mathrm{Bar}}(A) \to A$) is a quasi-isomorphism of dg A-bimodules was already shown at the end of the second and the beginning of the third paragraphs of this section. It remains to prove that $\mathrm{Bar}(A)$ (or $\overline{\mathrm{Bar}}(A)$) is a semi-free dg bimodule over A. Let us prove it for the nonreduced bar resolution, the case of the reduced one being analogous. Since A is a semi-free dg k-module, the same applies to the dg k-module $A[1]$, and to the tensor products $A[1]^{\otimes n}$. This in turn implies that the dg A-bimodule $A \otimes A[1]^{\otimes n} \otimes A$ (provided only with the differential induced by d_A, *i.e.* d_0 given before) is semi-free. The proof ends by using the last property of semi-free modules given in the antepenultimate paragraph of the previous subsection by noting that the previous dg A-bimodule $A \otimes A[1]^{\otimes n} \otimes A$ is isomorphic to the quotient T_n/T_{n-1}, for the increasing Hausdorff and exhaustive filtration $\{T_n\}_{n\in\mathbb{N}_0}$ of $\mathrm{Bar}(A)$ given by the dg A-bimodules whose underlying graded modules are $T_n = \oplus_{i=0}^{n} A \otimes A[1]^{\otimes i} \otimes A$ for all $n \in \mathbb{N}_0$. □

Let now M be a dg bimodule over A. The *Hochschild homology* $H_\bullet(A, M)$ of A with coefficients on M is just the homology of the complex $M \otimes_{A^e} \mathrm{Bar}(A)$, or equivalently, $M \otimes_{A^e} \overline{\mathrm{Bar}}(A)$, with differential $d_M \otimes_{A^e} \mathrm{id}_{B_\bullet(A)} + \mathrm{id}_M \otimes_{A^e} b'$, or $d_M \otimes_{A^e} \mathrm{id}_{\bar{B}_\bullet(A)} + \mathrm{id}_M \otimes_{A^e} \bar{b}'$, respectively. We recall the canonical identification $\Phi_{A,M} : M \otimes_{A^e} \overline{\mathrm{Bar}}(A) \to M \otimes T(s(\bar{A}))$ of the form $m \otimes_{A^e} a_0[a_1|\dots|a_n]a_{n+1} \mapsto (-1)^{\deg a_{n+1}(\deg m + (\sum_{i=0}^{n} \deg a_i) + n)} a_{n+1} m a_0 \otimes [a_1|\dots|a_n]$. By means of the former isomorphism we induce a differential of the form $D'_0 + D'_1$ on $M \otimes T(s(\bar{A}))$ given by

$$\begin{aligned} D'_0(m \otimes [a_1|\dots|a_n]) =& d_M(m) \otimes [a_1|\dots|a_n] \\ &- \sum_{i=1}^{n} (-1)^{\tilde{\epsilon}_i} m \otimes [a_1|\dots|d_A(a_i)|\dots|a_n] \\ &+ \sum_{i=2}^{n} (-1)^{\tilde{\epsilon}_i} m \otimes [a_1|\dots|a_{i-1}a_i|\dots|a_n], \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} D'_1(m \otimes [a_1|\dots|a_n]) =& (-1)^{\deg m} m a_1 \otimes [a_2|\dots|a_n] \\ &- (-1)^{\tilde{\epsilon}_n(\deg a_n + 1)} a_n m \otimes [a_1|\dots|a_{n-1}], \end{aligned} \tag{1.2}$$

where $\tilde{\epsilon}_i = \deg m + (\sum_{j=1}^{i-1} \deg a_j) - i + 1$, and as usual the expression $[a_1|\dots|a_n]$ vanishes if $a_i = \lambda 1_A$, for some $\lambda \in k$ and $i \in \{1, \dots, n\}$. The same expression of the differential hold for the nonreduced bar complex. Note that our expression of differential coincides with the corresponding one of [21], Subsection 2.1, if one understands in their equation before (10), and following their notation, that the sum is indexed over $i = 0, \dots, p$, and $\sum_{k<i}(|a_k| + 1) + 1$ (*i.e.* $\sum_{k=0}^{i-1}(|a_k| + 1) + 1$) is in fact $|a_0| + (|a_1 + 1|) + \dots + (|a_{i-1}| + 1)$, so it vanishes if $i = 0$, by the principle of summing over the empty set (here $|\ \ |$ is our cohomological degree). Note that the expressions written before to interpret $\sum_{k<i}(|a_k|+1)+1$ coincide for $i \geq 1$. Accordingly, if we regard the convention of [19], Section 2, (2.2), and also following their notation, η_j should be understood as $((|a_0| - 1) + |a_1| + \dots + |a_{i-1}|) + 1$ (following the interpretation consistent with their identity (2.1) and not with their definition before (1.1)) and not as $\sum_{k=0}^{j-1}(|a_k|)$ (here $|\ \ |$ is our cohomological degree plus one). The difference between the two expressions is only apparent when $j = 0$, for the latter gives 0, being a sum indexed over an empty set. With this interpretation, our differential would just be $b - \delta$ instead of $b + \delta$ in the notation of that article.

Analogously, the *Hochschild cohomology* $H^\bullet(A, M)$ of A with coefficients on M is given by the cohomology of the complex $\mathcal{H}om_{A^e}(\mathrm{Bar}(A), M)$, or equivalently, $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), M)$, with differential $f \mapsto d_M \circ f - (-1)^{degf} f \circ b'$, or $f \mapsto d_M \circ f - (-1)^{degf} f \circ \bar{b}'$, respectively. More explicitly, using the canonical identification $\Phi^{A,M} : \mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), M) \to \mathcal{H}om(T(s(\bar{A})), M)$ given by $\Phi^{A,M}(f)([a_1|\dots|a_n]) = f(1_A[a_1|\dots|a_n]1_A)$, we get that the latter complex induces a differential given by a sum $D_0 + D_1$, whose value at $f \in \mathcal{H}om(T(s(\bar{A})), M)$ is

$$\begin{aligned} D_0(f)([a_1|\dots|a_n]) =& d_M(f([a_1|\dots|a_n])) \\ &+ \sum_{i=1}^{n} (-1)^{\bar{\epsilon}_i} f([a_1|\dots|d_A(a_i)|\dots|a_n]) \\ &- \sum_{i=2}^{n} (-1)^{\bar{\epsilon}_i} f([a_1|\dots|a_{i-1}a_i|\dots|a_n]), \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} D_1(f)([a_1|\dots|a_n]) =& -(-1)^{\deg a_1 \deg f - \deg f} a_1 f([a_2|\dots|a_n]) \\ &+ (-1)^{\bar{\epsilon}_n} f([a_1|\dots|a_{n-1}]) a_n, \end{aligned} \tag{1.4}$$

where $\bar{\epsilon}_i = \deg f + (\sum_{j=1}^{i-1} \deg a_j) - i + 1$, and as before it is supposed that the expression $[a_1|\dots|a_n]$ vanishes if $a_i = \lambda 1_A$, for some $\lambda \in k$ and $i \in \{1, \dots, n\}$. The same expression of the differential holds for the nonreduced bar complex. Notice that our expression of differential does coincide with the corresponding one of [19], Section 1, Definition 1.1, or [21], Subsection 2.2, p. 80.

1.4 The bar construction and Hochschild (co)homology of augmented dg algebras

We are interested in the Hochschild (co)homology $H^\bullet(A, A)$ of A with coefficients on the same dg algebra A (also denoted by $HH^\bullet(A)$) under the slightly stronger assumption of A being an augmented dg algebra. We recall that a unitary dg algebra A is called *augmented* if there is a morphism $\epsilon_A : A \to k$ of unitary dg algebras. In this case, $I_A = \mathrm{Ker}(\epsilon_A)$ is called the *augmentation ideal* of A. A morphism of augmented dg algebras $f : A \to A'$ is a morphism of unitary dg algebras such that $\epsilon_{A'} \circ f = \epsilon_A$. Analogously, a counitary dg coalgebra C is said to be *coaugmented* if there exists a morphism of counitary dg coalgebras $\eta_C : k \to C$. We will usually denote the coaugmentation cokernel $C/\mathrm{Im}(\eta_C)$ of C by J_C, which is a (nonunitary) dg coalgebra. As in the case of augmented dg algebras, J_C is canonically identified with $\mathrm{Ker}(\epsilon_C)$, and under that identification the coproduct of $\mathrm{Ker}(\epsilon_C)$ is given by $\Delta_C(c) - 1_C \otimes c - c \otimes 1_C$, for $c \in \mathrm{Ker}(\epsilon_C)$, where $1_C = \eta_C(1_k)$. We shall denote such an element by $\Delta_{\mathrm{Ker}(\epsilon_C)}(c)$, or $c_{\bar{(1)}} \otimes c_{\bar{(2)}}$. A morphism of coaugmented dg coalgebras $f : C' \to C$ is a morphism of counitary dg coalgebras such that $f \circ \eta_{C'} = \eta_C$.

1.4.1 Twists of (augmented) dg algebras

One natural question is whether we may "perturb" the differential d_A of an augmented dg algebra A such that the resulting dg k-module is still an augmented dg algebra for the previous product, unit and augmentation. We will be concerned with the much simpler issue of finding a homogeneous element $a \in A^{(1,0_{G'})}$ such that $d_{A,a} = d_A + \mathrm{ad}(a)$, where $\mathrm{ad}(a) : A \to A$ is the morphism of graded k-modules of degree $(1, 0_{G'})$ given by $a' \mapsto [a, a'] = aa' - (-1)^{\deg a'} a'a$, is a differential, *i.e.* $d_{A,a}^2 = 0$, and A is an augmented dg algebra for the same product, unit and augmentation as before but new differential $d_{A,a}$. It is clear that the map $d_{A,a}$ is a differential if and only if $d_A(a) + aa = d_A(a) + [a, a]/2$ lies in the (graded) center of A (the latter identity having meaning if the characteristic of k is different from 2), and in particular if $d_A(a) + aa = 0$, which is called the *Maurer-Cartan equation* for a (this can be done in fact for any dg Lie algebra if the characteristic is different from 2). It is trivially verified that $d_{A,a}$ is always a derivation, that $d_{A,a}(1_A) = 0$

and $\epsilon_A \circ d_{A,a} = 0$, so A is an augmented algebra for $d_{A,a}$ if and only if the latter is a differential. This is always the case if a satisfies the Maurer-Cartan equation. The procedure of obtaining $d_{A,a}$ from d_A and an element a satisfying the Maurer-Cartan equation is usually called a *twist* of the differential, and the new augmented dg algebra is called the *twisted augmented dg algebra of* (A, d_A) *by* a. Moreover, we see that this latter twisting construction is natural, in the sense that if $f : A \to A'$ is a morphism of augmented dg algebras and $a \in A^{(1,0_{G'})}$ satisfies the Maurer-Cartan equation, then $f(a) \in (A')^{(1,0_{G'})}$ also satisfies the Maurer-Cartan equation and f can be regarded as a morphism of augmented dg algebras for A and A' provided with the new differentials $d_{A,a}$ and $d_{A',f(a)}$, respectively.

Suppose further that we are given a dg A-bimodule M of an augmented dg algebra A. Let us consider an element $a \in A^{(1,0_{G'})}$ solution to the Maurer-Cartan equation, so we may regard the new augmented dg algebra structure on A given by only changing the differential by $d_{A,a}$. One may wonder how to twist the differential d_M of M in order to still define a dg bimodule with the same action map $A \otimes M \otimes A \to M$ over the new augmented dg algebra A provided with the differential $d_{A,a}$. It is clear that the only condition one needs to verify for the new differential on M is the Leibniz identity, for the others are automatic. Furthermore, it is easily verified that the new differential $d_{M,a}$ given by $d_M + \mathrm{ad}(a)$, where $\mathrm{ad}(a) : M \to M$ is the morphism of graded k-modules of degree $(1, 0_{G'})$ given by $m \mapsto am - (-1)^{\deg m} ma$, satisfies the Leibniz identity and thus defines on M the structure of a dg bimodule over the augmented dg algebra A provided with the differential $d_{A,a}$. If M and N are two dg bimodules over the augmented dg algebra A with differential d_A, and $g : M \to N$ is a morphism of dg bimodules, then it is also a morphism of dg bimodules over the augmented dg algebra A provided with the differential $d_{A,a}$, when we regard M and N with differentials $d_{M,a}$ and $d_{N,a}$, respectively.

1.4.2 The convolution algebra and the tensor product module

The following constructions are well-known (see [20], Lemme 1.35). Let C be a coaugmented dg coalgebra a A an augmented dg algebra. Consider the dg k-module given by $\mathcal{H}om(C, A)$. It is in fact an augmented dg algebra with product given by

$$\phi * \psi = \mu_A \circ (\phi \otimes \psi) \circ \Delta_C,$$

unit $\eta_A \circ \epsilon_C$ and augmentation $\phi \mapsto (\epsilon_A \circ \phi \circ \eta_C)(1_k)$. We remark that, using the Sweedler notation, the coproduct can be written as

$$(\phi * \psi)(c) = (-1)^{\deg \psi \deg c_{(1)}} \phi(c_{(1)}) \psi(c_{(2)}),$$

for $c \in C$. Note that the previous construction is natural, *i.e.* if $f' : C' \to C$ is a morphism of coaugmented dg coalgebras and $f : A \to A'$ is a morphism of augmented dg algebras, then the morphism of dg k-modules $\mathcal{H}om(f', f) : \mathcal{H}om(C, A) \to \mathcal{H}om(C', A')$ given by $\phi \mapsto f \circ \phi \circ f'$ is in fact a morphism of augmented dg algebras. If k is semisimple, it is clear that if f' and f are quasi-isomorphisms, then $\mathcal{H}om(f', f)$ is so (see [3], §5.2. Cor. 1, and §2.5, Ex. 4).

Moreover, given M any dg bimodule over A we see that the dg k-module given by the tensor product $M \otimes C$ has a structure of a dg bimodule over $\mathcal{H}om(C, A)$. The action is given by

$$\phi \cdot (m \otimes c) \cdot \psi = (-1)^{\epsilon} \phi(c_{(3)}).m.\psi(c_{(1)}) \otimes c_{(2)},$$

where $m \in M$, $c \in C$, $\phi, \psi \in \mathcal{H}om(C, A)$, and

$$\epsilon = \deg \psi \deg c + \deg c_{(3)}(\deg m + \deg c_{(1)} + \deg c_{(2)} + \deg \psi).$$

If $f' : C' \to C$ is a morphism of coaugmented dg coalgebras and $g : M' \to M$ is a morphism of dg bimodules over A, then $g \otimes f'$ is a morphism of dg bimodules over $\mathcal{H}om(C, A)$, where $M' \otimes C'$ has the structure of dg bimodule over $\mathcal{H}om(C, A)$ given by the restriction of scalar through $\mathcal{H}om(f', \mathrm{id}_A) : \mathcal{H}om(C, A) \to \mathcal{H}om(C', A)$. Provided k is Von Neumann regular (in particular, this holds if k is semisimple), if f' and g are quasi-isomorphisms, then $g \otimes f'$ is also (see [3], §4.7. Théo. 3).

Suppose that M has in fact two graded-commuting dg A-bimodule structures, *i.e.* M is a dg A^e-bimodule (*e.g.* $M = A^e$). In this case one may use one of the dg bimodule structures over A on M to induce the dg bimodule structure over $\mathcal{H}om(C, A)$ on $M \otimes C$, whereas the second dg bimodule structure over A on M gives in fact a dg A-bimodule structure on $M \otimes C$ by the formula

$$a(m \otimes c)a' = (-1)^{\deg a' \deg c}(ama') \otimes c,$$

where we remark that we are using the second dg A-bimodule structure on M. Moreover, both structures are compatible, *i.e.* $M \otimes C$ has in fact a dg bimodule over $\mathcal{H}om(C, A) \otimes A$.

1.4.3 The twisted convolution algebra and the twisted tensor product

A solution $\tau \in \mathcal{H}om(C, A)$ to the Maurer-Cartan equation on the augmented dg algebra $\mathcal{H}om(C, A)$ which also satisfies that $\epsilon_C \circ \tau = 0$ and $\tau \circ \eta_A = 0$ is called a *twisting cochain* (some authors call these *admissible* twisting cochains, because for them the term twisting cochain is any solution of the Maurer-Cartan equation of $\mathcal{H}om(C, A)$. See for *e.g.* [14], Déf. 2.2.1.1). We will denote the augmented dg algebra $\mathcal{H}om(C, A)$ with the twisted differential $d_{\mathcal{H}om(C,A),\tau}$ by $\mathcal{H}om^{\tau}(C, A)$. By the last paragraph of the two previous subsubsections, it is easy to see that the twist construction is natural, *i.e.* given $\tau \in \mathcal{H}om(C, A)$ a twisting cochain, a morphism of coaugmented dg coalgebras $f' : C' \to C$ and a morphism of augmented dg algebras $f : A \to A'$, $\mathcal{H}om(f', f)(\tau) \in \mathcal{H}om(C', A')$ is a twisting cochain and $\mathcal{H}om(f', f)$ induces a morphism of augmented dg algebras from $\mathcal{H}om^{\tau}(C, A)$ to $\mathcal{H}om^{f \circ \tau \circ f'}(C', A')$. Even for k semisimple and f and f' quasi-isomorphisms, the latter map need not be a quasi-isomorphism. A typical example would be as follows. Consider the quasi-isomorphism $\mathcal{H}om(B^+(A_+), A_+) \to \mathcal{H}om(k, A_+) \simeq A_+$ of augmented dg algebras given by composition with the canonical injection $k \subseteq B^+(A_+)$, where A is a unitary algebra and A_+ is the augmented algebra recalled in the fifth paragraph of Subsubsection 1.4.5 (the fact that it is a quasi-isomorphism follows from the comments in that paragraph). It is clear that the image of the twisting cochain τ_{A_+} under the previous mapping is zero, so we get a morphism $\mathcal{H}om^{\tau_{A_+}}(B^+(A_+), A_+) \to A_+$ of augmented dg algebras. Taking cohomology we obtain the morphism $HH^{\bullet}(A_+) \to A_+$ given by the composition of the canonical projection $HH^{\bullet}(A_+) \to HH^0(A_+) \simeq \mathcal{Z}(A_+)$ together with the inclusion of the center $\mathcal{Z}(A_+)$ of A_+ inside A_+. If A is noncommutative we see that the previous map in cohomology is not an isomorphism.

