A simple note on the Yoneda (co)algebra of a monomial algebra

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Abstract

If $A = TV/\langle R\rangle$ is a monomial K-algebra, it is well-known that $\mathrm{Tor}_p^A(K,K)$ is isomorphic to the space $V^{(p-1)}$ of (Anick) $(p-1)$-chains for $p \geq 1$. The goal of this short note is to show that the next result follows directly from well-established theorems on A_∞-algebras, without computations: there is an A_∞-coalgebra model on $\mathrm{Tor}_\bullet^A(K,K)$ satisfying that, for $n \geq 3$ and $c \in V^{(p)}$, $\Delta_n(c)$ is a linear combination of $c_1 \otimes \cdots \otimes c_n$, where $c_i \in V^{(p_i)}$, $p_1 + \cdots + p_n = p-1$ and $c_1 \dots c_n = c$. The proof follows essentially from noticing that the Merkulov procedure is compatible with an extra grading over a suitable category. By a simple argument based on a result by Keller we immediately deduce that some of these coefficients are ± 1.

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1 The results

This article arose from discussions with A. Solotar and M. Suárez-Álvarez in 2014, and more recently with V. Dotsenko and P. Tamaroff, on the A_∞-algebra structure on the Yoneda algebra of a monomial algebra. I want to thank them for the exchange and in particular the last two for lately renewing my interest in the problem. My aim is to explain some results describing such A_∞-algebras that do not seem to be well-known, but follow rather easily from the general theory, and were meant to be included in the Master thesis of my former student E. Sérandon in 2016.

In what follows, K will denote a finite product of r copies of a field k. By *module* we will mean a (not necessarily symmetric) bimodule over K (see [3], Section 2). All unadorned tensor products $\otimes$ will be over K, unless otherwise stated. For the conventions on A_∞-(co)algebras we refer the reader to [5], Subsection 2.1.

Let M be a small category with a finite set of objects $\{o_1, \dots, o_r\}$. As usual, we denote the set of all arrows of M by M itself, the composition by $\star$, and the identity of o_i by e_i. We remark that $m' \star m''$ implies that m' and m'' are composable morphisms. Let ${}^M\mathrm{Mod}$ be the category of modules V provided with an M-grading (*i.e.* a decomposition of modules $V = \oplus_{m\in M} V_m$) and linear morphisms preserving the degree. This is a monoidal category with the tensor product $V \otimes W$ whose m-th homogeneous component is $\oplus_{m'\star m''=m} V_{m'} \otimes W_{m''}$, and the unit $K = \oplus_{i=1}^r k_{e_i}$, where $e_j.k_{e_i} = k_{e_i}.e_j = \delta_{i,j} k_{e_i}$. Furthermore, it is easy to see that ${}^M\mathrm{Mod}$ is a semisimple category. We say that a unitary A_∞-algebra $(A, m_\bullet)$ *has an M-grading* if $(A, m_\bullet)$ is a unitary A_∞-algebra in the monoidal category ${}^M\mathrm{Mod}$. The same applies to M-graded augmented A_∞-algebras, and to morphisms of M-graded unitary or augmented A_∞-algebras. Moreover, the definitions of M-graded counitary and coaugmented A_∞-coalgebra as well as the morphisms between them are also clear.

Proposition 1.1. *Let $A = TV/\langle R\rangle$ is a monomial algebra over a field k, i.e. V is a module of finite dimension over k and R is a space of relations of monomial type. Then, there is a small category $(M, \star)$ with r objects such that A is an M-graded unitary algebra with $\dim_k(A_m) \leq 1$, for all $m \in M$.*

Proof. Let $\mathcal{B}$ be a basis of the underlying vector space of V such that $e_j.v.e_i$ vanishes or it is v, for all $v \in \mathcal{B}$ and all $i, j \in \{1, \dots, r\}$, and define M as the free small category generated by $\mathcal{B}$. Note that TV identifies with the unitary semigroup algebra associated with M. Given $m \in M$, set A_m as the vector subspace of A generated by the element $\bar{m}$ of A given as the image of $m \in TV$ under $TV \to A$. It is clear that $A = \oplus_{m\in M} A_m$ is an M-grading of A and $\dim_k(A_m) \leq 1$, for all $m \in M$. □

The next result follows directly from the definition of the bar construction.

Fact 1.2. *If A is an augmented A_∞-algebra over K with an M-grading, then the coaugmented dg coalgebra $B^+(A)$ given by the bar construction is M-graded for the canonically induced grading.*

We present now the main result of this short note.

