Applications of one-point extensions to compute the A_∞-(co)module structure of several Ext (resp., Tor) groups

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Abstract

Let A be a nonnegatively graded connected algebra over a noncommutative separable k-algebra K, and let M be a bounded below graded right A-module. If we denote by T the A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K,K)$, we know that there exists an A_∞-comodule structure on $T' = \mathrm{Tor}^A_\bullet(M,K)$ over T. The structure of the A_∞-algebra $E = \mathcal{E}xt^\bullet_A(K,K)$ and the corresponding A_∞-module on $E' = \mathcal{E}xt^\bullet_A(M,K)$ are just obtained by taking the bigraded dual. In this article we prove that there is partial description of the A_∞-comodule T' over T and of the structure of A_∞-module E' over E, similar to and also generalizing the partial description of the A_∞-algebra structure on E given by Keller's higher-multiplication theorem in [19]. We also provide a criterion to check if a given A_∞-comodule structure on T' is a model by regarding if the associated twisted tensor product is a minimal projective resolution of M, analogous to a theorem of B. Keller explained by the author of this article in [9]. Finally, we give an application of this result by computing the A_∞-module structure on E' for any generalized Koszul algebra A and any generalized Koszul module M.

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1 Introduction

The aim of this article is to study the A_∞-(co)module structure of $\mathcal{E}xt^\bullet_A(M,K)$ (resp., $\mathrm{Tor}^A_\bullet(M,K)$) over $\mathcal{E}xt^\bullet_A(K,K)$ (resp., $\mathrm{Tor}^A_\bullet(K,K)$) for a bounded below graded right module M over a nonnegatively graded connected algebra A over K, where K is a (not necessarily commutative) separable k-algebra and k is a field. We stress that both A and M are assumed to be locally finite dimensional over k. Since the structure of right A_∞-module structure of $\mathcal{E}xt^\bullet_A(M,K)$ is just the bigraded dual of the A_∞-comodule on $\mathrm{Tor}^A_\bullet(M,K)$, we may focus on one of the two. The two main results we have proved are the following:

1. There is partial description of the A_∞-comodule structure on $\mathrm{Tor}^A_\bullet(M,K)$, similar to the partial description of the A_∞-algebra structure on $\mathcal{E}xt^\bullet_A(K,K)$ given by Keller's higher-multiplication theorem in [19], Thm. A and Cor. B (see Theorem 3.1 and Remark 3.2).

2. There is a very simple manner to check if a given A_∞-comodule structure on $\mathrm{Tor}^A_\bullet(M,K)$ is a model (*i.e.* it is induced by the dg comodule $B^+(M,A)$ over the dg coalgebra $B^+(A)$, where $B^+(A)$ and $B^+(M,A)$ denote the corresponding bar constructions) by regarding if the associated twisted tensor

product is a minimal projective resolution of M, analogous to a theorem of B. Keller explained by the author of this article in [9], Thm. 4.7 (see Theorem 4.2).

Using the last result we have explicitly computed the A_∞-module structure over $\mathrm{Ext}^\bullet_A(k,k)$ on $\mathrm{Ext}^\bullet_A(M,k)$ for a generalized Koszul module M over a generalized Koszul algebra A over a field k, extending the main result of [8]. We emphasize that the hypothesis that K is a separable k-algebra rather than just a field is motivated by situations appearing in representation theory (see for instance [3, 6, 7]), and in particular it allows to naturally consider useful constructions like one-point extensions, which are used to prove the previous statements 1 and 2.

The structure of the article is as follows. In Section 2 we provide the preliminary results we shall later use in the sequel. In particular, in Subsection 2.1 we recall the basic definitions and properties about nonnegatively graded connected algebras $A = TV/(R)$ over a separable k-algebra K and their graded modules. All of the definitions and properties are standard and well-known to the experts, with possible exception of the definitions and properties concerning the sections of the canonical projection $TV \to A$ and the space of relations R (*e.g.* Proposition 2.4), even though some of them are rather simple statements (see Facts 2.1, 2.2, 2.3 and 2.7). In Subsection 2.2 we provide the precise statement of the dual version for dg coalgebras of the well-known construction of an A_∞-algebra structure on a (certain) subspace of a dg algebra done by S. Merkulov in [20] (see Theorem 2.8). We omit the proof of our statement for we believe it follows the same argument as the original one (see the note [11] for a detailed proof). In Subsection 2.3 we apply the previous result to briefly review the proof of Keller's higher-(co)multiplication theorem, following the pattern given in [19]. The only reason we have given the proof is to reduce some steps of the original one and to shed some light on the required hypotheses on the section of the canonical projection $TV \to A$ and of the space of relations R, which are somehow hidden in the original presentation [19] and which make use of the results of Subsection 2.1. Subsection 2.4 is devoted to state some remarks about the Yoneda algebras $\mathrm{Ext}^\bullet_A(K,K)$ and $\mathcal{E}xt^\bullet_A(K,K)$. More precisely we prove that the left and right versions of the Yoneda algebra (given by considering K as a left or right module, respectively) coincide when K is separable (see Proposition 2.13), and we give a counterexample when K is only required to be semisimple (see Example 2.12). They are direct consequences of basic results in representation theory, so they should be well-known among the experts but since we could not find any precise reference in the literature (this is not discussed for instance in [7], or in [6], Section 9) beyond the easy case given by assuming that $K = k$ is a field (see [22], Ch. 1, Section 1), we believe that they may be of use. We close the section with Subsection 2.5, where we recall the definition of one-point extension of an algebra by a module, but we also introduce the dual version: a one-point extension of an A_∞-coalgebra by an A_∞-comodule. The properties established in Propositions 2.16 and 2.18 are easily deduced from the definitions.

After having precisely stated and organised all the required definitions and results in Section 2, we prove in Section 3 the first main result of the article, which we referred to in the previous item 1. The main idea of the proof is to utilize our version of Keller's higher-comultiplication theorem for a particular one-point extension. In Section 4 we prove the second main result of this work, mentioned in item 2. The proof is also based on making use of a particular one-point extension. It is fair to say that the two main results of the article are somehow direct consequences of the preparatory results stated in Section 2. Finally, in the last section 5 we give an application by computing the A_∞-module structure of $\mathrm{Ext}^\bullet_A(M,k)$ over $\mathrm{Ext}^\bullet_A(k,k)$ for an s-Koszul module M over an s-Koszul algebra A over a field k, extending one of the main results of [8]. Our methods are completely different from those of the mentioned article.

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2 Preliminaries on basic algebraic structures

In what follows, k will denote a field and K will be a noncommutative unitary k-algebra. The term *module* (sometimes decorated by adjectives such as graded, or dg) will denote a (not necessarily symmetric) bimodule over K (correspondingly decorated), such that the induced bimodule structure over k is symmetric, and we denote the category that they form by $\mathscr{C}_K$. All morphisms between modules will be K-linear on both sides (and satisfy further requirements if the modules are decorated as before), unless otherwise stated. In order to emphasize the action on both sides we shall sometimes call the objects of the previous category K-*bimodules*. This will be specially used when we need to consider other k-algebras, such as $K \times K$ in Subsection 2.5. We follow the convention on gradings and definitions given in [10], with K instead of A.

We also recall that, if $V = \oplus_{(n,m)\in\mathbb{Z}^2} V^{(n,m)}$ is a bigraded module, $V[(p,q)]$ is the bigraded module whose (n,m)-th homogeneous component $V[(p,q)]^{(n,m)}$ is given by $V^{(p+n,q+m)}$, for all $n, m \in \mathbb{Z}$, and it is called the *shift* of V. In order to reduce our notation and if it is clear that we are referring to the first (resp., second) degree, which we usually call cohomological (resp., Adams), we shall denote the shift $V[(p,0)]$ (resp., $V[(0,p)]$) simply by $V[p]$. We recall that we can switch from cohomological (upper) grading to homological (lower) grading by the obvious relation $V^n = V_{-n}$, for $n \in \mathbb{Z}$, and where the Adams degree does not change. One trivially sees that all the standard definitions recalled in [10] of graded or dg (co)algebra, or even A_∞-(co)algebra, eventually provided with an Adams grading, and (co)modules over them make perfect sense in the monoidal category $\mathscr{C}_K$ of graded modules (correspondingly, provided with an Adams grading) together with the tensor product $\otimes_K$. All unadorned tensor products $\otimes$ and morphism spaces $\mathrm{Hom}(-,-)$ would be over K, unless otherwise stated.

Finally, $\mathbb{N}$ will denote the set of (strictly) positive integers, whereas $\mathbb{N}_0$ will be the set of nonnegative integers. Similarly, for $N \in \mathbb{N}$ we denote by $\mathbb{N}_{\geq N}$ the set of positive integers greater than or equal to N. Of course, similar notation could be used for other inequality signs.

2.1 Nonnegatively graded connected algebras

We now recall the main definitions of the theory of nonnegatively graded connected algebras. Even though we are working in a slightly more general context, for our algebras are defined over a noncommutative k-algebra K, all the basic constructions are still possible and the main results still hold. We refer the reader to [9], Section 2, and the references therein.

Except for some minor examples and comments, we shall suppose from now on that $K^{\mathrm{e}} = K \otimes_k K^{\mathrm{op}}$ is semisimple (this holds for instance if K coincides with k, K is a product of finite separable extensions of k, or a group algebra kG, where the characteristic of k does not divide the order of G). We recall that this is equivalent to the fact that K is separable over k (see [23], Thm. 9.2.11, and [5], Cor. (7.8)), so in particular K is also semisimple and finite dimensional over k. Let A be a nonnegatively graded connected algebra over K. We recall that this means that A is a graded K-bimodule $A = \oplus_{n\in\mathbb{N}_0} A^n$, where $A^0 = K$ (as K-bimodules), together with a morphism $\mu_A : A \otimes A \to A$ of K-bimodules and an element $1_A \in A^0$ such that μ_A is associative and 1_A is a unit for it. We remark that A is considered to be concentrated in cohomological degree zero and the grading A indicated previously is the Adams grading. As noted before, since the k-algebra K is not necessarily commutative, this definition is not completely standard but it has been considered before (see for instance [3, 14], but also [7]). We shall also assume that A is *locally finite dimensional (over k)*, *i.e.* the dimension of the k-vector space A^n is finite, for all $n \in \mathbb{N}$. Let $A_+ = \oplus_{n\in\mathbb{N}} A^n$.

