Hochschild (co)homology of Koszul dual pairs

Estanislao Herscovich *

Abstract

In this article we prove that the Tamarkin-Tsygan calculus of an Adams connected augmented dg algebra and of its Koszul dual are dual to each other. This uses the fact that the Hochschild cohomology and homology may be regarded as a twisted convolution dg algebra and of some twisted tensor product, respectively. As an immediate application of this latter point of view we also show that the cup product on Hochschild cohomology and the cap product on Hochschild homology of a Koszul algebra is directly computed from the coalgebra structure of $\mathrm{Tor}^A_\bullet(k,k)$ (the first of these results is proved differently in [2]).

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1 Introduction

The aim of this work is to highlight some aspects of the relation between the Hochschild (co)homology and the Koszul theory of Adams connected augmented (dg) algebras. The main result we have proved is the following. Consider an Adams connected dg algebra A over a field k, and denote by $E(A)$ its Koszul dual $B^+(A)^\#$ (where $(-)^\#$ is the graded dual). Then, there are an isomorphism of Gerstenhaber algebras $HH^\bullet(E(A)) \simeq HH^\bullet(A)$ (see Theorems 3.3 and 3.4) and an isomorphism of Gerstenhaber modules $HH_\bullet(E(A)) \simeq HH_\bullet(A)^\#$ over $HH^\bullet(E(A)) \simeq HH^\bullet(A)$ (see Theorems 3.3 and 3.6) such that the Connes' map on $HH_\bullet(E(A))$ is minus the dual of the Connes' map on $HH_\bullet(A)$ under the previous identification (see Proposition 3.5). We say in this case that the Tamarkin-Tsygan calculus of A and of its Koszul dual $E(A)$ are *dual* (see Corollary 3.7).

We would like to remark that the results stated before follow from a description of the Hochschild (co)homology of an augmented dg algebra given by a general twisting procedure, which is recalled in Subsection 2.1. We would have supposed that this twisting construction was widely known by the experts, but some articles appearing in the literature could imply that this is not exactly the case. In fact, the other interesting result of the article, included in Subsection 2.2, states that the dg algebra $C^\bullet(A,A)$ computing the Hochschild cohomology of any Koszul algebra A is quasi-isomorphic to the twisted hom construction $\mathcal{H}om^\tau(C,A)$, where C is the graded coalgebra $\mathrm{Tor}^A_\bullet(k,k)$ and $\tau : C \to A$ is a particular twisting cochain. As a consequence, $H^\bullet(\mathcal{H}om^\tau(C,A))$ is isomorphic to $HH^\bullet(A)$ as graded algebras. This is in fact the main result of [2], stated at the introduction, p. 443, or after as Theorem 2.3, which is based on the construction of a chain map $\Delta : P_\bullet \to P_\bullet \otimes_A P_\bullet$

*Institut Fourier, Université Grenoble Alpes, France. On leave of absence from Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Argentina, and CONICET (Argentina).

lifting the identity of A, where $P_\bullet$ is the minimal projective resolution of the A-bimodule A. Our proof is simpler, for we do not need to make any homological computation, and exploits a different point of view: the twisted procedure we mentioned previously. Moreover, using the same ideas we also show that, given an A-bimodule M, the dg bimodule $M \otimes_\tau C$ over $\mathcal{H}om^\tau(C, A)$ is quasi-isomorphic to the dg bimodule $C_\bullet(A, M)$ over $C^\bullet(A, A)$. This tells us that $H_\bullet(M \otimes_\tau C)$ is isomorphic to $H_\bullet(A, M)$ as graded bimodules over $HH^\bullet(A)$ (see Proposition 2.2). This result does not seem to have been known.

We now briefly discuss similar results that appeared in the literature stating isomorphisms between the Hochschild (co)homology of a dg algebra A and of the corresponding dual $E(A)$. If $A = TV/\langle R \rangle$ is a Koszul algebra and $A^! = T(V^*)/\langle R^\perp \rangle$, where V^* is concentrated in cohomological degree 1 and $R^\perp \subseteq (V^{\otimes 2})^* \simeq (V^*)^{\otimes 2}$ is the annihilator of $R \subseteq V^{\otimes 2}$, it was already observed by B. Feigin and B. Tsygan that there is in fact a duality pair between the Hochschild homology groups of A and $A^!$ (only regarded as graded vector spaces), and we remark that $A^!$ and $E(A)$ are quasi-isomorphic dg algebras. More precisely, the mentioned duality can be directly deduced from (or following the lines of) the isomorphism between the corresponding cyclic homology groups given in [4], Thm. 2.4.1, where the authors further suppose that the base field k has characteristic zero, even though this assumption is not strictly necessary if we are interested only in the Hochschild homology groups (see [16] for a more detailed analysis on the corresponding gradings). Furthermore, an isomorphism of graded algebras between the Hochschild cohomology groups $HH^\bullet(A^!)$ and $HH^\bullet(A)$ in case A is also a Koszul algebra was announced by R.-O. Buchweitz in the Conference on Representation Theory held at Canberra on July 2003. A more general result was proved by B. Keller in the preprint [14], where he showed that there is in fact a quasi-isomorphism of B_∞-algebras between the corresponding Hochschild cohomology cochain complexes. The isomorphism deduced by Keller is more general than ours, even though the map we constructed is in principle different from the one he obtained, and he only treated the case of cohomology and not of homology. On the other hand, Y. Félix, L. Menichi and J.-C. Thomas proved in [7], Prop. 5.1 and 5.3, that given a simply connected coaugmented dg coalgebra C over a field, there is a isomorphism of Gerstenhaber algebras between the Hochschild cohomology $HH^\bullet(C^\#)$ of the graded dual $C^\#$ of C and the Hochschild cohomology $HH^\bullet(\Omega^+(C))$ of the cobar construction of C. By specializing this result to the case $A = C^\#$, and using the obvious result $\Omega^+(C) \simeq E(A)$, we get that the Hochschild cohomology of A and $E(A)$ are isomorphic as Gerstenhaber algebras. The simply connectedness assumption is however completely unusual in the realm of representation theory of algebras. It is fair to say that our main results, Theorems 3.3, 3.4 and 3.6, get rid of the simply connectedness assumption but impose other grading hypotheses that are more typical in representation theory. Moreover, we also study the case of Hochschild homology, which was not considered in [7].

The article is organised as follows. In Section 2 we recall some basic tools. Subsection 2.1 is devoted to present the (supposedly well-known) relation between Hochschild (co)homology theory and twists of augmented dg algebras (see Fact 2.1). In Subsection 2.2 we show a general result that allows to compute the multiplicative structure on the Hochschild cohomology of a Koszul algebra and the corresponding module structure on the Hochschild homology, which in our situation is just a consequence of the way we presented the Hochschild (co)homology complexes (see Proposition 2.2). We believe the experts should not be surprised by the contents of these two subsections, but we have included them due to the simple characterization of the previously referred multiplicative structures, that we were unable to find in the literature: the proof of the piece of Proposition 2.2 considering Hochschild cohomology is done “without computations” (and it is different from the one given in [2]), and the statement concerning Hochschild homology seems to be new. Subsection 2.3 states the well-known duality between the bar and cobar constructions, that was only included due to some inaccuracies in the literature, and Subsection 2.4 recalls a resolution of the dg bimodule $\Omega^+(C)$, where C is a cocomplete coaugmented dg coalgebra, which is only present due to its later use. The preliminaries conclude with Subsection 2.5, where we recall the Gerstenhaber bracket on Hochschild cohomology of any augmented dg algebra, but we also introduce the analogous bracket on the Cartier-Doi cohomology of a coaugmented dg coalgebra. The existence of this last bracket is by no means unexpected but we have included it because we utilize it in the sequel. The main reason for these last three subsections is to establish the notation and basic constructions that will be later used.

In Section 3 we use the tools appearing in the previous section in order to prove the main result of the article: the Tamarkin-Tsygan calculus of an Adams connected augmented dg algebra and of its Koszul dual are dual.

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2 Preliminaries

We will fix throughout the article a field k. We shall follow the conventions on dg (co)algebras and their dg (co)modules, the bar resolution of a dg algebra, the bar and cobar constructions, and twisting of dg algebras and their dg modules recalled in [11], to which we refer together with the references therein. In particular these objects will be graded with respect to $G = \mathbb{Z} \times \mathbb{Z}$, where the first grading is called *(co)homological* and the second one is called *Adams*. When applying the Koszul sign rule, we will only take the cohomological degree into account (for a nice exposition on basic homological algebra of dg modules over dg algebras and a detailed account on the sign rule we refer to [1]). Given $n_0 \in \mathbb{Z}$ and $V = \oplus_{(n,m)\in\mathbb{Z}^2} V^{(n,m)}$ any graded object (decorated perhaps with further adjectives), we shall denote by $V[n_0]$ the *shift* of V, defined as $V[n_0]^{(n,m)} = V^{(n+n_0,m)}$ for all $n, m \in \mathbb{Z}$, and by $s_{V,n_0} : V \to V[n_0]$ the morphism of cohomological degree $-n_0$, called the *suspension on V of degree n_0*, whose underlying set-theoretic map is the identity. In this work we shall never consider shifts of the Adams degree. If $n_0 = 1$ we will just write s_V and call it the *suspension on V*. We shall say that a graded vector space $V = \oplus_{(m,n)\in\mathbb{Z}^2} V^{(m,n)}$ over k is *Adams connected* if $V^{(0,0)} = k$ and $V^{\wedge} = \oplus_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} V^{(m,n)}$ is concentrated in either strictly positive or strictly negative Adams degrees, and each homogeneous component of $V^{\wedge}$ of a fixed Adams degree d (but including all cohomological degrees) is locally finite dimensional (*cf.* [18], Def. 2.1).

2.1 Hochschild (co)homology as a twisted construction

In this subsection we recall the basic definitions of Hochschild cohomology and homology of an augmented dg algebra. We refer the reader to [8] and [23] for standard references.

The following constructions are standard (see [12], but also [15] and [21] for further details). We recall that, given a *Maurer-Cartan element a* of an augmented dg algebra (Λ, d_Λ), i.e. $a \in \Lambda^{(1,0)}$ and $d_\Lambda(a) + a^2 = 0$, one may consider the *twisted* augmented dg algebra $(\Lambda, d_{\Lambda,a})$, where $d_{\Lambda,a} = d_\Lambda + \mathrm{ad}(a)$ and the remaining algebraic structure of Λ remains unchanged. If (M, d_M) is a dg Λ-bimodule, the *twisted* dg bimodule $(M, d_{M,a})$ over the twisting of Λ is defined analogously. Given an augmented dg algebra A and a coaugmented dg coalgebra C, let $\Lambda = \mathcal{H}om(C, A)$ be the augmented dg algebra provided with the convolution product and the obvious unit and augmentation defined from C and A. A *twisting cochain* τ is a Maurer-Cartan element of Λ satisfying that $\epsilon_A \circ \tau = \tau \circ \eta_C = 0$, where ϵ_A is the augmentation of A and η_C is the coaugmentation of C. We shall denote the twisted dg algebra of Λ by $\mathcal{H}om^{\tau}(C,A)$, and call it the *twisted convolution algebra*. Moreover, if M is a dg A-bimodule, $M \otimes C$ is a dg bimodule over Λ for the action given by

$$\phi \cdot (m \otimes c) \cdot \psi = (-1)^{\epsilon} \phi(c_{(3)}).m.\psi(c_{(1)}) \otimes c_{(2)},$$

where $m \in M$, $c \in C$, $\Delta_C^{(3)}(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ is the usual Sweedler notation for the iterated application of the coproduct of C, $\phi, \psi \in \mathcal{H}om(C,A)$, and $\epsilon = \deg \psi \deg c + \deg c_{(3)}(\deg m + \deg c_{(1)} + \deg c_{(2)} + \deg \psi)$. The twist of the previous dg bimodule will be denoted by $M \otimes_{\tau} C$, that will be called the *twisted tensor product*. Note that it is a dg bimodule over $\mathcal{H}om^{\tau}(C,A)$.