Given M any dg bimodule over A, and a twisting cochain τ in $\mathcal{H}om(C, A)$, we see that we may twist the differential $d_{M \otimes C}$ of the tensor product $M \otimes C$, which is a dg bimodule over the augmented dg algebra $\mathcal{H}om(C, A)$, in order to obtain the dg bimodule provided with the differential $d_{M \otimes C, \tau}$ over the augmented dg algebra $\mathcal{H}om^{\tau}(C, A)$. We shall denote this new dg bimodule by $M \otimes_{\tau} C$. If $f'' : C' \to C$ is a morphism of coaugmented dg coalgebras and $g : M' \to M'$ is a morphism of dg bimodules over A, then $g \otimes f'$ is a morphism of dg bimodules over $\mathcal{H}om^{\tau}(C, A)$ from $M' \otimes_{\tau \circ f'} C'$ to $M \otimes_{\tau} C$, where we regard $M' \otimes_{\tau \circ f'} C'$ as a dg bimodule over $\mathcal{H}om^{\tau}(C, A)$ by means of the morphism of augmented dg algebras $\mathcal{H}om(f', \mathrm{id}_A) : \mathcal{H}om^{\tau}(C, A) \to \mathcal{H}om^{\tau \circ f'}(C', A)$. If M has two graded-commuting dg A-bimodule structures, so $M \otimes C$ is a dg bimodule over $\mathcal{H}om(C, A) \otimes A$, one notices that the previous twisting construction implies that $M \otimes_\tau C$ has in fact a dg bimodule structure over $\mathcal{H}om^\tau(C, A) \otimes A$.

1.4.4 The bar constructions

We point out the well-known fact that the dg k-module given by $k \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ is further a coaugmented dg coalgebra, called the *(reduced or normalized) bar construction* of A, and it is denoted by $B^+(A)$ (see [7], Section 19, but also [14], Notation 2.2.1.4, which we follow, though we do not use the same sign conventions). Note that in this case the reduced bar resolution $\overline{\mathrm{Bar}}(A)$ can be equivalently presented as the graded A-bimodule $\oplus_{i\in\mathbb{N}_0} A \otimes I_A[1]^{\otimes i} \otimes A$ provided with a differential given by the same expression, the isomorphism being induced by the obvious map $I_A \to A/k$. Moreover, using this identification we may construct an explicit quasi-inverse to the canonical quasi-isomorphism $\mathrm{Bar}(A) \to \overline{\mathrm{Bar}}(A)$ explained in the third paragraph of Subsection 1.3. Indeed, it is easy to check that the map

$$\oplus_{i\in\mathbb{N}_0} A \otimes I_A[1]^{\otimes i} \otimes A \to \oplus_{i\in\mathbb{N}_0} A \otimes A[1]^{\otimes i} \otimes A$$

induced by the inclusion $I_A[1] \subseteq A[1]$ is such a quasi-isomorphism. Now, using the obvious isomorphism

$$k \otimes_{A^e} \overline{\mathrm{Bar}}(A) \simeq \bigoplus_{i\in\mathbb{N}_0} I_A[1]^{\otimes i},$$

induced by the identification explained previously, the coproduct is given by the usual deconcatenation

$$\Delta([a_1|\dots|a_n]) = \sum_{i=0}^{n} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n],$$

where we set $[a_i|\dots|a_j] = 1_{B^+(A)}$ if $i > j$, for $1_{B^+(A)}$ the image of 1_k under the canonical inclusion $k = A[1]^{\otimes 0} \subseteq B^+(A)$. The differential, denoted by B, is trivially seen to be of the form

$$\begin{aligned} B([a_1|\dots|a_n]) = &- \sum_{i=1}^{n} (-1)^{\epsilon_i} [a_1|\dots|d_A(a_i)|\dots|a_n] \\ &+ \sum_{i=2}^{n} (-1)^{\epsilon_i} [a_1|\dots|a_{i-1}a_i|\dots|a_n], \end{aligned}$$

where $\epsilon_i = (\sum_{j=1}^{i-1} \deg a_j) - i + 1$. One checks easily that it is a coderivation. The counit is given by the canonical projection $B^+(A) \to I_A[1]^{\otimes 0} = k$, and the coaugmentation is defined as the obvious inclusion $k = I_A[1]^{\otimes 0} \subseteq B^+(A)$. Since $B^+(A)$ is a coaugmented tensor graded coalgebra, it is *cocomplete*. The image of its differential B lies inside the augmentation kernel $\mathrm{Ker}(\epsilon_{B^+(A)})$ of $B^+(A)$, so B is thus uniquely determined by $\pi_1 \circ B$, where $\pi_1 : B^+(A) \to I_A[1]$ is the canonical projection (see [14], Lemme 1.1.2.2, Sections 2.1.1 and 2.1.2, and Notation 2.2.1.4). We recall that a coaugmented graded coalgebra C is cocomplete (or sometimes denoted as *conilpotent*) if the (quotient) nonunitary graded coalgebra $J_C = C/\mathrm{Im}(\eta_C)$ satisfies that its *primitive filtration* given by

$$\mathrm{Ker}(\Delta_{J_C}) \subseteq \mathrm{Ker}(\Delta^{(3)}_{J_C}) \subseteq \dots \subseteq \mathrm{Ker}(\Delta^{(n)}_{J_C}) \subseteq \dots$$

is exhausting, *i.e.* its union is J_C. This composition map $\pi_1 \circ B$ is written as the sum of two terms $b_1 : I_A[1] \to I_A[1]$ and $b_2 : I_A[1]^{\otimes 2} \to I_A[1]$ given by $sa \mapsto -sda$

and $sa \otimes sb \mapsto (-1)^{\deg a+1} s(ab)$, for $a, b \in A$ homogeneous, resp. We note thus that $b_1 = -s_{I_A} \circ d_A \circ s_{I_A}^{-1}$ and $b_2 = -s_{I_A} \circ \mu_A \circ (s_{I_A}^{\otimes 2})^{-1}$. We warn the reader that even though we have set up our sign conventions for the differential of the bar construction in order to agree with several in the literature (*e.g.* [7], Section 19, or [15], Section 8), and in particular they coincide with the "universally" accepted conventions in case our dg algebra is a plain algebra, they differ from others (*e.g.* those in the thesis [14] of K. Lefèvre-Hasegawa, Ch. 1 and 2). By very definition, the cohomology of $B^+(A)$ is the Tor group $\mathrm{Tor}^A_\bullet(k, k)$, where we recall that we should switch from cohomological (upper) grading to homological (lower) grading by the obvious relation $V^n = V_{-n}$, for $n \in \mathbb{Z}$, and where the Adams degree does not change. We also note that, under this identification a morphism of cohomological degree n becomes a morphism of homological degree $-n$.

We remark that there is a *nonreduced (or unnormalized) bar construction* $\tilde{B}^+(A)$ of A given as a dg module over k by $k \otimes_{A^e} \mathrm{Bar}(A)$, which in turn is canonically isomorphic to $\oplus_{i \in \mathbb{N}_0} A[1]^{\otimes i}$ as graded k-modules. It is also a coaugmented dg coalgebra, and the formulas for the coproduct, the counit and the coaugmentation are the same as for the reduced bar construction. Its underlying coaugmented graded coalgebra structure is thus the one of a coaugmented tensor graded coalgebra. The explicit form of the differential, which we denote by $\tilde{B}$, is however different from the reduced case, namely, $\tilde{B}([]) = 0$ and for $n \in \mathbb{N}$ we have that

$$\begin{aligned}\tilde{B}([a_1|\dots|a_n]) = &-\sum_{i=1}^{n} (-1)^{\epsilon_i} [a_1|\dots|d_A(a_i)|\dots|a_n] \\ &+ \sum_{i=2}^{n} (-1)^{\epsilon_i} [a_1|\dots|a_{i-1}a_i|\dots|a_n] \\ &+ \epsilon_A(a_1)[a_2|\dots|a_n] - (-1)^{\epsilon_n} \epsilon(a_n)[a_1|\dots|a_{n-1}],\end{aligned}$$

where $\epsilon_i = (\sum_{j=1}^{i-1} \deg a_j) - i + 1$, and $a_1, \dots, a_n \in A$. A rather long computation shows it is in fact a coderivation. It is not difficult to verify that the image of $\tilde{B}$ lies inside the augmentation kernel of $\tilde{B}^+(A)$, so it is uniquely determined by its composition with the canonical projection $p_1 : \tilde{B}^+(A) \to A[1]$, which is just the sum of two terms $\tilde{b}_1 : A[1] \to A[1]$ and $\tilde{b}_2 : A[1]^{\otimes 2} \to A[1]$ given by $sa \mapsto -sda$ and $sa \otimes sb \mapsto (-1)^{\deg a+1} s(ab) + \epsilon_A(a)sb - (-1)^{\deg a+1}\epsilon_A(b)sa$, for $a, b \in A$ homogeneous, resp. Note that the definition just given before does not coincide with the one given in [8], for our differential is different (due to the last two terms in the explicit expression of $\tilde{B}$). In particular, their definition of nonreduced bar construction of a unitary dg algebra is always quasi-isomorphic to k, whereas in our case it will be quasi-isomorphic to the reduced bar construction.

Moreover, the quasi-isomorphism $\mathrm{Bar}(A) \to \overline{\mathrm{Bar}}(A)$ of dg A-bimodules given by taking quotients and recalled in the third paragraph of Subsection 1.3 induces a quasi-isomorphism $\tilde{B}^+(A) \to B^+(A)$ of dg modules over k also given by taking quotients. By the explicit expressions for the coproduct, the counit and the coaugmentation for both bar constructions the previous map is in fact a quasi-isomorphism of coaugmented dg coalgebras. Furthermore, the quasi-isomorphism $\overline{\mathrm{Bar}}(A) \to \mathrm{Bar}(A)$ of dg A-bimodules induced by the inclusion $I_A[1] \subseteq A[1]$ also induces a quasi-isomorphism of coaugmented dg coalgebras $B^+(A) \to \tilde{B}^+(A)$, which is the quasi-inverse to the map $\tilde{B}^+(A) \to B^+(A)$.

We also want to stress the fact that the bar constructions are functorial. Indeed, if $f : A \to A'$ is a morphism of augmented dg algebras, the unique morphism of coaugmented graded coalgebras $B^+(f) : B^+(A) \to B^+(A')$ whose composition with the canonical projection $\pi_1' : B^+(A') \to I_{A'}[1]$, where $I_{A'}$ is the augmentation kernel of A', is given by the composition of the canonical projection

$\pi_1 : B^+(A) \to I_A[1]$, where I_A is the augmentation kernel of A, together with $-f^{[1]}_{[1]}$ commutes with the differentials, so it gives a morphism of coaugmented dg coalgebras. It can be described explicitly as $B^+(f)([a_1|\dots|a_n]) = [f(a_1)|\dots|f(a_n)]$, for $n \in \mathbb{N}_0$ and $a_1, \dots, a_n \in A$. We remark that the minus sign in front of the $f^{[1]}_{[1]}$ was not quite arbitrary: had we chosen the plus sign, then we should have added a $(-1)^n$ sign to the previous expression of $B^+(f)$. This would imply in particular that $B^+(\mathrm{id}_A)$ would be different from $\mathrm{id}_{B^+(A)}$, so it would not be a functorial choice. The explicit expression of the corresponding morphism of coaugmented dg coalgebras $\tilde{B}^+(f) : \tilde{B}^+(A) \to \tilde{B}^+(A')$ for the nonreduced bar construction is the same as for the reduced one. Moreover, the quasi-isomorphisms $\tilde{B}^+(A) \to B^+(A)$ and $B^+(A) \to \tilde{B}^+(A)$ of coaugmented dg coalgebras described in the previous paragraph yield in fact natural transformations. It is not difficult to show that, given a morphism of augmented dg algebras $f : A \to A'$ over a Von Neumann regular ring k, f is a quasi-isomorphism if and only if $B^+(f)$ (or $\tilde{B}^+(f)$) is also a quasi-isomorphism (see [7], Section 19, Ex. 1, p. 271, or [14], Lemme 1.3.2.2, and Lemme 1.3.2.3, (a) and (b)).

If A is an augmented dg algebra and we consider its bar construction $B^+(A)$, there is in fact a *(normalized) universal twisting cochain* $\tau_A : B^+(A) \to A$ of A whose restriction to $I_A[1]^{\otimes n}$ vanishes if $n \neq 1$ and such that its restriction to $I_A[1]$ is given by minus the composition of canonical inclusion $I_A[1] \subseteq A[1]$ with s_A^{-1}. Analogously, we also have a a *unnormalized universal twisting cochain* $\tilde{\tau}_A : \tilde{B}^+(A) \to A$ of A whose restriction to $A[1]^{\otimes n}$ vanishes if $n \neq 1$ and such that its restriction to $A[1]$ is given by the composition of minus s_A^{-1}, the projection $A \to I_A$ (which can be described by the composition of the canonical projection $A \to A/k.1_A$ with the identification $A/k.1_A \simeq I_A$) and the inclusion $I_A \to A$. Note that, if $f : A \to A'$ is morphism of augmented dg algebras, then the morphism $B^+(f)$ satisfies that

$$f \circ \tau_A = \tau_{A'} \circ B^+(f) \tag{1.5}$$

and the same expression holds for the corresponding morphism $\tilde{B}^+(f)$ by considering the unnormalized universal twisting cochains.

1.4.5 The cobar constructions

Even though an analogous definition of the (un)reduced bar resolution for coaugmented dg coalgebras would be possible (*e.g.* following the lines of Drinfeld's idea), we would recall a more *ad-hoc* definition of the (un)reduced cobar constructions, without passing through the corresponding cobar resolutions. Given a coaugmented dg coalgebra C, $\Omega^+(C)$ denotes the augmented dg algebra called the *(reduced or normalized) cobar construction* of C (see *e.g.* [7], Section 19, or [15], Def. 8.4). Its underlying augmented graded algebra structure is given by the tensor algebra on the graded vector space $J_C[-1] = C/\operatorname{Im}(\eta_C)[-1]$, with the product defined by concatenation, unit given by the inclusion of k and the obvious augmentation given by the canonical projection onto k. Since the algebra is free, a differential D is uniquely determined by its restriction to $J_C[-1]$, which we take to be the sum of two terms

$$-s^{-1}_{J_C[-1]} \circ d_C \circ s_{J_C[-1]}$$

and

$$(s^{\otimes 2}_{J_C[-1]})^{-1} \circ \Delta_{J_C} \circ s_{J_C[-1]}.$$

If $n \in \mathbb{N}$, an element of the form $s^{-1}_{J_C[-1]}(\bar{c}_1) \otimes \dots \otimes s^{-1}_{J_C[-1]}(\bar{c}_n)$ will be typically denoted by $\langle c_1|\dots|c_n\rangle$, where $c_1, \dots, c_n \in C$. Analogously, we may denote by $\langle\rangle$

the unit $1_{\Omega^+(C)}$ of the algebra $\Omega^+(C)$. We may now write the differential D more explicitly as

$$\begin{aligned} D(\langle c_1|\dots|c_n\rangle) = &- \sum_{i=1}^{n}(-1)^{\epsilon_i}\langle c_1|\dots|c_{i-1}|d_C(c_i)|c_{i+1}|\dots|c_n\rangle \\ &+ \sum_{i=1}^{n}(-1)^{\epsilon_i+\deg c^-_{i,(1)}+1}\langle c_1|\dots|c_{i-1}|c^-_{i,(1)}|c^-_{i,(2)}|c_{i+1}|\dots|c_n\rangle, \end{aligned}$$

where $\Delta_{J_C}(c_i) = c^-_{i,(1)} \otimes c^-_{i,(2)}$ and $\epsilon_i = (\sum_{j=1}^{i-1}\deg c_j) - i + 1$. Note the coincidence with the sign in [15], Def. 8.4, but the difference with the one given in [8], Section 2.7. Notice that, using the identification between J_C and $\mathrm{Ker}(\epsilon_C)$ given in the first paragraph of Subsection 1.4, we may have equivalently presented the reduced cobar construction of C as the tensor algebra on the graded vector space $\mathrm{Ker}(\epsilon_C)$, with the product defined by concatenation, unit given by the inclusion of k and the obvious augmentation given by the canonical projection onto k. The differential given by the same expression as before, but one should use $\Delta_{\mathrm{Ker}(\epsilon_C)}$, defined in the first paragraph of Subsection 1.4, instead of Δ_{J_C}.

