ARITHMETICITY OF THE COUWENBERG-HECKMAN-LOOIJENGA LATTICES

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Abstract. We study the arithmeticity of the Couwenberg-Heckman-Looijenga lattices in $PU(n,1)$, and show that they contain a non-arithmetic lattice in $PU(3,1)$ which is not commensurable to the non-arithmetic Deligne-Mostow lattice in $PU(3,1)$. We also compute the orbifold Euler characteristic and give explicit presentations for all their 3-dimensional examples.

1. Introduction

Lattices in the isometry groups of most symmetric spaces of non-compact type are arithmetic, due to celebrated superrigidity results by Margulis (symmetric spaces of higher rank), Corlette, and Gromov-Schoen (quaternionic hyperbolic spaces, and the octonionic hyperbolic plane).

For small values of $n$, it is fairly easy to construct non-arithmetic lattices in $SO(n,1)$ by using Coxeter polyhedra (there is a simple criterion to determine the arithmeticity of these groups). For $n$ large enough, there are no Coxeter polytopes in $H^n_R$, but there are non-arithmetic lattices in $SO(n,1)$ for arbitrary $n$ by a beautiful construction due to Gromov and Piatetski-Shapiro. With a bit of extra work, one easily sees that this construction produces infinitely many commensurability classes of non-arithmetic lattices in any dimension. Note that the general structure of lattices in $SO(n,1)$ remains mysterious.

The situation much less clear for lattices in $SU(n,1)$, $n \geq 2$, which is (up to finite index) the isometry group of complex hyperbolic space $H^n_C$. Here there is currently no analogue of the Gromov-Piatetski-Shapiro construction (due to the non-constant curvature of $H^n_C$, there exist no real totally geodesic hypersurfaces, so there is no reasonable gluing interface to construct hybrids). In fact, only finitely many commensurability classes of non-arithmetic lattices in $SU(n,1)$ are currently known, only for very low values of $n$.

The first examples in $SU(2,1)$ were due to Mostow [17], and his construction was soon generalized to produce several more examples in $SU(2,1)$, and a single one in $SU(3,1)$ see [7]. For some decades, the Deligne-Mostow examples were the only known examples, even though some alternative constructions were given, see [22] for instance. To this day, it is still unknown whether there exist non-arithmetic lattices in $SU(n,1)$ for any $n > 3$.

A slightly different construction was given by Hirzebruch (see [1]), based on the equality case in the Miyaoka-Yau inequality, i.e. an orbifold version of the fact that a compact complex surface $X$ of general type with $c_2^2(X) = 3c_2(X)$ is covered by the ball. Given such
an $X$, the existence of a lattice $\Gamma$ in $PU(2,1)$ such that $X = \Gamma \setminus \mathbb{B}^2$ is guaranteed, but it is not obvious how to describe the lattice explicitly (the existence of a Kähler-Einstein metric is obtained by showing existence of a solution to a Monge-Ampère equation).

In fact, the arithmetic structure of the Hirzebruch examples seems not to have been worked out anywhere in the literature, apart from a small number of examples where coincidences with some arithmetic groups were found (see the work of Holzapfel [14], [15], and also the more recent [9]).

The Deligne-Mostow construction and the Barthel-Hirzebruch-Höfer construction were given a common generalization by Couwenberg, Heckman and Looijenga [5], but their work barely brushes the discussion of arithmeticity (they mention that the Coxeter examples are arithmetic, without any details). It was recently observed [9] that some of the non-arithmetic lattices in $PU(2,1)$ produced by the author, Parker and Paupert [12] were in fact conjugate to some specific Couwenberg-Heckman-Looijenga lattices.

The main goal of the present paper is to give a systematic study the arithmeticity of the Couwenberg-Heckman-Looijenga lattices. The CHL lattices in $PU(2,1)$ were already mentioned in [9] and [10]. For completeness, we review this in the form following result, stated using the notation from [11]. We write $C(G, p)$ for the CHL lattice derived from the Shephard-Todd group $G$, generated by complex reflections by angle $2\pi/p$.

**Theorem 1.**

1. The lattices $C(G_{23}, p)$, $p = 3, 4, 5, 10$ are arithmetic, and isomorphic to the corresponding sporadic groups $S(\sigma_{10}, p)$.
2. For each $p = 3, 4, 5, 6, 8, 12$, the lattice $C(G_{24}, p)$ is isomorphic to the sporadic groups $S(\sigma_4, p)$. It is arithmetic only for $p = 3$.
3. The lattices $C(G_{25}, p)$, $p = 5, 6, 7, 8, 9, 10, 12, 18$ are isomorphic to the corresponding Livné lattices, and they are all arithmetic, except for $C(G_{25}, 9)$.
4. The lattices $C(G_{26}, p)$, $p = 4, 5, 6, 8, 12$ are commensurable to some Deligne-Mostow groups, and also to the corresponding Thompson groups $T(S_4, p)$. They are all arithmetic.
5. The lattices $C(G_{27}, p)$, $p = 3, 4, 5$ are isomorphic to the corresponding Thompson groups $T(S_2, p)$. The groups $C(G_{27}, 4)$ and $C(G_{27}, 5)$ are not arithmetic, they are not commensurable to each other, and they are not commensurable to any Deligne-Mostow lattice either.