Let A be an augmented dg algebra with unit $\eta_A : k \to A$ (and we define $1_A = \eta_A(A)$), differential d_A of cohomological degree 1 and Adams degree zero, and augmentation ϵ_A, whose kernel is denoted by I_A. Consider the augmented dg algebra $\mathcal{H}om(B^+(A),A)$, where $B^+(A)$ denotes the reduced bar construction of A (see for instance [17], Def. 8.1), provided with the convolution product and the differential D_0 induced by that of A and $B^+(A)$, *i.e.*

$$\begin{aligned} D_0(f)([a_1|\dots|a_n]) =& d_A(f([a_1|\dots|a_n])) + \sum_{i=1}^{n} (-1)^{\bar{\epsilon}_i} f([a_1|\dots|d_A(a_i)|\dots|a_n]) \\ &- \sum_{i=2}^{n} (-1)^{\bar{\epsilon}_i} f([a_1|\dots|a_{i-1}a_i|\dots|a_n]), \end{aligned} \tag{2.1}$$

where $\bar{\epsilon}_i = \deg f + (\sum_{j=1}^{i-1} \deg a_j) - i + 1$. The unit and augmentation of $\mathcal{H}om(B^+(A),A)$ are the obvious ones. Let us consider $\tau_A : B^+(A) \to A$ the universal twisting cochain of A whose restriction to $I_A[1]^{\otimes n}$ vanishes if $n \neq 1$ and such that its restriction to $I_A[1]$ is given by the composition of minus the canonical inclusion $I_A[1] \subseteq A[1]$ with s_A^{-1}. It is easy to check that the twist of the differential D_0 of $\mathcal{H}om(B^+(A),A)$ by τ_A gives precisely the differential of the complex $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A),A)$ computing the Hochschild cohomology of A, where $\overline{\mathrm{Bar}}(A)$ denotes the reduced bar resolution of A. Indeed, the map $D_1(f) = \mathrm{ad}(\tau_A)(f)$ coincides with

$$\begin{aligned} D_1(f)([a_1|\dots|a_n]) =& -(-1)^{\deg a_1 \deg f - \deg f} a_1 f([a_2|\dots|a_n]) \\ &+ (-1)^{\bar{\epsilon}_n} f([a_1|\dots|a_{n-1}])a_n. \end{aligned} \tag{2.2}$$

Hence, $\mathcal{H}om^{\tau_A}(B^+(A),A)$ is canonically identified with the complex computing the Hochschild cohomology of A. It is also trivial to verify that the *cup product* on $\mathcal{H}om^{\tau_A}(B^+(A),A)$ (see [3], Ch. XI, §4 and 6) coincides with the convolution product on $\mathcal{H}om^{\tau_A}(B^+(A),A)$.

The Hochschild homology of A with coefficients in a dg A-bimodule (M,d_M) can be regarded in a similar fashion. The tensor product $M \otimes B^+(A)$ has a differential D_0' induced by the ones of A and of $B^+(A)$ given by

$$\begin{aligned} D_0'(m \otimes [a_1|\dots|a_n]) =& d_M(m) \otimes [a_1|\dots|a_n] - \sum_{i=1}^{n} (-1)^{\tilde{\epsilon}_i} m \otimes [a_1|\dots|d_A(a_i)|\dots|a_n] \\ &+ \sum_{i=2}^{n} (-1)^{\tilde{\epsilon}_i} m \otimes [a_1|\dots|a_{i-1}a_i|\dots|a_n], \end{aligned} \tag{2.3}$$

where $\tilde{\epsilon}_i = \deg m + (\sum_{j=1}^{i-1} \deg a_j) - i + 1$. It is a dg bimodule over $\mathcal{H}om(B^+(A), A)$. The twist of the differential of $M \otimes B^+(A)$ by τ_A gives precisely the differential of $M \otimes_{A^e} \overline{\mathrm{Bar}}(A)$, since $D_1'(m \otimes [a_1|\dots|a_n]) = \mathrm{ad}(\tau_A)(m \otimes [a_1|\dots|a_n])$ is given by

$$\begin{aligned} D_1'(m \otimes [a_1|\dots|a_n]) =& (-1)^{\deg m} m a_1 \otimes [a_2|\dots|a_n] \\ &- (-1)^{\tilde{\epsilon}_n(\deg a_n + 1)} a_n m \otimes [a_1|\dots|a_{n-1}]. \end{aligned} \tag{2.4}$$

Thus, the complex $M \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ computing the Hochschild homology of A with coefficients in M is canonically identified with $M \otimes_{\tau_A} B^+(A)$. It is also trivial to verify that the left and right *cap products* on $M \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ over $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ (see [3], Ch. XI, §4 and 6) coincide with the left and right actions of the algebra $\mathcal{H}om^{\tau_A}(B^+(A), A)$ on $M \otimes_{\tau_A} B^+(A)$.

We remark that, for $M = A$, the graded bimodule $HH_\bullet(A)$ over $HH^\bullet(A)$ is graded symmetric, as one may easily deduce as follows. Indeed, as noted in the literature (see for instance [19], (10)), given any dg bimodule N over A there are obvious "cup" actions (on the left and on the right) of $HH^\bullet(A)$ on $H^\bullet(A, N)$, which coincide with the product of the subspaces $HH^\bullet(A)$ and $H^\bullet(A, N)$ regarded inside $HH^\bullet(A[N])$ (where $A[N]$ is the dg algebra with underlying dg module given by $A \oplus N$, the usual product $(a, n) \cdot (a', n') = (aa', an' + na')$, unit $(1_A, 0_N)$ and augmentation $(a, n) \mapsto \epsilon_A(a)$). Since $HH^\bullet(A[N])$ is a graded commutative algebra, we see that $H^\bullet(A, N)$ is a graded symmetric bimodule over $HH^\bullet(A)$. Furthermore, there is an isomorphism $H^\bullet(A, A^\#) \simeq HH_\bullet(A)^\#$ of bimodules over $HH^\bullet(A)$ (see for instance [19], Lemma 16), so $HH_\bullet(A)$ is graded symmetric.

We summarize our previous comments in the following result.

Fact 2.1. *Let A be an augmented dg algebra over k, and let τ_A denote the universal twisting cochain of A. Then,*

(i) the complex $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ computing Hochschild cohomology is canonically identified with $\mathcal{H}om^{\tau_A}(B^+(A), A)$. Moreover, the cup product on the first complex coincides exactly with the convolution product on the latter.

(ii) the complex $M \otimes_{A^e} \overline{\mathrm{Bar}}(A)$ computing the Hochschild homology of A with coefficients in M is canonically identified with the twisted tensor product $M \otimes_{\tau_A} B^+(A)$. Furthermore, the bimodule structure of the first complex over $\mathcal{H}om_{A^e}(\overline{\mathrm{Bar}}(A), A)$ given by the cap products coincides exactly with the bimodule structure of the latter complex over $\mathcal{H}om^{\tau_A}(B^+(A), A)$.

We see that the previous description of Hochschild homology and cohomology groups is by no means accidental. Indeed, it is a direct consequence of the definitions as soon as one notes that the reduced bar resolution $\overline{\mathrm{Bar}}(A)$ of A is canonically identified (as complexes of vector spaces) with $A^e \otimes_{\tau_A} B^+(A)$, where A^e is a dg A-bimodule with the *outer structure* given by $a(a' \otimes b')b = (aa') \otimes (b'b)$, for $a, a', b, b' \in A$. The identification isomorphism is given by

$$(a_{n+1} \otimes a_0) \otimes [a_1|\dots|a_n] \mapsto (-1)^{\deg a_{n+1}(\deg a_0 + \epsilon)} a_0 [a_1|\dots|a_n] a_{n+1}, \tag{2.5}$$

where $\epsilon = (\sum_{i=1}^n \deg a_i) - n$. Consider the dg A-bimodule structure on $A^e \otimes_{\tau_A} B^+(A)$ coming from the *inner structure* of A^e given by

$$a(a' \otimes b')b = (-1)^{(\deg a' \deg a + \deg b \deg b' + \deg a \deg b)} (a'b) \otimes (ab'),$$

for $a, a', b, b' \in A$. This induces a structure of dg bimodule on the twisted tensor product $A^e \otimes_{\tau_A} B^+(A)$ over the algebra $\mathcal{H}om^{\tau_A}(B^+(A), A) \otimes A$. By means of this structure the previous identification gives in fact an isomorphism of dg bimodules over

$\mathcal{H}om^{\tau_A}(B^+(A), A) \otimes A$. If we apply the functors $\mathcal{H}om_{A^e}(-, A)$ and $A \otimes_{A^e} (-)$ to the identification (2.5) we get precisely the description of the complexes computing the Hochschild cohomology and homology described in the previous fact. We also recall that, since k is a field, $\overline{\mathrm{Bar}}(A)$ is a semifree resolution of the dg A-bimodule A (see [5], Lemma 4.3).

2.2 Application to the Hochschild (co)homology of Koszul algebras

In this subsection we show that the description of Hochschild (co)homology given by Fact 2.1 allows us to compute the augmented graded algebra structure on the Hochschild cohomology $HH^{\bullet}(A)$ of a (quadratic) Koszul algebra A and the corresponding graded bimodule structure on the Hochschild homology $H_{\bullet}(A, M)$ with coefficients in any A-bimodule M. In particular, we give a simpler proof of the main result of [2] (stated at the introduction, p. 443, or after as Theorem 2.3), where they have computed the algebra structure of the Hochschild cohomology. Indeed, the theorem of the mentioned article is just the (basis-dependent expression of the) of the convolution product of $\mathcal{H}om^{\tau}(C, A)$, where $C = \mathrm{Tor}^A_{\bullet}(k, k)$. It seems that the result concerning homology has not been observed so far.