There is also a *nonreduced (or unnormalized) cobar construction* $\tilde{\Omega}^+(C)$ of C, whose underlying augmented graded algebra structure is given by the tensor algebra on the graded vector space $C[-1]$, with the product defined by concatenation, unit given by the inclusion of k and the obvious augmentation given by the canonical projection onto k. As in the reduced case, an element of the form $s^{-1}_{C[-1]}(c_1) \otimes \dots \otimes s^{-1}_{C[-1]}(c_n)$ will be usually denoted by $\langle c_1|\dots|c_n\rangle$, where $c_1,\dots,c_n \in C$. The explicit expression of the differential is given by

$$\begin{aligned} \tilde{D}(\langle c_1|\dots|c_n\rangle) = &- \sum_{i=1}^{n}(-1)^{\epsilon_i}\langle c_1|\dots|c_{i-1}|d_C(c_i)|c_{i+1}|\dots|c_n\rangle \\ &+ \sum_{i=1}^{n}(-1)^{\epsilon_i+\deg c_{i,(1)}+1}\langle c_1|\dots|c_{i-1}|c_{i,(1)}|c_{i,(2)}|c_{i+1}|\dots|c_n\rangle \\ &+ \langle 1_C|c_1|\dots|c_n\rangle - (-1)^{\epsilon_n}\langle c_1|\dots|c_n|1_C\rangle, \end{aligned}$$

where $\Delta_C(c_i) = c_{i,(1)} \otimes c_{i,(2)}$. One may easily check that it is a derivation of square zero. As for the bar construction, there is a quasi-isomorphism of augmented dg algebras $\tilde{\Omega}^+(C) \to \Omega^+(C)$ given by the direct sum of the maps $(-q^{[1]}_{[1]})^{\otimes n}$, for $n \in \mathbb{N}_0$, where q denotes the canonical projection $C \to C/\mathrm{Im}(\eta_C)$. Note that by convention, the underlying maps of $-q^{[1]}_{[1]}$ and of q coincide. Furthermore, the previous quasi-isomorphism has a quasi-inverse given by the morphism of augmented dg algebras $\Omega^+(C) \to \tilde{\Omega}^+(C)$ induced by the inclusion $\mathrm{Ker}(\epsilon_C)[-1] \subseteq C[-1]$, where we have also used the identification between J_C and $\mathrm{Ker}(\epsilon_C)$ given in the first paragraph of this subsection. Notice again that our definition of nonreduced cobar construction differs from the one given for instance in [8], Subsection 2.7.

If C is a coaugmented dg coalgebra and we consider its cobar construction $\Omega^+(C)$, there is in fact a *universal (normalized) twisting cochain* $\tau^C : C \to \Omega^+(C)$ of C, given by the composition of the canonical projection $C \to C/\mathrm{Im}(\eta_C)$, $s^{-1}_{C/\mathrm{Im}(\eta_C)[-1]}$ and the canonical inclusion of $C/\mathrm{Im}(\eta_C)[-1]$ inside $\Omega^+(C)$. Analogously, we also have a a *universal (unnormalized) twisting cochain* $\tau^C : C \to \tilde{\Omega}^+(C)$ of C given by the composition of the projection $C \to \mathrm{Ker}(\epsilon_C)$ (given by the composition of the canonical projection $C \to C/\mathrm{Im}(\eta_C)$ and the identification $C/\mathrm{Im}(\eta_C) \simeq \mathrm{Ker}(\epsilon_C)$), the canonical inclusion $\mathrm{Ker}(\epsilon_C) \subseteq C$, $s^{-1}_{C[-1]}$ and the canonical inclusion of $C[-1]$ inside $\tilde{\Omega}^+(C)$. The adjective for the twisting cochains τ_A and τ^C is justified. Indeed, given an augmented dg algebra A and a coaugmented dg coalgebra C, if

$\mathrm{Tw}(C, A)$ denote the set of twisting cochains from C to A, we have canonical maps

$$\mathrm{Hom}_{\text{aug-dg-alg}}(\Omega^+(C), A) \to \mathrm{Tw}(C, A) \leftarrow \mathrm{Hom}_{\text{coaug-dg-coalg}}(C, B^+(A)), \qquad (1.6)$$

where the left map is given by $g \mapsto g \circ \tau^C$ and the right one by $f \mapsto \tau_A \circ f$, the first space of homomorphism denotes the set of morphisms of augmented dg algebras and the last one the set of morphisms of coaugmented dg coalgebras. It is clear that left map is a bijection, and the same holds for the right one provided C is cocomplete (see [14], Lemme 2.2.1.5). Moreover these maps are clearly seen to be natural.

We also remark that the cobar constructions are functorial. Given $f : C \to C'$ a morphism of coaugmented dg coalgebras, the unique morphism of augmented graded algebras $\Omega^+(f) : \Omega^+(C) \to \Omega^+(C')$ whose restriction to $J_C[-1]$, where J_C is the coaugmentation cokernel of C, is given by the composition of $-f^{[-1]}_{[-1]}$ with the canonical inclusion $J_{C'}[-1] \to \Omega^+(C')$, where $J_{C'}$ is the coaugmentation cokernel of C', commutes with the differentials, so it gives a morphism of augmented dg algebras. It is given explicitly by $\Omega^+(f)(\langle c_1|\dots|c_n\rangle) = \langle f(c_1)|\dots|f(c_n)\rangle$, for $n \in \mathbb{N}_0$ and $c_1, \dots, c_n \in C$. As for the bar construction we stress that the minus sign in front of the $f^{[-1]}_{[-1]}$ was not really arbitrary, since a plus sign would have added a $(-1)^n$ sign to the previous expression of $\Omega^+(f)$, so in particular $\Omega^+(\mathrm{id}_C)$ would be different from $\mathrm{id}_{\Omega^+(C)}$ and it would not be a functorial choice. The explicit expression of the corresponding morphism of augmented dg algebras $\tilde{\Omega}^+(f) : \tilde{\Omega}^+(C) \to \tilde{\Omega}^+(C')$ for the nonreduced cobar construction is the same as for the reduced one. Moreover, the quasi-isomorphisms $\tilde{\Omega}^+(C) \to \Omega^+(C)$ and $\Omega^+(C) \to \tilde{\Omega}^+(C)$ of augmented dg algebras described in the previous paragraph yield in fact natural transformations. Note that, if $f : C \to C'$ is a morphism of coaugmented dg coalgebras, then

$$\tau^C \circ f = \Omega^+(f) \circ \tau^{C'} \qquad (1.7)$$

and the same expression holds for the corresponding morphism $\tilde{\Omega}^+(f)$ by considering the unnormalized universal twisting cochains.

We would like to note that, in contrast with the property of preservation of quasi-isomorphisms of the bar construction(s) for augmented dg algebras, it may occur that a morphism of coaugmented dg coalgebras $f : C \to C'$ is a quasi-isomorphism such that $\Omega^+(f)$ (or $\tilde{\Omega}^+(f)$) is not. A typical example of such a situation may be constructed as follows. We will also assume in the rest of this subsubsection that k is a semisimple. Take A a unitary dg algebra and consider the augmented dg algebra A_+ whose underlying dg k-module is given by $A \oplus k$, and with product $(a, \lambda) \cdot (a', \lambda') = (aa' + a\lambda' + \lambda a, \lambda\lambda')$, unit η_{A_+} given the canonical inclusion $k \subseteq A_+$ of the form $\lambda \mapsto (0_A, \lambda)$ and augmentation ϵ_{A_+} given by the canonical projection $A_+ \to k$. It is easy to check that $B^+(\eta_{A_+})$ and $B^+(\epsilon_{A_+})$ are in fact quasi-isomorphisms of coaugmented dg coalgebras, for the underlying dg k-module of $B^+(A_+)$ is just $k \oplus \mathrm{Bar}(A)[2]$, but $\Omega^+(B^+(\eta_{A_+}))$ and $\Omega^+(B^+(\epsilon_{A_+}))$ are not, since $\Omega^+(B^+(A_+))$ is quasi-isomorphic to A_+. We remark however that if C and C' are cocomplete, and either $\Omega^+(f)$ or $\tilde{\Omega}^+(f)$ is a quasi-isomorphism, then f is also (see for instance [14], Lemme 1.3.2.2, and Lemme 1.3.2.3, (a) and (c). For this last item see also the corresponding errata). Following the terminology of K. Lefèvre-Hasegawa (see [14], the definitions before Thm. 1.2.1.2), we will say that morphism of cocomplete coaugmented dg coalgebras $f : C \to C'$ is a *weak equivalence* if $\Omega^+(f)$ (or, equivalently, $\tilde{\Omega}^+(f)$) is a quasi-isomorphism. There is a standard criterion for a morphism f of cocomplete coaugmented dg coalgebras to be a weak equivalence, which we now recall. In order to do so, we need to introduce the following definitions. We say that a coaugmented dg coalgebra C

has an *admissible filtration* (or that C is *admissibly filtered*), if there is an exhaustive increasing sequence $\{C_i\}_{i\in\mathbb{N}_0}$ of coaugmented dg subcoalgebras of C, such that $C_0 = k.1_C$. The primitive filtration of a cocomplete coaugmented dg coalgebra C, given by $C_i = \mathrm{Ker}(\Delta_{J_C}^{(i+1)}) \oplus k.1_C$, if $i \in \mathbb{N}$ and $C_0 = k.1_C$, is admissible, by its very definition. It is clear in this case that it is in fact a filtration of C by coaugmented dg coalgebras, which induces a filtration of augmented dg algebras on $\Omega^+(C)$, that induces in turn an admissible filtration on $B^+(\Omega^+(C))$, which is called the *C-primitive filtration*. Given a coaugmented dg coalgebra C with an admissible filtration, we may construct the *associated graded object*,

$$\mathrm{Gr}_{C_\bullet}(C) = \bigoplus_{i\in\mathbb{N}_0} C_i/C_{i-1},$$

that may be in principle regarded as a graded k-module over the grading group $G \times \mathbb{Z}$, where the last factor comes from the index i of the filtration, that will be called the *filtration grading*. Now, we further provide with the unique differential induced by that of C which preserves the filtration grading, and it thus becomes a dg module over k. Given two coaugmented dg coalgebras C and C' provided with admissible filtrations $\{C_i\}_{i\in\mathbb{N}_0}$ and $\{C'_i\}_{i\in\mathbb{N}_0}$, resp., a *filtered morphism* is a morphism $f : C \to C'$ of coaugmented dg coalgebras such that $f(C_i) \subseteq C'_i$, for all $i \in \mathbb{N}_0$. It further induces a unique morphism $\mathrm{Gr}(f) : \mathrm{Gr}_{C_\bullet}(C) \to \mathrm{Gr}_{C'_\bullet}(C')$ of dg k-modules preserving the filtration grading, called the *associated graded morphism*. We say that f is a *filtered quasi-isomorphism* if the associated graded morphism $\mathrm{Gr}(f)$ is a quasi-isomorphism. Given a filtered morphism $f : C \to C'$ of cocomplete coaugmented dg coalgebras provided with admissible filtrations, it can be proved that f is a weak equivalence if it is a filtered quasi-isomorphism (see [14], Lemme 1.3.2.2). Moreover, given a morphism $f : A \to A'$ of augmented dg algebras, $B^+(f)$ is a filtered morphism of coaugmented dg coalgebras, where $B^+(A)$ and $B^+(A')$ are provided with the primitive filtration (see [14], Lemme 1.3.2.3, (a)). Note that these filtrations are admissible, for both $B^+(A)$ and $B^+(A')$ are cocomplete. The same comments apply to the nonreduced bar construction, and moreover the quasi-isomorphisms $\tilde{B}^+(A) \to B^+(A)$ and $B^+(A) \to \tilde{B}^+(A)$ described in the previous subsection are filtered for the primitive filtrations of the cocomplete dg coalgebras $\tilde{B}^+(A)$ and $B^+(A)$, so the former quasi-isomorphisms are in fact weak equivalences.

Finally, we recall that the canonical map $\beta_A : \Omega^+(B^+(A)) \to A$ given by $\langle\rangle \mapsto 1_A$ and $\langle\omega_1|\dots|\omega_n\rangle \mapsto (-1)^n s_{I_A}^{-1}(\pi_1(\omega_1))\dots s_{I_A}^{-1}(\pi_1(\omega_n))$ if $n \in \mathbb{N}$, where $\pi_1 : B^+(A) \to I_A[1]$ is the canonical projection and $\omega_1, \dots, \omega_n$ are elements in the coaugmentation cokernel of $B^+(A)$, is a quasi-isomorphism of augmented dg algebras (see [7], Section 19, Ex. 2, or [13], Section II.4, Thm. II.4.4, or [20], Th. 2.28). It is the unique morphism of augmented dg algebras satisfying that the composition of $\tau^{B^+(A)}$ with it is τ_A. We have also the morphism of coaugmented dg coalgebras $\beta^C : C \to B^+(\Omega^+(C))$ given by the unique such morphism that its composition with $\tau_{\Omega^+(C)} : B^+(\Omega^+(C)) \to \Omega^+(C)$ is τ^C. Hence β^C sends 1_C to $1_{B^+(\Omega^+(C))}$, and for $c \in \mathrm{Ker}(\epsilon_C)$, it satisfies that

$$\beta^C(c) = -[\langle c\rangle] + \sum_{n\in\mathbb{N}_{\geq 2}} (-1)^n [\langle c^-_{(1)}\rangle|\dots|\langle c^-_{(n)}\rangle],$$

where $\Delta^{(n)}_{\mathrm{Ker}(\epsilon_C)}(c) = c^-_{(1)} \otimes \dots \otimes c^-_{(n)}$ is the iterated coproduct of the comultiplication indicated in the first paragraph of Subsection 1.4. If C is cocomplete, β^C is a filtered quasi-isomorphism, where C is provided with the primitive filtration and $B^+(\Omega^+(C))$ with the C-primitive one (see [14], Lemme 1.3.2.3, (c), and the corresponding errata).

The previous comments may be used in order to provide a simpler resolution than the bar resolution of an augmented dg algebra A, when $A = \Omega^+(C)$, where C is a cocomplete coaugmented dg coalgebra. In this case, we already know that the reduced bar resolution of $\Omega^+(C)$ is isomorphic to $\Omega^+(C)^e \otimes_{\tau_{\Omega^+(C)}} B^+(\Omega^+(C))$. However, since $\beta^C : C \to B^+(\Omega^+(C))$ is a quasi-isomorphism of coaugmented dg coalgebras which satisfies by definition that $\tau_{\Omega^+(C)} \circ \beta^C = \tau^C$, then

$$\mathrm{id}_{\Omega^+(C)^e} \otimes \beta^C : \Omega^+(C)^e \otimes_{\tau^C} C \to \Omega^+(C)^e \otimes_{\tau_{\Omega^+(C)}} B^+(\Omega^+(C))$$

is a quasi-isomorphism (see [14], Prop. 2.2.4.1). Furthermore, the map is clearly a morphism of dg bimodules over $\Omega^+(C)$, where the action by the last tensor factor is given by the inner structure of $\Omega^+(C)^e$. Using the identification $\Omega^+(C)^e \otimes_{\tau^C} C \to \Omega^+(C) \otimes C \otimes \Omega^+(C)$ of graded bimodules over $\Omega^+(C)$ given by $\omega' \otimes \omega \otimes c \mapsto (-1)^{\deg \omega'(\deg \omega + \deg c)} \omega \otimes c \otimes \omega'$, the differential of the latter space becomes

$$\begin{aligned} \omega \otimes c \otimes \omega' \mapsto & D(\omega) \otimes c \otimes \omega' + (-1)^{\deg \omega} \omega \otimes d_C(c) \otimes \omega' \\ & + (-1)^{\deg \omega + \deg c} \omega \otimes c \otimes D(\omega') \\ & + (-1)^{\deg \omega + \deg c_{(1)}} \omega \otimes c_{(1)} \otimes \tau^C(c_{(2)}) \omega' \\ & - (-1)^{\deg \omega} \omega \tau^C(c_{(1)}) \otimes c_{(2)} \otimes \omega', \end{aligned}$$

where $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$. Interestingly, a quasi-inverse for the previous map $\mathrm{id}_{\Omega^+(C)^e} \otimes \beta^C$ can be easily constructed (see [22], Théo. 1.4). Indeed, the morphism

$$\gamma^C : \overline{\mathrm{Bar}}(\Omega^+(C)) \to \Omega^+(C) \otimes C \otimes \Omega^+(C)$$

of graded $\Omega^+(C)$-bimodules satisfying that $\gamma^C(\omega_0[]\omega_1) = \omega_0 \otimes 1_C \otimes \omega_1$ and

$$\gamma^C(\omega_0[\omega_1|\dots|\omega_m]\omega_{m+1}) = \delta_{m,1} \sum_{j=1}^{n} (-1)^{\varepsilon_j + 1} \omega_0 \langle c_1|\dots|c_{j-1}\rangle \otimes c_j \otimes \langle c_{j+1}|\dots|c_n\rangle \omega_2$$

if $m \in \mathbb{N}$, where $\omega_1 = \langle c_1|\dots|c_n\rangle$ and $\varepsilon_j = (\sum_{l=1}^{j-1} \deg c_l) + j - 1$, is in fact a morphism of dg bimodules over $\Omega^+(C)$, and it satisfies that it is a left inverse of $\mathrm{id}_{\Omega^+(C)^e} \otimes \beta^C$ (after using the canonical identifications explained before).