Since $K = A/A_+$ we see that K can also be regarded as a graded A-bimodule concentrated in degree zero. We want to remark that, in particular, Lemmas 2.1 and 2.2 of [9] are still valid replacing k by K (and the referred proofs in [4] hold *verbatim*). Moreover, all the comments in the two paragraphs before and the two after Lemma 2.2 of [9] also hold in this situation if one replaces k by K and "graded vector space" by "graded module". The same applies to the last paragraph of [9], Section 2, where the beginning of the minimal projective resolution of the graded (left or right) module K is explained. We want to remark that all the results mentioned in the three previous sentences also hold if one only supposes that K is semisimple. However, there is a major difference with the case when K^{e} is semisimple: if we assume that K is semisimple the maps involved are not necessarily K-linear on both sides, so the morphisms will not necessarily lie in the monoidal category $\mathscr{C}_K$ of K-bimodules we mentioned before. On the other hand, if we assume that K^{e} is semisimple then all the constructed morphisms will be of K-bimodules. Although this may not sound very striking, we may state the following derived pathology: there exist semisimple k-algebras K and noncommutative graded connected algebras A over K for which $\mathrm{Ext}^{\bullet}_A({}_AK, {}_AK)$ is not isomorphic to $\mathrm{Ext}^{\bullet}_A(K_A, K_A)$ as K-bimodules (see Example 2.12). We will prove that this cannot happen if K^{e} is semisimple (see Proposition 2.13), which also gives another reason for our setting of work (see however Remark 2.15).

A graded submodule $V \subseteq A_+$ is called a *module of irreducible generators* if the composition of the previous inclusion with the canonical projection $A_+ \to A_+/(A_+ \cdot A_+)$ is bijective. Note that V is also locally finite dimensional over k. The previous inclusion induces a unique morphism $\pi : TV \to A$ of unitary graded algebras, where $TV = \oplus_{n\in\mathbb{N}_0} V^{\otimes n}$ is the *tensor algebra* whose product is given by concatenation. We remark that $V^{\otimes 0} = K$. By the locally finiteness assumption on A we see that π is surjective, for, given any $n \in \mathbb{N}$, the descending chain of modules $(A_+ \cdot \dots \cdot A_+) \cap A^n$ of A^n given by increasing the number of factors A_+ stabilizes by a simple dimension argument. On the other hand, we have by definition the internal direct sum $A_+ = V \oplus A_+ \cdot A_+$ of graded modules and denote by $q : A_+ \to A_+/(A_+ \cdot A_+) \simeq V$ and $q' : A_+ \to A_+ \cdot A_+$ the associated projections. For each $n \in \mathbb{N}_0$, define the canonical projection

$$\Pi_n : TV \to V^{\otimes n}. \tag{2.1}$$

Moreover, by the semisimplicity assumption on K^{e}, we see that we can consider a section $s : A \to TV$ of π satisfying that

(i) $s(1_A) = 1_{TV}$,

(ii) $s(A_+) \subseteq (TV)_+$,

(iii) $s|_V : V \subseteq A \to TV$ coincides with the canonical inclusion.

The first two conditions always hold but the third one is not necessarily true, and a section of π satisfying it will be called *standard*. From now, we shall only consider standard sections of π, unless otherwise stated.

We will need the following result in the sequel.

Fact 2.1. *Let $s : A \to TV$ be a standard section of π and $t : A_+ \to A \otimes V$ be a morphism of graded modules. Then, for all $n > 2$, we have that*

$$\Pi_n \circ (s \otimes \mathrm{id}_V) \circ t = \Pi_n \circ (s \otimes \mathrm{id}_V) \circ (q' \otimes \mathrm{id}_V) \circ t,$$

where $q' : A_+ \to A_+ \cdot A_+$ is the canonical projection.

Proof. Let $a \in A_+$ and write $t(a) = a_1 + a_2 + a_3$, where $a_1 \in K \otimes V$, $a_1 \in V \otimes V$ and $a_2 \in (A_+ \cdot A_+) \otimes V$. Since $n > 2$ and s is standard, $(\Pi_n \circ (s \otimes \mathrm{id}_V))(a_i)$ trivially vanishes for $i = 1, 2$, which proves the claim. $\square$

Since A is a graded TV-module by means of π, one would be tempted to choose s as a morphism of (say right) TV-modules, but this is not possible unless the kernel of π is trivial. Indeed, if s is such a morphism and $R \neq 0$ is a space of relations of A (see the definition after Remark 2.6), take $\alpha = \sum_{i \in J} \alpha_i v_i$ a nonzero element of R with $v_i \in V$, for all $i \in J$, forming a linearly independent set and all $\alpha_i \in TV$ nonzero. By definition of space of relations we see that there exists $i \in J$ such that $\alpha_i \notin \mathrm{Ker}(\pi)$. Then

$$0 = (s \circ \pi)(\alpha) = s\big(\sum \pi(\alpha_i) v_i\big) = \sum s\big(\pi(\alpha_i)\big) v_i.$$

Since the set $\{v_i : i \in J\}$ is linearly independent we conclude that $s(\pi(\alpha_i))$ vanishes for all $i \in J$, which in turn implies that $\pi(\alpha_i)$ vanishes for all $i \in J$, because s is injective, and we obtain thus a contradiction. There are however weaker properties of interest.

Let us define $\tilde{s}_n$ as the morphism of graded modules given by the composition

$$A \xrightarrow{s} TV \longrightarrow \bigoplus_{m \geq n} V^{\otimes m} = TV \otimes V^{\otimes n} \xrightarrow{\pi \otimes \mathrm{id}_V^{\otimes n}} A \otimes V^{\otimes n}, \tag{2.2}$$

where the second map is the canonical projection. We say that s is *good (on the right)* if $(\tilde{s}_n \otimes \mathrm{id}_V^{\otimes m}) \circ \tilde{s}_m = \tilde{s}_{n+m}$, for all $n, m \in \mathbb{N}$. There is an analogous left version of good section. The following result follows from a direct recursive argument.

Fact 2.2. *Let $s : A \to TV$ be a standard section of π. Then, s is good if and only if $\tilde{s}_{n+1} = (\tilde{s}_n \otimes \mathrm{id}_V) \circ \tilde{s}_1$, for all $n \in \mathbb{N}$.*

Note that, if s is a morphism of right TV-modules, then it is good, so this last property is weaker.

Let $t : A_+ \to A \otimes V$ be a section of the canonical projection $A \otimes V \to A_+$ induced by the product μ_A of A. We say that a standard section $s : A \to TV$ of π is *compatible* with t if

$$\begin{tikzcd}
A_+ \arrow[dr, "s|_{A_+}"] \arrow[dd, "t"'] & \\
& TV \otimes V \\
A \otimes V \arrow[ur, "s \otimes \mathrm{id}_V"'] &
\end{tikzcd}$$

The following result is clear.

Fact 2.3. *Let $t : A_+ \to A \otimes V$ be a section of the canonical projection $A \otimes V \to A_+$ induced by the product μ_A. Then there exists a unique section $s : A \to TV$ of π compatible with t.*

Moreover, the section s can be directly computed from t as follows. Set $A_{[1]} = A_+$ and $t_{[1]} = t$. Recursively, having defined $A_{[1]} \supseteq \cdots \supseteq A_{[n]}$ and $t_{[1]}, \dots, t_{[n]}$, where $t_{[i]} : A_{[i]} \to A \otimes V^{\otimes i}$, for $1 \leq i \leq n$, set $A_{[n+1]} = t_{[n]}^{-1}(A_+ \otimes V^{\otimes n})$ and $t_{[n+1]} = (t \otimes \mathrm{id}_V^{\otimes n}) \circ t_{[n]}$. Define $V_{[n]} = t_{[n]}^{-1}(A^0 \otimes V^{\otimes n})$, for all $n \in \mathbb{N}$. It is easy to prove that $A_{[n]}$ is the internal direct sum of the modules $A_{[n+1]}$ and $V_{[n]}$, and, by an inductive argument, A_+ is the internal direct sum of the graded submodules $\{V_{[n]}\}_{n \in \mathbb{N}}$. Define now $s|_{A_+} : A_+ \to (TV)_+$ as the unique morphism of graded modules satisfying that $s|_{V_{[n]}} = t_{[n]}$. The previous map is the unique section $s : A \to TV$ of π compatible with t.

Even though the authors of [19] have explained it in different terms, they have precisely considered a section s of π compatible with a given (fixed) section $t : A_+ \to V \otimes A$ of the canonical projection $V \otimes A \to A_+$ induced by the product μ_A, which is tantamount to the fact that s is good (on the left), as we will now see.

The following result shows that the two previously introduced properties are equivalent.

Proposition 2.4. *Let A be a nonnegatively graded algebra, V be a space of irreducible generators of A and $\pi : TV \to A$ be the induced morphism of graded algebras. The following statements are equivalent for a standard section s of π:*

(i) $\mathrm{Im}(s|_{A_+}) \subseteq \mathrm{Im}(s) \otimes V$,

(ii) s is good (on the right),

(iii) s is compatible with a section $t : A_+ \to A \otimes V$ of the canonical projection $A \otimes V \to A_+$ induced by the product of A.

Proof. Let us assume the property (i). We want to prove that s is good, or, equivalently, the characterization given in Fact 2.2. We will prove first the following intermediate result.

Lemma 2.5. *If s satisfies property (i), then the following diagram*

$$\begin{tikzcd} & & & A \otimes V \arrow[dd, "s\otimes \mathrm{id}_V"] \\ A \arrow[r, "s"] & TV \arrow[r] & (TV)_+ \arrow[ur, "\pi\otimes\mathrm{id}_V"] \arrow[dr, "\mathrm{id}_{(TV)_+}"'] & \\ & & & TV \otimes V \end{tikzcd}$$

is commutative, where the second horizontal map is the canonical projection.