Let $A = TV/\langle R \rangle$ be a (quadratic) Koszul algebra over a field k, and consider the coaugmented dg coalgebra $\mathrm{Tor}^A_{\bullet}(k, k)$ with zero differential, where the cohomological degree of the component $\mathrm{Tor}^A_i(k, k)$ is $-i$ (see for instance [20], Ch. 1, Section 1, pp. 4–5). The structure of this coaugmented dg coalgebra and its quasi-isomorphism to the bar construction $B^+(A)$, that we will briefly recall, is well-known and lies in the heart of the Koszul property. We have the isomorphisms of Adams graded vector spaces $\mathrm{Tor}^A_0(k, k) \simeq k$, $\mathrm{Tor}^A_1(k, k) \simeq V$, and

$$\mathrm{Tor}^A_i(k, k) \simeq \bigcap_{j=0}^{i-2} V^{\otimes j} \otimes R \otimes V^{\otimes (i-j)},$$

for $i \in \mathbb{N}_{\geq 2}$. Let us denote the latter intersection space by C_i, if $i \in \mathbb{N}_{\geq 2}$, and set $C_1 = V$ and $C_0 = k$. We identify $C = \oplus_{i \in \mathbb{N}_0} C_i$ with $\mathrm{Tor}^A_{\bullet}(k, k)$. Under this identification the coproduct Δ is given as follows. The composition of the restriction of Δ to C_i with the canonical projection onto $C_{i'} \otimes C_{i''}$, where $i = i' + i''$, is the canonical inclusion $C_i \subseteq C_{i'} \otimes C_{i''}$ of Adams graded vector subspaces of TV. The counit is given by the canonical projection of C onto $C_0 = k$, and the cougmentation by the usual inclusion of $k = C_0$ inside C. It is clear that C is cocomplete. We recall that the canonical map $f : C \to B^+(A)$ induced by the inclusions

$$C_i \subseteq V^{\otimes i} \to I_A[1]^{\otimes i},$$

for $i \in \mathbb{N}_0$, is a quasi-isomorphism of cocomplete coaugmented dg coalgebras (this is in fact equivalent to the Koszul property on A). In this case it is trivial to verify that the twisting cochain $\tau = \tau_A \circ f$ is given by the map whose restriction to V is minus the canonical inclusion $V \to A$, and the restriction to $V^{\otimes i}$ vanishes for $i \in \mathbb{N}_0 \setminus \{1\}$. We have the following direct consequence of the description of the Hochschild (co)homology given in the previous subsection.

Proposition 2.2. *Let A be a Koszul algebra over a field k, M be an A-bimodule and $C = \mathrm{Tor}^A_{\bullet}(k, k)$ be the Tor coalgebra recalled previously, provided with the quasi-isomorphism of coaugmented dg coalgebras $f : C \to B^+(A)$. Set $\tau = \tau_A \circ f$. Then, the map $\mathcal{H}om(f, A)$ gives a quasi-isomorphism of augmented dg algebras from $\mathcal{H}om^{\tau_A}(B^+(A), A)$ to $\mathcal{H}om^{\tau}(C, A)$, and $\mathrm{id}_M \otimes f$ is a quasi-isomorphism of dg bimodules over $\mathcal{H}om^{\tau_A}(B^+(A), A)$ from $M \otimes_{\tau} C$ to $M \otimes_{\tau_A} B^+(A)$, where $M \otimes_{\tau} C$ has a structure of dg bimodule over $\mathcal{H}om^{\tau_A}(B^+(A), A)$ by* *means of the morphism of dg algebras $\mathcal{H}om(f,A)$. We have thus an isomorphism of graded algebras $HH^\bullet(A) \to H^\bullet(\mathcal{H}om^\tau(C,A))$ and an isomorphism $H_\bullet(M \otimes_\tau C) \to H_\bullet(A,M)$ of graded bimodules over $HH^\bullet(A)$, where $H_\bullet(M \otimes_\tau C)$ is a graded bimodule over $HH^\bullet(A)$ by means of the previous isomorphism of graded algebras.*

Proof. We first note that $A^e \otimes_\tau C$ together with the map $A^e \otimes C_0 = A^e \to A$ given by the product of A is a minimal projective resolution of the A-bimodule A. Moreover, the mapping $\mathrm{id}_{A^e} \otimes f : A^e \otimes_\tau C \to A^e \otimes_{\tau_A} B^+(A)$ is a morphism of resolutions, so a homotopical equivalence. Since $\mathcal{H}om(f,A)$ coincides with the map given by applying the functor $\mathcal{H}om_{A^e}(-,A)$ to $\mathrm{id}_{A^e} \otimes f$, $\mathcal{H}om(f,A)$ is a quasi-isomorphism of augmented dg algebras from $\mathcal{H}om^{\tau_A}(B^+(A),A)$ to $\mathcal{H}om^\tau(C,A)$, so it induces an isomorphism of augmented graded algebras from $HH^\bullet(A)$ to $H^\bullet(\mathcal{H}om^\tau(C,A))$. Analogously, we have a quasi-isomorphism of dg bimodules over $\mathcal{H}om^{\tau_A}(B^+(A),A)$ from $M \otimes_\tau C$ to $M \otimes_{\tau_A} B^+(A)$ defined by $\mathrm{id}_M \otimes f$, where $M \otimes_\tau C$ has a structure of dg bimodule over $\mathcal{H}om^{\tau_A}(B^+(A),A)$ by means of the morphism of dg algebras $\mathcal{H}om(f,A)$. The map $\mathrm{id}_M \otimes f$ is indeed a quasi-isomorphism, because it coincides with the map given by applying the functor $M \otimes_{A^e} (-)$ to $\mathrm{id}_{A^e} \otimes f$. Hence, we get an isomorphism of graded bimodules over $HH^\bullet(A)$ from $H_\bullet(M \otimes_\tau C)$ to $H_\bullet(A,M)$, where the structure of graded bimodule over $HH^\bullet(A)$ of $H_\bullet(M \otimes_\tau C)$ is given by the previous isomorphism $HH^\bullet(A) \to H^\bullet(\mathcal{H}om^\tau(C,A))$. □

2.3 Usual duality between the bar and cobar construction

In this subsection we shall prove the usual duality relations between bar and cobar constructions. We follow the sign conventions for the bar and cobar constructions given in [17], Section 8, even though we denote the elements of the cobar constructions following the notation of [7] or [8]. These results are well-known among the experts but we prefer to give the precise statements since there are some minor imprecisions in some parts of the literature.

A graded or dg vector space M is *locally finite dimensional* if each of the homogeneous components of M is finite dimensional. Note that in this case the graded dual $M^\#$ is also locally finite dimensional and the canonical map $\iota_M : M \to (M^\#)^\#$ given by $\iota_M(m)(f) = (-1)^{\deg m \deg f} f(m)$, for $m \in M$ and $f \in M^\#$ homogeneous, is an isomorphism of graded or dg vector spaces, respectively. Furthermore, if M and N are locally finite dimensional, then the map $\iota_{M,N} : M^\# \otimes N^\# \to (M \otimes N)^\#$ defined as $\iota_{M,N}(\phi \otimes \psi)(m \otimes n) = (-1)^{\deg \psi \deg m} \phi(m)\psi(n)$ is an isomorphism in the corresponding category. Given $p, p' \in \mathbb{Z}$, we shall also use the isomorphism of dg modules from $\mathcal{H}om(M[p], N[p'])$ to $\mathcal{H}om(M,N)[p'-p]$, which we denote by $H_{M,N,p,p'}$, given by $f \mapsto s_{\mathcal{H}om(M,N),p'-p}(\mathcal{H}om(s_{M,p}, s^{-1}_{N,p'})(f))$, for $f \in \mathcal{H}om(M[p], N[p'])$ (*cf.* [1], Subsubsection 3.1.8). The underlying set-theoretic map is thus the identity times a $(-1)^{(\deg f + p')p}$ sign.

We now recall the well-known fact that if D is a coaugmented dg coalgebra, the graded dual $D^\#$ has the structure of an augmented dg algebra, where the product is given by $\Delta_D^\# \circ \iota_{D,D}$, the unit is ϵ_D and the augmentation is defined as $\omega \mapsto \omega(\eta_D(1_k))$, for $\omega \in D^\#$. Conversely if Λ is a locally finite dimensional augmented dg algebra, then the graded dual $\Lambda^\#$ has the structure of a (locally finite dimensional) coaugmented dg coalgebra, where the product is given by $\iota^{-1}_{\Lambda,\Lambda} \circ \mu^\#_\Lambda$, the counit is $\omega \mapsto \omega(\eta_\Lambda(1_k))$, for $\omega \in \Lambda^\#$, and the coaugmentation is defined as $1_k \mapsto \epsilon_\Lambda$. Note that in this latter case the morphism $\iota_\Lambda : \Lambda \to (\Lambda^\#)^\#$ is an isomorphism of augmented dg algebras. Analogously, if D is a locally finite dimensional coaugmented dg coalgebra, then $\iota_D : D \to (D^\#)^\#$ is an isomorphism of coaugmented dg coalgebras.

The main duality properties we shall use between the bar and cobar constructions are the following ones (see [6], Section 19, Ex. 3, p. 272, and [17], Lemma 8.6, (c). We remark however that in the last reference one should further impose that ΩC and BA are locally finite dimensional, where we are using the notation of that article. The same correction would apply to [18], Lemma 1.15).

Proposition 2.3. *There is a natural isomorphism $\Omega^+((-)^\#) \to B^+(-)^\#$, in the category of augmented dg algebras Λ such that $B^+(\Lambda)$ is locally finite dimensional.*

Analogously, there is a natural isomorphism $B^+((-)^\#) \to \Omega^+(-)^\#$ in the category of coaugmented dg coalgebras D such that $\Omega^+(D)$ is locally finite dimensional.

Proof. Let Λ be an augmented dg algebra such that $B^+(\Lambda)$ is locally finite dimensional. Notice that Λ is *a fortiori* locally finite dimensional, and denote by I_Λ its augmentation ideal. Note that the graded dual of the inclusion $I_\Lambda \subseteq \Lambda$ induces an isomorphism $J_{\Lambda^\#} \simeq I_\Lambda^\#$, where $J_{\Lambda^\#}$ denotes the cokernel of the coaugmentation of $\Lambda^\#$. Define the isomorphism $j_\Lambda : \Omega^+(\Lambda^\#) \to B^+(\Lambda)^\#$ as the unique one satisfying that its restriction to $J_{\Lambda^\#}[-1] \simeq (I_\Lambda^\#)[-1]$ is the composition of the inverse of the isomorphism $H_{I_\Lambda,k,1,0} : (I_\Lambda[1])^\# \to (I_\Lambda^\#)[-1]$ with the graded dual of the canonical projection $B^+(\Lambda) \to I_\Lambda[1]$ (using the identification $I_\Lambda^\# \simeq J_{\Lambda^\#}$ explained before). We remark that the choice of signs is exactly the one in order to make our map commute with the differentials. We will provide an explicit expression of this isomorphism. The morphism j_Λ sends $\langle\rangle \in \Omega^+(\Lambda^\#)$ to the canonical projection $B^+(\Lambda) \to k$, and for $n \in \mathbb{N}$ and $\omega_1, \dots, \omega_n \in I_\Lambda^\#$ it sends $\langle\omega_1|\dots|\omega_n\rangle$ to the linear functional

$$[\lambda_1|\dots|\lambda_m] \mapsto (-1)^\epsilon \delta_{n,m}\omega_1(\lambda_1)\dots\omega_n(\lambda_n),$$

where $\lambda_1, \dots, \lambda_m \in \Lambda$, $\delta_{n,m}$ is the Kronecker delta symbol, and

$$\begin{aligned}\epsilon = &\deg\omega_1 + \dots + \deg\omega_n + n + (\deg\omega_2 + 1)(\deg\lambda_1 + 1)\\ &+ \dots + (\deg\omega_n + 1)(\deg\lambda_1 + \dots + \deg\lambda_{n-1} + n - 1).\end{aligned}$$