It turns out that, for $n \geq 3$, all the (non Deligne-Mostow) CHL lattices in $PU(n, 1)$ are arithmetic except for one. Recall that the Deligne-Mostow lattices appear in the CHL list in the infinite families of arrangements of type $A_n$ and $B_n$. We refer to the other arrangements (the ones that are not in these two infinite families) as exceptional arrangements, and to the corresponding groups as exceptional complex reflection groups.

**Theorem 2.** Let $\Gamma$ be a CHL lattice derived from an exceptional reflection group $G$. Then $\Gamma$ is arithmetic, unless $\Gamma = C(G_{29}, 3)$.

More precisely, we state the following.

**Theorem 3.** The lattice $C(G_{29}, 3)$ is a non-arithmetic, non-cocompact lattice, with adjoint trace field $\mathbb{Q}(\sqrt{3})$. It is not commensurable to any Deligne-Mostow lattice.
Recall that the Deligne-Mostow list of lattices contains only one non-arithmetic lattice in $PU(n,1)$ with $n \geq 3$, namely the lattice $\Gamma_\mu$ for $\mu = (3,3,3,3,5,7)/12$; so the main additional content of Theorem 3 is the claim that $\mathcal{C}(G_{29}, 3)$ is not commensurable to that specific $\Gamma_\mu$.

Given the commensurability analysis in [11], putting together all known non-arithmetic lattices in $PU(n,1)$, we see that there are currently 22 known commensurability classes in $PU(2,1)$, and 2 commensurability classes in $PU(3,1)$.

The basic tool for proving these results is the knowledge of explicit presentations of the braid groups associated to the Shephard-Todd groups (see the conjectural statements in [3], later proved in [2]). Using braid relations between the generators, we study the irreducible representations of the corresponding braid groups that send the generators to complex reflections of the appropriate angle (the values of the angle for the discrete holonomy groups in Couwenberg-Heckman-Looijenga are tabulated, see section 8 of [5]).

It turns out there are finitely many such representations, and the finite number is usually very small. Basic geometric considerations (using cocompactness or basic discreteness arguments) allow us to single out (a group conjugate to) the Couwenberg-Heckman-Looijenga holonomy group. Along the way, we find explicit matrices for generators for the holonomy groups, which may be of independent interest (but was not given in [5]).

Each holonomy group preserves an explicit Hermitian form (by irreducibility, such an invariant Hermitian form is unique up to scaling). The strategy for determining arithmeticity is then to

1. Find a coordinate change in order for the matrix of the Hermitian form to have entries in a simple number field;
2. Compute traces, and check that the above number field is as small as possible;
3. Find coordinates where the matrices of the generators are actually algebraic integers.

Step 2 allows one to determine the adjoint trace field, which is a well known commensurability invariant for lattices (in fact for Zariski dense groups). It is not completely obvious that step 3 can always be achieved, even though it is strongly believed to be the case for any lattice in $PU(n,1)$ (for cocompact lattices, this follows from very recent work of Esnault and Groechenig [13]). Recall that there are so-called quasi-arithmetic lattices in $SO(n,1)$ for every $n$, i.e. lattices where arithmeticity fails only by failure of integrality. We will work out integrality by a case by case analysis in the appendix.

This part of the paper requires some delicate arguments. One is the proof that $\mathcal{C}(G_{29}, 3)$ is not commensurable to the Deligne-Mostow non-arithmetic lattice in $PU(3,1)$. Indeed, the two groups have the same rough commensurability invariants (cocompactness, non-arithmeticity index, and adjoint trace fields). We use an explicit description of the cusps of these two lattices, and show that the cusps themselves are not commensurable.

Another delicate part is the determination of the representations for (the braid group associated to) the Shephard-Todd group $G_{31}$. This group is not well-generated, in the sense that it is not generated by the right number of reflections for the ambient dimension. This forced us to use slightly heavier computational tools.
We finish the paper by discussing presentations in terms of generators and relations and orbifold Euler characteristics. The fact that one can work out explicit presentations was already mentioned by Couwenberg, Heckman and Looijenga (see Theorem 7.1 in [5]). This depends on the knowledge of the presentations for braid groups that were worked out by Broué, Malle, Rouquier [4], Bessis and Michel [3].

The orbifold Euler characteristics are obtained by identifying the quotients as pairs \((X, \Delta)\) where \(X\) is an explicit normal space birational to \(\mathbb{P}^n\), and \(\Delta\) is an explicit \(\mathbb{Q}\)-divisor. The description of these pairs can be deduced from the results in [5]. We then compute

\[
\frac{1}{(n+1)^{n-1}}c_1^{orb}(X, \Delta)^n = \frac{(-1)^n}{(n+1)^{n-1}}(K_X + \Delta)^n,
\]

which is equal to \(c_n^{orb}(X, \Delta)\), which in turn is the orbifold Euler characteristic. The latter is of course related to the covolume of the corresponding ball quotient by a universal multiplicative constant, namely

\[
Vol(\Gamma \backslash \mathbb{B}^n) = \frac{(-4\pi)^n}{(n+1)!} \chi^{orb}(X, \Delta),
\]

if the metric is normalized to have holomorphic sectional curvature \(-1\). We work out the volumes only for the 3-dimensional examples, even though in principle the same method can be applied in higher dimensions.