Proof. Note that the commutativity of the diagram is equivalent to the fact that $(f \otimes \mathrm{id}_V) \circ \bar{\Pi}_1 \circ s$ vanishes, where $f = \mathrm{id}_{TV} - s \circ \pi$ and $\bar{\Pi}_n : TV \to (TV)_+^n$ denotes the canonical projection. Note that f is an idempotent endomorphism of TV whose image is the kernel of π. It is trivial to see that $f \circ s = 0$, since s is a section of π. Hence $((f \otimes \mathrm{id}_V) \circ \bar{\Pi}_1)(\mathrm{Im}(s)) = (f \otimes \mathrm{id}_V)(\mathrm{Im}(s|_{A_+}))$. Note that $(f \otimes \mathrm{id}_V)$ is an idempotent endomorphism of $(TV)_+$ whose image is $\mathrm{Ker}(\pi) \otimes V$, and whose kernel is $\mathrm{Im}(s) \otimes V$. Since the latter includes $\mathrm{Im}(s|_{A_+})$, by hypothesis, we see that $(f \otimes \mathrm{id}_V)(\mathrm{Im}(s|_{A_+})) = 0$, which proves the lemma. □

From the previous lemma we see that $\tilde{s}_{n+1} = (\tilde{s}_n \otimes \mathrm{id}_V) \circ \tilde{s}_1$, for all $n \in \mathbb{N}$, since this equality is tantamount to the commutativity of

$$\begin{tikzcd} & & & A \otimes V \arrow[dr, "s\otimes\mathrm{id}_V"] & & & \\ A \arrow[r, "s"] & TV \arrow[r, "\bar{\Pi}_1"] & (TV)_+ \arrow[ur, "\pi\otimes\mathrm{id}_V"] \arrow[rr, "\mathrm{id}_{(TV)_+}"'] & & TV \otimes V \arrow[r, "\bar{\Pi}_n\otimes\mathrm{id}_V"] & TV \otimes V^{\otimes(n+1)} \arrow[r, "\pi\otimes\mathrm{id}_V^{\otimes(n+1)}"] & A \otimes V^{\otimes(n+1)} \end{tikzcd}$$

Hence, Fact 2.2 tells us that s is good.

Let us now assume that property (ii) holds, *i.e.* s is good. Define the morphism $t = \tilde{s}_1|_{A_+}$, or, equivalently, t is given by the composition

$$A_+ \xrightarrow{s|_{A_+}} (TV)_+ = TV \otimes V \xrightarrow{\pi\otimes\mathrm{id}_V} A \otimes V.$$

It is trivial to verify that t is a section of the canonical projection $A \otimes V \to A_+$ induced by the product A. Moreover, since s is good, we get that s is compatible with t, which proves (iii).

We shall now suppose that condition (iii) holds. We have to prove that the same is true for (i). By Fact 2.3, the compatibility of s and t yields that s is completely determined by t. Since $s(V) = A^0 \otimes V$, it suffices to prove that $s(V_{[n+1]}) \subseteq s(V_{[n]}) \otimes V$, for all $n \in \mathbb{N}$. Moreover, the Fact 2.3 also describes explicitly the image of s. Indeed, from the definition of the maps stated there we see that $t(V_{[n+1]}) \subseteq V_{[n]} \otimes V$ and $t_{[n+1]} = (t_{[n]} \otimes \mathrm{id}_V) \circ t|_{V_{[n+1]}}$, for all $n \in \mathbb{N}$. Hence, we get that $s(V_{[n+1]}) \subseteq s(V_{[n]}) \otimes V$, for all $n \in \mathbb{N}$, as was to be shown. The proposition is thus proved. □

Remark 2.6. *Using the first of the equivalent conditions of the previous proposition it is easy to see that a section of π is not necessarily good (or compatible with a section of the canonical projection $A \otimes V \to A_+$ induced by the product of A). For instance, for $K = k$, take $V = \mathrm{span}_k\langle x, y\rangle$ concentrated in degree 1, $A = TV/(R)$, where $R = \mathrm{span}_k\langle yx\rangle$, and (partially) define $s : A \to TV$ to be the section of π such that $s(\bar{x}^2) = x^2$, $s(\bar{y}^2) = y^2$, $s(\bar{x}\bar{y}) = xy$, and $s(\bar{x}^3) = x^3 + yx^2$. Then $s(\bar{x}^3) \notin \mathrm{Im}(s) \otimes V$, for yx does not lie in the image of s.*

We recall that a *space of relations* R of A is a graded submodule of the kernel $\mathrm{Ker}(\pi)$ of π satisfying that the composition of the inclusion $R \subseteq \mathrm{Ker}(\pi)$ together with the canonical projection

$$\mathrm{Ker}(\pi) \to \mathrm{Ker}(\pi)/\big((TV)_+ \cdot \mathrm{Ker}(\pi) + \mathrm{Ker}(\pi) \cdot (TV)_+\big)$$

is an isomorphism. Since TV is locally finite dimensional, we see that the ideal generated by R coincides with $\mathrm{Ker}(\pi)$. Let $s : A \to TV$ be a standard section of π. We say that a space of relations R is *(right-)compatible* with s if R lies in the kernel of the map $(\mathrm{id}_{TV} - s \circ \pi) \otimes \mathrm{id}_V : (TV)_+ \to (TV)_+$. Given a non-negatively graded connected algebra of the form $A = TV/(R)$ together with a standard section s of π, we can always change the space of relations R for another one R' (so we still have $A = TV/(R')$) satisfying that R' is compatible with s (*cf.* [19], Lemma 5.2). Indeed, take for instance $R' = ((s \circ \pi) \otimes \mathrm{id}_V)(R)$. We also note that the compatibility condition is not automatically true. For example, set $K = k$, $V = \mathrm{span}_k\langle x, y, z\rangle$ concentrated in degree 1, $A = TV/(R)$, where $R = \mathrm{span}_k\langle yx, z^3 + yxz\rangle$, and consider $s : A \to TV$ to be a section of π satisfying that $s(\bar{z}^2) = z^2$. Then $((\mathrm{id}_{TV} - s \circ \pi) \otimes \mathrm{id}_V)(z^3 + yxz) = yxz \neq 0$, so R cannot be compatible with the given section.

We also recall that the beginning of the minimal projective resolution of the right A-module $K = A/A_+$ is given by

$$\cdots \to R \otimes A \xrightarrow{d_2} V \otimes A \xrightarrow{d_1} A \xrightarrow{d_0} K, \tag{2.3}$$

where d_0 is the canonical projection, d_1 is the restriction of minus the product of A to $V \otimes A$, and d_2 is the restriction to $R \otimes A$ of the map $(TV)_+ \otimes A \to A \otimes V$ given by the composition

$$(TV)_+ \otimes A = V \otimes TV \otimes A \xrightarrow{\mathrm{id}_V \otimes \pi \otimes \mathrm{id}_A} V \otimes A \otimes A \xrightarrow{\mathrm{id}_V \otimes \mu_A} V \otimes A.$$

The usual *reduced bar resolution* of the right A-module K is defined as the tensor product $K \otimes_A \overline{\mathrm{Bar}}(A)$, where $\overline{\mathrm{Bar}}(A)$ is the reduced bar resolution of the A-bimodule A (see [12], Subsection 2.1). The beginning of this resolution is given by

$$\cdots \to A_+^{\otimes 2} \otimes A \xrightarrow{B_2} A_+ \otimes A \xrightarrow{B_1} A \xrightarrow{B_0} K, \tag{2.4}$$

where B_0 is the canonical projection, B_1 is the restriction of minus the product of A to $A_+ \otimes A$, and B_2 is induced by

$$a_0 \otimes a_1 \otimes a_2 \mapsto -a_0 a_1 \otimes a_2 + a_0 \otimes a_1 a_2.$$

A comparison morphism from the resolution (2.3) to (2.4) is given as follows.

$$\begin{tikzcd} \dots \arrow[r] & R \otimes A \arrow[r, "d_2"] \arrow[d, "\sigma"] & V \otimes A \arrow[r, "d_1"] \arrow[d, "\mathrm{inc}"] & A \arrow[r, "d_0"] \arrow[d, equal] & K \arrow[r] \arrow[d, equal] & 0 \\ \dots \arrow[r] & A_+^{\otimes 2} \otimes A \arrow[r, "B_2"] & A_+ \otimes A \arrow[r, "B_1"] & A \arrow[r, "B_0"] & K \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

where σ is the restriction to $R \otimes A$ of the map $(TV)_+ \otimes A \to A_+^{\otimes 2} \otimes A$ defined as

$$v_1 \dots v_n \otimes a \mapsto \sum_{i=2}^{n} \pi(v_1 \dots v_{i-1}) \otimes \pi(v_i) \otimes \pi(v_{i+1} \dots v_n).a,$$

where $v_1, \dots, v_n \in V$ and $a \in A$. Let us denote by $\bar{\sigma} : R \to A_+^{\otimes 2}$ the map given by $\sigma \otimes_A \mathrm{id}_K$. Note that $\bar{\sigma}(R) \subseteq A_+ \otimes V$.

Fact 2.7. *Let $s : A \to TV$ be a standard section of π and R a space of relations of A. Then, for all $n > 2$, we have that*

$$\big(\Pi_n \circ (s \otimes \mathrm{id}_V)\big)|_{\bar{\sigma}(R)} = \big(\Pi_n \circ (s \otimes \mathrm{id}_V) \circ (q' \otimes \mathrm{id}_V)\big)|_{\bar{\sigma}(R)},$$

where $q' : A_+ \to A_+ \cdot A_+$ is the canonical projection and we consider $\bar{\sigma}(R)$ as a subset of $A_+ \otimes V$.

Proof. Let $r \in R$ and write $r = r_1 + r_2$, where $r_1 \in V^{\otimes 2}$, $r_2 \in (TV)_+ \cdot (TV)_+ \cdot (TV)_+$. Since $n > 3$ and s is standard, $(\Pi_n \circ (s \otimes \mathrm{id}_V))(\bar{\sigma}(r_1))$ trivially vanishes, which proves the claim. $\square$

2.2 Merkulov construction

We shall briefly present the dual procedure to the one introduced by S. Merkulov in [20] to produce an A_∞-algebra structure from a particular data on a dg submodule of a dg algebra. In our case, we produce an A_∞-coalgebra structure on a quotient dg module of a dg coalgebra. We also note that, even though the results of the article of Merkulov are stated for vector spaces, they clearly hold (by exactly the same arguments) in our more general situation of bimodules over the k-algebra K.