Analogously, let D be a coaugmented dg coalgebra such that $\Omega^+(D)$ is locally finite dimensional. Then D is also locally finite dimensional. Define the isomorphism $j^D : B^+(D^\#) \to \Omega^+(D)^\#$ as follows. It sends $[] \in B^+(D^\#)$ to the canonical projection $\Omega^+(D) \to k$, and, for $n \in \mathbb{N}$ and $\rho_1, \dots, \rho_n \in I_{D^\#}$, it sends $[\rho_1|\dots|\rho_n]$ to the linear functional

$$\langle\theta_1|\dots|\theta_m\rangle \mapsto (-1)^\epsilon \delta_{n,m}\rho_1(\theta_1)\dots\rho_n(\theta_n),$$

where $\theta_1, \dots, \theta_m \in D$, $\delta_{n,m}$ is the Kronecker delta, and

$$\begin{aligned}\epsilon = &\deg\rho_1 + \dots + \deg\rho_n + (\deg\rho_2 + 1)(\deg\theta_1 + 1)\\ &+ \dots + (\deg\rho_n + 1)(\deg\theta_1 + \dots + \deg\theta_{n-1} + n - 1).\end{aligned}$$

The naturality of the morphisms follows easily from the definitions. □

Let us consider an augmented dg algebra Λ. We denote by $\beta_\Lambda : \Omega^+(B^+(\Lambda)) \to \Lambda$ the canonical quasi-isomorphism of augmented dg algebras given by $\langle\rangle \mapsto 1_\Lambda$ and $\langle\omega_1|\dots|\omega_n\rangle \mapsto (-1)^n s_{I_\Lambda}^{-1}(\pi_1(\omega_1))\dots s_{I_\Lambda}^{-1}(\pi_1(\omega_n))$ if $n \in \mathbb{N}$, where $\pi_1 : B^+(\Lambda) \to I_\Lambda[1]$ is the canonical projection and $\omega_1, \dots, \omega_n$ are elements in the coaugmentation cokernel of $B^+(\Lambda)$ (see [6], Section 19, Ex. 2, or [12], Section II.4, Thm. II.4.4, or [21], Th. 2.28). If D is a cocomplete coaugmented dg coalgebra we denote by $\tau^D : D \to \Omega^+(D)$ the *couniversal twisting cochain of D* given by the composition of the canonical projection $D \to J_D$, $s_{J_D[-1]}^{-1}$ and the canonical inclusion of $J_D[-1]$ inside $\Omega^+(D)$, where J_D is the cokernel of the coaugmentation of D, and we define $\beta^D : D \to B^+(\Omega^+(D))$ as the unique morphism of coaugmented dg coalgebras such that its composition with $\tau_{\Omega^+(D)} : B^+(\Omega^+(D)) \to \Omega^+(D)$ is τ^D. Hence β^D sends 1_D to $1_{B^+(\Omega^+(D))}$, and for $d \in \mathrm{Ker}(\epsilon_D)$, it satisfies that

$$\beta^D(d) = -[\langle d\rangle] + \sum_{n\in\mathbb{N}_{\geq 2}} (-1)^n [\langle d_{(1)}^-\rangle|\dots|\langle d_{(n)}^-\rangle],$$

where $\Delta^{(n)}_{\mathrm{Ker}(\epsilon_D)}(d) = d^-_{(1)} \otimes \cdots \otimes d^-_{(n)}$ is the iterated coproduct of the comultiplication of $\mathrm{Ker}(\epsilon_D)$. It can be proved that β^D is a filtered quasi-isomorphism, so a weak equivalence, where D is provided with the primitive filtration and $B^+(\Omega^+(D))$ with the C-primitive one (see [15], Lemme 1.3.2.3, (c), and the corresponding errata).

Proposition 2.4. *Assume the same hypotheses as in the previous proposition. The previous natural isomorphisms satisfy that*

$$\begin{aligned} j^D &= \Omega^+(\iota_D)^\# \circ (j_{D^\#})^\# \circ \iota_{B^+(D^\#)}, & (j^D)^\# \circ \iota_{\Omega^+(D)} &= j_{D^\#} \circ \Omega^+(\iota_D), \\ j_\Lambda &= B^+(\iota_\Lambda)^\# \circ (j^{\Lambda^\#})^\# \circ \iota_{\Omega^+(\Lambda^\#)}, & (j_\Lambda)^\# \circ \iota_{B^+(\Lambda)} &= j^{\Lambda^\#} \circ B^+(\iota_D). \end{aligned} \tag{2.6}$$

Moreover, we have that

$$\begin{aligned} \tau_{D^\#} &= (\tau^D)^\# \circ j^D, \\ \tau^\#_\Lambda &= j_\Lambda \circ \tau^{\Lambda^\#}. \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \beta_{D^\#} &= (\beta^D)^\# \circ j_{\Omega^+(D)} \circ \Omega^+(j^D), \\ \beta^\#_\Lambda &= j^{B^+(\Lambda)} \circ B^+(j_\Lambda) \circ \beta^{\Lambda^\#}. \end{aligned} \tag{2.8}$$

Proof. The first set of four of displayed equations in (2.6) follow easily from the explicit definitions of the natural isomorphisms. The same is true for (2.7). We will only show how to prove the second one of (2.8), for the first one is analogous. We first note that, since β_Λ is the unique morphism of augmented dg algebras such that the composition of $\tau^{B^+(\Lambda)}$ with it is τ_Λ, by taking duals $\beta^\#_\Lambda$ is the unique morphism of coaugmented dg coalgebras such that its composition with $(\tau^{B^+(\Lambda)})^\#$ is $\tau^\#_\Lambda$. It thus suffices to prove that the composition of the right member of the second equation of (2.8) with $(\tau^{B^+(\Lambda)})^\#$ is $\tau^\#_\Lambda$. By the first identity of (2.7) for $D = B^+(\Lambda)$, we get that this composition is $\tau_{B^+(\Lambda)^\#} \circ B^+(j_\Lambda) \circ \beta^{\Lambda^\#}$. Now, using the naturality of the universal twisting cochains for the morphism $f = j_\Lambda$, we get that the latter composition coincides with $j_\Lambda \circ \tau_{\Omega^+(\Lambda^\#)} \circ \beta^{\Lambda^\#}$, which is equal to $j_\Lambda \circ \tau^{\Lambda^\#}$. The second identity of (2.7) gives the claim. □

We remark that a quasi-isomorphism $f : C \to D$ of cocomplete coaugmented dg coalgebras whose cobar constructions are locally finite dimensional is a weak equivalence (the converse is always true). Indeed, f induces a quasi-isomorphism of augmented dg algebras $f^\# : D^\# \to C^\#$, giving a quasi-isomorphisms $B^+(f^\#) : B^+(D^\#) \to B^+(C^\#)$. The map $(\Omega^+(f))^\# : \Omega^+(D)^\# \to \Omega^+(C)^\#$ is a quasi-isomorphism of coaugmented dg coalgebras by Proposition 2.3. Hence, we obtain that $\Omega^+(f) : \Omega^+(C) \to \Omega^+(D)$ is a quasi-isomorphism of augmented dg algebras, which tells us that f is a weak equivalence.

2.4 A smaller resolution

In this subsection we will recall a resolution of any augmented dg algebra of the form $\Omega^+(C)$, where C is a cocomplete coaugmented dg coalgebra, that is simpler than the reduced bar resolution $\Omega^+(C)^e \otimes_{\tau_{\Omega^+(C)}} B^+(\Omega^+(C))$.

Consider the dg $\Omega^+(C)$-bimodule $\Omega^+(C)^e \otimes_{\tau^C} C$ provided with the morphism to $\Omega^+(C)$ given by $\omega' \otimes \omega \otimes c \to (-1)^{\deg \omega' \deg \omega} \epsilon_C(c) \omega.\omega'$, where the twisting cochain τ^C utilizes the outer structure of $\Omega^+(C)^e$, and the $\Omega^+(C)$-bimodule $\Omega^+(C)^e \otimes_{\tau^C} C$ is obtained from the inner structure. By the identification $\Omega^+(C)^e \otimes_{\tau^C} C \to$

$\Omega^+(C)\otimes C\otimes\Omega^+(C)$ of graded bimodules over $\Omega^+(C)$ defined as $\omega'\otimes\omega\otimes c\mapsto(-1)^{\deg\omega'(\deg\omega+\deg c)}\omega\otimes c\otimes\omega'$, the differential of the codomain becomes

$$\begin{aligned}\omega\otimes c\otimes\omega'\mapsto & D(\omega)\otimes c\otimes\omega'+(-1)^{\deg\omega}\omega\otimes d_C(c)\otimes\omega'+(-1)^{\deg\omega+\deg c}\omega\otimes c\otimes D(\omega')\\ &+(-1)^{\deg\omega+\deg c_{(1)}}\omega\otimes c_{(1)}\otimes\tau^C(c_{(2)})\omega'-(-1)^{\deg\omega}\omega\tau^C(c_{(1)})\otimes c_{(2)}\otimes\omega',\end{aligned}$$

where $\Delta_C(c)=c_{(1)}\otimes c_{(2)}$.

Note that the map

$$\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C:\Omega^+(C)^e\otimes_{\tau^C}C\to\Omega^+(C)^e\otimes_{\tau_{\Omega^+(C)}}B^+(\Omega^+(C))$$

is a morphism of resolutions of dg bimodules over $\Omega^+(C)$. Consider also the map

$$\gamma^C:\overline{\mathrm{Bar}}(\Omega^+(C))\to\Omega^+(C)\otimes C\otimes\Omega^+(C) \tag{2.9}$$

of graded $\Omega^+(C)$-bimodules satisfying that $\gamma^C(\omega_0[]\omega_1)=\omega_0\otimes 1_C\otimes\omega_1$ and

$$\gamma^C(\omega_0[\omega_1|\dots|\omega_m]\omega_{m+1})=\delta_{m,1}\sum_{j=1}^{n}(-1)^{\varepsilon_j+1}\omega_0\langle c_1|\dots|c_{j-1}\rangle\otimes c_j\otimes\langle c_{j+1}|\dots|c_n\rangle\omega_2$$

if $m\in\mathbb{N}$, where $\omega_1=\langle c_1|\dots|c_n\rangle$ and $\varepsilon_j=(\sum_{l=1}^{j-1}\deg c_l)+j-1$. It is clearly a morphism of resolutions of dg $\Omega^+(C)$-bimodules and a left inverse of $\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C$ (after using the canonical identifications explained before). We have further the following result (*cf.* [23], Théo. 1.4).