The computations of \((K_X + \Delta)^3\) depend on the detailed properties of the combinatorics of the hyperplane arrangements given by the mirrors in 4-dimensional Shephard-Todd groups. We list these combinatorial properties in section 11 in the form of tables, since we could not find all of it in the literature (they can be checked fairly easily using modern computer technology).

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## 2. The Couwenberg-Heckman-Looijenga lattices

In [5], Couwenberg-Heckman-Looijenga give a fairly general construction of affine structures on the complement of hyperplane arrangements in projective space, parametrized by the holonomy around the hyperplanes in the arrangement. They also give necessary and sufficient conditions for the completion of that structure to be an orbifold (i.e. the holonomy is discrete, and the completion is a quotient of the appropriate complex space form).

It is unclear how often these conditions are satisfied, but there is a somewhat large list of examples associated to finite unitary groups generated by complex reflections (these were classified by Shephard and Todd [20]). That list contains a lot of the previously known examples of reflective lattices in \(PU(n, 1)\), namely the Deligne-Mostow lattices [7], as well
as the ones constructed by Barthel, Hirzebruch and Höfer [1]. Note that some examples in [11] are still not covered by the Couwenberg-Heckman-Looijenga construction, see [10].

The Couwenberg-Heckman-Looijenga lattices are described by giving:

- an irreducible Shephard-Todd group $G$;
- a positive integer for each orbit of mirrors of complex reflections in $G$.

We denote by $C(G, p_1, \ldots, p_k)$ the corresponding group. In this paper, we only consider the exceptional Shephard-Todd groups, since the other ones are covered by Deligne-Mostow theory.

It turns out (exceptional) Shephard-Todd groups have at most two orbits of mirrors, so we only take $k \leq 2$. In fact, there is a single orbit of mirrors (i.e. $k = 1$) for all but one group, namely $G = G_{28}$, which is isomorphic to the Coxeter group $F_4$.

The CHL structures are obtained as structures on $V^0 = V \setminus \mathcal{H}$ invariant under the action of the finite group $G$, where $\mathcal{H}$ is the union of mirrors of $G$. In particular, by construction, the holonomy group is a quotient of $\pi_1(\mathbb{P}(V^0/G))$, which is often called a braid group.

It is reasonably easy (especially using modern computer technology, and more so in low dimensions) to write down explicit group presentations in terms of generators and relations for the Shephard-Todd groups. This was done by Coxeter, see [6] and also the Appendix 2 in [4] for a convenient list.

This gives presentations for some quotients of the braid group $\pi_1(\mathbb{P}(V^0/G))$, namely the orbifold fundamental group of the quotient $\mathbb{P}(V/G)$, but it is not completely obvious to deduce a presentation for $\pi_1(\mathbb{P}(V^0/G))$. Roughly speaking, one would like to cancel the relations expressing the order of reflections, and keep the braid relations, but this is of course not well-defined. Presentations $\pi_1(\mathbb{P}(V^0/G))$ were proposed in [3], and a proof of the Bessis-Michel conjectural statements follows from the results in [2].

Note that the Bessis-Michel presentations are given so that the generators correspond to suitably chosen simple loops around hyperplanes in the arrangement. It follows that the Couwenberg-Heckman-Looijenga holonomy groups $C(G, p)$ (resp. $C(G, p_1, p_2)$) are homomorphic images of the braid group $\pi_1(\mathbb{P}(V^0/G))$, such that the corresponding homomorphism maps the Bessis-Michel generators to complex reflections of angle $2\pi/p$ (resp. $2\pi/p_1$ and $2\pi/p_2$).

For arrangements of type $A_n$ or $B_n$, the corresponding lattices are commensurable to Deligne-Mostow lattices, and the list is a bit too long to be reproduced here, see p.157-159 of [5], and also section 10 in this paper for the 3-dimensional case.

The other CHL lattices are listed in Tables 1 through 6 (bold-face means cocompact, red means non-arithmetic).

3. Basic facts about complex reflections.

Recall that a complex reflection in $H^n_C$ is one that can be written as an isometry of $\mathbb{C}^{n-1}$ of the form

$$x \mapsto x + (\zeta - 1)\frac{(x, v)}{(v, v)}v,$$

where $v$ is a mirror vector and $\zeta$ is a complex number.
Table 1. 2-dimensional CHL lattices. Note that $G_{25}$ and $G_{26}$ are not listed because they yield Deligne-Mostow groups.