Let (C, Δ_C, d_C) be a dg coalgebra provided with an Adams grading and let $(C, d_C) \twoheadrightarrow (W, d_W)$ be a dg module quotient of C respecting the Adams degree. We denote by K the kernel of the previous quotient. We assume that there is a linear map $Q : C \to C[-1]$ of total degree zero satisfying that $\mathrm{id}_C - [d_C, Q]$ vanishes on K, where $C[-1]$ denotes the shift of the cohomological degree whereas the Adams degree remains unchanged, and $[d_C, Q] = d_C \circ Q + Q \circ d_C$ is the graded commutator. For all $n \geq 2$, define $\gamma_n : C \to C^{\otimes n}$ as follows. Setting formally $\gamma_1 = -Q^{-1}$, define

$$\gamma_n = \sum_{i=1}^{n-1} (-1)^{i+1} \big((\gamma_{n-i} \circ Q) \otimes (\gamma_i \circ Q)\big) \circ \Delta_C \tag{2.6}$$

for $n \geq 2$. We shall say that Q is *admissible* if the family $\{\gamma_n\}_{n \in \mathbb{N}_{\geq 2}}$ is locally finite, *i.e.* if the induced map $C \to \prod_{n \geq 2} C^{\otimes n}$ factors through the canonical inclusion $\oplus_{n \geq 2} C^{\otimes n} \to \prod_{n \geq 2} C^{\otimes n}$. We have the following result, whose first part is just the dual of [20], Thm. 3.4, whereas the rest is a slightly more general version of the dual of [19], Prop. 2.3 and Lemma 2.5.

Theorem 2.8. *Let (C, Δ_C, d_C) be a dg coalgebra provided with an Adams grading and let $\rho : (C, d_C) \to (W, d_W)$ be a quotient dg module of C (respecting the Adams degree) with kernel K. Suppose there is an admissible linear map $Q : C \to C[-1]$ of total degree zero, where $C[-1]$ denotes the shift of the cohomological degree, satisfying that $\mathrm{id}_C - [d_C, Q]$ vanishes on K. For all $n \in \mathbb{N}$, define $\Delta_n : W \to W^{\otimes n}$ as follows. Set $\Delta_1 = d_W$ and Δ_n to be the unique map satisfying that $\Delta_n \circ \rho = \rho^{\otimes n} \circ \gamma_n \circ (\mathrm{id}_C - [d_C, Q])$, for $n \geq 2$. Then, $(W, \Delta_\bullet)$ is an Adams graded A_∞-coalgebra.*

Define the collection $f_\bullet : C \to W$, where $f_n : C \to W^{\otimes n}$ is the linear map of homological degree $n-1$ and Adams degree zero given by $f_n = -\rho^{\otimes n} \circ \gamma_n \circ Q$, for $n \in \mathbb{N}$. Then $f_\bullet$ is a morphism of Adams graded A_∞-coalgebras.

Furthermore, assume C has a counit ϵ_C, there a linear map $\epsilon_W : W \to K$ such that $\epsilon_W \circ \rho = \epsilon_C$, $Q \circ Q = 0$ and $\rho \circ Q = 0$. Then, ϵ_W is a strict counit of the Adams graded A_∞-coalgebra $(W, \Delta_\bullet)$, and $f_\bullet : C \to W$ is a morphism of strictly counitary Adams graded A_∞-coalgebras.

Proof. For the first part, the proof is dual to one given in [20]. The second and third assertions are proved by similar arguments. The interested reader is referred to [11] to find all the details. □

The structure of A_∞-coalgebra on W given by the previous theorem will be called a *Merkulov model* on W, or simply a *model*. As in the case of dg algebras, note that the result stated in the first two paragraphs of the previous theorem is slightly more general than those obtained in perturbation theory, since we did not impose the standard assumption of a SDR (see [15], Section 6).

We will be interested in the following situation. Recall that $K^{\mathrm{e}} = K \otimes_k K^{\mathrm{op}}$ is semisimple and let W be the cohomology $H(C)$ of C, ρ be the composition of a linear retraction (of degree zero) r of the canonical inclusion $Z(C) \to C$ with the standard projection $p' : Z(C) \to H(C)$, and let $i : H(C) \to Z(C)$ be section of the latter map. Note that $\rho \circ d_C = 0$. We remark that all these maps are K^{e}-linear of zero degree. Set Q as a homotopy between the map $i \circ p' \circ r$ projecting C onto $i(H(C))$ and the identity map of C, dual to the one explained in [19], Section 2, which we briefly explain: consider the decomposition $C = B(C) \oplus i(H(C)) \oplus L$ of graded modules (preserving the Adams grading), where $B(C)$ denotes the image of d_C and we have chosen $L \subseteq \mathrm{Ker}(\rho)$, and define Q as the unique linear map whose restriction to $i(H(C)) \oplus L$ is trivial and $Q|_{B(C)}$ is the inverse of the linear map $d_C|_L : L \to B(C)$. It is trivial to note that it satisfies $Q \circ Q = \rho \circ Q = 0$. The existence of the linear maps i and Q is assured by the separability hypothesis of K^{e}.

A coaugmented A_∞-coalgebra C provided with an Adams grading is said to be *Adams connected* if the cokernel J_C of the coaugmentation η_C is concentrated in either strictly positive or strictly negative Adams degrees, and each homogeneous component of J_C of a fixed Adams degree d (but all homological degrees) is locally finite dimensional (*cf.* [18], Def. 2.1). In fact this definition makes sense for any bigraded module V if $V = K \oplus V'$, where K lies in total degree $(0, 0)$ and V' satisfies the same condition as J_C above.

Addendum 2.9. *Consider the same assumptions as in the previous theorem (including those of the last paragraph), except the admissibility condition on Q. In that case all the statements of the theorem hold with exception of the local finiteness of the maps $\Delta_\bullet$ and $f_\bullet$. If C is Adams connected, then it is cocomplete. Moreover, the family $\{\gamma_n\}_{n\geq 2}$ is also locally finite. Indeed, by definition* (2.6) *we see that $\{\gamma_n\}_{n\geq 2}$ is locally finite if and only if $\{f_n\}_{n\in\mathbb{N}}$ is so, and the Adams connectedness assumption together with the fact that $f_\bullet$ satisfy the equations of a strictly counitary morphism of A_∞-coalgebras tell us that the latter family is locally finite if and only its restriction to J_C and corestriction to J_W is so, which follows trivially from the Adams connectedness hypothesis. Note that in this case W is a a fortiori*

Adams connected. If we suppose further that $\rho : C \to W$ is a quasi-isomorphism of dg modules, then $f_\bullet : C \to W$ is a quasi-isomorphism of Adams graded coaugmented A_∞-coalgebras, whose bigraded dual is a quasi-isomorphism of Adams graded augmented A_∞-algebras.

2.3 Keller's higher-(co)multiplication theorem

In this subsection we will briefly review the proof of the dual result (of a slight generalization) of Keller's higher-multiplication theorem, that ascertains the coaugmented A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(K,K)$ for a nonnegatively graded connected algebra A over the separable k-algebra K. In fact this result follows from [19], Thm. A and Cor. B, but it can also be proved using the dual version of the Merkulov construction. The reason to briefly sketch the proof is due to some slight changes and minor improvements, which were not fully addressed in the nice article we mentioned previously. In particular, we highlighted the hypotheses of the section s and the space of relations R chosen, which are not completely clear in the source we indicated. Furthermore, we believe that our point of view is somehow simpler and cleaner, for we do not need to consider duals during the proof: the Yoneda algebra $\mathrm{Ext}^\bullet_A(K,K)$ (resp., $\mathcal{E}xt^\bullet_A(K,K)$) is obtained as the (resp., bigraded) dual of $\mathrm{Tor}^A_\bullet(K,K)$ so we only need to dualize once at the end. The prominent role of the A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K,K)$ with respect to the Yoneda algebra is also evidenced by its use computing the algebraic structure of Hochschild (co)homology (see [13], Thm. 4.3).

Let $A = TV/(R)$ be a nonnegatively graded connected algebra and locally finite dimensional over k, which is considered to be concentrated in cohomological degree zero and the Adams grading is the nonnegative one. We suppose that s is a good section of the canonical projection and R is compatible with s. Denote by t the section defined as $\tilde{s}_1|_{A_+}$ (see (2.2)). Let $C = B^+(A)$ be the (reduced) bar construction of the augmented Adams graded algebra A (see [12], for a complete account of the sign and grading conventions we follow). Note that C is Adams connected. We shall use the notation stated in the antepenultimate paragraph of the previous subsection, and shall identify the underlying module of $H(C)_n$ with the module of generators of the minimal projective resolution $P_\bullet$ of the right module K, *i.e.* $P_n = H(C)_n \otimes A$, for all $n \in \mathbb{N}_0$. Moreover, we choose the section $i : H(C) \to Z(C)$ such that its restriction to $H(C)_0 = K$ and $H(C)_1 = V$ is the canonical inclusion, and the restriction to $H(C)_2 = R$ is given by $\bar{\sigma}$, defined before Fact 2.7. By Theorem 2.8 we get a strictly counitary A_∞-coalgebra structure on the homology $H(C)$ of C, and a morphism of strictly counitary A_∞-coalgebras from C to $H(C)$. Furthermore, by grading considerations one sees that both C and $H(C)$ are coaugmented, and the previous morphism respect the coaugmentations, and is in fact a quasi-isomorphism of coaugmented A_∞-coalgebras.

By using the explicit comparison morphism (2.5), we get that the homogeneous component of C of homological degree 1 can be decomposed as $C_1 = B(C)_1 \oplus i(H(C)_1) \oplus L_1$, where $B(C)_1 = A_+ \cdot A_+$, $i(H(C)_1) = V$ and $L_1 = 0$, whereas the homogeneous component of C of homological degree 2 can be decomposed as $C_2 = B(C)_2 \oplus i(H(C)_2) \oplus L_2$, where $i(H(C)_2) = \bar{\sigma}(R)$ and we choose $L_2(C) = t(A_+ \cdot A_+)$, because $-t|_{A_+ \cdot A_+} : A_+ \to A_+ \otimes V \subseteq A_+ \otimes A_+$ is a section of $(d_C)|_{C_2} : C_2 \to C_1$, which is the restriction of minus the multiplication of A (see [19], Lemma 5.3). Moreover, dually to [19], Lemma 5.4, one can partially define the map Q such that $Q|_{C_1} : C_1 \to C_2$ is given by the composition of the projection $C_1 \to B(C)_1$ and the restriction of the map $-t$ to $A_+ \cdot A_+$. Note that the image of $Q|_{C_1}$ lies inside of $A_+ \otimes V$.