1.4.6 Usual duality between the bar and cobar construction

We will assume in this subsubsection that k is a field. A graded or dg vector space M is *locally finite dimensional* if each of the homogeneous components of M is finite dimensional. Note that in this case the graded dual $M^{\#}$ is also locally finite dimensional and the canonical map $\iota_M : M \to (M^{\#})^{\#}$ given by $\iota_M(m)(f) = (-1)^{\deg m \deg f} f(m)$, for $m \in M$ and $f \in M^{\#}$ homogeneous, is an isomorphism of graded or dg vector spaces. Furthermore, if M and N are locally finite dimensional, then $\iota_{M,N}$ is an isomorphism in the corresponding category.

We now recall the well-known fact that if C is a coaugmented dg coalgebra, the graded dual $C^{\#}$ has a structure of augmented dg algebra, where the product is given by $\Delta_C^{\#} \circ \iota_{C,C}$, unit ϵ_C and augmentation given by $\omega \mapsto \omega(\eta_C(1_k))$, for $\omega \in C^{\#}$. Conversely if A is a locally finite dimensional augmented dg algebra, then the graded dual $A^{\#}$ has a structure of (locally finite dimensional) coaugmented dg coalgebra, where the product is given by $\iota_{A,A}^{-1} \circ \mu_A^{\#}$, counit given by $\omega \mapsto \omega(\eta_A(1_k))$, for $\omega \in A^{\#}$, and coaugmentation given by $1_k \mapsto \epsilon_A$. Note that in this latter case the morphism $\iota_A : A \to (A^{\#})^{\#}$ is in fact an isomorphism of augmented dg algebras. Analogously, if C is a locally finite dimensional coaugmented dg coalgebra, then $\iota_C : C \to (C^{\#})^{\#}$ is also an isomorphism of coaugmented dg coalgebras.

The main duality properties we shall use between the bar and cobar constructions are the following ones. We shall state them for the reduced bar and cobar construction, though the exact same results *mutatis mutandi* are obtained for the nonreduced ones. In particular, all the analogous morphisms for the nonreduced cases will be denoted just adding a tilde over the corresponding nonreduced ones that will be explicitly stated in the rest of the subsubsection. If Λ is locally finite dimensional augmented dg algebra such that $B^+(\Lambda)$ is also locally finite dimensional, then $B^+(\Lambda)^\#$ is canonically isomorphic to $\Omega^+(\Lambda^\#)$, as augmented dg algebras (see [15], Lemma 8.6, (c), where, using the notation of that article, one should further impose that ΩC and BA are locally finite dimensional. The same correction would apply to [16], Lemma 1.15. They are more or less a consequence of [7], Section 19, Ex. 3, p. 272). The isomorphism $j_\Lambda : \Omega^+(\Lambda^\#) \to B^+(\Lambda)^\#$ is the unique one satisfying that its restriction to the coaugmentation cokernel $J_{\lambda^\#} \simeq I_\Lambda^\#$ of $\Lambda^\#$ is minus the graded dual of the canonical projection $B^+(\Lambda) \to I_\Lambda[1]$, where I_Λ is the augmentation ideal of Λ (using the identification $I_\Lambda^\# \simeq J_{\lambda^\#}$ induced by the graded dual of the inclusion $I_\Lambda \subseteq \Lambda$, and the isomorphism $H_{\Lambda,k,(1,0_{G'}),0_G} : (\Lambda^\#)[-1] \to (\Lambda[1])^\#$ of dg modules explained in the sixth paragraph of Subsection 1.1), and it may be explicitly given as follows. We remark that the choice of signs is exactly the one in order to make our map commute with the differentials. We will provide an explicit expression of this isomorphism. For $n \in \mathbb{N}$ and $\omega_1, \dots, \omega_n \in I_\Lambda^\#$ the morphism sends $\langle \omega_1 | \dots | \omega_n \rangle$ to the linear functional

$$[\lambda_1 | \dots | \lambda_m] \mapsto (-1)^\epsilon \delta_{n,m} \omega_1(\lambda_1) \dots \omega_n(\lambda_n),$$

where $\lambda_1, \dots, \lambda_m \in \Lambda$, $\delta_{n,m}$ is the Dirac delta sign,

$$\begin{aligned} \epsilon = &\deg \omega_1 + \dots + \deg \omega_n + n + (\deg \omega_2 + 1)(\deg \lambda_1 + 1) \\ &+ \dots + (\deg \omega_n + 1)(\deg \lambda_1 + \dots + \deg \lambda_{n-1} + n - 1), \end{aligned}$$

and it sends $1 \in \Omega^+(\Lambda^\#)$ to the canonical projection $B^+(\Lambda) \to k$. We remark that the previous morphism is in fact a natural isomorphism between the functors $\Omega^+((-)^\#)$ and $B^+(-)^\#$, *i.e.* if $f : \Lambda \to \Lambda'$ is a morphism of augmented dg algebras, then it can be directly verified from the explicit expression of the morphisms involved that we have the commutative diagram

$$\begin{array}{ccc} \Omega^+((\Lambda')^\#) & \xrightarrow{j_{\Lambda'}} & B^+(\Lambda')^\# \\ \big\downarrow{\scriptstyle \Omega^+(f^\#)} & & \big\downarrow{\scriptstyle B^+(f)^\#} \\ \Omega^+(\Lambda^\#) & \xrightarrow{j_\Lambda} & B^+(\Lambda)^\# \end{array}$$

Analogously, if D is locally finite dimensional coaugmented dg coalgebra such that $\Omega^+(D)$ is also locally finite dimensional, then $B^+(D^\#)$ is canonically isomorphic to $\Omega^+(D)^\#$, as coaugmented dg coalgebras (see [15], Lemma 8.6, (c), where the analogous corrections to the ones indicated before apply). Indeed, the isomorphism $j^D : B^+(D^\#) \to \Omega^+(D)^\#$ may be explicitly given as follows. For $n \in \mathbb{N}$ and $\rho_1, \dots, \rho_n \in I_{D^\#}$ the morphism sends $[\rho_1 | \dots | \rho_n]$ to the linear functional

$$\langle \theta_1 | \dots | \theta_m \rangle \mapsto (-1)^\epsilon \delta_{n,m} \rho_1(\theta_1) \dots \rho_n(\theta_n),$$

where $\theta_1, \dots, \theta_m \in D$, $\delta_{n,m}$ is the Dirac delta sign,

$$\begin{aligned} \epsilon = &\deg \rho_1 + \dots + \deg \rho_n + (\deg \rho_2 + 1)(\deg \theta_1 + 1) \\ &+ \dots + (\deg \rho_n + 1)(\deg \theta_1 + \dots + \deg \theta_{n-1} + n - 1), \end{aligned}$$

and it sends $1 \in B^+(D^\#)$ to the canonical projection $\Omega^+(D) \to k$. Again, if $f : D \to D'$ is a morphism of coaugmented dg coalgebras, then it can be directly verified from the explicit expression of the morphisms involved that we have the commutative diagram

$$\begin{array}{ccc} B^+((D')^\#) & \xrightarrow{j^{D'}} & \Omega^+(D')^\# \\ \downarrow{\scriptstyle B^+(f^\#)} & & \downarrow{\scriptstyle \Omega^+(f)^\#} \\ B^+(D^\#) & \xrightarrow{j^D} & \Omega^+(D)^\# \end{array}$$

Moreover, a straighforward computation shows that

$$\begin{aligned} j^D &= \Omega^+(\iota_D)^\# \circ (j_{D^\#})^\# \circ \iota_{B^+(D^\#)}, \\ j_\Lambda &= B^+(\iota_\Lambda)^\# \circ (j^{\Lambda^\#})^\# \circ \iota_{\Omega^+(\Lambda^\#)}. \end{aligned} \tag{1.8}$$

Using the easy fact that $\iota_M^\# \circ \iota_{M^\#} = \mathrm{id}_{M^\#}$, for any dg k-module M, these identities imply that

$$\begin{aligned} (j^D)^\# \circ \iota_{\Omega^+(D)} &= j_{D^\#} \circ \Omega^+(\iota_D), \\ (j_\Lambda)^\# \circ \iota_{B^+(\Lambda)} &= j^{\Lambda^\#} \circ B^+(\iota_D). \end{aligned} \tag{1.9}$$

On the other hand, it is trivial to verify from the expressions of the morphisms involved that

$$\begin{aligned} \tau_{D^\#} &= (\tau^D)^\# \circ j^D, \\ \tau_\Lambda^\# &= j_\Lambda \circ \tau^{\Lambda^\#}. \end{aligned} \tag{1.10}$$

These equations in turn imply the following identities

$$\begin{aligned} \beta_{D^\#} &= (\beta^D)^\# \circ j_{\Omega^+(D)} \circ \Omega^+(j^D), \\ \beta_\Lambda^\# &= j^{B^+(\Lambda)} \circ B^+(j_\Lambda) \circ \beta^{\Lambda^\#}. \end{aligned} \tag{1.11}$$

Indeed, let us show how to prove the second one, for the first one is analogous. We first note that, since β_Λ is the unique morphism of augmented dg algebras such that the composition of $\tau^{B^+(\Lambda)}$ with it is τ_Λ, by taking duals $\beta_\Lambda^\#$ is the unique morphism of coaugmented dg coalgebras such that its composition with $(\tau^{B^+(\Lambda)})^\#$ is $\tau_\Lambda^\#$. It thus suffices to prove that the composition of the right member with $(\tau^{B^+(\Lambda)})^\#$ is $\tau_\Lambda^\#$. By the first identity of (1.10) for $D = B^+(\Lambda)$ we get that this composition is $\tau_{B^+(\Lambda)^\#} \circ B^+(j_\Lambda) \circ \beta^{\Lambda^\#}$. Now, (1.5) for the morphism $f = j_\Lambda$ tells us that the latter composition coincides with $j_\Lambda \circ \tau_{\Omega^+(\Lambda^\#)} \circ \beta^{\Lambda^\#}$, which is equal to $j_\Lambda \circ \tau^{\Lambda^\#}$. The second identity of (1.10) gives the claim.

We remark that under these assumptions a quasi-isomorphism of cocomplete coaugmented dg coalgebras is a weak equivalence (the converse is always true), for a quasi-isomorphism $C \to D$ induces a quasi-isomorphism of augmented dg algebras $D^\# \to C^\#$, which induces a quasi-isomorphisms between the bar constructions $B^+(D^\#) \to B^+(C^\#)$, and by the previously recalled isomorphism we get a quasi-isomorphism $\Omega^+(D)^\# \to \Omega^+(C)^\#$ of coaugmented dg coalgebras. Taking duals again we obtain a quasi-isomorphism $\Omega^+(C) \to \Omega^+(D)$ of augmented dg algebras, so a weak equivalence $C \to D$.

1.4.7 Hochschild (co)homology as a twisted construction

Now, given an augmented dg algebra A, we may consider the augmented dg algebra $\mathcal{H}om(B^+(A), A)$. It is easy to verify that its underlying graded k-module coincides with the corresponding one computing the Hochschild cohomology, but the underlying dg k-module is different. Indeed, the differential of $\mathcal{H}om(B^+(A), A)$ is given by D_0 in (1.3). However, a twist of the differential of $\mathcal{H}om(B^+(A), A)$ will give us precisely the differential of $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$: it is easy to check that $d_{\mathcal{H}om(B^+(A),A),\tau_A}$ coincides with the differential $D_0 + D_1$ of the dg k-module computing the Hochschild cohomology of A given by (1.3) and (1.4), since it is clear that $D_1(f) = \mathrm{ad}(\tau_A)(f)$. In other words, the dg k-module $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ is canonically identified with $\mathcal{H}om^{\tau_A}(B^+(A), A)$. The latter has further the structure of a(n augmented) dg algebra, whose multiplication is usually called *cup product* (see [4], Ch. XI, §4 and 6), so the Hochschild cohomology $HH^\bullet(A)$ thus becomes an augmented graded algebra. All the previous comments apply *mutatis mutandi* as well to $\tilde{B}^+(A)$ instead of $B^+(A)$. Moreover, if k is semisimple, the canonical quasi-isomorphisms $\tilde{B}^+(A) \to B^+(A)$ and $B^+(A) \to \tilde{B}^+(A)$ of coaugmented dg coalgebras induce quasi-isomorphisms of augmented dg algebras

$$\mathcal{H}om(B^+(A), A) \to \mathcal{H}om(\tilde{B}^+(A), A) \text{ and } \mathcal{H}om(\tilde{B}^+(A), A) \to \mathcal{H}om(B^+(A), A)$$

sending τ_A to $\tilde{\tau}_A$ and $\tilde{\tau}_A$ to τ_A, respectively. Moreover, they also induce quasi-isomorphisms of augmented dg algebras

$$\mathcal{H}om^{\tau_A}(B^+(A), A) \to \mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A)$$

and

$$\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A) \to \mathcal{H}om^{\tau_A}(B^+(A), A).$$

Indeed, the two maps are obtained by applying the functor $\mathcal{H}om_{A^e}(-, A)$ to the quasi-isomorphisms $\mathrm{Bar}(A) \to \overline{\mathrm{Bar}}(A)$ and $\overline{\mathrm{Bar}}(A) \to \mathrm{Bar}(A)$, respectively, and using the canonical identifications explained in the last paragraph of Subsection 1.3.

The Hochschild homology of an augmented dg algebra A can be regarded in a similar fashion. The underlying graded k-module $A \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ computing the Hochschild homology of A coincides with $A \otimes B^+(A)$, though the dg k-module structure is different, for the differential of $A \otimes B^+(A)$ the former coincides with D'_0 given in (1.1). However, the twist of the differential of $A \otimes B^+(A)$ will give us precisely the differential of $A \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ as for the cohomology. We see that $d_{A \otimes B^+(A),\tau_A}$ coincides with the differential $D'_0 + D'_1$ of the dg k-module computing the Hochschild homology of A given by (1.1) and (1.2), since it is easily verified that D'_1 coincides with the action of $\mathrm{ad}(\tau_A)$ on $A \otimes B^+(A)$. Thus, the dg k-module $A \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ is canonically identified with $A \otimes_{\tau_A} B^+(A)$. By the last paragraph of Subsubsection 1.4.3, the latter has further the structure of a dg bimodule over $\mathcal{H}om^{\tau_A}(B^+(A), A)$. Either the left or the right action (or both of them together) may be called *cap product* (see [4], Ch. XI, §4 and 6), so the Hochschild homology $HH_\bullet(A)$ in turn becomes a graded bimodule over the augmented graded algebra given by Hochschild cohomology $HH^\bullet(A)$. Again, all the previous comments apply *mutatis mutandi* to $\tilde{B}^+(A)$ instead of $B^+(A)$. If k is Von Neumann regular, the canonical quasi-isomorphism $\tilde{B}^+(A) \to B^+(A)$ (resp., $B^+(A) \to \tilde{B}^+(A)$) of coaugmented dg coalgebras induces a quasi-isomorphism of dg bimodules $A \otimes \tilde{B}^+(A) \to A \otimes B^+(A)$ over $\mathcal{H}om(B^+(A), A)$ (resp., $A \otimes B^+(A) \to A \otimes \tilde{B}^+(A)$ over $\mathcal{H}om(\tilde{B}^+(A), A)$), where the domain has a structure of bimodule over $\mathcal{H}om(B^+(A), A)$ (resp., $\mathcal{H}om(\tilde{B}^+(A), A)$) by $\mathcal{H}om(B^+(A), A) \to$

$\mathcal{H}om(\tilde{B}^+(A), A)$ (resp., $\mathcal{H}om(\tilde{B}^+(A), A) \to \mathcal{H}om(B^+(A), A)$), which is the morphism of augmented dg algebras explained in the previous paragraph. Furthermore, we have a quasi-isomorphism of dg bimodules $A \otimes_{\tilde{\tau}_A} \tilde{B}^+(A) \to A \otimes_{\tau_A} B^+(A)$ (resp., $A \otimes_{\tau_A} B^+(A) \to A \otimes_{\tilde{\tau}_A} \tilde{B}^+(A)$) over the algebra $\mathcal{H}om^{\tau_A}(B^+(A), A)$ (resp., $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A)$), where the domain has a structure of bimodule over the algebra $\mathcal{H}om^{\tau_A}(B^+(A), A)$ (resp., $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A)$) given by $\mathcal{H}om^{\tau_A}(B^+(A), A) \to \mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A)$ (resp., given by $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A) \to \mathcal{H}om^{\tau_A}(B^+(A), A)$) of augmented dg algebras explained in the previous paragraph. Indeed, the previous maps follow by applying the functor $A \otimes_{A^e} (-)$ to the quasi-isomorphisms $\mathrm{Bar}(A) \to \overline{\mathrm{Bar}}(A)$ and $\overline{\mathrm{Bar}}(A) \to \mathrm{Bar}(A)$, respectively, and using the canonical identifications explained in the penultimate paragraph of Subsection 1.3.

We remark that in fact the graded bimodule $HH_\bullet(A)$ over $HH^\bullet(A)$ is (graded) symmetric, as one may easily deduce as follows. Indeed, as noted in the literature (see for instance [18], (10) and the proof of Lemma 16), $H^\bullet(A, A^\#)$ is a symmetric graded bimodule over $HH^\bullet(A)$ (for the action in fact comes from the cup product on $HH^\bullet(A[M])$, where $A[M]$ is the dg algebra with underlying dg module given by $A \oplus M$, the usual product $(a, m) \cdot (a', m') = (aa', am' + ma')$, unit $(1_A, 0_M)$ and augmentation $(a, m) \mapsto \epsilon_A(a)$), which is isomorphic to $HH_\bullet(A)^\#$.