<table>
<thead>
<tr>
<th>Shephard-Todd</th>
<th>Other description</th>
<th>Values of $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{23}$</td>
<td>Coxeter $H_3, S(\sigma_{10}, p)$</td>
<td>$3, 4, 5, 10$</td>
</tr>
<tr>
<td>$G_{24}$</td>
<td>$S(\sigma_4, p)$</td>
<td>$3, 4, 5, 6, 8, 12$</td>
</tr>
<tr>
<td>$G_{27}$</td>
<td>$T(S_2, p)$</td>
<td>$3, 4, 5$</td>
</tr>
</tbody>
</table>

Table 2. 3-dimensional CHL lattices. The group $G_{32}$ is not listed, since it is also $A_3$ and yields Deligne-Mostow groups.

<table>
<thead>
<tr>
<th>Shephard-Todd</th>
<th>Other description</th>
<th>Values of $p$ (resp. $(p_1, p_2)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{28}$</td>
<td>Coxeter $F_4$</td>
<td>$(2, 4), (2, 5), (2, 6), (2, 8), (2, 12)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(3, 3), (3, 4), (3, 6), (3, 12)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(4, 4), (6, 6)$</td>
</tr>
<tr>
<td>$G_{29}$</td>
<td></td>
<td>$3, 4$</td>
</tr>
<tr>
<td>$G_{30}$</td>
<td></td>
<td>$3, 5$</td>
</tr>
<tr>
<td>$G_{31}$</td>
<td></td>
<td>$3, 5$</td>
</tr>
</tbody>
</table>

Table 3. 4-dimensional CHL lattices.

<table>
<thead>
<tr>
<th>Shephard-Todd</th>
<th>Other description</th>
<th>Values of $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{33}$</td>
<td></td>
<td>$3$</td>
</tr>
</tbody>
</table>

Table 4. 5-dimensional CHL lattices.

for some vector $v$ with $\langle v, v \rangle \neq 0$, and some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. By an isometry, we mean a linear transformation that preserves the Hermitian inner product $\langle \cdot, \cdot \rangle$, which we assume linear on the first factor, and antilinear on the second factor.

Note that scaling the vector $v$ does not change the above transformation, so $v$ is not unique. The reflection fixes pointwise the complex-linear subspace $v^\perp = \{w \in \mathbb{C}^{n-1} : \langle w, v \rangle = 0\}$ and it acts on $\mathbb{C}v$ by multiplication by $\zeta$. In this paper, we will only consider reflections of finite order, i.e. $\zeta$ will actually be a root of unity. We will always assume that $v$ is a positive vector, i.e. $\langle v, v \rangle > 0$, in which case the restriction of the Hermitian form to $v^\perp$ has signature $(n - 1, 1)$, hence the set of negative vectors in $v^\perp$ projects down to a (totally geodesic) copy of $H_{\mathbb{C}^{n-1}}$. Moreover since we are free to scale $v$, we can (and will almost always) assume $\langle v, v \rangle = 1$.

For $k \in \mathbb{N}^*$, two group elements $a$ and $b$ are said to satisfy a braid relation of length $k$ if

$$(ab)^{k/2} = (ba)^{k/2}.$$
In that case, we write $br_k(a, b)$. When $k$ is odd, the notation $(ab)^{k/2}$ stands for an alternating product $a \cdot b \ldots b \cdot a$ with $2k$ factors. Note that when $br_k(a, b)$ holds, $br_{nk}(a, b)$ also holds for any $n \geq 1$. The smallest $k$ such that $br_k(a, b)$ holds is called the braid length of $a$ and $b$, which we denote by $br(a, b)$.

It is often convenient to describe reflection groups by a (complex) Coxeter diagrams. The diagram is attached to a generating set of reflections; it has a vertex for each generating complex reflection (labelled by the order of the corresponding reflection if that order is $\geq 3$), and an edge joining two vertices if the corresponding reflections satisfy a braid relation of length $\geq 3$ (labelled by the corresponding braid length).

One should beware that such a diagram does not usually determine the group uniquely (up to conjugation in $PU(n, 1)$). In particular, when the Coxeter diagram is not a tree, loops produce continuous families of representations of the corresponding Artin braid group (see [17] for instance).

A basic observation is that when generators $R_1, \ldots, R_m$ form a cycle in the Coxeter graph, using the ambiguity in $v$ in (1), we can choose the corresponding vectors $v_1, \ldots, v_m$ so that the inner products $\langle v_1, v_2 \rangle, \ldots, \langle v_m-1, v_m \rangle$ are all real, but then the product $\langle v_m, v_1 \rangle$ is usually not real. The argument of

$$\langle v_1, v_2 \rangle \cdot \langle v_2, v_3 \rangle \cdots \langle v_{m-1}, v_m \rangle \cdot \langle v_m, v_1 \rangle$$

is called the phase shift corresponding to that cycle in the Coxeter graph.

### 4. Arithmeticity

We will use the following arithmeticity criterion, which is proved in [17] (see also [7]). We refer to it as the Vinberg/Mostow arithmeticity criterion.