Theorem 1.3. *Let $A = TV/\langle R\rangle$ be a monomial K-algebra and let M be the small category defined in Proposition 1.1. Then, there is an M-graded coaugmented A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(K, K)$ together with a quasi-equivalence from it to the M-graded coaugmented dg coalgebra $B^+(A)$.*

Proof. We first remark that [4], Thm. 4.5, holds *verbatim* if we replace Adams grading by M-grading, since ${}^M\mathrm{Mod}$ is a semisimple category. Using a grading argument based on the fact that both $B^+(A)$ and $\mathrm{Tor}^A_\bullet(K, K)$ are Adams connected modules (see [5], Section 2, for the definition for vector spaces), we see that the operator Q in [4], Thm. 4.5, is locally finite (see [3], Addendum 2.9). Hence, applying [4], Thm. 4.5, to the coaugmented dg coalgebra $B^+(A)$, which projects onto its homology $\mathrm{Tor}^A_\bullet(K, K)$, we see that the latter has a structure of M-graded coaugmented A_∞-coalgebra. Moreover, by the same theorem, there is a quasi-isomorphism of coaugmented A_∞-coalgebras from $B^+(A)$ to $\mathrm{Tor}^A_\bullet(K, K)$, which is trivially a quasi-equivalence by a grading argument. □

Remark 1.4. *The previous theorem and its proof hold more generally for any M-graded K-algebra A that is* connected, *i.e. $A_{e_i} = k$ for all $i \in \{1, \dots, r\}$, and such that A/K has a compatible (strictly) positive grading. This occurs e.g. if there is a functor $\ell : M \to \mathbb{N}_0$ such that $\ell(m) = 0$ if and only if m is an identity of M, where the monoid $\mathbb{N}_0$ is regarded as a category with one object.*

The result in the abstract is obtained from the previous theorem by identifying $\mathrm{Tor}^A_p(K, K)$ with the module $V^{(p-1)}$ generated by the (Anick) $(p-1)$-chains for $p \geq 1$ (see [1], Lemma 3.3, for the case K is a field, and [2], Thm 4.1, for the general case), *i.e.* given $c \in V^{(p)}$ and $n \geq 3$,

$$\Delta_n(c) = \sum_{\substack{c_i \in V^{(p_i)},\, c_1 \dots c_n = c \\ p_i \in \mathbb{N}_0,\, p_1 + \dots + p_n = p-1}} \lambda_{(c_1 \otimes \dots \otimes c_n)} c_1 \otimes \dots \otimes c_n, \text{ where } \lambda_{(c_1 \otimes \dots \otimes c_n)} \in k. \tag{1.1}$$

Note that Δ_2 is given by the usual coproduct of $\mathrm{Tor}^A_\bullet(K, K)$. The (left or right) dual of this A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(K, K)$ gives an A_∞-algebra model on $\mathrm{Ext}^\bullet_A(K, K)$ (see [3], Prop. 2.13).

With no extra effort we can say a little more about the coefficients in (1.1) [1].

1. P. Tamaroff has told me that, by carefully choosing the SDR data for $B^+(A)$ and following all the steps in the recursive Merkulov procedure, he can even prove that all nonzero coefficients are ± 1, at least if K is a field (see [7]). Our results are not so general but they are immediate, since we did not need to look at the interior of the Merkulov construction.

Theorem 1.5. *Assume the same hypotheses as in the previous theorem. Given* $c \in V^{(p)}$, $n \geq 3$, *and* $c_i \in V^{(p_i)}$ $(p_i \in \mathbb{N}_0)$ *such that* $c_1 \dots c_n = c$, $p_1 + \dots + p_n = p - 1$ *and* $p = p_j + 1$ *for some* $j \in \{1, \dots, n\}$, *then* $\lambda_{(c_1 \otimes \dots \otimes c_n)} = \pm 1$.

Proof. By [5], Thm. 4.2, (or [3], Thm. 4.1) the twisted tensor product $A^e \otimes_\tau C$ is isomorphic to the minimal projective resolution of the regular A-bimodule A, where $C = \mathrm{Tor}^A_\bullet(K, K)$ is the previous coaugmented A_∞-algebra and τ is the twisting cochain given in that theorem. Comparing the differential of $A^e \otimes_\tau C$ given in [5], (4.1), with the one in [2], Thm. 4.1, (see also [6], Section 3), it follows that the mentioned coefficient is ± 1. □

Remark 1.6. *In the examples, the computation of the remaining coefficients in* (1.1) *is in general rather simple to carry out, by imposing that the Stasheff identities are fulfilled.*

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