We summarize our previous comments in the following result.

Fact 1.2. *Let A be an augmented dg algebra over k, and let τ_A denote the universal twisting cochain of A. Then,*

(i) the dg k-module $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ computing Hochschild cohomology is canonically identified with $\mathcal{H}om^{\tau_A}(B^+(A), A)$. Moreover, the cup product on the first complex coincides exactly with the convolution product on the latter.

(ii) the dg k-module $A \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ computing Hochschild homology is canonically identified with the twisted tensor product $A \otimes_{\tau_A} B^+(A)$. Furthermore, the symmetric bimodule structure of the first complex over $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ given by the cap product coincides exactly with the symmetric bimodule structure of the latter complex over $\mathcal{H}om^{\tau_A}(B^+(A), A)$.

We see that the previous description of Hochschild homology and cohomology groups is by no mean accidental. Indeed, it is a direct consequence of the definitions once one notes that the reduced (resp., nonreduced) bar resolution of A is canonically identified (as dg k-modules) with $A^e \otimes_{\tau_A} B^+(A)$ (resp., $A^e \otimes_{\tilde{\tau}_A} \tilde{B}^+(A)$), where A^e is a dg A-bimodule with the *outer structure* of A^e given by $a(a' \otimes b')b = (aa') \otimes (b'b)$, for $a, a', b, b' \in A$. The identification isomorphism is given by $(a_{n+1} \otimes a_0) \otimes [a_1|\dots|a_n] \mapsto (-1)^{\deg a_{n+1}(\deg a_0 + \epsilon)} a_0[a_1|\dots|a_n]a_{n+1}$, where $\epsilon = (\sum_{i=1}^n \deg a_i) - n$. Consider the dg A-bimodule structure of $A^e \otimes_{\tau_A} B^+(A)$ (resp., $A^e \otimes_{\tilde{\tau}_A} \tilde{B}^+(A)$) coming from the *inner structure* of A^e given by

$$a(a' \otimes b')b = (-1)^{(\deg a' \deg a + \deg b \deg b' + \deg a \deg b)}(a'b) \otimes (ab'),$$

for $a, a', b, b' \in A$. By the last paragraphs of the Subsubsections 1.4.2 and 1.4.3, it induces a structure of dg bimodule over the algebra $\mathcal{H}om^{\tau_A}(B^+(A), A) \otimes A$ (resp., $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A) \otimes A$) on the twisted tensor product $A^e \otimes_{\tau_A} B^+(A)$ (resp., $A^e \otimes_{\tilde{\tau}_A} \tilde{B}^+(A)$). By means of this structure the previous identification gives in fact an isomorphism of dg bimodules over $\mathcal{H}om^{\tau_A}(B^+(A), A) \otimes A$ (resp., $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A), A) \otimes A$). If we apply the functors $\mathcal{H}om_{A^e}(-, A)$ and $A \otimes_{A^e} (-)$ to the previous identifications of the bar constructions we get precisely the description of the dg k-modules computing Hochschild cohomology and homology as described in the previous two paragraphs.

1.4.8 Gerstenhaber brackets

We recall that given an augmented dg algebra A, the graded module $\mathrm{Der}(A)$ over k of *derivations* of A is the graded submodule of $\mathcal{H}om(A, A)$ given by sums of homogeneous maps $d : A \to A$ satisfying that $\mu_A \circ (d \otimes \mathrm{id}_A + \mathrm{id}_A \otimes d) = d \circ \mu_A$ and $d(1_A) = 0$. Analogously, given a coaugmented dg coalgebra C, the graded module $\mathrm{Coder}(C)$ over k of *coderivations* of C is the graded submodule of $\mathcal{H}om(C, C)$ given by sums of homogeneous maps $d : C \to C$ satisfying that $(d \otimes \mathrm{id}_C + \mathrm{id}_C \otimes d) \circ \Delta_C = \Delta_C \circ d$ and $\epsilon_C \circ d = 0$. Note that, by composing the first of the previous identities with η_C, we get that $d \circ \eta_C(1_k)$ is a *primitive* element of C, *i.e.* an element $c \in C$ satisfying that $\Delta_C(C) = 1_C \otimes c + c \otimes 1_C$, where we recall that $1_C = \eta_C(1_k)$. In the particular case of a coaugmented dg coalgebra C whose underlying coaugmented graded coalgebra structure is a coaugmented tensor coalgebra on a graded module V, which is usually denoted by T^cV, we note that the primitive elements are given by $V \subseteq T^cV$. Note that both $\mathrm{Der}(A)$ and $\mathrm{Coder}(C)$ are graded Lie algebras with the bracket given by the graded commutator.

As noted by E. Getzler in [11], Prop. 1.3., for an augmented dg algebra A, the graded module $\mathcal{H}om(\tilde{B}^+(A), A)[1] \simeq \mathcal{H}om(\tilde{B}^+(A), A[1])$ over k is isomorphic to the graded module $\mathrm{Coder}(\tilde{B}^+(A))$ of coderivations of the coaugmented dg coalgebra $\tilde{B}^+(A)$. Indeed, the isomorphism δ_A sends $s\phi$, where $\phi \in \mathcal{H}om(\tilde{B}^+(A), A)$ to the coderivation $\delta_A(s\phi)$ satisfying that $\delta_A(s\phi)(1_{\tilde{B}^+(A)}) = s_A(\phi(1_{\tilde{B}^+(A)}))$ and, for $n \in \mathbb{N}$, $\pi_j(\delta_A(s\phi)([a_1|\dots|a_n]))$ is given by

$$\begin{cases} \displaystyle\sum_{i=1}^{j} (-1)^{(\deg\phi-1)\epsilon_i}[a_1|\dots|a_{i-1}|\phi([a_i|\dots|a_{i+n-j}])|a_{i+n-j+1}|\dots|a_n], & \text{if } 1 \leq j \leq n, \\ \displaystyle\sum_{i=1}^{n+1} (-1)^{(\deg\phi-1)\epsilon_i}[a_1|\dots|a_{i-1}|\phi(1_{B^+(A)})|a_i|\dots|a_n], & \text{if } j = n+1, \\ 0, & \text{if } j > n+1, \end{cases}$$

where $\epsilon_i = (\sum_{j=1}^{i-1} \deg a_j) - i + 1$ and $\pi_j : \tilde{B}^+(A) \to A[1]^{\otimes j}$ is the canonical projection. It is rather usual to provide the explicit expression of $\pi_j(\delta_A(\phi)([a_1|\dots|a_n]))$ only by the first of the previous lines, for the others are easily obtained as a typical abuse of (or "extended") notation. The inverse of δ_A is given by sending a coderivation $d \in \mathrm{Coder}(\tilde{B}^+(A))$ to the morphism $s_A^{-1} \circ \pi_1 \circ d$.

Also, given a cocomplete coaugmented dg coalgebra C we have a canonical isomorphism between the graded module $\mathcal{H}om(C, \tilde{\Omega}^+(C))[1] \simeq \mathcal{H}om(C[-1], \tilde{\Omega}^+(C))$ over k and the graded module $\mathrm{Der}(\tilde{\Omega}^+(C))$ of derivations of the augmented dg algebra $\tilde{\Omega}^+(C)$. Indeed, the isomorphism δ^C sends $s\psi$, for $\psi \in \mathcal{H}om^{\tilde{\tau}_A}(C, \tilde{\Omega}^+(C))$ to the derivation $\delta^C(s\psi)$ satisfying that $\delta^C(s\psi)(1_{\tilde{\Omega}^+(C)}) = \psi(1_A)$ and, for $n \in \mathbb{N}$, is given by

$$\delta^C(s\psi)(\langle c_1|\dots|c_n\rangle) = \sum_{i=1}^{n+1} (-1)^{(\deg\psi-1)(\bar{\epsilon}_i+1))}\langle c_1|\dots|c_{i-1}\rangle\psi(c_i)\langle c_{i+1}|\dots|c_n\rangle,$$

where $\bar{\epsilon}_i = (\sum_{j=1}^{i-1} \deg c_j) - i + 1$. The inverse is given by sending a derivation $d \in \mathrm{Der}(\tilde{\Omega}^+(C))$ to the morphism $(-1)^{\deg d} d|_{C[-1]} \circ s_{C[-1]}^{-1}$.

From the previous identifications one obtains graded Lie algebra structures on both $\mathcal{H}om(\tilde{B}^+(A), A)[1]$ and $\mathcal{H}om(C, \tilde{\Omega}^+(C))[1]$. These bracket will be called the *Gerstenhaber brackets*, for the first of these was introduced by M. Gerstenhaber in the seminal work [10]. For $\phi \in \mathcal{H}om(A[1]^{\otimes n}, A)$ and $\phi' \in \mathcal{H}om(A[1]^{\otimes m}, A)$, with $n, m \in \mathbb{N}_0$, the Gerstenhaber bracket is given by $[\phi, \phi'] \in \mathcal{H}om(A[1]^{\otimes(n+m-1)}, A)$, sending $[a_1|\dots|a_{n+m-1}]$ to

$$\begin{aligned}&\sum_{i=0}^{n-1}(-1)^{(\deg\phi'-1)\epsilon_{i+1}}\phi([a_1|\dots|a_i|\phi'([a_{i+1}|\dots|a_{i+m}])|a_{i+m+1}|\dots|a_{n+m-1}])\\&-\sum_{i=0}^{m-1}(-1)^{\epsilon'_i}\phi'([a_1|\dots|a_i|\phi([a_{i+1}|\dots|a_{i+n}])|a_{i+n+1}|\dots|a_{n+m-1}]),\end{aligned}\tag{1.12}$$

where $\epsilon_{i+1}=(\sum_{j=1}^{i}\deg a_j)-i$ and $\epsilon'_i=(\deg\phi-1)(\epsilon_{i+1}+\deg\phi'-1)$ (see [21], Subsection 2.2). Note that the notation implies that if $n=0$ the left sum of the right member vanishes and we should consider $\phi(1_{\tilde{B}^+(A)})$ in the right one, whereas if $m=0$ the right sum is zero and we should have $\phi'(1_{\tilde{B}^+(A)})$ in the left one. It is straightforward to prove that the differential of $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A),A)$ is in fact given by $\phi\mapsto[s_A^{-1}\circ\pi_1\circ\tilde{B},\phi]$. This in turn implies that $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A),A)[1]$ is in fact a dg Lie algebra. Moreover, the expression for the Gerstenhaber bracket (1.12) may be also applied to elements of $\mathcal{H}om(B^+(A),A)[1]$, which also becomes a dg Lie algebra such that the quasi-isomorphisms $\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A),A)\to\mathcal{H}om^{\tau_A}(B^+(A),A)$ and $\mathcal{H}om^{\tau_A}(B^+(A),A)\to\mathcal{H}om^{\tilde{\tau}_A}(\tilde{B}^+(A),A)$ recalled in the first paragraph of Subsubsection 1.4.7 induce in fact morphisms of dg Lie algebras.

Analogously, let $\psi\in\mathcal{H}om(C,C[-1]^{\otimes n})$ and $\psi'\in\mathcal{H}om(C,C[-1]^{\otimes m})$, where $n,m\in\mathbb{N}_0$. We will use the (Sweedler-alike) notation $\psi(c)=\langle c^{\psi}_{(1)}|\dots|c^{\psi}_{(n)}\rangle$ and $\psi'(c)=\langle c^{\psi'}_{(1)}|\dots|c^{\psi'}_{(m)}\rangle$, for $c\in C$, where the sum is omitted for simplicity. Then, the Gerstenhaber bracket is given by the element $[\psi,\psi']\in\mathcal{H}om(C,C[-1]^{\otimes(n+m-1)})$ sending $c\in C$ to

$$\begin{aligned}&\sum_{i=0}^{m-1}(-1)^{(\deg\psi-1)\epsilon'_{i+1}}\langle c^{\psi'}_{(1)}|\dots|c^{\psi'}_{(i)}\rangle\psi(c^{\psi'}_{(i+1)})\langle c^{\psi'}_{(i+2)}|\dots|c^{\psi'}_{(m)}\rangle\\&-\sum_{i=0}^{n-1}(-1)^{(\deg\psi'-1)(\epsilon_{i+1}+\deg\psi-1)}\langle c^{\psi}_{(1)}|\dots|c^{\psi}_{(i)}\rangle\psi'(c^{\psi}_{(i+1)})\langle c^{\psi}_{(i+2)}|\dots|c^{\psi}_{(n)}\rangle,\end{aligned}\tag{1.13}$$

where $\epsilon_{i+1}=(\sum_{j=1}^{i}\deg c^{\psi}_{(j)})-i$ and $\epsilon'_{i+1}=(\sum_{j=1}^{i}\deg c^{\psi'}_{(j)})-i$. It can be proved along the same lines as for the case of algebras that this gives a dg Lie algebra structure on $\mathcal{H}om(C,\tilde{\Omega}^+(C))[1]$. Moreover, the expression for the Gerstenhaber bracket (1.13) may be also applied to elements of $\mathcal{H}om(C,\Omega^+(C))[1]$, which also becomes a dg Lie algebra such that the quasi-isomorphisms $\mathcal{H}om^{\tilde{\tau}^A}(C,\tilde{\Omega}^+(C))\to\mathcal{H}om^{\tau^C}(C,\Omega^+(C))$ and $\mathcal{H}om^{\tau^C}(C,\Omega^+(C))\to\mathcal{H}om^{\tilde{\tau}^A}(C,\tilde{\Omega}^+(C))$ induced by those recalled in the second paragraph of Subsubsection 1.4.5 are in fact morphisms of dg Lie algebras.

Let us assume that k is a field and C is a locally finite dimensional coaugmented dg coalgebra. By the comments of the previous subsubsection we know that $C^{\#}$ is a locally finite dimensional augmented dg algebra. Moreover, the map $\mathrm{Coder}(C)\to\mathrm{Der}(C^{\#})^{\mathrm{op}}$ given by $\phi\mapsto(-1)^{\deg\phi}\phi^{\#}$ is an isomorphism of graded Lie algebras, where we recall that the *opposite* graded Lie algebra $\mathfrak{g}^{\mathrm{op}}$ of a graded Lie algebra $\mathfrak{g}$ with bracket $[\,,]$ has the same underlying graded k-module structure and *opposite bracket* $[\,,]_{\mathrm{op}}$ given by $[x,y]_{\mathrm{op}}=(-1)^{\deg x\deg y}[y,x](=-[x,y])$, for $x,y\in\mathfrak{g}$ homogeneous. In such situations we may usually say that the previous map is an *anti-isomorphism of dg Lie algebras*. Analogously, if A is a locally finite dimensional augmented dg algebra, then $A^{\#}$ is a locally finite dimensional coaugmented dg coalgebra, and $\mathrm{Der}(A)\to\mathrm{Coder}(A^{\#})^{\mathrm{op}}$ given by $\phi\mapsto\phi^{\#}$ is an isomorphism of graded Lie algebras. Since the map $\mathfrak{g}\to\mathfrak{g}^{\mathrm{op}}$ given by $x\mapsto-x$ is an isomorphism of graded Lie algebras, we get in fact isomorphisms of graded Lie algebras $\mathrm{Coder}(C) \to \mathrm{Der}(C^{\#})$ and $\mathrm{Der}(A) \to \mathrm{Coder}(A^{\#})$ given by $\phi \mapsto -\phi^{\#}$, for $\phi \in \mathrm{Coder}(C)$ or $\phi \in \mathrm{Der}(A)$, respectively.

1.5 Generalities on A_∞-(co)algebras

For the following definitions we refer to [20], Chapitre 3, Section 3.1 (or also [14], Déf. 1.2.1.1, 1.2.1.8, using the obvious equivalences between non(co)unitary objects and (co)augmented ones), even though we do not follow the same sign conventions and they do not consider any Adams grading (see for instance [17] for several uses of Adams grading in A_∞-algebra theory). We first recall that an *augmented A_∞-algebra* structure on a cohomological graded vector space A is the following data:

(i) a collection of maps $m_i : A^{\otimes i} \to A$ for $i \in \mathbb{N}$ of cohomological degree $2 - i$ and Adams degree zero satisfying the *Stasheff identities* given by

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}_A^{\otimes r} \otimes m_s \otimes \mathrm{id}_A^{\otimes t}) = 0, \tag{1.14}$$

for $n \in \mathbb{N}$, where $\mathcal{I}_n = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r+s+t = n\}$. We shall denote the (sum of) morphism(s) on the left by $\mathrm{SI}^{m_\bullet}(n)$.

(ii) a map $\eta_A : k \to A$ of complete degree 0_G such that

$$m_i \circ (\mathrm{id}_A^{\otimes r} \otimes \eta_A \otimes \mathrm{id}_A^{\otimes t})$$

vanishes for all $i \neq 2$ and all $r, t \geq 0$ such that $r + 1 + t = i$, and

$$m_2 \circ (\mathrm{id}_A \otimes \eta_A) = \mathrm{id}_A = m_2 \circ (\eta_A \otimes \mathrm{id}_A).$$

(iii) a map $\epsilon_A : A \to k$ of complete degree 0_G such that $\epsilon_A \circ \eta_A = \mathrm{id}_k$, $\epsilon \circ m_2 = \epsilon_A^{\otimes 2}$, and $\epsilon_A \circ m_i = 0$, for all $i \in \mathbb{N} \setminus \{2\}$.