**Theorem 4.** Let $H$ be a Hermitian form of signature $(n, 1)$, defined over a CM field $L \supset K$. Let $\Gamma$ be a lattice in $SU(H, O_L)$, such that $\text{tr} \text{Ad} \Gamma = K$. Then $\Gamma$ is arithmetic if and only if $H^\sigma$ is definite for every $\sigma \in \text{Gal}(L/L)$ acting non-trivially on $K$.

Recall that a CM field is a purely imaginary quadratic extension of a totally real number field, we denote by $K$ the totally real field and by $L$ the imaginary quadratic extension. As usual, $O_L$ denotes the ring of algebraic integers. Note that not every lattice is commensurable to a lattice in the above statement (for the general case, one needs to consider division algebras over a CM field), which are sometimes called lattices of simplest type, or of Kazhdan type.
Because of the fact that the adjoint representation of a unitary representation $\rho$ is isomorphic to the tensor product $\rho \otimes \overline{\rho}$, we have
\[
\text{tr} \text{Ad}_\gamma = |\text{tr} \gamma|^2,
\]
which we will repeatedly use in the sequel.

5. Explicit generators and arithmeticity

The goal of this section is to give explicit matrix generators for the CHL lattices, as well as explicit Hermitian forms. This will be used to apply the arithmeticity criterion stated in section 4. We only work on lattices derived from exceptional complex reflection groups acting on $\mathbb{C}^{n+1}$ with $n \geq 3$, which give an action on $\mathbb{P}^n$ with $n \geq 3$. Indeed, non-exceptional ones yield Deligne-Mostow groups (explicit matrices can easily be deduced from [7], see also [21]); 2-dimensional examples have been studied elsewhere (see [11], for instance).

We go through a somewhat painful case by case analysis in sections 5.1 through 5.4.

5.1. Lattices derived from $G_{28}$. Recall that $G_{28}$ has two orbits of mirrors of reflections, see section 11 for more details on the combinatorics, hence the corresponding CHL lattices depend on two integer parameters. We denote the corresponding groups by $\mathcal{C}(G_{28}, p, q)$.

We call $r_1, \ldots, r_4$ generators of $G_{28}$, numbered according to the numbering of the nodes in Figure 5. The orbits of mirrors can easily be checked to be represented by the mirrors of $r_1$ and $r_4$, say. Indeed, $r_1$ and $r_2$ are conjugate since $b r_3 (r_1, r_2)$. Similarly $r_3$ and $r_4$ are in the same orbit. It is not completely obvious that $r_1$ and $r_4$ are not conjugate in the group.

Couwenberg-Heckman-Looijenga show that there exist lattices in $PU(3,1)$ that map $r_1, r_2$ to complex reflections of order $p$ and $r_3, r_4$ to complex reflections of order $q$, for $(p,q)$ given by $(2, q)$, $q = 4, 5, 6, 8, 12$, $(3, q)$ for $q = 3, 4, 6, 12$, $(4, 4)$ and $(6, 6)$.

For a generic value of $p, q$, we set up the Hermitian form as
\[
\begin{pmatrix}
1 & \alpha & 0 & 0 \\
\alpha & 1 & \beta & 0 \\
0 & \beta & 1 & \gamma \\
0 & 0 & \gamma & 1
\end{pmatrix},
\]
and the generators are given by
\[
R_1 = \begin{pmatrix}
z & \alpha(z-1) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
R_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\alpha(z-1) & z & \beta(z-1) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
R_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \beta(w-1) & w & \gamma(w-1) \\
0 & 0 & 0 & 1
\end{pmatrix},
R_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \gamma(w-1) & w
\end{pmatrix},
\]
where $z = e^{2\pi i/p}$, $w = e^{2\pi i/q}$. 
It turns out the braid relation $br(R_1, R_2) = 3$ is equivalent to $|\alpha| = \frac{1}{2\sin \frac{p}{2}}$ (see [17] for instance). Similarly $br(R_3, R_4) = 3$ is equivalent to $|\gamma| = \frac{1}{2\sin \frac{p}{2}}$.

One checks that $br(R_2, R_3) = 4$ is equivalent to $\beta^2 = z + w \frac{z + w}{z + w - 1 - zw}$.

The case $\beta = 0$ is irrelevant here, since it would correspond to a local monodromy group generated by two commuting reflections, hence the local structure of the arrangement would be a pair of transverse hyperplanes, which is not the case here (there are actually four mirrors containing $v_j \perp v_k \cap v_j \perp v_k$, where $v_j$ is orthogonal to the mirror of $r_j$).

Since the Coxeter diagram contains no loop, we can choose $\alpha, \beta, \gamma$ to be real, and we can actually assume

$$\alpha = \frac{1}{2\sin \frac{p}{2}}, \quad \beta = \sqrt{\frac{z + w}{z + w - 1 - zw}}, \quad \gamma = \frac{1}{2\sin \frac{q}{2}}.$$  

This gives a description of explicit generators (for some group conjugate in $PU(3, 1)$ to $C(G_{28}, p, q)$), but the field generated by the entries of these generators is not always the smallest possible (and the entries are not usually algebraic integers).