Since γ_n has homological degree $n-2$, we see that $\rho^{\otimes n} \circ \gamma_n|_{C_m} : C_m \to V^{\otimes n}$ vanishes for $m \neq 2$ (note that the proof of the dual result given in [19], Lemma 5.5, item (a), could have been done using the same simple reasons). The proof of the following result is parallel to that of [19], Lemma 5.5, item (b).

Fact 2.10. *The fact s is a good section implies that, for all $n \geq 2$, $g_n = \mathfrak{p}^{\otimes n} \circ \rho^{\otimes n} \circ \gamma_n \circ Q|_{C_1} : C_1 \to V^{\otimes n}$, where $\mathfrak{p} : H(C) \to V$ denotes the canonical projection, coincides with the map h_n defined as minus the composition*

$$C_1 \xrightarrow{q'} A_+ \cdot A_+ \xrightarrow{s|_{A_+ \cdot A_+}} (TV)_+ \xrightarrow{\Pi_n} V^{\otimes n},$$

where the first and the third maps are the corresponding canonical projections.

Proof. By (2.6), the fact that Δ_C is given by deconcatenation and $Q|_V = 0$, we see that

$$(\rho^{\otimes n} \circ \gamma_n)|_{A_+ \otimes V} = -\Big((\rho^{\otimes(n-1)} \circ \gamma_{n-1} \circ Q)|_{A_+} \otimes \mathrm{id}_V\Big) \tag{2.7}$$

for all $n \geq 2$. This implies that $g_n = (g_{n-1} \otimes \mathrm{id}_V) \circ t \circ q'$ for all $n \geq 2$, where we set $g_1 = -q$ and q is the canonical projection $A_+ \to V$. By definition of h_2 and the previous identity, we have that $h_2 = -\Pi_2 \circ s \circ q' = -\Pi_2 \circ (s \otimes \mathrm{id}_V) \circ t \circ q' = -(q \otimes \mathrm{id}_V) \circ t \circ q' = g_2$, because the restriction of s to V is the canonical inclusion into TV. Fact 2.1 together with the property that s is good immediately imply that h_n satisfies the same recursive relation as g_n, and the result follows. $\square$

The previous fact implies the following result, which is just the dual version of [19], Thm. A and Cor. B. The proof follows the same pattern as the one given for [19], Lemma 5.5, item (c).

Theorem 2.11. *Let $A = TV/(R)$ be a nonnegatively graded connected algebra over K and locally finite dimensional over k, which is considered to be concentrated in cohomological degree zero and the Adams grading is the nonnegative one. We choose a good section s of the canonical projection $\pi : TV \to A$ and we assume R is compatible with s. Then, there is a structure of minimal coaugmented A_∞-coalgebra on $D = \mathrm{Tor}^A_\bullet(K, K)$, such that, if $\mathfrak{p} : D \to V$ denotes the canonical projection, then, for all $n \geq 2$, $\mathfrak{p}^{\otimes n} \circ \Delta_n|_R : R \to V^{\otimes n}$ coincides with*

$$R \to (TV)_+ \to V^{\otimes n},$$

where the first map is the canonical inclusion and the second one is the standard projection.

Proof. Let $C = B^+(A)$ be the coaugmented dg coalgebra given by the reduced bar construction. Since $D = \mathrm{Tor}^A_\bullet(K, K) = H(C)$, by Theorem 2.8 and Addendum 2.9, D has a coaugmented A_∞-coalgebra structure $\Delta_\bullet$, for C is Adams connected. For all $n \geq 2$, define $\ell_n = \mathfrak{p}^{\otimes n} \circ \Delta_n|_R : R \to V^{\otimes n}$ and κ_n as

$$R \longrightarrow (TV)_+ \xrightarrow{\Pi_n} V^{\otimes n},$$

where the first map is the canonical inclusion and the second the standard projection. By definition of Δ_n, we see that, for all $n \geq 2$, $\mathfrak{p}^{\otimes n} \circ \Delta_n|_R : R \to V^{\otimes n}$ is given by $\mathfrak{p}^{\otimes n} \circ \rho^{\otimes n} \circ \gamma_n \circ \bar{\sigma} : R \to V^{\otimes n}$, where $\bar{\sigma}$ is defined before Fact 2.7. By (2.7) we see that $\ell_n = -(g_{n-1} \otimes \mathrm{id}_V) \circ \bar{\sigma}$, for all $n \geq 2$. Using the previous identity, and the definition of $\bar{\sigma}$ we get that $\ell_2 = \kappa_2$. If $n > 2$, Fact 2.10 tells us that $\ell_n = -(h_{n-1} \otimes \mathrm{id}_V) \circ \bar{\sigma}$. The compatibility between R and s, and Fact 2.7 imply in turn that $\ell_n = \kappa_n$, as was to be shown. $\square$

2.4 Some remarks on the Yoneda algebra

We note that there are different possible definitions of Yoneda algebra, given by taking the graded dual with respect to the homological degree or the bigraded dual with respect to both degrees. This will give in principle two different notions of Yoneda algebras, usually denoted by $\mathrm{Ext}^{\bullet}_{A}(K,K)$ and $\mathcal{E}xt^{\bullet}_{A}(K,K)$, respectively. We remark however, that if A is homologically smooth over K, then both constructions coincide (see [21], Corollary 2.4.4). On the other hand, for the Yoneda algebra $\mathrm{Ext}^{\bullet}_{A}(K,K)$ one may also consider K either as a left or a right A-module, which gives in principle two other different possibilities. In what follows we shall concentrate on the Yoneda algebra $\mathrm{Ext}^{\bullet}_{A}(K,K)$ but similar remarks hold for the Yoneda algebra $\mathcal{E}xt^{\bullet}_{A}(K,K)$. By usual homological arguments, if K is regarded as a left module, the Yoneda algebra is just $\mathrm{Hom}({}_{K}\mathrm{Tor}^{A}_{\bullet}(K,K), {}_{K}K)$, where $\mathrm{Tor}^{A}_{\bullet}(K,K)$ is regarded as a left module over K. Using the remaining right module structures, the previous dual is in fact a graded K-bimodule. Analogously, one also obtains a graded K-bimodule on $\mathrm{Hom}(\mathrm{Tor}^{A}_{\bullet}(K,K)_{K}, K_{K})$, where $\mathrm{Tor}^{A}_{\bullet}(K,K)$ is regarded as a right module over K.

As indicated in Subsection 2.1, if we only assume that K is semisimple, then both versions of the Yoneda algebra are not necessarily isomorphic (even as k-vector spaces), as the following simple example shows.

Example 2.12. *Let k be any field and let F be an infinite dimensional extension of k (for instance $k = \mathbb{Q}$ and F its algebraic closure). Define $K = k \times F$. Note that K is not separable, so a fortiori K^{e} is not semisimple. We denote by $p_1 : E \to k$ and $p_2 : E \to F$ the two canonical projections. Set V the K-bimodule given by the abelian group F provided with the actions $\lambda \cdot v \cdot \lambda' = p_1(\lambda).v.p_2(\lambda')$, for $v \in V$, $\lambda, \lambda' \in K$, where the last products are those of the field F. Let $A = TV$. In this case the left and right versions of the Yoneda algebra, $\mathrm{Ext}^{\bullet}_{A}({}_{A}K, {}_{A}K)$ and $\mathrm{Ext}^{\bullet}_{A}(K_{A}, K_{A})$, resp., only consist of two homogeneous components. By general arguments we have the following isomorphisms of K-bimodules $\mathrm{Ext}^{1}_{A}({}_{A}K, {}_{A}K) \simeq \mathrm{Hom}({}_{K}V, {}_{K}K) \simeq \mathrm{Hom}_{k}(F,k)$, where the K-bimodule structures appear if one uses the remaining K-module structures on the right. On the other hand, we obtain the chain of isomorphisms of K-bimodules $\mathrm{Ext}^{1}_{A}(K_{A}, K_{A}) \simeq \mathrm{Hom}(V_{K}, K_{K}) \simeq \mathrm{Hom}_{k}(k,F)$, where the K-bimodule structures follows from using the remaining K-module structures on the left. Since F is an infinite dimensional vector space over k, we know that there can be no k-linear isomorphism between F and its dual, so the left and right versions of the Yoneda algebras are not even isomorphic as k-vector spaces.*

Proposition 2.13. *Assume that K^{e} is semisimple. Let A be a nonnegatively graded connected algebra over K and locally finite dimensional over k. Then the Adams graded augmented A_∞-algebras $\mathrm{Ext}^{\bullet}_{A}({}_{A}K, {}_{A}K)$ and $\mathrm{Ext}^{\bullet}_{A}(K_{A}, K_{A})$ are quasi-isomorphic.*

Proof. It suffices to prove that the augmented A_∞-algebras given as the two duals $\mathrm{Hom}({}_{K}\mathrm{Tor}^{A}_{\bullet}(K,K), {}_{K}K)$ and $\mathrm{Hom}(\mathrm{Tor}^{A}_{\bullet}(K,K)_{K}, K_{K})$ of any fixed coaugmented A_∞-coalgebra structure on $\mathrm{Tor}^{A}_{\bullet}(K,K)$ are strictly isomorphic. This is a direct consequence of the following lemma, which is just an exercise in representation theory (*cf.* [1], §31, Exercise 7), but we provide the proof for completeness.

Lemma 2.14. *We assume the same hypotheses as in the proposition. We recall that $\mathscr{C}_K$ denotes the category of K-bimodules such that the action of k on both sides is symmetric. Then the functors $L, G : \mathscr{C}_K \to \mathscr{C}_K$ given by*

$$L(-) = \mathrm{Hom}\left({}_{K}(-), {}_{K}K\right) \text{ and } R(-) = \mathrm{Hom}\left((-)_{K}, K_{K}\right)$$

are naturally isomorphic.