It is further called *minimal* if m_1 vanishes. If we do not assume the items (ii) and (iii) in the definition, then A is called an *A_∞-algebra*, to which we sometimes refer as nonunitary, to stress the fact that it does not necessarily have a unit. If it also satisfies (ii), the *A_∞-algebra* is called *(strictly) unitary*.

We recall that a family of linear maps $\{f_i : C \to C_i\}_{i\in\mathbb{N}}$, where C and C_i, for $i \in \mathbb{N}$, are vector spaces, is called *locally finite* if, for all $c \in C$, there exists a finite subset $S \subseteq \mathbb{N}$, which depends on c, such that $f_i(c)$ vanishes for all $i \in \mathbb{N} \setminus S$. An *coaugmented A_∞-coalgebra* structure on a homological graded vector space C is the following data:

(i) a locally finite collection of maps $\Delta_i : C \to C^{\otimes i}$ for $i \in \mathbb{N}$ of homological degree $i - 2$ and Adams degree zero satisfying the following identities

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{rs+t} (\mathrm{id}_C^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta_{r+1+t} = 0, \tag{1.15}$$

for $n \in \mathbb{N}$.

(ii) a map $\epsilon_C : C \to k$ of complete degree 0_G such that

$$(\mathrm{id}_C^{\otimes r} \otimes \epsilon_C \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta_i$$

vanishes for all $i \neq 2$ and all $r, t \geq 0$ such that $r + 1 + t = i$, and

$$(\mathrm{id}_C \otimes \epsilon) \circ \Delta_2 = \mathrm{id}_C = (\epsilon_C \otimes \mathrm{id}_C) \circ \Delta_2.$$

(iii) a map $\eta_C : k \to C$ of complete degree 0_G such that $\epsilon_C \circ \eta_C = \mathrm{id}_k$, $\Delta_2 \circ \eta_C(1_k) = \eta_C(1_k)^{\otimes 2}$, and $\Delta_i \circ \eta_C(1_k) = 0$, for all $i \in \mathbb{N} \setminus \{2\}$.

We shall usually denote $\eta_C(1_k)$ by 1_C. An Adams graded coaugmented A_∞-coalgebra C is called *minimal* if $\Delta_1 = 0$. Note that the condition that the family $\{\Delta_n\}_{n\in\mathbb{N}}$ is locally finite follows from the other data if we further suppose that $\mathrm{Ker}(\epsilon)$ is positively graded for the Adams degree. Again, an A_∞*-coalgebra* is defined as the graded k-module C provided with the maps $\{\Delta_i\}_{i\in\mathbb{N}}$ satisfying the identities of the first item, and it is called *(strictly) counitary* if it further satisfies condition (ii).

Note that an A_∞-algebra A is *a fortiori* a dg k-module where the differential is given by m_1. Analogously, a A_∞-coalgebra C is also a dg k-module for the differential Δ_1. Moreover, an augmented dg algebra structure on A is tantamount to an augmented A_∞-algebra structure with vanishing *higher multiplications* m_n for $n \geq 3$, where the differential is m_1 and the multiplication is m_2. In the same manner, a coaugmented dg coalgebra structure on C is equivalent to a coaugmented A_∞-coalgebra structure with vanishing *higher comultiplications* Δ_n for $n \geq 3$, where the differential is Δ_1 and the coproduct is Δ_2.

As for the case of augmented dg algebras, given an augmented A_∞-algebra A there exists a coaugmented dg coalgebra $B^+(A)$, called the *(reduced) bar construction of A*. Its underlying graded coalgebra structure is given by the tensor coalgebra $\oplus_{i\in\mathbb{N}_0} I_A[1]^{\otimes i}$, where $I_A = \mathrm{Ker}(\epsilon_A)$. As before, if $n \in \mathbb{N}$ we will typically denote an element $s(\bar{a}_1) \otimes \cdots \otimes s(\bar{a}_n) \in I_A[1]^{\otimes n}$ in the form $[a_1|\dots|a_n]$, where $a_1, \dots, a_n \in A$, $\bar{a} \in A/k \simeq I_A$ denotes the canonical projection of an element $a \in A$, and $s : I_A \to I_A[1]$ is the canonical morphism of degree -1 recalled in the third paragraph of Subsection 1.1. The coproduct is thus given by the usual deconcatenation

$$\Delta([a_1|\dots|a_n]) = \sum_{i=0}^{n} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n],$$

where we set $[a_i|\dots|a_j] = 1_{B^+(A)}$ if $i > j$, for $1_{B^+(A)}$ the image of 1_k under the canonical inclusion $k = I_A[1]^{\otimes 0} \subseteq B^+(A)$, which may be also denoted by $[]$. The counit is defined as the canonical projection $B^+(A) \to I_A[1]^{\otimes 0} = k$, and the coaugmentation is given by the obvious inclusion $k = I_A[1]^{\otimes 0} \subseteq B^+(A)$. We recall that since $B^+(A)$ is a coaugmented tensor graded coalgebra, it is cocomplete, and its differential B is defined as follows. It is the unique coderivation whose image lies inside the augmentation kernel $\mathrm{Ker}(\epsilon_{B^+(A)})$ of $B^+(A)$, so B is thus uniquely determined by $\pi_1 \circ B$, where $\pi_1 : B^+(A) \to I_A[1]$ is the canonical projection (see [14], Lemme 1.1.2.2, Sections 2.1.1 and 2.1.2, and Notation 2.2.1.4), such that this composition map is given by the sum $b = \sum_{i\in\mathbb{N}} b_i$, where $b_i : I_A[1]^{\otimes i} \to I_A[1]$ is defined as $b_i = -s_{I_A} \circ m_i \circ (s_{I_A}^{\otimes i})^{-1}$. In fact, equation (1.14) is precisely the condition for this coderivation to be a differential. Our convention for the bar construction clearly coincides with the one given for augmented dg algebras in the case the higher multiplications vanish, but it differs from others in the literature (*e.g.* those in the thesis [14] of K. Lefèvre-Hasegawa, Ch. 1 and 2).

Dually, given a coaugmented A_∞-coalgebra C there exists an augmented dg algebra $\Omega^+(C)$, called (reduced) cobar construction of C. Its underlying graded algebra structure is given by the tensor algebra $\oplus_{i\in\mathbb{N}_0} J_C[-1]^{\otimes i}$, where $J_C = \mathrm{Coker}(\eta_C)$. As before, if $n \in \mathbb{N}$ we will typically denote an element $s^{-1}(\bar{c}_1) \otimes \cdots \otimes s(\bar{c}_n) \in J_C[-1]^{\otimes n}$ in the form $\langle c_1|\dots|c_n\rangle$, where $c_1, \dots, c_n \in C$, $\bar{c} \in J_C$ denotes the canonical projection of an element $c \in C$, and $s : J_C[-1] \to J_C$ is the canonical morphism of degree -1 recalled in the third paragraph of Subsection 1.1. The unit is given by the obvious inclusion $k = J_C[-1]^{\otimes 0} \subseteq \Omega^+(C)$, and we denote the image of 1_k under the previous map either by $1_{\Omega^+(C)}$ or by $\langle\rangle$. The augmentation is defined as the canonical projection $\Omega^+(C) \to J_C[-1]^{\otimes 0} = k$. The differential D of $\Omega^+(C)$ is defined as follows. Since $\Omega^+(C)$ is graded free algebra, it is the unique derivation D whose composition d with the canonical injection $J_C[-1] \to \Omega^+(C)$, where we define $d = \sum_{i\in\mathbb{N}} d_i$ for $d_i : J_C[-1] \to J_C[-1]^{\otimes i}$ given by $d_i = (-1)^i (s_{J_C[-1]}^{\otimes i})^{-1} \circ \Delta_i \circ s_{J_C[-1]}$. Note again that (1.15) is exactly the condition for this derivation to be a differential. As before, our definition for the cobar construction coincides with the one given for coaugmented dg coalgebras in the case the higher comultiplications Δ_i for $i \geq 3$ vanish, but it differs from others in the literature (*e.g.* those in the thesis [14] of K. Lefèvre-Hasegawa, Ch. 1 and 2).

We will also be particularly interested in the case that (co)augmented A_∞-(co)algebras are *Adams connected*, in the sense introduced in [16], Def. 2.1, *i.e.* a (co)augmented A_∞-(co)algebra A (resp., C) where the grading group G is $\mathbb{Z}\times\mathbb{Z}$ such that its augmentation kernel I_A (resp., J_C) satisfies that $\oplus_{n\in\mathbb{Z}} I_A^{(n,m)}$ (resp., $\oplus_{n\in\mathbb{Z}} J_C^{(n,m)}$) is finite dimensional, for all $m \in G' = \mathbb{Z}$, and either $I_A^{(n,m)}$ (resp., $J_C^{(n,m)}$) vanishes for all $n \in \mathbb{Z}$ and all $m \geq 0$, or $I_A^{(n,m)}$ (resp., $J_C^{(n,m)}$) vanishes for all $n \in \mathbb{Z}$ and all $m \leq 0$. By the previously cited article, an Adams connected augmented A_∞-algebra is also locally finite dimensional, and its Koszul dual $E(A) = B^+(A)^\#$ is also locally finite dimensional and also Adams connected (see [16], Lemma 2.2).

A *morphism of augmented A_∞-algebras* $f_\bullet : A \to B$ between two augmented A_∞-algebras A and B is a collection of morphisms of the underlying graded k-modules $f_i : A^{\otimes n} \to B$ of complete degree $(1-i, 0_{G'})$ for $i \in \mathbb{N}$ such that

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}_A^{\otimes r} \otimes m_s^A \otimes \mathrm{id}_A^{\otimes t}) = \sum_{q\in\mathbb{N}} \sum_{\bar{i}\in\mathbb{N}^{q,n}} (-1)^w m_q^B \circ (f_{i_1} \otimes \cdots \otimes f_{i_q}), \tag{1.16}$$

where $w = \sum_{j=1}^q (q-j)(i_j-1)$ and $\mathbb{N}^{q,n}$ is the subset of $\mathbb{N}^q$ of elements $\bar{i} = (i_1, \dots, i_q)$ such that $|\bar{i}| = i_1 + \cdots + i_q = n$. We also assume that $f_1(1_A) = 1_B$, for all $i \geq 2$ we have that $f_i(a_1, \dots, a_i)$ vanishes if there exists $j \in \{1, \dots, i\}$ such that $a_j = 1_A$, and that $\epsilon_B \circ f_1 = \epsilon_A$ and $\epsilon_B \circ f_i$ vanishes for $i \geq 2$. If we do not suppose this last collection of extra-assumptions the family of maps $\{f_i\}_{i\in\mathbb{N}}$ is only called a *morphism of A_∞-algebras*. We shall denote the (sum of) morphism(s) of the left (resp., right) member of (1.16) by $\mathrm{MI}^{m_\bullet}(n)_l$ (resp., $\mathrm{MI}^{m_\bullet}(n)_r$). Notice that f_1 is a morphism of dg k-modules for the underlying structures on A and B. We say that a morphism $f_\bullet$ is *strict* if f_i vanishes for $i \geq 2$.

Dually, a *morphism of coaugmented A_∞-coalgebras* $f_\bullet : C \to D$ between two coaugmented A_∞-coalgebras C and D is a locally finite collection of morphisms of the underlying graded k-modules $f_i : C \to D^{\otimes i}$ of homological degree $i-1$ and Adams degree zero for $i \in \mathbb{N}$ such that

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{rs+t} (\mathrm{id}_D^{\otimes r} \otimes \Delta_s^D \otimes \mathrm{id}_D^{\otimes t}) \circ f_{r+1+t} = \sum_{q\in\mathbb{N}} \sum_{\bar{i}\in\mathbb{N}^{q,n}} (-1)^{w'} (f_{i_1} \otimes \cdots \otimes f_{i_q}) \circ \Delta_q^C, \tag{1.17}$$

where $w' = \sum_{j=1}^q (j-1)(i_j+1)$. We also suppose that $\epsilon_D \circ f_1 = \epsilon_D$, for all $i \geq 2$ and $j \in \{1, \dots, i\}$ we have that $(\mathrm{id}_D^{\otimes(j-1)} \otimes \epsilon_D \otimes \mathrm{id}_D^{\otimes(i-j)}) \circ f_i$ vanishes, and that $f_1 \circ \eta_C = \eta_D$ and $f_i \circ \eta_C$ vanishes for $i \geq 2$. If we do not suppose this last collection of extra-assumptions the family of maps $\{f_i\}_{i\in\mathbb{N}}$ is only called a *morphism of A_∞-coalgebras*. Notice that f_1 is also a morphism of dg k-modules for the underlying structures on C and D. In this case we also say that a morphism $f_\bullet$ is *strict* if f_i vanishes for $i \geq 2$.

Given $f_\bullet : A \to B$ a morphism of augmented A_∞-algebras, it induces a morphism of coaugmented dg coalgebras $B^+(f_\bullet) : B^+(A) \to B^+(B)$ between the bar constructions as follows. We first note that the unitarity condition on $f_\bullet$ implies that it is completely determined by the induced collection of morphism $I_A^{\otimes i} \to I_B$, which we are going to denote also by f_i, for $i \in \mathbb{N}$. The morphism $B^+(f_\bullet)$ being of coaugmented graded coalgebras implies that it sends $1_{B^+(A)}$ to $1_{B^+(B)}$, and the coaugmentation cokernel of $B^+(A)$ to the coaugmentation cokernel of $B^+(B)$. Moreover, since $B^+(B)$ is a cocomplete graded coalgebra, such a morphism of graded coalgebras is completely determined by the composition $\pi_1^B \circ B^+(f_\bullet) : B^+(A) \to I_B[1]$, which vanishes on $1_{B^+(A)}$. The latter composition is thus given by a sum $\sum_{i\in\mathbb{N}} F_i$, where $F_i : I_A[1]^{\otimes i} \to I_B[1]$, which we define to be $F_i = s_{I_B} \circ f_i \circ (s_{I_A}^{\otimes i})^{-1}$, for $i \in \mathbb{N}$. In fact, (1.16) is precisely the condition for this morphism to commute with the differentials.

Dually, given $f_\bullet : C \to D$ a morphism of coaugmented A_∞-coalgebras, it induces a morphism of augmented dg algebras $\Omega^+(f_\bullet) : \Omega^+(C) \to \Omega^+(D)$ between the cobar constructions as follows. We first note that the counitarity condition on $f_\bullet$ implies that it is completely detemined by the induced collection of morphism $J_C \to J_D^{\otimes i}$, which we are going to denote also by f_i, for $i \in \mathbb{N}$. We suppose that it sends $1_{B^+(A)}$ to $1_{B^+(B)}$, and the augmentation kernel of $\Omega^+(C)$ to the augmentation kernel of $\Omega^+(D)$. Moreover, since $\Omega^+(C)$ is a free graded algebra, such a morphism $\Omega^+(f_\bullet)$ of graded algebras is completely determined by the composition of the canonical inclusion $J_C[-1] \to \Omega^+(C)$ with it. Let us denote this latter composition by F. Hence, $F = \sum_{i\in\mathbb{N}} F_i$, where $F_i : J_C[-1] \to J_D[-1]^{\otimes i}$, which we define to be $F_i = (-1)^{i+1}(s_{J_D[-1]}^{\otimes i})^{-1} \circ f_i \circ s_{J_C[-1]}$, for $i \in \mathbb{N}$. As expected, (1.17) is precisely the condition for this morphism to commute with the differentials. We remark that our definition of $B^+(f_\bullet)$ ($\Omega^+(f_\bullet)$) agrees with the corresponding one for (co)augmented dg (co)algebras in that case if the morphism $f_\bullet$ is further assumed to be strict.

A morphism of (co)augmented A_∞-(co)algebras is called a *quasi-isomorphism* if the map f_1 is so. The same definition may be stated for the nonaugmented case. We further say that a morphism $f_\bullet$ of coaugmented A_∞-coalgebras is a *weak equivalence* provided its cobar $\Omega^+(f_\bullet)$ is a quasi-isomorphism of augmented dg algebras. This is not completely standard since we are not assuming that there is any model structure, but it will be convenient for our purposes. Note that a morphism of augmented A_∞-algebras is a quasi-isomorphism if and only if $B^+(f_\bullet)$ is also (see [20], Thm. 3.25). We refer to [20], Sections 3.2 and 3.3 (Déf. 3.3, 3.4, and 3.11), or [14], Sections 1.2 and 1.3 for more details on these definitions (though we follow a different sign convention), and we remark that these morphisms are supposed to preserve the Adams degree (*cf.* [17], Section 2).

Notice that if C is a coaugmented dg coalgebra and A is an augmented A_∞-algebra, the dg k-module $\mathcal{H}om(C, A)$ has in fact a structure of augmented A_∞-algebra where m_1 is given by the usual differential $d_{\mathcal{H}om(C,A)}(\phi) = m_1^A \circ \phi - (-1)^{\deg \phi}\phi \circ d_C$. Indeed, if we further define

$$m_n(\phi_1 \otimes \cdots \otimes \phi_n) = m_n^A \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta_C^{(n)},$$

for $n \geq 2$, $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$ and $\epsilon_{\mathcal{H}om(C,A)}(\phi) = \epsilon_A \circ \phi \circ \eta_C(1_k)$, it is easily verified that they provide the structure of augmented A_∞-algebra on $\mathcal{H}om(C, A)$. Furthermore, if $f_\bullet : A \to B$ is a morphism of augmented A_∞-algebras, then the collection of morphisms

$$f_n^* : \mathcal{H}om(C, A)^{\otimes n} \to \mathcal{H}om(C, B),$$

for $n \in \mathbb{N}$, of graded k-modules of complete degree $(1, 0_{G'})$ given by $f_1^*(\phi) = f_1 \circ \phi$, and

$$f_n^*(\phi_1 \otimes \cdots \otimes \phi_n) = f_n \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta_C^{(n)},$$

for $n \geq 2$, is a morphism of augmented A_∞-algebras.