### 5.1.1. Arithmetcity for the case $p = 2$.

In this case $\alpha = 1/2$, $\beta = 1/\sqrt{2}$. The upper left $3 \times 3$ submatrix of $H$ is positive definite, so the form has signature either $(3, 1), (3, 0)$ or $(4, 0)$ depending on the sign of the determinant, which is given by

$$\det H = \frac{1 - 3\gamma^2}{4}.$$  

This is zero for $q = 3$, and negative for all other values $q = 4, 5, 6, 8, 12$ (recall that $\gamma = e^{2\pi i/q}$).

In order to give nice matrices, we consider the change of coordinates given by $Q = diag(\sqrt{2}, \sqrt{2}, 1, \frac{1 - 1}{2\sin \frac{q}{2}})$. One checks directly that

$$Q^*HQ = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 1 & \frac{1}{\gamma q - 1} \\
0 & 0 & \frac{1}{\gamma q - 1} & 1
\end{pmatrix}.$$  

From this, it is clear that the generators of the group can be written as matrices in $\mathbb{Q}(\zeta_q)$, and one easily checks that their entries are algebraic integers (in that same cyclotomic field).

**Proposition 1.** The group $C(G_{28}, 2, q)$ has adjoint trace field given by $\mathbb{Q}(\cos \frac{2\pi q}{q})$, and these groups are all arithmetic.
Proof: Let $\Gamma = C(G_{28}, 2, q)$ and $K = \mathbb{Q}(\operatorname{tr} \operatorname{Ad} \Gamma)$. The previous paragraph implies that $K \subset \mathbb{Q}(\cos \frac{2\pi}{q})$. From the matrices given above for the $R_j$, we have

$$|\operatorname{tr} R_3|^2 = |\zeta_q + 3|^2 = 10 + 2\cos \frac{2\pi}{q},$$

so we also have the other inclusion $\mathbb{Q}(\cos \frac{2\pi}{q}) \subset K$.

Now the arithmeticity criterion in section 4 says that the group is arithmetic if and only if for every non-trivial Galois automorphism $\sigma \in \operatorname{Gal}(K)$,

$$1 - 3(\gamma^2_\sigma)^2 > 0.$$ 

These automorphisms are of course very simple to list. Indeed, since automorphisms of $\mathbb{Q}(\zeta_q)$ map $\zeta_q$ to $\zeta_q^k$ for some $k$ prime to $q$, the nontrivial Galois conjugates of $\gamma^2 = \frac{1}{4\sin^2 \frac{\pi}{q}} = \frac{1}{2 - 2\cos \frac{2\pi}{q}}$ by $\frac{1}{2 - 2\cos \frac{2\pi}{q}}$ for some $k$ prime to $q$.

\[\square\]

5.1.2. Arithmeticity for the case $p = 3$, $q = 3$. Using the freedom in choosing vectors parametrizing complex reflections (see section 3), we can use the Hermitian form

$$H = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 1 & \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & 1 & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 1 \end{pmatrix},$$

where $\omega = \frac{1 + \sqrt{3}}{2}$. Indeed, note that $|\omega - 1| = \sqrt{3}$, and $1/2 \sin \frac{\pi}{p} = 1/\sqrt{3}$.

Proposition 2. The group $C(G_{28}, 3, 3)$ has adjoint trace field $\mathbb{Q}$, and it is arithmetic.

Proof: In order to get entries in a nice number field, we use $Q = \operatorname{diag}(\sqrt{2}, \frac{1}{\sqrt{2}}, \zeta_{12}^5, \zeta_{12})$. This gives a Hermitian form defined over $Q(\omega)$, namely

$$Q^* HQ = \begin{pmatrix} \frac{1}{2} & \frac{\omega - 1}{3} & 0 & 0 \\ \frac{\omega - 1}{3} & \frac{1}{3} & \frac{1 - \omega}{3} & 0 \\ 0 & \frac{1 - \omega}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$ 

This implies that the adjoint trace field $\mathbb{Q}(\operatorname{tr} \operatorname{Ad} \Gamma)$ is $\mathbb{Q}(\omega) \cap \mathbb{R} = \mathbb{Q}$.

Moreover, one checks that the corresponding matrices $Q^{-1} R_j Q$, $j = 1, \ldots, 4$ have algebraic integer entries. Hence we can use Theorem 4 to check arithmeticity, and the group is clearly arithmetic since there are no non-trivial Galois conjugates at all.

\[\square\]

5.1.3. Arithmeticity for the case $p = 3$, $q = 4$. For this and the next few groups, we use change of coordinates inspired by Deligne-Mostow theory (but we find little use of explaining the details of how we found them, which would take us far afield).

From the shape of the Hermitian form given in the beginning of the section for generic $p, q$ (see equation (2)) we perform a change of coordinates using $Q = \operatorname{diag}(\sqrt{3 + \sqrt{3}}, \sqrt{3 - \sqrt{3}}, \sqrt{2}, 1)$. 