Proof. Let $t : K \to k$ be the reduced trace of the separable extension K/k (see [5], §7D). By [5], Prop. (7.41), it induces an isomorphism $K \simeq \mathrm{Hom}_{k}(K,k)$ of K-bimodules (since the action of k is symmetric it does not matter on which side we impose the k-linearity). We have a natural isomorphism from L to R given by the composition of the natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_K({}_K M, {}_K K) &\simeq \mathrm{Hom}_K({}_K M, {}_K \mathrm{Hom}_k(K,k)) \simeq \mathrm{Hom}_k({}_k K \otimes_K M, {}_k k) \\ &\simeq \mathrm{Hom}_k({}_k M, {}_k k) = \mathrm{Hom}_k(M_k, k_k) \simeq \mathrm{Hom}_k(M \otimes_K K_k, k_k) \\ &\simeq \mathrm{Hom}_K(M_K, \mathrm{Hom}_k(K,k)_K) \simeq \mathrm{Hom}_K(M_K, K_K)\end{aligned}$$

in $\mathscr{C}_K$, where we have used the adjunction between the Hom and the tensor product and the obvious isomorphisms $K \otimes_K M \simeq M \simeq M \otimes_K K$. □

□

Remark 2.15. *Note that the proof of the previous lemma and of the preceding proposition hold under the weaker assumption that K is symmetric Frobenius.*

2.5 One-point extensions

Given three (possibly bigraded) K-bimodules X, Y, and Z, we shall consider the (correspondingly bigraded) bimodule structure over $K^2 = K \times K$ on $X \oplus Y \oplus Z$, which we shall denote by $\mathrm{LT}(X,Y,Z)$, defined by

$$(\lambda'_1, \lambda_1) \cdot (x,y,z) \cdot (\lambda'_2, \lambda_2) = (\lambda'_1.x.\lambda'_2, \lambda_1.y.\lambda'_2, \lambda_1.z.\lambda_2),$$

for all $\lambda'_1, \lambda_1, \lambda'_2, \lambda_2 \in K$, $x \in X$, $y \in Y$ and $z \in Z$.

Given a nonnegatively graded connected algebra A over K and a positively graded right A-module M, the *one-point extension* $A[M]$ is the graded algebra over $K^2 = K \times K$ whose underlying graded bimodule structure over K^2 is given by $\mathrm{LT}(A, M, K)$, where the last copy of K is concentrated in degree zero. The product is given by $(a, m, \lambda) \cdot (a', m', \lambda') = (a.a', m.a' + \lambda.m', \lambda.\lambda')$, for all $a, a' \in A$, $m, m' \in M$, and $\lambda, \lambda' \in K$. The hypothesis that M is positively graded is only needed to assure that $A[M]$ is still nonnegatively graded and connected.

Dually, we will need to introduce a (generalized) dual version of one-point extensions. Let $(C, \Delta^C_\bullet)$ be an Adams graded coaugmented A_∞-coalgebra over K and N be a right A_∞-comodule over C provided with an Adams grading, *i.e.* N is a bigraded module over K together with morphisms $\Delta^N_n : N \to N \otimes C^{\otimes(n-1)}$, for all $n \in \mathbb{N}$, of degree homological degree $n-2$ and Adams degree zero such that the Stasheff identity $\mathrm{SI}(n)$

$$\sum_{(r,s,t)\in\mathcal{I}_{n-1}} (-1)^{rs+s+t}(\mathrm{id}_N \otimes \mathrm{id}_C^{\otimes r} \otimes \Delta^C_s \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta^N_{r+t+2} + \sum_{t=0}^{n-1} (-1)^t (\Delta^N_{n-t} \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta^N_{1+t} = 0 \quad (2.8)$$

holds for all $n \in \mathbb{N}$, where $\mathcal{I}_m = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r+s+t = m\}$. We also impose that $(\mathrm{id}_N \otimes \epsilon_C) \circ \Delta^N_2 = \mathrm{id}_N$, and $(\mathrm{id}_N \otimes \mathrm{id}_C^{\otimes i} \otimes \epsilon_C \otimes \mathrm{id}_C^{\otimes(n-2-i)}) \circ \Delta^N_n = 0$, for all $n \geq 3$ and $i \in \{0, \dots, n-2\}$, where ϵ_C is the counit of C.

The *one-point extension* $C[N]$ is the Adams graded coaugmented A_∞-coalgebra over K^2 whose underlying graded K^2-bimodule structure is given by $\mathrm{LT}(C, N, K)$, where the last summand lies in homological and Adams degree 0. The strict counit is given by $(c, x, \lambda) \mapsto (\epsilon_C(c), \lambda)$, for ϵ_C the counit of C, and the coaugmentation $K^2 \to C[N]$ is defined as $(\lambda, \lambda') \mapsto (\eta_C(\lambda), 0, \lambda')$, where η_C denotes the coaugmentation of C. Note that we have the obvious isomorphism

$$C[N]^{\otimes_{K^2} n} \simeq \mathrm{LT}\Big(C^{\otimes n}, \big(\bigoplus_{i=0}^{n-1} N \otimes C^{\otimes i}\big), K\Big) \quad (2.9)$$

of graded K^2-bimodules, for all $n \in \mathbb{N}$. Furthermore, we can regard K as an A_∞-coalgebra with $\Delta_n^K = 0$ for $n \neq 2$ and $\Delta_2^K(1_K) = 1_K \otimes 1_K$. The A_∞-coalgebra structure of the one-point extension is $\Delta_n^{C[N]}(c, x, \lambda) = (\Delta_n^C(c), \Delta_n^N(x), \Delta_n^K(\lambda))$, by means of (2.9), for all $n \in \mathbb{N}$.

The following result is a direct consequence of the dual Merkulov construction in Theorem 2.8, the definition of A_∞-comodule and of one-point extension for A_∞-coalgebras.

Proposition 2.16. *Let D be an Adams graded coaugmented dg coalgebra and U be a right dg comodule over D provided with an Adams grading. Then, the underlying bigraded K^2-bimodule of the homology $H(D[U])$ of the one-point extension $D[U]$ is canonically isomorphic to $\mathrm{LT}(H(D), H(U), K)$. Moreover, a Merkulov model on $H(D[U])$ naturally induces a structure of Adams graded coaugmented A_∞-coalgebra on $H(D)$, which is also a model, and a structure of right A_∞-comodule over $H(D)$ on $H(U)$ provided with an Adams grading, such that the isomorphism between $H(D[U])$ and $H(D)[H(U)]$ mentioned previously is of Adams graded coaugmented A_∞-coalgebras.*

Remark 2.17. *Note that the previous proposition states precisely that, if one can obtain an A_∞-coalgebra structure on the one-point extension $D[U]$ of D by U, we can simultaneously apply the Merkulov construction to the quotients $H(D)$ and $H(U)$ of D and U, respectively, in such a manner that $H(U)$ is an A_∞-comodule over the coaugmented A_∞-coalgebra $H(D)$. Moreover, $H(D)$ and $H(U)$ are quasi-isomorphic to D and U, respectively.*

On the other hand, the next statement follows directly from the definition of bar construction (see for instance [16], Section 2.3.3, for the definition of $B^+(M, A)$).

Proposition 2.18. *Let A be a nonnegatively graded connected algebra A over the separable k-algebra K and let M be a positively graded right A-module, both locally finite dimensional over k. Then the (reduced) cobar construction $B^+(A[M])$ of the one-point extension K^2-algebra $A[M]$ is isomorphic to the one-point extension $B^+(A)[B^+(M, A)]$, where $B^+(M, A)$ denotes the bar construction of the right A-module M, which is canonically a (cofree) right dg comodule over the coaugmented dg coalgebra $B^+(A)$.*

To avoid any ambiguity we remark that the underlying bigraded module of $B^+(M, A)$ is $M[1] \otimes B^+(A)$, and not $M \otimes B^+(A)$, as in [17], Section 8. Our sign convention is the same as the one explained in [13], Subsection 2.2. In particular, note that $H_\bullet(B^+(M, A)) = \mathrm{Tor}^A_{\bullet-1}(M, K)$.

3 The A_∞-comodule structure of the Tor and the A_∞-module structure of the Ext

We need the following simple definition. If $(U, \Delta_\bullet^U)$ is a right A_∞-comodule over an A_∞-coalgebra D and $U[\ell]$ denotes the cohomological shift of U by some $\ell \in \mathbb{Z}$, *i.e.* $U[\ell]_n = U_{n-\ell}$, where we are considering only the homological degree, then $U[\ell]$ is also a right A_∞-comodule over D with the comultiplications $\Delta_n^{U[\ell]} : U[\ell] \to U[\ell] \otimes D^{\otimes(n-1)}$ given by $\Delta_n^{U[\ell]}(s(u)) = (-1)^{n\ell}(s \otimes \mathrm{id}_D^{\otimes(n-1)}) \circ \Delta_n^U(u)$, for all $u \in U$, where $s : U \to U[\ell]$ denotes the shift morphism whose underlying set theoretic map is the identity.

Let M be a bounded below graded right module over a nonnegatively graded algebra A, where both are supposed to be locally finite dimensional graded k-vector spaces. By combining Propositions 2.16 and 2.18 we get the structure of A_∞-comodule over $H(B^+(A))$ on $H(B^+(M, A))$. Since $H(B^+(A))$ and $H(B^+(M, A))$

compute $\mathrm{Tor}^A_\bullet(K,K)$ and $\mathrm{Tor}^A_\bullet(M,K)[1]$, resp., we get a right A_∞-comodule structure on $\mathrm{Tor}^A_\bullet(M,K)$ over $\mathrm{Tor}^A_\bullet(K,K)$. Furthermore, by the obvious isomorphism (*cf.* for instance [22], Chapter 1, Section 1)

$$\mathrm{Ext}^\bullet_A(M,K) = \mathrm{Hom}\big({}_K\,\mathrm{Tor}^A_\bullet(M,K), {}_K K\big), \tag{3.1}$$

the dual of the previous A_∞-comodule on $\mathrm{Tor}^A_\bullet(M,K)$ gives an A_∞-module structure on $\mathrm{Ext}^\bullet_A(M,K)$ over the augmented A_∞-algebra $\mathrm{Ext}^\bullet_A(K,K)$, which is the graded dual of $\mathrm{Tor}^A_\bullet(K,K)$. There is an analogous statement for the A_∞-module $\mathcal{E}xt^\bullet_A(M,K)$ over the Adams graded augmented A_∞-algebra $\mathcal{E}xt^\bullet_A(K,K)$.