Fact 2.5. *The maps γ^C and $\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C$ are homotopy inverses to each other, so $\Omega^+(C)\otimes C\otimes\Omega^+(C)$ is a homotopically projective resolution of the dg $\Omega^+(C)$-bimodule $\Omega^+(C)$.*

Proof. Since $\beta^C:C\to B^+(\Omega^+(C))$ is a quasi-isomorphism of coaugmented dg coalgebras which satisfies by definition that $\tau_{\Omega^+(C)}\circ\beta^C=\tau^C$, the map $\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C$ is a quasi-isomorphism (see [15], Prop. 2.2.4.1). Combining this with the fact that γ^C is a left inverse of $\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C$, we get that the former map is also a quasi-isomorphism. In order to prove our claim, it suffices to show that $\mathrm{id}_{\overline{\mathrm{Bar}}(A)}$ is homotopic to $(\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C)\circ\gamma^C$. As the latter is a quasi-isomorphism on a semifree dg $\Omega^+(C)$-bimodule, it is an homotopy equivalence (see [1], Cor. 9.5.3). Let h be its homotopy inverse. The statement follows from the next chain of homotopy equivalences

$$\begin{aligned}(\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C)\circ\gamma^C &\sim(\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C)\circ\gamma^C\circ(\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C)\circ\gamma^C\circ h\\ &=(\mathrm{id}_{\Omega^+(C)^e}\otimes\beta^C)\circ\gamma^C\circ h\sim\mathrm{id}_{\overline{\mathrm{Bar}}(A)}.\end{aligned}$$

□

2.5 Gerstenhaber brackets

We recall that, given an augmented dg algebra A, there is a graded Lie algebra structure on $\mathcal{H}om^{\tau_A}(B^+(A),A)[1]$, provided with the *Gerstenhaber bracket* introduced by M. Gerstenhaber in his article [9]. Consider the map

$$\delta_A:\mathrm{Coder}(B^+(A))\ltimes(B^+(A)^{\#}[1])\to\mathcal{H}om^{\tau_A}(B^+(A),A)[1] \tag{2.10}$$

given by $\delta_A(T,s_{B^+(A)^{\#}}(\lambda))=s(s_{I_A}^{-1}\circ\pi_1\circ T+1_A.\lambda)$, where s is the suspenson on $\mathcal{H}om^{\tau_A}(B^+(A),A)$, $\pi_1:B^+(A)\to I_A[1]$ is the canonical projection, $T\in\mathrm{Coder}(B^+(A))$, $\lambda\in B^+(A)^{\#}$. Its domain is a dg Lie algebra, where $\mathrm{Coder}(B^+(A))$ is provided with the graded commutator and acts in the abelian graded Lie algebra $B^+(A)^{\#}[1]$ by the shift of the dual action, and the differential is the obvious one. The map δ_A is a morphism of complexes and clearly bijective (*cf.* [15], Lemme 1.1.2.2). This induces the structure of a dg Lie algebra on $\mathcal{H}om^{\tau_A}(B^+(A),A)[1]$, that we mentioned previously. An explicit expression of the Gerstenhaber bracket can be found in [22], Subsection 2.2. This result is analogous to the one remarked by E. Getzler in [10], Prop. 1.3, where he uses the nonreduced bar construction of A. However, the one we stated does not seem to have been observed.

Analogously, given a coaugmented dg coalgebra C, $\mathcal{H}om^{\tau^C}(C,\Omega^+(C))[1]$ is a dg Lie algebra as follows. We have the morphism

$$\delta^C : \mathrm{Der}(\Omega^+(C)) \ltimes (\Omega^+(C)[1]) \to \mathcal{H}om^{\tau^C}(C,\Omega^+(C))[1] \tag{2.11}$$

given by $\delta^C(T, s_{\Omega^+(C)}(\omega)) = s((-1)^{\deg T} T \circ s^{-1}_{J_C[-1]} \circ \pi + \omega.\epsilon_C)$, where s denotes the suspension on the corresponding twisted convolution algebra , $\pi : C \to J_C = C/\mathrm{Im}(\eta_C)$ is the canonical projection, $T \in \mathrm{Der}(\Omega^+(C))$ and $\omega \in \Omega^+(C)$. Its domain is a dg Lie algebra, where $\mathrm{Der}(\Omega^+(C))$ has the graded commutator and acts in the abelian graded Lie algebra $\Omega^+(C)[1]$ by the shift of the regular action, and the differential is the obvious one. It is clearly bijective (*cf.* [15], Lemme 1.1.2.1), and a morphism of complexes of vector spaces. By means of δ^C we define a structure of dg Lie algebra on $\mathcal{H}om^{\tau^C}(C,\Omega^+(C))[1]$. We give an explicit expression of the bracket. Let $\psi \in \mathcal{H}om(C, J_C[-1]^{\otimes n})$ and $\psi' \in \mathcal{H}om(C, J_C[-1]^{\otimes m})$, for $n,m \in \mathbb{N}_0$. We will use the notation

$$\psi(c) = \langle c^{\psi}_{(1)}|\dots|c^{\psi}_{(n)}\rangle \text{ and } \psi'(c) = \langle c^{\psi'}_{(1)}|\dots|c^{\psi'}_{(m)}\rangle,$$

for $c \in C$, where the sum is omitted for simplicity. Then, the bracket $[\psi,\psi'] \in \mathcal{H}om(C, J_C[-1]^{\otimes(n+m-1)})$ sends $c \in C$ to

$$\begin{aligned} &\sum_{i=0}^{m-1} (-1)^{(\deg\psi-1)\epsilon'_{i+1}} \langle c^{\psi'}_{(1)}|\dots|c^{\psi'}_{(i)}\rangle \psi(c^{\psi'}_{(i+1)}) \langle c^{\psi'}_{(i+2)}|\dots|c^{\psi'}_{(m)}\rangle \\ &- \sum_{i=0}^{n-1} (-1)^{(\deg\psi'-1)(\epsilon_{i+1}+\deg\psi-1)} \langle c^{\psi}_{(1)}|\dots|c^{\psi}_{(i)}\rangle \psi'(c^{\psi}_{(i+1)}) \langle c^{\psi}_{(i+2)}|\dots|c^{\psi}_{(n)}\rangle, \end{aligned} \tag{2.12}$$

where $\epsilon_{i+1} = (\sum_{j=1}^{i} \deg c^{\psi}_{(j)}) - i$ and $\epsilon'_{i+1} = (\sum_{j=1}^{i} \deg c^{\psi'}_{(j)}) - i$.

3 Hochschild (co)homology of Koszul dual dg algebras

3.1 Preparatory results

Before stating the following result we recall that given an augmented dg algebra Λ and a dg bimodule M over Λ, the graded dual $M^{\#}$ is also a dg bimodule over Λ provided with the action

$$(\lambda \cdot f \cdot \lambda')(m) = (-1)^{\deg\lambda(\deg f + \deg\lambda' + \deg m)} f(\lambda' m \lambda),$$

where $m \in M$, $f \in M^{\#}$ and $\lambda, \lambda' \in \Lambda$ are homogeneous.

The following result is straightforward to prove but somehow tedious.

Proposition 3.1. *Given a locally finite dimensional coaugmented dg coalgebra C and a locally finite dimensional augmented dg algebra A, there is an isomorphism of augmented dg algebras*

$$\mathcal{H}om(C,A) \to \mathcal{H}om(A^{\#}, C^{\#})$$

given by taking dual $\phi \mapsto \phi^{\#}$, the inverse being defined as $\psi \mapsto \iota_A^{-1} \circ \psi^{\#} \circ \iota_C$. Via this isomorphism, the dg $\mathcal{H}om(A^{\#}, C^{\#})$-bimodule $C^{\#} \otimes A^{\#}$ becomes a dg $\mathcal{H}om(C, A)$-bimodule. Moreover, the canonical nondegenerate pairing

$$\beta : (C^{\#} \otimes A^{\#}) \otimes (A \otimes C) \to k$$

given by $(g \otimes f) \otimes (a \otimes c) \to g(c)f(a)$, for $a \in A$, $c \in C$, $g \in C^{\#}$ and $f \in A^{\#}$ homogeneous, is a morphism of dg vector spaces which is also $\mathcal{H}om(C, A)^e$-balanced, i.e.

$$\beta((g \otimes f).(\phi \otimes \psi) \otimes (a \otimes c)) = \beta((g \otimes f) \otimes (\phi \otimes \psi).(a \otimes c)),$$

where $\phi \otimes \psi \in \mathcal{H}om(C, A)^e$, and we are using the obvious equivalence between dg bimodules and (either left or right) dg modules over the enveloping algebra. This further implies that there is an isomorphism of dg $\mathcal{H}om(C, A)$-bimodules between $C^{\#} \otimes A^{\#}$ and $(A \otimes C)^{\#}$, given explicitly by $(g \otimes f) \mapsto ((a \otimes c) \mapsto f(a)g(c))$.

We also twist the previous result by a twisting cochain $\tau \in \mathcal{H}om(C, A)$. This implies that $\tau^{\#}$ is a twisting cochain of $\mathcal{H}om(A^{\#}, C^{\#})$ and that the map $\phi \mapsto \phi^{\#}$ provides an isomorphism of augmented dg algebras

$$\mathcal{H}om^{\tau}(C, A) \to \mathcal{H}om^{\tau^{\#}}(A^{\#}, C^{\#}).$$

Furthermore, the canonical pairing β considered before can be regarded as a pairing on the underlying graded vector spaces

$$\beta : (C^{\#} \otimes_{\tau^{\#}} A^{\#}) \otimes (A \otimes_{\tau} C) \to k,$$

which is still $\mathcal{H}om(C, A)^e$-balanced, for the graded bimodule structures have not changed. It is easily verified that it commutes with the new differentials, taking into account the previous isomorphism of augmented dg algebras and that the differential twists are given in terms of the bimodule structure. Hence β is in fact a morphism of dg bimodules, which in turn implies that the previous isomorphism of dg $\mathcal{H}om(C, A)$-bimodules between $C^{\#} \otimes A^{\#}$ and $(A \otimes C)^{\#}$ induces an isomorphism of dg $\mathcal{H}om^{\tau}(C, A)$-bimodules between $C^{\#} \otimes_{\tau^{\#}} A^{\#}$ and $(A \otimes_{\tau} C)^{\#}$.

We collect the previous remarks in the following statement.