This gives

\[ K = Q^*HQ = \begin{pmatrix} 3 + \sqrt{3} & 1 & 0 & 0 \\ 1 & 3 - \sqrt{3} & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

This gives generators with entries in \( \mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\zeta_{12}) \), but the corresponding reflections do not have algebraic integer entries.

Now consider \( J = R_2R_3R_4 \) and \( f_1 = \mu e_1, f_2 = e_2, f_3 = Je_2, f_4 = J^{-1}e_2 \), where \( \mu = (-2 + \zeta_{12} + \zeta_{12}^2 - \zeta_{12}^3)/2 \). Working in the basis \( f_1, f_2, f_3, f_4 \), we get the generators to have algebraic integer entries.

**Proposition 3.** The group \( C(G_{28}, 3, 4) \) has adjoint trace field given by \( \mathbb{Q}(\sqrt{3}) \), and it is arithmetic.

**Proof:** We denote by \( \Gamma = C(G_{28}, 3, 4) \) and \( K = \text{tr} \text{Ad} \Gamma \). Because of the previous discussion, we have \( K \subset \mathbb{Q}(\zeta_{12}) \cap \mathbb{R} = \mathbb{Q}(\sqrt{3}) \). We compute \( |\text{tr}(R_2R_3)^2| = 8 + 4\sqrt{3} \), so we also get the other inclusion and \( K = \mathbb{Q}(\sqrt{3}) \).

There is only one non-trivial Galois conjugate for \( K \), obtained by replacing \( \sqrt{3} \) by \( -\sqrt{3} \) in the matrix of equation (3), which makes the form definite. Arithmeticity then follows from Theorem 4. \( \square \)

5.1.4. **Arithmeticity for the case** \( p = 3, q = 6 \). From the generic shape of the Hermitian form (equation (2)), using \( Q = \text{diag}(\sqrt{3}, 1, 1, 1) \), we get

\[ Q^*HQ = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \]

so we can write the generators as matrices with entries in \( \mathbb{Q}(\omega) \), hence \( \mathbb{Q}(\text{tr} \text{Ad} \Gamma) = \mathbb{Q} \).

One also checks that the corresponding complex reflections have algebraic integer entries, so we get:

**Proposition 4.** The lattice \( C(G_{28}, 3, 6) \) has adjoint trace field \( \mathbb{Q} \), and it is arithmetic.

5.1.5. **Arithmeticity for the case** \( p = 3, q = 12 \). From the generic shape, using \( Q = \text{diag}(1, \sqrt{3}, \frac{1}{\sqrt{3}+\sqrt{3}}, \frac{1}{\sqrt{3}+\sqrt{3}}) \), we get

\[ Q^*HQ = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{3-\sqrt{3}}{6} & 1 \\ 0 & 0 & 1 & 3 - \sqrt{3} \end{pmatrix}, \]

so we can write the generators as matrices with entries in \( \mathbb{Q}(\zeta_{12}) \), hence \( \mathbb{Q}(\text{tr} \text{Ad} \Gamma) \subset \mathbb{Q}(\sqrt{3}) \).

One computes \( |\text{tr}(R_2R_3)|^2 = 8 + \sqrt{3} \), so \( \mathbb{Q}(\text{tr} \text{Ad} \Gamma) = \mathbb{Q}(\sqrt{3}) \).
Proposition 5. The lattice $\mathcal{C}(G_{28}, 3, 12)$ has adjoint trace field $\mathbb{Q}(\sqrt{3})$, and it is arithmetic.

Proof: The adjoint trace field was already discussed above. In order to prove arithmeticity, we need to exhibit integral matrices. Consider $J = R_2 R_3 R_4$ and $f_1 = (\omega - 1)e_1$, $f_2 = e_2$, $f_3 = Je_2$, $f_4 = J^{-1}e_2$. One checks that the matrices for generators in that basis are integral.

Now there is only one non-trivial Galois conjugate of $H$, and one checks that replacing $\sqrt{3}$ by $-\sqrt{3}$ in $H$ makes the form definite, hence the group is arithmetic. \(\square\)

5.1.6. Arithmeticity for the case $p = 4$, $q = 4$. Using $Q = \text{diag}(\sqrt{2}, 1, 1, \sqrt{2})$, we get
\[
Q^*HQ = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix},
\]
so we can write the generators as matrices with entries in $\mathbb{Q}(i)$, hence $\mathbb{Q}(\text{tr} \text{Ad} \Gamma) = \mathbb{Q}$.

Proposition 6. The lattice $\mathcal{C}(G_{28}, 4, 4)$ has adjoint trace field $\mathbb{Q}$, and it is arithmetic.

Proof: Once again, we need to prove integrality. Just as in the previous case, one can take $J = R_2 R_3 R_4$ and $f_1 = (\omega - 1)e_1$, $f_2 = e_2$, $f_3 = Je_2$, $f_4 = J^{-1}e_2$. One checks that the matrices for generators in the basis $f_1, f_2, f_3, f_4$ are integral.