By means of the projection $\pi : TV \to A$ we see that M is also a right TV-module. Let $W \subseteq M$ be a space of generators of M, *i.e.* a graded submodule W satisfying that the composition of the canonical inclusion together with the canonical projection $M \to M/(M.A_+)$ is an isomorphism. Note that W is locally finite dimensional. We thus get a surjective morphism $\pi^M : W \otimes TV \to M$ of right modules over TV. Moreover we suppose that $S \subseteq \mathrm{Ker}(\pi^M)$ is a space of relations of M, *i.e.* a graded submodule satisfying that the composition of the inclusion $S \subseteq \mathrm{Ker}(\pi^M)$ together with the projection

$$\mathrm{Ker}(\pi^M) \to \mathrm{Ker}(\pi^M)/\big(\mathrm{Ker}(\pi^M)\cdot TV_+\big)$$

is an isomorphism. It is easy to see that S generates the right TV-module $\mathrm{Ker}(\pi^M)$. All the previous definitions are in fact easily obtained from the one-point extension algebra $A[M]$. Indeed, the space of generators of the latter is precisely given by $\mathrm{LT}(V,W,0) \subseteq A[M]$, and the space of relations is exactly the K^2-subbimodule $\mathrm{LT}(R,S,0)$ of the tensor algebra over K^2 generated by $\mathrm{LT}(V,W,0)$.

Let s be a good section of $\pi : TV \to A$. We shall say that a section $s' : M \to W \otimes TV$ is *good (on the right, with respect to s)* if the induced section of $T_{K^2}(\mathrm{LT}(V,W,0)) \to A[M]$ is good. Analogously, if R is compatible with s, we shall say that S is *compatible* with s' if $\mathrm{LT}(R,S,0)$ is compatible with the section of $T_{K^2}(\mathrm{LT}(V,W,0)) \to A[M]$ induced by s and s'.

The following result is a generalization of Theorem 2.11 to the case of the A_∞-comodules $\mathrm{Tor}^A_\bullet(M,K)$.

Theorem 3.1. *Let $A = TV/(R)$ be a nonnegatively graded connected K-algebra, and let $M = (W \otimes TV)/(S)$ be a bounded below graded right A-module, both locally finite dimensional over k and considered to be concentrated in cohomological degree zero and whose Adams grading is the bounded below one. Suppose we have chosen good sections s and s' of $TV \to A$ and $W \otimes TV \to M$, respectively, and that R and S are compatible with s and s', respectively. We utilize the model on $D = \mathrm{Tor}^A_\bullet(K,K)$ indicated in Theorem 2.11. Then, there exists a structure of A_∞-comodule on $N = \mathrm{Tor}^A_\bullet(M,K)$ over C, such that, if $\mathfrak{p} : D \to V$ and $\mathfrak{p}' : N \to W$ denote the canonical projections, then, for all $n \geq 2$, $(\mathfrak{p}' \otimes \mathfrak{p}^{\otimes(n-1)}) \circ \Delta^N_n|_S : S \to W \otimes V^{\otimes(n-1)}$ coincides with $(-1)^n$ times*

$$S \to W \otimes (TV)_+ \to W \otimes V^{\otimes(n-1)},$$

where the first map is the canonical inclusion and the second one is the standard projection, and where we recall that W and S are canonically identified with $\mathrm{Tor}^A_0(M,K)$ and $\mathrm{Tor}^A_1(M,K)$, respectively.

Proof. Since M is bounded below, there is an Adams degree shift $M[p]$ of M such that $M[p]$ is positively graded, where we recall that $M[p]^m = M^{p+m}$, for all $m \in \mathbb{Z}$. Note that, if $M' = M[p]$, we have that $\mathrm{Tor}^A_\bullet(M,K) = \mathrm{Tor}^A_\bullet(M',K)[-p]$, where we have shifted only the Adams degree but the homological one remains unchanged, and the A_∞-comodule structures of both groups are precisely the same. The result now follows by Theorem 2.11 applied to the one-point extension $A[M']$ together with Propositions 2.16 and 2.18. □

Remark 3.2. *By taking the (resp., bigraded) graded dual of the last map indicated in the theorem (see* (3.1)*) we get a partial description of the structure of the right A_∞-module of $\mathrm{Ext}^\bullet_A(M,K)$ (resp., $\mathcal{E}xt^\bullet_A(M,K)$) over the (resp., Adams graded) augmented A_∞-algebra $\mathrm{Ext}^\bullet_A(K,K)$ (resp., $\mathcal{E}xt^\bullet_A(K,K)$). This result does not seem to have been observed so far.*

4 On a generalization of a theorem of Keller

We recall the following theorem, that was announced by B. Keller at the X ICRA of Toronto, Canada, in 2002. For the definitions and notation used we refer the reader to [13], Thm. 4.2.

Theorem 4.1. *Let C be a minimal (i.e. $\Delta_1 = 0$) coaugmented A_∞-coalgebra and A be a nonnegatively (Adams) graded connected algebra, which we regard in zero (co)homological degree, locally finite dimensional over k. Then, the following are equivalent:*

(i) There is a quasi-isomorphism of minimal coaugmented A_∞-coalgebras

$$C \to \mathrm{Tor}^A_\bullet(K,K).$$

(ii) There is a twisting cochain $\tau : C \to A$ such that the twisted tensor product $C \otimes_\tau A_{\epsilon_A}$ is a minimal projective resolution of the trivial right A-module K, where A_{ϵ_A} denotes the A-bimodule structure on A with the action induced by the augmentation ϵ_A of A on the right and with the standard action on the left.

Proof. The proof given in [9], Thm. 4.7, extends straightforward in this case where K is not necessarily a field. We have introduced a slight change in the presentation, since our first item is in principle different from the first item of the mentioned theorem. To see that the item (i) in this article is equivalent to item (i) of [9], Thm. 4.7, we only need to observe that there is a quasi-isomorphism $C \to \mathrm{Tor}^A_\bullet(K,K)$ of Adams graded coaugmented A_∞-coalgebras if and only there is a quasi-isomorphism of Adams graded augmented A_∞-algebras $\mathcal{E}xt^\bullet_A(K,K) \to C^\#$, given by taking the corresponding bigraded dual, and the theorem follows. $\square$

We can in fact extend the previous theorem to include right modules. Given a twisting cochain from the A_∞-coalgebra to the algebra, the definition of twisted tensor product of an A_∞-comodule over an A_∞-coalgebra and an algebra is analogous to the one recalled in [13], Subsection 3.4, and can also be deduced by considering one-point extensions.

Theorem 4.2. *Let A be a nonnegatively graded connected K-algebra, and let M be a bounded below graded right A-module, both considered to be concentrated in cohomological degree zero and whose Adams grading is the remaining one. We assume that both of them are locally finite dimensional over k. Let C be the minimal coaugmented A_∞-coalgebra $\mathrm{Tor}^A_\bullet(K,K)$ and let W be a minimal (i.e. $\Delta^W_1 = 0$) A_∞-comodule over C. We denote τ the associated twisting cochain given by the previous theorem. Then the following are equivalent:*

(i) There is a quasi-isomorphism of right A_∞-comodules

$$W \to \mathrm{Tor}^A_\bullet(M,K)$$

over $\mathrm{Tor}^A_\bullet(K,K)$.

(ii) The twisted tensor product $W \otimes_\tau A_{\epsilon_A}$ is a minimal projective resolution of the right A-module M, where A_{ϵ_A} denotes the A-bimodule structure on A with the action induced by the augmentation ϵ_A of A on the right and with the standard action on the left.

(iii) There is a quasi-isomorphism of right A_∞-modules

$$\mathcal{E}xt_A^\bullet(M,K) \to W^\#$$

over $\mathcal{E}xt_A^\bullet(K,K)$.

Proof. As explained in the proof of Theorem 3.1, by taking an adequate shift on the Adams degree of M, we may assume without loss of generality that M is positively graded, since conditions (i), (ii) and (iii) are compatible with shifts on the Adams degree. We suppose thus that M is positively graded. We shall prove that (i) is equivalent to (ii), for the equivalence between (i) and (iii) follows directly by taking the bigraded dual functor $(-)^\#$. By Proposition 2.18, the condition (i) is equivalent to the fact that the A_∞-coalgebra $C[(W[1])]$ is quasi-isomorphic to $\mathrm{Tor}_\bullet^{A[M]}(K^2,K^2)$. By Theorem 4.1, the last statement is tantamount to the fact that $C[(W[1])] \otimes_{\hat{\tau}} A[M]_{\epsilon_{A[M]}}$ is a minimal projective resolution of K^2, for some twisting cochain $\hat{\tau} : C[(W[1])] \to A[M]$. By grading considerations, any such twisting cochain $\hat{\tau}$ is given by $(c, s(w), \lambda) \mapsto (\tau_1(c), \tau_2(s(w)), 0)$, for $\tau_1 : C \to A$ a twisting cochain and a map $\tau_2 : W[1] \to M$ of cohomological degree 1. By the previous theorem, changing the quasi-isomorphism between $C[(W[1])]$ and $\mathrm{Tor}_\bullet^{A[M]}(K^2,K^2)$ if needed, we may assume without loss of generality that $\tau_1 = \tau$. The only possible nonzero component of τ_2 is thus the restriction $(\tau_2)|_{W_0}$, where W_0 denotes the component of W of homological degree zero. The fact that $C[(W[1])] \otimes_{\hat{\tau}} A[M]_{\epsilon_{A[M]}}$ is a minimal projective resolution of K^2 is tantamount to the fact that $C \otimes_\tau A_{\epsilon_A}$ is a minimal projective resolution of K and the map $W \otimes_\tau A_{\epsilon_A} \to M$ of complexes of modules given by $w \otimes a \mapsto \tau_2(w).a$ is a quasi-isomorphism. We obtain thus the condition (ii) of the statement. By reversing the last steps, it is also clear that the condition (ii) implies the existence of a twisting cochain $\hat{\tau} : C[(W[1])] \to A[M]$, so it implies (i). The theorem follows. □