Dually, if C is a coaugmented A_∞-coalgebra and A is an augmented dg algebra, the dg k-module $\mathcal{H}om(C,A)$ has in fact a structure of augmented A_∞-algebra where m_1 is given by the usual differential $d_{\mathcal{H}om(C,A)}(\phi) = d_A \circ \phi - (-1)^{\deg \phi} \phi \circ \Delta_1^C$. In this case the rest of the structure is given by

$$m_n(\phi_1 \otimes \cdots \otimes \phi_n) = (-1)^{n(\deg \phi_1 + \cdots + \deg \phi_n + 1)} \mu_A^{(n)} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta_n^C,$$

for $n \geq 2$, $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$ and $\epsilon_{\mathcal{H}om(C,A)}(\phi) = \epsilon_A \circ \phi \circ \eta_C(1_k)$. Moreover, if $f_\bullet : C \to D$ is a morphism of coaugmented A_∞-coalgebras, then the collection of morphisms

$$(f_n)_* : \mathcal{H}om(D,A)^{\otimes n} \to \mathcal{H}om(C,A), \tag{1.18}$$

for $n \in \mathbb{N}$, of graded k-modules of complete degree $(1, 0_{G'})$ given by $(f_1)_*(\phi) = \phi \circ f_1$, and

$$(f_n)_*(\phi_1 \otimes \cdots \otimes \phi_n) = (-1)^{(n-1)(\deg \phi_1 + \cdots + \phi_n)} \mu_A^{(n)} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ f_n,$$

for $n \geq 2$, is a morphism of augmented A_∞-algebras.

If $f_\bullet : A \to A'$ and $g_\bullet : A' \to B$ are morphisms of augmented A_∞-algebras, the composition $g_\bullet \circ f_\bullet$ is the morphism of augmented A_∞-algebras given by the collection of maps $\{(g \circ f)_n : A^{\otimes n} \to B\}_{n \in \mathbb{N}}$ defined as

$$(g \circ f)_n = \sum_{q \in \mathbb{N}} \sum_{\bar{i} \in \mathbb{N}^{q,n}} (-1)^w g_q \circ (f_{i_1} \otimes \cdots \otimes f_{i_q}), \tag{1.19}$$

where $w = \sum_{j=1}^q (q-j)(i_j - 1)$. Dually, if $f_\bullet : C \to C'$ and $g_\bullet : C' \to D$ are morphisms of coaugmented A_∞-coalgebras, the composition $g_\bullet \circ f_\bullet$ is the morphism of coaugmented A_∞-coalgebras given by the collection of maps $\{(g \circ f)_n : C \to D^{\otimes n}\}_{n \in \mathbb{N}}$ of the form

$$(g \circ f)_n = \sum_{q \in \mathbb{N}} \sum_{\bar{i} \in \mathbb{N}^{q,n}} (-1)^{w'} (g_{i_1} \otimes \cdots \otimes g_{i_q}) \circ f_q, \tag{1.20}$$

where $w' = \sum_{j=1}^q (j-1)(i_j + 1)$.

We remark that the previous construction defines an augmented A_∞-algebra structure on the graded dual $C^\#$ of C. If C is Adams connected, we see that $\Omega^+(C)^\#$ is isomorphic to $B^+(C^\#)$ (using the isomorphism j^C defined in Subsubsection 1.4.6). In this case we get that a quasi-isomorphism of Adams connected coaugmented A_∞-coalgebras is a weak equivalence (the converse is also true), for a quasi-isomorphism $C \to D$ induces a quasi-isomorphism of augmented A_∞-algebras $D^\# \to C^\#$, which induces a quasi-isomorphisms between the bar constructions $B^+(D^\#) \to B^+(C^\#)$, and by the previously recalled isomorphism we get a quasi-isomorphism $\Omega^+(D)^\# \to \Omega^+(C)^\#$ of coaugmented dg coalgebras. Taking duals again we obtain a quasi-isomorphism $\Omega^+(C) \to \Omega^+(D)$ of augmented dg algebras, so a weak equivalence $C \to D$.

For the following definitions we refer to [14], Ch. 2, Section 5. Given an augmented A_∞-algebra A, an *A_∞-bimodule over A* is a graded k-module M provided with morphisms $m_{p,q}^M : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M$ of complete degree $(1-(p+q), 0_{G'})$, for each $p, q \in \mathbb{N}_0$ satisfying the following identity on morphisms from $A^{\otimes n'} \otimes M \otimes A^{\otimes n''}$ to M given by

$$\sum_{(r,s,t) \in \mathcal{I}_{n'+n''+1}} (-1)^{r+st} \tilde{m}_{r,t}^M \circ (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) = 0, \tag{1.21}$$

for all $n', n'' \in \mathbb{N}$, where we recall that $\mathcal{I}_n = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r+s+t = n\}$, and where $\tilde{m}_s$ is interpreted as the corresponding multiplication map m_s of A if either $r+s \leq n'$ or $s+t \leq n''$, or $\tilde{m}_s$ is understood as $m^M_{n'-r,n''-t}$ else. In the first case, $\tilde{m}^M_{r,t}$ is $m^M_{n'-s+1,n''}$ if $r+s \leq n'$ or $m^M_{n',n''-s+1}$ if $s+t \leq n''$, and it is $m^M_{r,t}$ else. We have omitted the subindex (A or M) on the identity morphisms for it depends on the indices r, s, t, and it is clearly deduced from the previous explanation. We also assume that M satisfies that $m^M_{p,q} \circ (\mathrm{id}^{\otimes r} \otimes \eta_A \otimes \mathrm{id}^{\otimes t})$ vanishes for $r \neq p$ and $(p,q) \notin \{(0,1),(1,0)\}$, and that $m^M_{1,0} \circ (\eta_A \otimes \mathrm{id}_M) = \mathrm{id}_M = m^M_{0,1} \circ (\mathrm{id}_M \otimes \eta_A)$. Note that an augmented A_∞-algebra is also an A_∞-bimodule for the structure maps $m_{p,q} = m_{p+q+1}$, where $p,q \in \mathbb{N}_0$. There are also obvious notions of left and right A_∞-modules but we will not need them.

Given an augmented A_∞-algebra A and an A_∞-bimodule over A, one can construct the *bar construction* of M, which we shall denote by $B^+(A,M,A)$. Its underlying graded vector space is $B^+(A) \otimes M[1] \otimes B^+(A)$, regarded as a cofree graded counitary bicomodule over $B^+(A)$, provided with a the unique bicoderivation B_M satisfying the following condition. Since $B^+(A) \otimes M[1] \otimes B^+(A)$ is a cofree graded bicomodule, a bicoderivation is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \mathrm{id}_{M[1]} \otimes \epsilon_{B^+(A)}$, which is a sum of mappings of the form $b_{p,q} : I_A[1]^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes p} \to M[1]$, for $p,q \in \mathbb{N}_0$, each of degree $(1-p-q, 0_{G'})$. Define $b_{p,q} : I_A[1]^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes q} \to M[1]$ as the unique map satisfying that $s_M^{-1} \circ m_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}} = -b_{p,q} \circ (s_{I_A}^{\otimes p} \otimes s_M \otimes s_{I_A}^{\otimes q})$, for all $p,q \in \mathbb{N}_0$. Note that the unitarity conditions on the morphisms $\{m_{p,q}\}_{p,q \in \mathbb{N}_0}$ imply that they are uniquely determined by their restrictions $\{m_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}}\}_{p,q \in \mathbb{N}_0}$. It is easy to see that the equations (1.21) are equivalent to the fact that B_M is a differential, *i.e.* $B_M \circ B_M = 0$.

It is also convenient to recall the *unshifted bar construction* of M, which we shall denote by $B^u(A,M,A)$. Its underlying graded vector space is $B^+(A) \otimes M \otimes B^+(A)$, regarded as a cofree graded counitary bicomodule over $B^+(A)$, provided with a the unique bicoderivation B^u_M satisfying the following condition. Again, since $B^+(A) \otimes M \otimes B^+(A)$ is a cofree graded bicomodule, a bicoderivation is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \mathrm{id}_M \otimes \epsilon_{B^+(A)}$, which is a sum of mappings of the form $b^u_{p,q} : I_A[1]^{\otimes p} \otimes M \otimes I_A[1]^{\otimes p} \to M$, for $p,q \in \mathbb{N}_0$, each of degree $(1-p-q, 0_{G'})$. Define $b^u_{p,q} : I_A[1]^{\otimes p} \otimes M \otimes I_A[1]^{\otimes q} \to M$ as the unique map satisfying that $(-1)^p m_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}} = b^u_{p,q} \circ (s_{I_A}^{\otimes p} \otimes \mathrm{id}_M \otimes s_{I_A}^{\otimes q})$, for all $p,q \in \mathbb{N}_0$. Again, the equations (1.21) are equivalent to the fact that B^u_M is a differential, *i.e.* $B^u_M \circ B^u_M = 0$. Note that

$$s_M \circ b^u_{p,q} = -b_{p,q} \circ (\mathrm{id}_{I_A}^{\otimes p} \otimes s_M \otimes \mathrm{id}_{I_A}^{\otimes q}),$$

for all $p,q \in \mathbb{N}_0$.

Given an A_∞-bimodule $(M, m^M_{\bullet,\bullet})$ over an A_∞-algebra A, we define the *shifted* A_∞*-bimodule* $M[1]$ as follows. The map $m^{M[1]}_{p,q} : A^{\otimes p} \otimes M[1] \otimes A^{\otimes q} \to M[1]$ is uniquely defined by

$$m^{M[1]}_{p,q} \circ (\mathrm{id}_A^{\otimes p} \otimes s_M \otimes \mathrm{id}_A^{\otimes q}) = -(-1)^{p+q} s_M \circ m^M_{p,q},$$

for all $p,q \in \mathbb{N}_0$. It is easy to verify that the maps $\{m^{M[1]}_{p,q}\}_{p,q \in \mathbb{N}_0}$ satisfy the unitarity condition and equations (1.21). Moreover, they obviously coincide with the conventions we considered for the case of (differential) graded algebras. Note that the previous definition is tantamount to set $B^u(A, M[1], A) = B^+(A,M,A)$.

Given two A_∞-bimodules M and N, a *morphism* $f_{\bullet,\bullet}$ *of* A_∞*-bimodules from* M *to* N is a collection of morphisms of graded k-modules $f_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to N$ for $p,q \in \mathbb{N}_0$ of complete degree $(-p-q, 0_{G'})$ satisfying the following identity on the space of morphisms from $A^{\otimes n'} \otimes M \otimes A^{\otimes n''}$ to N given by

$$\sum_{(r,s,t)\in\mathcal{I}_{n'+n''+1}} (-1)^{r+st} f_{r',t'} \circ (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) = \sum_{(a,k,l,b)\in\mathbb{N}_{0,n',n''}} (-1)^{b(-k-l)} m^N_{a,b} \circ (\mathrm{id}_A^{\otimes a} \otimes f_{k,l} \otimes \mathrm{id}_A^{\otimes b}), \tag{1.22}$$

where $\mathbb{N}_{0,n',n''}$ is the subset of $\mathbb{N}_0^4$ of elements (a,k,l,b) such that $a+k=n'$ and $l+b=n''$, and where we should understand $\tilde{m}_s$ as m^A_s if either $r+s \leq n'$ or $s+t \leq n''$, or as $m^M_{n'-r,n''-t}$ else. The indices (r',t') are completely determined from the previous cases. We also suppose that $f_{\bullet,\bullet}$ satisfies that $f_{p,q} \circ (\mathrm{id}^{\otimes r} \otimes \eta_A \otimes \mathrm{id}^{\otimes t})$ vanishes for $r \neq p$ and $(p,q) \notin \{(0,0)\}$. We say that it is *strict* if $f_{p,q}$ vanishes for all $(p,q) \neq (0,0)$.

Analogously, given two A_∞-bimodules M and N, it can be uniquely described by a morphism between the corresponding bar constructions. More precisely, let $f_{\bullet,\bullet}$ be a morphism of A_∞-bimodules from M to N. It gives a morphism of counitary dg bicomodules $B^+(f_{\bullet,\bullet}) : B^+(A,M,A) \to B^+(A,N,A)$ as follows. Since $B^+(A) \otimes M[1] \otimes B^+(A)$ is a cofree graded bicomodule, $B^+(f_{\bullet,\bullet})$ is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \mathrm{id}_{N[1]} \otimes \epsilon_{B^+(A)}$, which is a sum of maps $F_{p,q} : I_A^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes q} \to N[1]$. Define $F_{p,q} : I_A^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes q} \to N[1]$ as the unique map satisfying that $F_{p,q} \circ (s_{I_A}^{\otimes p} \otimes s_M \otimes s_{I_A}^{\otimes q}) = s_N \circ f_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}}$. Note that the unitarity conditions on the morphisms $\{f_{p,q}\}_{p,q\in\mathbb{N}_0}$ imply that they are uniquely determined by their restrictions $\{f_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}}\}_{p,q\in\mathbb{N}_0}$. It is easy to see that the equations (1.22) are equivalent to the fact that $B^+(f_{\bullet,\bullet})$ commutes with the differentials, *i.e.* $B^+(f_{\bullet,\bullet}) \circ B_M = B_N \circ B^+(f_{\bullet,\bullet})$. If one wanted to consider the unshifted bar constructions, then $f_{\bullet,\bullet} : M \to N$ is equivalently defined by the morphism $B^u(f_{\bullet,\bullet}) : B^u(A,M,A) \to B^u(A,N,A)$ of dg bicomodules given as follows. By the same arguments as before, $B^u(f_{\bullet,\bullet})$ is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \mathrm{id}_N \otimes \epsilon_{B^+(A)}$, which is also a sum of maps $F^u_{p,q} : I_A^{\otimes p} \otimes M \otimes I_A[1]^{\otimes q} \to N$, for $p,q \in \mathbb{N}_0$. In this case, set $F^u_{p,q} : I_A^{\otimes p} \otimes M \otimes I_A[1]^{\otimes q} \to N$ as the unique map satisfying that $F^u_{p,q} \circ (s_{I_A}^{\otimes p} \otimes \mathrm{id}_M \otimes s_{I_A}^{\otimes q}) = (-1)^p f_{p,q}|_{I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}}$. A similar argument as in the case for the usual bar construction shows that equations (1.22) are equivalent to the fact that $B^u(f_{\bullet,\bullet})$ commutes with the differentials, *i.e.* $B^u(f_{\bullet,\bullet}) \circ B^u_M = B^u_N \circ B^u(f_{\bullet,\bullet})$. Note that

$$s_M \circ F^u_{p,q} = F_{p,q} \circ (\mathrm{id}_{I_A}^{\otimes p} \otimes s_M \otimes \mathrm{id}_{I_A}^{\otimes q}),$$

for all $p,q \in \mathbb{N}_0$.

The *composition* of two morphisms $f_{\bullet,\bullet} : M \to N$ and $g_{\bullet,\bullet} : N \to P$ is given by the family of maps

$$(g \circ f)_{p,q} = \sum_{(a,k,l,b)\in\mathbb{N}_{0,p,q}} (-1)^{b(-k-l)} g_{a,b} \circ (\mathrm{id}_A^{\otimes a} \otimes f_{k,l} \otimes \mathrm{id}_A^{\otimes b}).$$

It is easy to verify that $B^+((g \circ f)_{\bullet,\bullet}) = B^+(g_{\bullet,\bullet}) \circ B^+(f_{\bullet,\bullet})$. If $f_\bullet : A \to B$ is a morphism of augmented A_∞-algebras and N is an A_∞-bimodule over B with structure maps $m_{\bullet,\bullet}$, then it can be easily regarded as an A_∞-bimodule over A via the maps $m'_{\bullet,\bullet}$ given by

$$m'_{p,q} = \sum_{r,s\in\mathbb{N}_0} \sum_{(\bar{i},\bar{j})\in\mathbb{N}^{r,p}\times\mathbb{N}^{s,q}} (-1)^{\varepsilon} m_{r,s} \circ (f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes \mathrm{id}_N \otimes f_{j_1} \otimes \cdots \otimes f_{j_s}), \tag{1.23}$$

where we recall that $\mathbb{N}^{m,n}$ is the subset of $\mathbb{N}^m$ of elements $\bar{i} = (i_1, \dots, i_m)$ such that $|\bar{i}| = i_1 + \dots + i_m = n$, and $\varepsilon = \sum_{u=1}^{r}(r+s+1-u)(i_u - 1) + \sum_{u=1}^{s}(s-u)(j_u - 1)$.