Proposition 3.2. *Given a locally finite dimensional coaugmented dg coalgebra C, a locally finite dimensional augmented dg algebra A and a twisting cochain $\tau \in \mathcal{H}om(C, A)$, then $\tau^{\#}$ is also a twisting cochain and there is an isomorphism of augmented dg algebras*

$$\mathcal{H}om^{\tau}(C, A) \to \mathcal{H}om^{\tau^{\#}}(A^{\#}, C^{\#})$$

given by taking dual $\phi \mapsto \phi^{\#}$, with inverse defined as $\psi \mapsto \iota_A^{-1} \circ \psi^{\#} \circ \iota_C$. Via this last isomorphism, the dg $\mathcal{H}om^{\tau^{\#}}(A^{\#}, C^{\#})$-bimodule $C^{\#} \otimes_{\tau^{\#}} A^{\#}$ becomes a dg $\mathcal{H}om^{\tau}(C, A)$-bimodule. Moreover, the canonical nondegenerate pairing

$$\beta : (C^{\#} \otimes_{\tau^{\#}} A^{\#}) \otimes (A \otimes_{\tau} C) \to k$$

given by $(g \otimes f) \otimes (a \otimes c) \to g(c)f(a)$, for $a \in A$, $c \in C$, $g \in C^{\#}$ and $f \in A^{\#}$ homogeneous, is $\mathcal{H}om^{\tau}(C, A)^e$-balanced, i.e.

$$\beta((g \otimes f).(\phi \otimes \psi) \otimes (a \otimes c)) = \beta((g \otimes f) \otimes (\phi \otimes \psi).(a \otimes c)),$$

where $\phi \otimes \psi \in \mathcal{H}om^{\tau}(C, A)^e$, and we are using the obvious equivalence between dg bimodules and (either left or right) dg modules over the enveloping algebra. This implies that there is an isomorphism of dg $\mathcal{H}om^{\tau}(C, A)$-bimodules between $C^{\#} \otimes_{\tau^{\#}} A^{\#}$ and $(A \otimes_{\tau} C)^{\#}$, given explicitly by $(g \otimes f) \mapsto ((a \otimes c) \mapsto f(a)g(c))$.

3.2 The relation between the cup and cap products of Koszul dual algebras

We will apply the previous result to obtain the isomorphism statements for the Hochschild homology and cohomology groups of Koszul dual pairs.

Following B. Keller, the *Koszul dual* of an augmented dg algebra A is defined as the augmented dg algebra $B^+(A)^\#$ (see [13], Section 10.2), and shall be denoted by $E(A)$. We have the following result, which may be regarded as the Koszul duality phenomenon for Hochschild (co)homology.

Theorem 3.3. *Let A be an augmented dg algebra which is assumed to be Adams connected. We have a quasi-isomorphism of augmented dg algebras*

$$\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A)) \to \mathcal{H}om^{\tau_A}(B^+(A), A),$$

which yields an isomorphism of augmented graded algebras $HH^\bullet(E(A)) \to HH^\bullet(A)$. We also have a quasi-isomorphism of dg $\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A))$-bimodules

$$(A \otimes_{\tau_A} B^+(A))^\# \to E(A) \otimes_{\tau_{E(A)}} B^+(E(A)),$$

where the domain has the structure of bimodule over $\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A))$ via the first morphism of this proposition. We obtain an isomorphism $HH_\bullet(A)^\# \to HH_\bullet(E(A))$ of graded bimodules over $HH^\bullet(E(A))$, where the domain has the structure of bimodule given by the isomorphism $HH^\bullet(E(A)) \to HH^\bullet(A)$.

Proof. We first remark that, by [18], Lemma 2.2, any Adams connected augmented dg algebra is also locally finite dimensional, and moreover, its Koszul dual $E(A)$ is locally finite dimensional and Adams connected. There is a quasi-isomorphism $\beta'_A : E(E(A)) \to A$ of augmented dg algebras given by $\beta_A \circ \Omega^+(\iota_{B^+(A)})^{-1} \circ j^{-1}_{E(A)}$. The second identity of (2.6) with $D = B^+(A)$ tells us that

$$\beta'_A = \beta_A \circ \iota^{-1}_{\Omega^+(B^+(A))} \circ ((j^{B^+(A)})^{-1})^\#.$$

Also note that the first equation of (2.6) for $D = B^+(A)$ yields

$$\iota^{-1}_{B^+(E(A))} \circ (\beta'_A)^\# = (j^{B^+(A)})^{-1} \circ \beta^\#_A. \tag{3.1}$$

We have thus a quasi-isomorphism $(\beta'_A)^\# : A^\# \to E(E(A))^\#$ of coaugmented dg coalgebras. Moreover, it can be easily checked that $\tau_{E(A)} \circ \iota^{-1}_{B^+(E(A))} \circ (\beta'_A)^\# = \tau^\#_A$. Indeed, by (3.1) and the second identity of (2.8) with $\Lambda = A$, the equation we want to prove is tantamount to

$$\tau_{E(A)} \circ B^+(j_A) \circ \beta^{A^\#} = \tau^\#_A.$$

The naturality of the twisting cochain tells us that the last equation coincides with

$$j_A \circ \tau_{\Omega^+(A^\#)} \circ \beta^{A^\#} = \tau^\#_A,$$

and by the second identity of (2.7) with $\Lambda = A$ it further reduces to

$$\tau_{\Omega^+(A^\#)} \circ \beta^{A^\#} = \tau^{A^\#},$$

which follows from the definition of $\beta^{A^\#}$. This implies that we have a morphism of augmented dg algebras

$$\mathcal{H}om(\iota^{-1}_{B^+(E(A))} \circ (\beta'_A)^\#, E(A)) : \mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A)) \to \mathcal{H}om^{\tau^\#_A}(A^\#, E(A)). \tag{3.2}$$

By Proposition 3.2 applied to $C = B^+(A)$ the latter augmented dg algebra is isomorphic to $\mathcal{H}om^{\tau_A}(B^+(A), A)$, which implies thus we have a morphism of augmented dg algebras of the form

$$\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A)) \to \mathcal{H}om^{\tau_A}(B^+(A), A). \tag{3.3}$$

We would like to remark that the map (3.2) is in fact a quasi-isomorphism. Indeed, the morphism

$$\mathcal{H}om(B^+(j_A^{-1}), j_A) \circ \mathcal{H}om_{\Omega^+(A^\#)^e}(\gamma^{A^\#}, \Omega^+(A^\#)) \circ \mathcal{H}om(A^\#, j_A^{-1}) \tag{3.4}$$

is a quasi-inverse of (3.2), where $\gamma^{A^\#}$ is given in (2.9). To prove this, consider the composition of the maps $\mathcal{H}om(B^+(E(A)), j_A)$, (3.2) and $\mathcal{H}om(A^\#, j_A^{-1})$, which is just $\mathcal{H}om((j^{B^+(A)})^{-1} \circ \beta_A^\#, \Omega^+(A^\#))$. Our claim is tantamount to prove that the latter is a quasi-inverse of

$$\mathcal{H}om(B^+(j_A^{-1}), \Omega^+(A^\#)) \circ \mathcal{H}om_{\Omega^+(A^\#)^e}(\gamma^{A^\#}, \Omega^+(A^\#)),$$

which is in turn equivalent to the fact that

$$\mathcal{H}om(B^+(j_A^{-1}) \circ (j^{B^+(A)})^{-1} \circ \beta_A^\#, \Omega^+(A^\#)) \text{ and } \mathcal{H}om_{\Omega^+(A^\#)^e}(\gamma^{A^\#}, \Omega^+(A^\#)) \tag{3.5}$$

are quasi-inverses to each other. By the second identity of (2.8) with $\Lambda = A$, the first of these two last maps coincides with $\mathcal{H}om(\beta^{A^\#}, \Omega^+(A^\#))$, which is clearly a quasi-inverse of the second map of (3.5). This implies in particular that there is an isomorphism of augmented graded algebras between the Hochschild cohomology rings $HH^\bullet(E(A)) \to HH^\bullet(A)$.

We may apply similar arguments to the tensor products. Indeed, by taking $C = B^+(A)$ in the previous proposition, we see that there is an isomorphism of dg bimodules over $\mathcal{H}om^\tau(B^+(A), A)$ of the form $E(A) \otimes_{\tau_A^\#} A^\# \to (A \otimes_{\tau_A} B^+(A))^\#$. On the other hand, the morphism of dg vector spaces

$$\mathrm{id}_{E(A)} \otimes ((j^{B^+(A)})^{-1} \circ \beta_A^\#) : E(A) \otimes_{\tau_A^\#} A^\# \to E(A) \otimes_{\tau_{E(A)}} B^+(E(A)) \tag{3.6}$$

is a quasi-isomorphism. Indeed, the second identity of (2.8) with $\Lambda = A$ and the properties of $\gamma^{A^\#}$ tell us that

$$(\mathrm{id}_{E(A)} \otimes_{E(A)^e} \gamma^{A^\#}) \circ (\mathrm{id}_{E(A)} \otimes B^+(j_A^{-1}))$$

is a quasi-inverse of (3.6).

The composition of the inverse of the isomorphism $E(A) \otimes_{\tau_A^\#} A^\# \to (A \otimes_{\tau_A} B^+(A))^\#$ of Proposition 3.2 with the previous map (3.6) thus gives a quasi-isomorphism

$$(A \otimes_{\tau_A} B^+(A))^\# \to E(A) \otimes_{\tau_{E(A)}} B^+(E(A)). \tag{3.7}$$

It is clear by the previous proposition that (3.7) is also a morphism of dg bimodules over $\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A))$, where the domain of the map has a bimodule structure given by the morphism of augmented dg algebras (3.3). Since homology commutes with taking duals we obtain an isomorphism $HH_\bullet(A)^\# \to HH_\bullet(E(A))$ of graded bimodules over $HH^\bullet(E(A))$, where the Hochschild homology $HH_\bullet(A)$ has the structure of bimodule over the Hochschild cohomology of $E(A)$ via the previously mentioned isomorphism $HH^\bullet(E(A)) \to HH^\bullet(A)$. □

3.3 Comparison to previous work in the literature

The isomorphism between the Hochschild homology groups of Koszul dual algebras has not received much attention, although in case that A is a (quadratic) Koszul algebra a linear isomorphism between $HH_\bullet(E(A))$ and $HH_\bullet(A)^\#$ (*i.e.* as graded modules over k) can be obtained from (or following the lines of) the isomorphism between the corresponding cyclic homology groups given in [4], Thm. 2.4.1. In that article the authors suppose k has characteristic zero, but this assumption is not necessary if dealing only with Hochschild homology groups (see [16] for a more detailed analysis on the corresponding gradings). An isomorphism of graded algebras between $HH^\bullet(E(A))$ and $HH^\bullet(A)$ in case A is a (quadratic) Koszul algebra was already announced by R.-O. Buchweitz in the Conference on Representation Theory, Canberra, in July 2003.

On the other hand, the morphism (3.3) for Hochschild cohomology has already appeared under somehow different assumptions. It is essentially the same as the one appearing in [7], Prop. 5.1, even though the situation they considered is different from ours as we now explain. Their construction uses two different versions of both the bar and cobar constructions (whereas we use just one), and their hypotheses on the grading are also distinct, for they have just one grading: the cohomological one. However, they need to assume that the coalgebra C (whose dual would be our algebra A) is not only connected but also *simply connected*, *i.e.* the coaugmentation cokernel of C lies in positive (homological) degrees greater than or equal to 2, which is a very harsh assumption for us, in order to obtain their quasi-isomorphism result. This is required in order to ascertain that their map Θ (the analogous of our morphism $j_{C^\#}$) given before Prop. 5.1 of that article is an isomorphism. Our Theorem 3.3 replaces the simply connectedness hypothesis by a more typical one in representation theory: the Adams connectedness assumption. In any case, if we further suppose that the coaugmented dg coalgebra C in their article is Adams connected, the morphism they obtain is essentially given by

$$\mathcal{H}om^{\tau_{\Omega^+(C)}}(B^+(\Omega^+(C)), \Omega^+(C)) \to \mathcal{H}om^{\tau_{C^\#}}(B^+(C^\#), C^\#), \tag{3.8}$$

which coincides (up to quasi-isomorphism) with the one given in (3.3) for $A = C^\#$ using the identification of $\Omega^+(C)$ with $E(A)$ given by $(j^C)^\# \circ \iota_{\Omega^+(C)}$. There is however no analogous statement in [7] of the part in Theorem 3.3 concerning Hochschild homology.