5 Application: Tor and Ext of generalized Koszul modules over generalized Koszul algebras

In this section we set $K = k$ and suppose that V is a finite dimensional vector space over k. Nothing prevents us from working with a general separable k-algebra K, for which the results are also true, but we have decided to consider the case K is the base field in order to simplify our exposition and because the results we shall refer to are originally formulated over a field. Let $A = TV/(R)$ be an s-homogeneous algebra for $s \in \mathbb{N}_{\geq 2}$, *i.e.* $R \subseteq V^{\otimes s}$. We say that a finitely generated graded right A-module M is called *generalized Koszul in the restricted sense* (or *restricted s-Koszul*, if we want to emphasize the degree of R) if its minimal projective resolution $P_\bullet$ satisfies that P_n is (a graded free module) generated in degree $\phi_s(n)$, for all $n \in \mathbb{N}_0$, where $\phi_s(2m) = sm$ and $\phi_s(2m+1) = sm+1$, for all $m \in \mathbb{N}_0$. More generally, M is called *generalized Koszul* (or *s-Koszul*) if there is shift $M[p]$ of the (Adams) grading of M such that $M[p]$ is restricted s-Koszul. Note that the previous terminology is not completely standard, for several authors impose that s-Koszul modules are necessarily generated in degree zero (so, they are what we called restricted s-Koszul). Since a shift of the grading of a module over A does not change its homology or the algebraic structures on it, we believe that these shift should also be allowed in the definitions (see Proposition 5.1). We recall that A is said to be *generalized Koszul* (or *s-Koszul*) if the trivial right A-module k is so.

We shall now apply Theorem 4.2 to obtain an explicit A_∞-module structure on $\mathrm{Ext}_A^\bullet(M,k)$ over the A_∞-algebra $\mathrm{Ext}_A^\bullet(k,k)$, for A an s-Koszul algebra and M an s-Koszul module. Since it is well-known that the higher products vanish in the case $s = 2$, we suppose from now on that $s > 2$ (even though the case $s = 2$ can also be treated with the techniques explained here). Let $A^! = T(V^*)/(R^\perp)$ be the homogeneous dual of A, where $R^\perp \subseteq (V^*)^{\otimes s} \simeq (V^{\otimes s})^*$ is the annihilator of R and we use the isomorphism $(V^*)^{\otimes n} \simeq (V^{\otimes n})^*$ given by sending $f_1 \otimes \cdots \otimes f_n \in (V^*)^{\otimes n}$ to the functional whose value at $v_1 \otimes \cdots \otimes v_n$ is $(-1)^{n(n-1)/2} f_1(v_1) \dots f_n(v_n)$. It is well-known that the fact that A is generalized Koszul implies that $\mathrm{Ext}_A^m(k,k) = A^!_{\phi_s(m)}$, for all $m \in \mathbb{N}_0$. Moreover, by [8], Thm. 6.5, the A_∞-algebra structure on the latter is also known. Of course, one can also prove that the A_∞-algebra structure on $\mathrm{Ext}_A^\bullet(k,k)$ stated by that theorem is a model on that space by means of Theorem 4.1. Indeed, the Koszul property of A implies that $C = \mathrm{Tor}^A_\bullet(k,k)$ satisfies that

$$C_p = \mathrm{Tor}^A_p(k,k) = \bigcap_{i=0}^{\phi_s(p-2)} V^{\otimes i} \otimes R \otimes V^{\otimes \left(\phi_s(p-2)-i\right)} \tag{5.1}$$

for $p \geq 2$, together with $C_0 = \mathrm{Tor}^A_0(k,k) = k$ and $C_1 = \mathrm{Tor}^A_1(k,k) = V$ (see [2], eq. (2.5)). We give to C the following A_∞-coalgebra structure. There are only two nonvanishing comultiplications, Δ_2 and Δ_s, which satisfy that

(i) $(\mathfrak{p}_{p_1} \otimes \mathfrak{p}_{p_2}) \circ \Delta_2|_{C_p}$ is the canonical inclusion if $p_1 + p_2 = p$ and $\phi_s(p_1) + \phi_s(p_2) = \phi_s(p)$ for $p_1, p_2, p \in \mathbb{N}_0$, and zero else;

(ii) $(\mathfrak{p}_{p_1} \otimes \cdots \otimes \mathfrak{p}_{p_s}) \circ \Delta_s|_{C_p}$ is the canonical inclusion if $p_1 + \cdots + p_s = p + s - 2$ and $\phi_s(p_1) + \cdots + \phi_s(p_s) = \phi_s(p)$ for $p_1, \dots, p_s, p \in \mathbb{N}$, and zero otherwise;

where $\mathfrak{p}_m : C \to C_m$ denotes the canonical projection. Note that the nonvanishing statement of item (i) implies that either p_1 or p_2 is even, whereas in the case of item (ii) it implies that $p_1, \dots, p_s$ are odd (and p even). It is now trivial to verify $C \otimes_\tau A_{\epsilon_A}$ is the minimal projective (Koszul) resolution of k_A, where $\tau : C \to A$ is the twisting cochain given by the composition of the canonical projection $C \to V$ together with minus the canonical inclusion. Finally, the A_∞-algebra structure on $\mathrm{Ext}_A^\bullet(k,k) = A^!_{\phi_s(\bullet)}$ given as the bigraded dual to the A_∞-coalgebra C (using the convention given in [13], Subsection 2.3 and the previously explained identifications) is precisely the one given in [8], Thm. 6.5.

Let $M = (W \otimes TV)/(S)$ be a graded right A-module that is generalized Koszul (so $S \subseteq W \otimes V$). Let us define $M^! = (W^* \otimes T(V^*))/(S^\perp)$, where $S^\perp \subseteq W^* \otimes V^* \simeq (W \otimes V)^*$ is the annihilator of S and the isomorphism used is the same as before. It is clear that $M^!$ is a right module over $A^!$, and it is a well-known fact that, if M is s-Koszul, then $\mathrm{Ext}_A^m(M,k) = M^!_{\phi_s(m)}$, for all $m \in \mathbb{N}_0$, and $N = \mathrm{Tor}^A_\bullet(M,k)$ satisfies that

$$N_p = \mathrm{Tor}^A_p(M,k) = \Big(\bigcap_{i=0}^{\phi_s(p-2)} W \otimes V^{\otimes i} \otimes R \otimes V^{\otimes \left(\phi_s(p-2)-i\right)} \Big) \cap \Big(S \otimes V^{\otimes \left(\phi_s(p)-1\right)} \Big) \tag{5.2}$$

for $p \geq 2$, together with $N_0 = \mathrm{Tor}^A_0(M,k) = W$ and $N_1 = \mathrm{Tor}^A_1(M,k) = S$ (this follows from a recursive argument on p, when describing the minimal projective resolution of M and imposing at the same time that M is s-Koszul; *cf.* [2], Section 2, for the case $M = k$).

The following result can be regarded as an extension of [8], Thm. 6.5.

Proposition 5.1. *Let A be an s-Koszul algebra for $s > 2$ and M a right s-Koszul module over A. Assume that $\mathrm{Ext}_A^\bullet(k,k) = A^!_{\phi_s(\bullet)}$ has the A_∞-algebra structure given in* [8], *Thm. 6.5. Then, the corresponding structure of right A_∞-module on $\mathrm{Ext}_A^\bullet(M,k) = M^!_{\phi_s(\bullet)}$ over $\mathrm{Ext}_A^\bullet(k,k)$ is given by*

$$m_2(f,g) = \begin{cases} f.g, & \text{if the homological degree of either } f \text{ or } g \text{ is even,} \\ 0, & \text{else,} \end{cases}$$

where $f \in M^!_{\phi_s(\bullet)}$, $g \in A^!_{\phi_s(\bullet)}$ and $f.g$ denotes the action of g on f, and

$$m_s(f, g_1, \dots, g_{s-1}) = \begin{cases} f.g_1 \dots g_{s-1}, & \text{if the homological degrees of} \\ & f \text{ and of } g_1, \dots, g_{s-1} \text{ are odd,} \\ 0, & \text{else,} \end{cases}$$

where $f \in M^!_{\phi_s(\bullet)}$ and $g_1, \dots, g_{s-1} \in A^!_{\phi_s(\bullet)}$. The other higher multiplications vanish.

Proof. Define the following right A_∞-comodule structure over $C = \mathrm{Tor}^A_\bullet(k,k)$ on $N = \mathrm{Tor}^A_\bullet(M,k)$ with only two nonvanishing comultiplications, Δ^N_2 and Δ^N_s, which satisfy that

(a) $(\mathfrak{p}'_{p_1} \otimes \mathfrak{p}_{p_2}) \circ \Delta^N_2|_{N_p}$ is the canonical inclusion if $p_1 + p_2 = p$ and $\phi_s(p_1) + \phi_s(p_2) = \phi_s(p)$ for $p_1, p_2, p \in \mathbb{N}_0$, and zero otherwise;

(b) $(\mathfrak{p}'_{p_1} \otimes \mathfrak{p}_{p_2} \otimes \cdots \otimes \mathfrak{p}_{p_s}) \circ \Delta^N_s|_{N_p}$ is the canonical inclusion if $p_1 + \cdots + p_s = p + s - 2$ and $\phi_s(p_1) + \cdots + \phi_s(p_s) = \phi_s(p)$ for $p_1, \dots, p_s, p \in \mathbb{N}$, and zero else;

where $\mathfrak{p}_m : C \to C_m$ and $\mathfrak{p}'_m : N \to N_m$ denote the canonical projections, and we have used (5.1) and (5.2). It is trivial to verify that the bigraded dual of N is just $M^!_{\phi_s(\bullet)}$ (as bigraded vector spaces) and that the induced right A_∞-module over the A_∞-algebra $A^!_{\phi_s(\bullet)}$ described in [8], Thm. 6.5 (see for instance [13], Subsection 4.2, for the explicit sign convention we use) is precisely the one stated in the proposition. Using the twisting cochain $\tau : C \to A$ described previously as the composition of the canonical projection $C \to V$ together with minus the canonical inclusion, we can form the twisted tensor product $W \otimes_\tau A$. A direct computation shows that $W \otimes_\tau A$ coincides with the minimal projective resolution of M, and the statement follows from Theorem 4.2. $\square$

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