Given an A_∞-bimodule M over an augmented A_∞-algebra A, we define its *(graded) dual A_∞-bimodule $M^\#$*, as follows. The underlying graded space is given by the usual graded dual $M^\#$ of M. The multiplication maps ${}^\# m_{p,q} : A^{\otimes p} \otimes M^\# \otimes A^{\otimes q} \to M^\#$ are defined by

$$ {}^\# m_{p,q}(a_1, \dots, a_p, \lambda, a'_1, \dots, a'_q)(m) = -(-1)^{\sigma'} \lambda\big(m_{q,p}(a'_1, \dots, a'_q, m, a_1, \dots, a_p)\big), \tag{1.24} $$

where

$$ \sigma' = (1+p+q)\deg\lambda + p + q + pq + \big(\sum_{j=1}^{p} \deg a_j\big)\big(\deg m + \deg\lambda + \sum_{i=1}^{q}\deg a'_i\big), $$

for all homogeneous $m \in M$, $\lambda \in M^\#$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in A$. It is clear that the maps $\{{}^\# m_{p,q}\}_{p,q\in\mathbb{N}_0}$ satisfy the required unitarity conditions, for the maps $\{m_{p,q}\}_{p,q\in\mathbb{N}_0}$ do so. Indeed, note first that the previous choice is equivalently described by the unshifted bar construction of $M^\#$, which is provided with the differential induced by the maps ${}^\# b^u_{p,q} : I_A[1]^{\otimes p} \otimes M^\# \otimes I_A[1]^{\otimes q} \to M^\#$ defined by

$$ \begin{aligned} {}^\# b^u_{p,q}(sa_1, \dots, &sa_p, \lambda, sa'_1, \dots, sa'_q)(m) \\ &= -(-1)^{\sigma''} \lambda\big(b^u_{q,p}(sa'_1, \dots, sa'_q, m, sa_1, \dots, sa_p)\big), \end{aligned} \tag{1.25} $$

where

$$ \sigma'' = \deg\lambda + \big(\sum_{j=1}^{p} 1 + \deg a_j\big)\big(\deg m + \deg\lambda + \sum_{i=1}^{q} 1 + \deg a'_i\big), $$

for all homogeneous $m \in M$, $\lambda \in M^\#$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in I_A$, and we have denoted $s_{I_A}(a)$ by sa. It is easy to verify that, if we further denote by $B^u_{M^\#}$ the unique coderivation induced on $B^u(A, M^\#, A)$ by the maps $\{{}^\# b^u_{p,q}\}_{p,q\in\mathbb{N}_0}$ and ${}^\# b^u = \sum_{p,q\in\mathbb{N}_0} {}^\# b^u_{p,q}$, and analogously for M, then

$$ {}^\# b^u \circ B^u_{M^\#}(sa_1, \dots, sa_p, \lambda, sa'_1, \dots, sa'_q)(m) = 0 $$

is precisely

$$ -(-1)^z \lambda(b^u_M \circ B^u_M(sa'_1, \dots, sa'_q, m, sa_1, \dots, sa_p)) = 0, $$

where $z = \sum_{i=1}^{q} 1 + \deg a'_i$, for all $p, q \in \mathbb{N}_0$, and all homogeneous $m \in M$, $\lambda \in M^\#$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in I_A$. This proves that (1.24) indeed defines an A_∞-bimodule on $M^\#$. Note that (1.24) coincides with the convention we considered for the case of (differential) graded bimodules over (differential) graded algebras. We remark that the coderivation of $B(A, M^\#, A)$ is given by the maps $\{{}^\# b_{p,q}\}_{p,q\in\mathbb{N}_0}$ defined as

$$ \begin{aligned} H_{M,k,-1,0}&({}^\# b_{p,q}(sa_1, \dots, sa_p, s\lambda, sa'_1, \dots, sa'_q))(s^{-1}m) \\ &= -(-1)^{\sigma'''} H_{M,k,1,0}(s^{-1}\lambda)\big(b_{q,p}(sa'_1, \dots, sa'_q, sm, sa_1, \dots, sa_p)\big), \end{aligned} \tag{1.26} $$

where

$$ \sigma''' = (1 + \deg\lambda) + \big(\sum_{j=1}^{p} 1 + \deg a_j\big)\big(\deg m + \deg\lambda + \sum_{i=1}^{q} 1 + \deg a'_i\big), $$

for all homogeneous $m \in M$, $\lambda \in M^{\#}$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in I_A$.

Moreover, the previous (graded) dual construction is in fact functorial. To wit, if $f_{\bullet,\bullet} : M \to N$ is a morphism of A_∞-bimodules over the augmented A_∞-algebra A, then it defines a morphism $f^{\#}_{\bullet,\bullet} : N^{\#} \to M^{\#}$ of A_∞-bimodules between the corresponding graded duals, as follows. The maps $f^{\#}_{p,q} : A^{\otimes p} \otimes N^{\#} \otimes A^{\otimes q} \to M^{\#}$ are defined by

$$f^{\#}_{p,q}(a_1, \dots, a_p, \lambda, a'_1, \dots, a'_q)(m) = (-1)^{\rho'} \lambda\big(f_{q,p}(a'_1, \dots, a'_q, m, a_1, \dots, a_p)\big), \quad (1.27)$$

where

$$\rho' = (p+q) \deg \lambda + p + q + pq + \big(\sum_{j=1}^{p} \deg a_j\big)\big(\deg m + \deg \lambda + \sum_{i=1}^{q} \deg a'_i\big),$$

for all homogeneous $m \in M$, $\lambda \in N^{\#}$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in A$. It is clear that the maps $\{f^{\#}_{p,q}\}_{p,q \in \mathbb{N}_0}$ satisfy the required unitarity conditions, for the maps $\{f_{p,q}\}_{p,q \in \mathbb{N}_0}$ do so. Indeed, note first that the previous choice is equivalently described by the morphism ${}^{\#}F^u_{p,q} : I_A[1]^{\otimes p} \otimes M^{\#} \otimes I_A[1]^{\otimes q} \to M^{\#}$ between the unshifted bar construction of $M^{\#}$ and $N^{\#}$, defined by

$$\begin{aligned} {}^{\#}F^u_{p,q}(sa_1, \dots, &sa_p, \lambda, sa'_1, \dots, sa'_q)(m) \\ &= (-1)^{\rho''} \lambda\big(F^u_{q,p}(sa'_1, \dots, sa'_q, m, sa_1, \dots, sa_p)\big), \end{aligned} \quad (1.28)$$

where

$$\rho'' = \big(\sum_{j=1}^{p} 1 + \deg a_j\big)\big(\deg m + \deg \lambda + \sum_{i=1}^{q} 1 + \deg a'_i\big),$$

for all homogeneous $m \in M$, $\lambda \in N^{\#}$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in I_A$. Let $F^u : B^u(A, M, A) \to B^u(A, N, A)$ and ${}^{\#}F^u : B^u(A, N^{\#}, A) \to B^u(A, M^{\#}, A)$ be the morphisms of bicomodules induced by $\{F^u_{p,q}\}_{p,q \in \mathbb{N}_0}$ and $\{{}^{\#}F^u_{p,q}\}_{p,q \in \mathbb{N}_0}$. Then, it is easy to verify that

$$(B_{M^{\#}} \circ {}^{\#}F^u - {}^{\#}F^u \circ B_{N^{\#}})(\omega)(m) = -(-1)^{K + \deg \lambda} \lambda((F^u \circ B_M - B_N \circ F^u)(\omega')),$$

where $\omega = sa_1 \otimes \dots \otimes sa_p \otimes \lambda \otimes sa'_1 \otimes \dots \otimes sa'_q$, $\omega' = sa'_1 \otimes \dots \otimes sa'_q \otimes m \otimes sa_1 \otimes \dots \otimes sa_p$ and

$$K = (\deg \lambda + \sum_{i=1}^{p} 1 + \deg a_i)(\deg m + \sum_{j=1}^{q} 1 + \deg a'_j).$$

Hence, ${}^{\#}F^u$ commutes with the differentials, as was to be shown.

From the previous notion we can provide a short definition of *d-cyclic augmented A_∞-algebra*, for $d \in \mathbb{Z}$. We define it as an augmented A_∞-algebra provided with a strict isomorphism $f : A \to A^{\#}[d]$ of A_∞-bimodules satisfying that $f(a)(b) = (-1)^{\deg a \deg b} f(b)(a)$, *i.e.* $f^{\#} = f$, as morphisms of graded vector spaces (see [5], Lemma 3.1). The more standard definition of d-cyclic (nonunitary, unitary, or augmented) A_∞-algebra can be recalled as follows. It is an A_∞-algebra $(A, m_\bullet)$ provided with a nondegenerate bilinear form γ on A of degree d (*i.e.* a morphism of graded vector spaces $\gamma : A \otimes A \to k$ of degree d) satisfying that $\gamma \circ \tau_{A,A} = \gamma$ and

$$\gamma(m_n(a_1, \dots, a_n), a_0) = (-1)^{n + (\deg a_0)(\sum_{i=1}^{n} \deg a_i)} \gamma(m_n(a_0, \dots, a_{n-1}), a_n), \quad (1.29)$$

for all homogeneous $a_0, \dots, a_n \in A$. It is clear to see that this last definition coincides with the first one when A is augmented or even just (strictly) unitary, by setting $\gamma(a, b) = (s^{-d}_{A^{\#}} f(a))(b)$, for all $a, b \in A$. Indeed, it is easy to see that (1.29) is tantamount to the fact that f is a strict morphism of A_∞-bimodules, whereas the nondegeneracy of γ is equivalent to f being bijective, and the symmetry property of γ is tantamount to $f^\# = f$. Define $\Gamma : A[1] \otimes A[1] \to k$ by $\Gamma \circ (s_A \otimes s_A) = \gamma$. Note that Γ has degree $d + 2$. Then, the super symmetry of γ is equivalent to $\Gamma \circ \tau_{A[1],A[1]} = -\Gamma$, and (1.29) is tantamount to

$$\Gamma(b_n(sa_1, \dots, sa_n), sa_0) = (-1)^{\deg sa_0 (\sum_{i=1}^{n} \deg sa_i)} \Gamma(b_n(sa_0, \dots, sa_{n-1}), sa_n), \tag{1.30}$$

for all homogeneous $a_0, \dots, a_n \in A$, where we have written sa_i instead of $s_A(a_i)$ to simplify.

Remark 1.3. *Let A be a unitary A_∞-algebra that is further assumed to be minimal, i.e. $m_1 = 0$, and let $f_{\bullet,\bullet} : A \to A^\#[d]$ be a quasi-isomorphism of A_∞-bimodules. Since A is minimal, $f_{0,0}$ is in fact an isomorphism of graded vector spaces. Moreover, by only considering the underlying unitary (associative) graded algebra structure of A given by m_2 and η_A, $f_{0,0}$ is in fact an isomorphism of graded bimodules, where $A^\#[d]$ has the underlying graded bimodule structure induced by its A_∞-bimodule structure. Hence,*

$$\begin{aligned} f_{0,0}(a)(b) = (af_{0,0}(1_A))(b) &= (-1)^{\deg a \deg b} f_{0,0}(1_A)(ba) \\ &= (-1)^{\deg a \deg b}(f_{0,0}(1_A)b)(a) = (-1)^{\deg a \deg b} f_{0,0}(b)(a), \end{aligned}$$

so $f_{0,0}^\# = f_{0,0}$. As a consequence, the symmetry condition in the definition of d-cyclic unitary (or augmented) A_∞-algebra A is superfluous if A is minimal.

If M is a dg A-bimodule over an augmented dg algebra A and C is a coaugmented A_∞-coalgebra, then $M \otimes C$ is in fact an A_∞-bimodule over $\mathcal{H}om(C, A)$ with the structure morphisms given by $m_{0,0}^M = d_M \otimes \mathrm{id}_C + \mathrm{id}_M \otimes \Delta_1^C$, and, for $p + q \geq 1$,

$$\begin{aligned} &m_{p,q}^{M \otimes C}(\phi_1 \otimes \dots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \dots \otimes \psi_q) \\ &\qquad = (-1)^{\epsilon'}(\phi_1(c_{(q+2)}) \dots \phi_p(c_{(q+p+1)})).m.(\psi_1(c_{(1)}) \dots \psi_q(c_{(q)})) \otimes c_{(q+1)}, \end{aligned}$$

where $\Delta_{p+q+1}^C(c) = c_{(1)} \otimes \dots \otimes c_{(p+q+1)}$, and

$$\begin{aligned} \epsilon' = pq + \deg c \deg m + (p+q+1)\Big(\sum_{i=1}^{p} \deg \phi_i + \sum_{j=1}^{q} \deg \psi_j\Big) + \sum_{\substack{1 \leq i \leq p \\ q+2 \leq i' \leq q+i}} \deg c_{(i')} \deg \phi_i \\ + \sum_{\substack{1 \leq j \leq q \\ 1 \leq j' < j}} \deg c_{(j')} \deg \psi_j + \Big(\deg m + \sum_{i=1}^{p} \deg c_{(q+1+i)} + \sum_{j=1}^{q} \deg \psi_j\Big)\Big(\sum_{j=1}^{q+1} \deg c_{(j)}\Big). \end{aligned}$$

This structure can be obtained as follows. First note that, given any dg bimodule N over A, it is easy to verify that $\mathcal{H}om(C, N)$ is an A_∞-bimodule over $\mathcal{H}om(C, A)$ via $m_{0,0}^{\mathcal{H}om(C,N)}(\omega) = d_N \circ \omega - (-1)^{|\omega|} \omega \circ \Delta_1^C$, and

$$\begin{aligned} &m_{p,q}^{\mathcal{H}om(C,N)}(\phi_1, \dots, \phi_p, \omega, \phi_{p+1}, \dots, \phi_{p+q}) \\ &= (-1)^{(p+q+1)(1+\deg \omega + \sum_{i=1}^{p+q} \deg \phi_i)} m_N^{p,q} \circ (\phi_1 \otimes \dots \otimes \phi_p \otimes \omega \otimes \phi_{p+1} \otimes \dots \otimes \phi_{p+q}) \circ \Delta_{p+q+1}^C, \end{aligned}$$

for all $p, q \in \mathbb{N}_0$ such that $p + q \geq 1$, where $m_N^{p,q} : A^{\otimes p} \otimes N \otimes A^{\otimes q} \to N$ denotes the successive application of the action of A on N, $\phi_1, \dots, \phi_{p+q} \in \mathcal{H}om(C, A)$ and $\omega \in \mathcal{H}om(C, N)$. Consider now A_∞-bimodule structure on $\mathcal{T} = \mathcal{H}om(C, M^\#)$, and take the graded dual A_∞-bimodule $\mathcal{T}^\#$ (over $\mathcal{H}om(C, A)$). The A_∞-bimodule structure on $M \otimes C$ over $\mathcal{H}om(C, A)$ is obtained by pulling back the structure on

$\mathcal{T}^{\#}$ via the canonical injection $i : M \otimes C \to \mathcal{T}^{\#}$ sending $m \otimes c$ to the functional that sends $\mu \in \mathcal{T}$ to $(-1)^w \mu(c)(m)$, where $w = \deg c \deg m + \deg c \deg \mu + \deg m \deg \mu$. Indeed, it is easy to see that the A_∞-bimodule $M \otimes C$ is the one obtained by imposing that i is a strict morphism of A_∞-bimodules over $\mathcal{H}om(C, A)$.

If M is only a left (resp., right) dg module over A, we may regard it as a dg A-bimodule by means of the augmentation ϵ_A, *i.e.* $a.m.a' = \epsilon_A(a')a.m$ (resp., $a.m.a' = \epsilon_A(a)m.a'$), so we may apply the previous construction. If $f : M \to N$ is a morphism of dg A-bimodules over an augmented dg algebra A and C is an coaugmented A_∞-coalgebra, then the map $f \otimes \mathrm{id}_C : M \otimes C \to N \otimes C$ is a strict morphism of A_∞-bimodules over $\mathcal{H}om(C, A)$. On the other hand, let $f_\bullet : C \to D$ be a morphism of coaugmented A_∞-coalgebras and let A be an augmented dg algebra. This induces a morphism of augmented A_∞-algebras $(f_\bullet)_* : \mathcal{H}om(D, A) \to \mathcal{H}om(C, A)$, as seen in (1.18). In particular, given any dg A-bimodule M, this allows to consider $M \otimes C$ as an A_∞-bimodule over $\mathcal{H}om(D, A)$ by means of (1.23). Then, the collection of morphisms

$$F_{p,q} : \mathcal{H}om(D, A)^{\otimes p} \otimes (M \otimes C) \otimes \mathcal{H}om(D, A)^{\otimes q} \to M \otimes D$$

given by

$$\begin{aligned} F_{p,q}(\phi_1 \otimes \cdots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \cdots \otimes \psi_q) \\ = (-1)^{\varepsilon'} (\phi_1(d_{(q+2)}) \dots \phi_p(d_{(q+p+1)})).m.(\psi_1(d_{(1)}) \dots \psi_q(d_{(q)})) \otimes d_{(q+1)}, \end{aligned}$$

where $f_{p+q+1}(c) = d_{(1)} \otimes \cdots \otimes d_{(p+q+1)}$, and

$$\begin{aligned} \varepsilon' = pq + \deg c \deg m + (p+q+1)\big(\sum_{i=1}^{p} \deg \phi_i + \sum_{j=1}^{q} \deg \psi_j\big) + \sum_{\substack{1 \le i \le p \\ q+2 \le i' \le q+i}} \deg d_{(i')} \deg \phi_i \\ + \sum_{\substack{1 \le j \le q \\ 1 \le j' < j}} \deg d_{(j')} \deg \psi_j + \big(\deg m + \sum_{i=1}^{p} \deg d_{(q+1+i)} + \sum_{j=1}^{q} \deg \psi_j\big)\big(\sum_{j=1}^{q+1} \deg d_{(j)}\big). \end{aligned}$$

defines a morphism of A_∞-bimodules over $\mathcal{H}om(D, A)$.

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