We want to remark that the authors of [7] have further proved in their Prop. 5.3 that the morphism (3.8) induces an isomorphism of graded Lie algebras for the Gerstenhaber bracket on the Hochschild cohomology rings, obtaining thus an isomorphism of Gerstenhaber algebras. We shall prove the same statement for our situation (using a slightly different but similar proof), but we shall also include Hochschild homology, which is not considered in [7].

3.4 The relation between the Gerstenhaber structures of Koszul dual algebras

3.4.1 The statement on the Hochschild cohomology

We first recall that the *opposite* dg Lie algebra $\mathfrak{g}^{\mathrm{op}}$ of a dg Lie algebra $\mathfrak{g}$ with bracket $[\,,]$ has the same underlying dg k-module structure but the *opposite bracket* $[\,,]_{\mathrm{op}}$ given by $[x,y]_{\mathrm{op}} = (-1)^{\deg x \deg y}[y,x](= -[x,y])$, for $x,y \in \mathfrak{g}$ homogeneous. Given two dg Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a map $\phi : \mathfrak{g} \to \mathfrak{h}$ of complexes of vector spaces is said to be an *anti-morphism of dg Lie algebras* if the induced map $\mathfrak{g} \to \mathfrak{h}^{\mathrm{op}}$, whose set-theoretic assignment is the same as ϕ, is a morphism of dg Lie algebras. Note that $\phi : \mathfrak{g} \to \mathfrak{h}$ is an anti-morphism of dg Lie algebras if and only if $-\phi : \mathfrak{g} \to \mathfrak{h}$ is a morphism of dg Lie algebras.

Theorem 3.4. *Let A be an augmented dg algebra which is assumed to be Adams connected. The quasi-isomorphism of augmented dg algebras*

$$\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A)) \to \mathcal{H}om^{\tau_A}(B^+(A), A)$$

given in Theorem 3.3 yields an isomorphism of augmented graded algebras

$$HH^\bullet(E(A)) \to HH^\bullet(A),$$

which is compatible with the Gerstenhaber brackets, so an isomorphism of Gerstenhaber algebras.

Proof. We shall prove that the isomorphism $HH^\bullet(E(A)) \to HH^\bullet(A)$ of Theorem 3.3 respects the Gerstenhaber bracket. As explained before, the previous isomorphism is induced by the map (3.3), which is the composition of $\mathcal{H}om((j^{B^+(A)})^{-1} \circ \beta_A^\#, E(A))$ together with the inverse of the map

$$\mathcal{H}om^{\tau_A}(B^+(A), A) \to \mathcal{H}om^{\tau_A^\#}(A^\#, E(A)) \tag{3.9}$$

given by $\phi \mapsto \phi^\#$. The proof consists in giving a quasi-inverse of (3.3) which is a morphism of dg Lie algebras. In order to do so, consider first the following commutative diagram of complexes of vector spaces

$$\begin{array}{ccc}
\mathcal{H}om^{\tau_A}(B^+(A), A)[1] & \xleftarrow{\ \delta_A\ } & \mathrm{Coder}(B^+(A)) \ltimes (B^+(A)^\#[1]) \\
\Big\downarrow{\scriptstyle (-)^\#[1]} & & \Big\downarrow{\scriptstyle (-)^\# \times \mathrm{id}_{B^+(A)^\#[1]}} \\
\mathcal{H}om^{\tau_A^\#}(A^\#, B^+(A)^\#)[1] & \xleftarrow{\ \delta'\ } & \mathrm{Der}(B^+(A)^\#) \ltimes (B^+(A)^\#[1]) \\
\Big\uparrow{\scriptstyle \mathcal{H}om(A^\#, j_A)[1]} & & \Big\uparrow{\scriptstyle \mathcal{H}om(j_A^{-1}, j_A) \times j_A[1]} \\
\mathcal{H}om^{\tau^{A^\#}}(A^\#, \Omega^+(A^\#))[1] & \xleftarrow{\ \delta^{A^\#}\ } & \mathrm{Der}(\Omega^+(A^\#)) \ltimes (\Omega^+(A^\#)[1])
\end{array}$$

where the map δ' sends $(T, s_{E(A)}(\Lambda))$ to $s((-1)^{\deg T+1} T \circ \pi_1^\# \circ (s_{I_A}^\#)^{-1} + \Lambda\epsilon_A)$, s is the suspension on the corresponding twisted convolution algebra, $\pi_1 : B^+(A) \to I_A[1]$ is the canonical projection, $T \in \mathrm{Der}(E(A))$ and $\Lambda \in E(A)$. The commutativity of the upper square implies that δ' is bijective. Its domain is a dg Lie algebra, where $\mathrm{Der}(E(A))$ has the graded commutator and acts in the abelian graded Lie algebra $E(A)[1]$ by the shift of the regular action, and the differential is the obvious one. It is clear that the upper right vertical map is an anti-isomorphism of dg Lie algebras, and that the lower right vertical map is an isomorphism of dg Lie algebras. We define thus a dg Lie algebra structure on $\mathcal{H}om^{\tau_A^\#}(A^\#, B^+(A)^\#)[1]$ as the unique one such that the middle horizontal map is an isomorphism of dg Lie algebras, which tells us that the upper left vertical map is an anti-isomorphism of Lie algebras, and that the lower left vertical map is an isomorphism of dg Lie algebras.

Furthermore, we also have the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}om^{\tau^{A^\#}}(A^\#, \Omega^+(A^\#))[1] & & \\
\Big\downarrow{\scriptstyle \gamma[1]} & & \\
\mathcal{H}om^{\tau_{\Omega^+(A^\#)}}(B^+(\Omega^+(A^\#)), \Omega^+(A^\#))[1] & \xleftarrow{\ \delta_{\Omega^+(A^\#)}\ } & \mathrm{Coder}(B^+(\Omega^+(A^\#))) \ltimes (B^+(\Omega^+(A^\#))^\#[1]) \\
\Big\downarrow{\scriptstyle \mathcal{H}om(B^+(j_A^{-1}), j_A)[1]} & & \Big\downarrow{\scriptstyle \mathcal{H}om(B^+(j_A^{-1}), B^+(j_A)) \times B^+(j_A^{-1})^\#} \\
\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)), E(A))[1] & \xleftarrow[\ \delta_{E(A)}\ ]{} & \mathrm{Coder}(B^+(E(A))) \ltimes (B^+(E(A))^\#[1])
\end{array}$$

where the morphism γ is induced by $\mathcal{H}om_{\Omega^+(A^\#)^e}(\gamma^{A^\#}, \Omega^+(A^\#))$ by making use of the identifications $\mathcal{H}om_{\Lambda^e}(\Lambda \otimes V \otimes \Lambda, \Lambda) \simeq \mathcal{H}om(V, \Lambda)$ of graded modules over k (for Λ an augmented dg algebra and V a graded module over k), which in our case are also compatible with the differentials involved. The lower right vertical map is clearly an isomorphism of dg Lie algebras, and the commutativity of the square implies that that the lower left vertical map is also an isomorphism of dg Lie algebras.

By the explicit expression of (2.9) and of the corresponding brackets it is a rather long but straightforward to show that the upper vertical map $\gamma[1]$ is in fact an anti-morphism of dg Lie algebras. The proof can be organised as follows. Given $\psi_i \in \mathcal{H}om(A^\#, (A^\#)^{\otimes p_i})$, for $p_i \in \mathbb{N}_0$ and $i = 0, 1$, then $\gamma(\psi_i) \in \mathcal{H}om(k.1_{B^+(\Omega^+(A^\#))} \oplus \Omega^+(A^\#), \Omega^+(A^\#))$, so the Gerstenhaber bracket of the elements $\gamma(\psi_i)$, $i = 0, 1$, is given by computing the graded commutator of endomorphisms of $\Omega^+(A^\#)[1]$ or the evaluation of such an endomorphism. We shall explain the first case, for the second is analogous. The graded commutator $[s(\gamma(\psi_0)), s(\gamma(\psi_1))]$ applied to an element $[\langle \lambda_1, \dots, \lambda_n \rangle]$, where $\lambda_j \in A^\#$, gives two kinds of terms:

1. for $i = 0, 1$, ψ_i is applied to λ_{j_i}, with $j_0 \neq j_1$;
2. there is $i \in \{0, 1\}$ such that ψ_i is applied to some λ_{j_i}, but ψ_{1-i} is applied to a tensor factor of $\psi_i(\lambda_{j_i})$.

A tedious but simple computation shows that all the elements of item 1 cancel, whereas (2.9) together with (2.12) tell us that the terms of item 2 are precisely those given by $-s(\gamma([\psi_1, \psi_2]))$ which is what we wanted to show. Note that γ is a quasi-isomorphism. Furthermore, the composition of the left vertical maps of the two previous diagrams gives the quasi-inverse to the morphism (3.3) given by the composition of (3.9) with (3.4). This proves that the latter map induces a morphism of graded Lie algebras on Hochschild cohomology, as claimed before. □

3.4.2 The statement on the Connes' map

We recall the following standard facts. Given Λ any augmented dg algebra, the Connes' map B_Λ is the endomorphism of the complex $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ of cohomological degree -1 given by

$$B_\Lambda(\lambda_0 \otimes [\lambda_1|\dots|\lambda_n]) = \sum_{i=0}^{n} (-1)^{\epsilon_{i+1}\epsilon^i} 1_\Lambda \otimes [\lambda_{i+1}|\dots|\lambda_n|\lambda_0|\dots|\lambda_i], \tag{3.10}$$

where $\lambda_j \in \Lambda$, for $j = 0, \dots, n$, $\epsilon_i = (\sum_{j=0}^{i-1} \deg \lambda_j) - i$ and $\epsilon^i = (\sum_{j=i+1}^{n} \deg \lambda_j) - n + i$ (see [22], Section 2.1, (11)). Define the augmented dg algebra $k[\varepsilon]/(\varepsilon^2)$, where ε has cohomological degree -1 and zero Adams degree, and the differential is trivial. It is clearly a graded commutative algebra. The Connes map B_Λ gives a left action of $k[\varepsilon]/(\varepsilon^2)$ on $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ by $\varepsilon \cdot v = B_\Lambda(v)$, and a right action by the usual expression $v \cdot \varepsilon = (-1)^{\deg v} \varepsilon \cdot v$, for $v \in \Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ homogeneous. This follows from the identities $B_\Lambda^2 = 0$ and $B_\Lambda \circ D' = -D' \circ B_\Lambda$, where D' is the differential of $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ (see [22], Prop 2.1). Moreover, these two actions clearly commute, so $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ is a dg bimodule over $k[\varepsilon]/(\varepsilon^2)$. As a consequence, $HH_\bullet(\Lambda)$ is a graded bimodule over $k[\varepsilon]/(\varepsilon^2)$. Finally, given a dg bimodule M over $k[\varepsilon]/(\varepsilon^2)$, we shall use the following convention (only of interest to us in the case $M = \Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$) for the left action (and thus right action by the usual Koszul sign rule) on the graded dual $M^\#$:

$$(\varepsilon \cdot f)(m) = -(-1)^{\deg f} f(\varepsilon m). \tag{3.11}$$

Note the choice of sign: it naturally comes from regarding $k[\varepsilon]/(\varepsilon^2)$ as the universal enveloping algebra of the abelian graded Lie algebra $k.\varepsilon$.

We have the following result.

Proposition 3.5. *Assume the same hypotheses as in the previous theorem. We assume that $HH_\bullet(A)^\#$ and $HH_\bullet(E(A))$ are graded bimodules over $k[\varepsilon]/(\varepsilon^2)$ as explained before. Then the isomorphism $HH_\bullet(A)^\# \to HH_\bullet(E(A))$ of Theorem 3.3 is of graded $k[\varepsilon]/(\varepsilon^2)$-modules on both sides.*

Proof. Since the action on the right is obtained from the action on the left, it suffices to proves the lemma for the left action. Moreover, by equation (3.11) the left action of $k[\varepsilon]/(\varepsilon^2)$ on $HH_\bullet(A)^\#$ is induced by minus the dual of the Connes' map B_A. On the other hand, we recall that the quasi-isomorphism of dg bimodules over $\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A), E(A))$ stated in Theorem 3.3 is given by $(\mathrm{id}_{E(A)} \otimes (B^+(j_A) \circ \beta^{A^\#})) \circ \tau_{A^\#, B^+(A)^\#} \circ \iota^{-1}_{A,B^+(A)}$, that we shall denote by ρ, where $\tau_{M,N} : M \otimes N \to N \otimes M$ is the *flip* morphism of dg modules defined as $\tau_{M,N}(m \otimes n) = (-1)^{\deg m \deg n} n \otimes m$, for all $m \in M$ and $n \in N$ homogeneous elements. Our claim is tantamount to the fact that the previous map ρ induces a morphism of left graded modules over $k[\varepsilon]/(\varepsilon^2)$ between the Hochschild homology groups $HH_\bullet(A)^\#$ and $HH_\bullet(E(A))$. We shall prove this by showing that a particular quasi-inverse of ρ is a morphism of left dg modules over $k[\varepsilon]/(\varepsilon^2)$. In order to do so, we consider

$$\iota_{A,B^+(A)} \circ \tau_{B^+(A)^\#, A^\#} \circ (j_A \otimes \mathrm{id}_{A^\#}) \circ (\mathrm{id}_{\Omega^+(A^\#)} \otimes_{\Omega^+(A^\#)^e} \gamma^{A^\#}) \circ (j_A^{-1} \otimes B^+(j_A^{-1})),$$

which we denote γ'. It is clear that γ' is a quasi-inverse of ρ, because the properties of the map $\gamma^{A^\#}$ stated in Subsection 2.4 imply that $\gamma' \circ \rho$ is the identity of $(A \otimes_{\tau_A} B^+(A))^\#$. It thus suffices to prove that γ' is a morphism of left dg modules over $k[\varepsilon]/(\varepsilon^2)$. This is equivalent to show that $\gamma' \circ B_{E(A)} = -B_A^\# \circ \gamma'$. By the explicit expression of the Connes map (3.10), it is evident that $(j_A \otimes B^+(j_A)) \circ B_{\Omega^+(A^\#)} = B_{E(A)} \circ (j_A \otimes B^+(j_A))$, so it suffices to show that $\gamma'' \circ B_{\Omega^+(A^\#)} = -B_A^\# \circ \gamma''$, where γ'' is given by

$$\iota_{A,B^+(A)} \circ \tau_{B^+(A)^\#, A^\#} \circ (j_A \otimes \mathrm{id}_{A^\#}) \circ (\mathrm{id}_{\Omega^+(A^\#)} \otimes_{\Omega^+(A^\#)^e} \gamma^{A^\#}).$$

Using the left dg module structure of $A \otimes_{\tau_A} B^+(A)$ over $k[\varepsilon]/(\varepsilon^2)$ given by $-B_A^\#$ and the bijection $\iota_{A,B^+(A)} \circ \tau_{B^+(A)^\#, A^\#} \circ (j_A \otimes \mathrm{id}_{A^\#})$, we obtain a unique left action of $k[\varepsilon]/(\varepsilon^2)$ on $\Omega^+(A^\#) \otimes A^\#$ such that the previous bijective map is an isomorphism of left dg modules over $k[\varepsilon]/(\varepsilon^2)$. It is explicitly given by

$$\varepsilon \cdot (\langle \omega_1 | \dots | \omega_m \rangle \otimes \omega_0) = \omega_0(1_A) \sum_{i=1}^{m} (-1)^{\bar{\epsilon}_{i+1} \bar{\epsilon}^i + \bar{\epsilon}_{m+1} + \deg \omega_i} \langle \omega_{i+1} | \dots | \omega_m | \omega_1 | \dots | \omega_{i-1} \rangle \otimes \omega_i, \tag{3.12}$$

where $\omega_i \in A^\#$ are homogeneous, for $j = 0, \dots, m$, $\bar{\epsilon}_i = (\sum_{j=1}^{i-1} \deg \omega_j) + i - 1$ and $\bar{\epsilon}^i = (\sum_{j=i+1}^{m} \deg \omega_j) + m - i$. We shall denote by B' the operator defined on $\Omega^+(A^\#) \otimes A^\#$ given by left multiplication by ε. Hence, γ' is a morphism of left dg modules over $k[\varepsilon]/(\varepsilon^2)$ if and only if

$$(\mathrm{id}_{\Omega^+(A^\#)} \otimes_{\Omega^+(A^\#)^e} \gamma^{A^\#}) \circ B_{\Omega^+(A^\#)} = B' \circ (\mathrm{id}_{\Omega^+(A^\#)} \otimes_{\Omega^+(A^\#)^e} \gamma^{A^\#}). \tag{3.13}$$

This can be easily proved as follows. From equations (2.9) and (3.12), the only non-trivial situation is when we evaluate (3.13) at an element of the form $\langle \omega_1 | \dots | \omega_m \rangle \otimes []$, for some $\omega_i \in A^\#$, $i = 1, \dots, m$. This latter case follows directly from the explicit expression of the maps given in (2.9), (3.10) and (3.12). □

3.4.3 The statement on the Hochschild homology

We recall that the left and right Lie actions on the Hochschild homology group $HH_\bullet(\Lambda)$ over the graded Lie algebra $HH^\bullet(\Lambda)[1]$, where Λ is any augmented dg algebra, are induced by the graded commutators of the corresponding left or right dg action of the augmented dg algebra $\mathcal{H}om^{\tau_\Lambda}(B^+(\Lambda),\Lambda)$ on $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ with the Connes map B_Λ. Indeed, the left Lie action of $\phi \in \mathcal{H}om^{\tau_\Lambda}(B^+(\Lambda),\Lambda)$ on $\bar{\lambda} \in \Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$, which we denote by $L_\phi(\bar{\lambda})$ is given by the graded commutator of the left action operator of $k[\varepsilon]/(\varepsilon^2)$ on $\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda)$ with the left action operator of the augmented dg algebra $\mathcal{H}om^{\tau_\Lambda}(B^+(\Lambda),\Lambda)$ on the same space, *i.e.* $L_\phi(\bar{\lambda}) = \varepsilon \cdot (\phi \cdot \bar{\lambda}) - (-1)^{\deg \phi}\phi \cdot (\varepsilon \cdot \bar{\lambda})$. The action on the dual space $(\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda))^\#$ is given by the usual formula in representation theory of dg Lie algebras $L_\phi(\bar{\lambda}') = -(-1)^{\deg \bar{\lambda}' \deg \phi} \bar{\lambda}' \circ L_\phi$, where $\bar{\lambda}' \in (\Lambda \otimes_{\tau_\Lambda} B^+(\Lambda))^\#$. For more details on these definitions and the fact that the usual dg bimodule structures together with the Lie module structures are part (with the Gerstenhaber algebra structure on Hochschild cohomology) of a *Tamarkin-Tsygan (pre)calculus*, we refer the reader to [22], Sections 2 and 4.

The following theorem is a direct consequence of the definitions recalled in the previous paragraph, Theorem 3.3 and Proposition 3.5.

Theorem 3.6. *Let A be an augmented dg algebra which is assumed to be Adams connected. The quasi-isomorphism of dg $\mathcal{H}om^{\tau_{E(A)}}(B^+(E(A)),E(A))$-bimodules*

$$(A \otimes_{\tau_A} B^+(A))^\# \to E(A) \otimes_{\tau_{E(A)}} B^+(E(A)),$$

stated in Theorem 3.3 gives an isomorphism $HH_\bullet(A)^\# \to HH_\bullet(E(A))$ of graded bimodules over $HH^\bullet(E(A))$ that is also compatible with the left actions of the graded Lie algebra $HH^\bullet(E(A))[1]$, where the domain has the structure of Lie module over $HH^\bullet(E(A))[1]$ via the isomorphism of graded Lie algebras $HH^\bullet(E(A))[1] \to HH^\bullet(A)[1]$. Furthermore, the Connes' map on $HH_\bullet(E(A))$ is minus the dual of the Connes' map on $HH_\bullet(A)$ under the previous identification of Hochschild homologies.

3.5 The final result: the duality of Tamarkin-Tsygan calculi of Koszul dual algebras

We can summarize the results in the previous subsections. In order to do so we shall introduce the following definition about Tamarkin-Tsygan calculi. We refer the reader to [22], Def. 4.3 and 4.4, for the basic definitions and notation. We will say that a Tamarkin-Tsygan calculus $(\tilde{H}^\bullet, \tilde{H}_\bullet, \tilde{d})$ is *dual* to another Tamarkin-Tsygan calculus $(H^\bullet, H_\bullet, d)$ if there is a pair (f,g) where $f : \tilde{H}^\bullet \to H^\bullet$ is an isomorphism of Gerstenhaber algebras, $g : H_\bullet^\# \to \tilde{H}_\bullet$ is an isomorphism of Gerstenhaber modules over $\tilde{H}^\bullet$ such that $\tilde{d} \circ g = -g \circ d^\#$, where $H_\bullet$ is a Gerstenhaber module over $\tilde{H}^\bullet$ via f. Note that the definition is clearly symmetric, since the previous duality implies that $(H^\bullet, H_\bullet, d)$ is dual to $(\tilde{H}^\bullet, \tilde{H}_\bullet, \tilde{d})$ via $(f^{-1}, \iota_{H_\bullet}^{-1} \circ g^\#)$.

Corollary 3.7. *Let A be an Adams connected augmented dg algebra. Then, the Tamarkin-Tsygan calculus of $E(A)$ is dual to the one of A.*

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Estanislao HERSCOVICH

Institut Joseph Fourier,
Université Grenoble Alpes,
Grenoble, FRANCE,
Estanislao.Herscovich@univ-grenoble-alpes.fr,

Departamento de Matemática,
FCEyN, UBA,
Buenos Aires, ARGENTINA,
eherscov@dm.uba.ar,