There is no non-trivial Galois conjugate, so arithmeticity follows. \(\square\)

5.1.7. Arithmeticity for the case $p = 6$, $q = 6$. Using $Q = \text{diag}(1, 1, \sqrt{2}, \sqrt{2})$, we get
\[
Q^*HQ = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 2 & 0 \\
0 & 2 & 2 & 2 \\
0 & 0 & 2 & 2
\end{pmatrix},
\]
so we can write the generators as matrices with entries in $\mathbb{Q}(\omega)$, hence $\mathbb{Q}(\text{tr} \text{Ad} \Gamma) = \mathbb{Q}$.

Proposition 7. The lattice $\mathcal{C}(G_{28}, 6, 6)$ has adjoint trace field $\mathbb{Q}$, and it is arithmetic.

Proof: Here no coordinate change is needed in order to find generators with algebraic integral entries, the above form does the job.

Since there is no non-trivial Galois conjugate, the group is arithmetic. \(\square\)

5.2. Lattices derived from $G_{29}$. It follows from the results by Broué, Malle, Rouquier [4] and Bessis and Michel [2], [3] that the corresponding braid group is given by
\[
B = \langle r_1, r_2, r_3, r_4 | \text{br}_2(r_1, r_3), \text{br}_2(r_1, r_4), \text{br}_3(r_1, r_2), \text{br}_3(r_2, r_3), \text{br}_3(r_3, r_4), \text{br}_4(r_2, r_4), \text{br}_4(r_3, r_2 r_4) \rangle.
\]

Couwenberg-Heckman-Looijenga show that there are lattices in $PU(3,1)$ that map every $r_j$ to a complex reflection $R_j$ of angle $2\pi/p$, where $p$ is either 3 or 4. We denote the corresponding groups by $\mathcal{C}(G_{29}, p)$. 
As before, we denote by \( v_j \) a polar vector to the mirror of \( R_j \). Note that these four vectors must be linearly independent, because the group generated by the \( R_j \) must act irreducibly on \( \mathbb{C}^4 \). By rescaling the vectors, we may assume

- \( \langle v_j, v_j \rangle = 1 \),
- \( \langle v_j, v_{j+1} \rangle \in \mathbb{R} \) for \( j = 1, 2, 3 \).

The braid relations \( \text{br}_2(r_j, r_k) \) impose \( \langle v_j, v_k \rangle = 0 \), and the braid condition \( \text{br}_3(r_j, r_{j+1}) \) translates into \( |\langle v_j, v_{j+1} \rangle| = \frac{1}{2\sin \frac{\pi}{2}} \), for \( j = 1, 2, 3 \).

### 5.2.1. The case \( p = 3 \)

We write \( \alpha = \langle v_2, v_4 \rangle \). In this case we have \( |\langle v_j, v_{j+1} \rangle| = 1/\sqrt{3} \), so we may assume \( H \) is given by

\[
(4) \quad \begin{pmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & \alpha \\
0 & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1
\end{pmatrix}.
\]

The \((2,1)\)-entry of the matrix \((R_2R_4)^2(R_4R_2)^{-2}\) is equal to

\[
3 \frac{i - \sqrt{3}}{2} |\alpha|^2 (3|\alpha|^2 - 2)^2,
\]

so the braid relation \( \text{br}_4(R_2, R_4) \) implies \( |\alpha|^2 = 0 \) or \( \frac{2}{3} \).

**Proposition 8.** In the CHL lattice \( \mathcal{C}(G_{29}, p) \), \( R_2 \) and \( R_4 \) do not commute.

**Proof:** Let \( S_2 \) and \( S_4 \) denote reflections in \( G_{29} \) acting on \( P^3 \) that correspond to \( R_2 \) and \( R_4 \). Then \( S_2 \) and \( S_4 \) generate a group of order 8, isomorphic to \( G(4,4,2) \). The branch locus of the quotient map \( \mathbb{C}^2 \to \mathbb{C}^2/G(4,4,2) \) has local analytic structure \( z_1^4 = z_2^2 \), which gives two tangent components.

In the CHL structure, the quotient has the same structure (at least locally near a generic point of the intersection of the mirrors), which precludes \( R_2 \) from commuting with \( R_4 \) (if they did, the branch locus would consist of two transverse smooth components).

From this point on, we assume \( |\alpha|^2 = 2/3 \). One checks by direct computation that the relation \( \text{br}_4(R_2, R_4) \) then holds.

One computes the following, where we write \( \zeta = e^{2\pi i/12} \), and \( \omega = \zeta^4 \).

\[
R_1 = \begin{pmatrix}
\omega & \zeta^5 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
R_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \zeta^5 & \omega & \zeta^3 \alpha(\omega-1) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
R_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \zeta^5 & \zeta^3 \\
0 & 0 & 0 & 1
\end{pmatrix},
R_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & \pi(\omega-1) & \zeta^5 & \omega
\end{pmatrix}.
\]

From the relation \( \text{br}_4(R_3, R_2R_4) \), we get two values of \( \alpha \), namely

\[
\alpha_1 = ((-1 + i)\sqrt{3} + 3(1 + i))/6 = (1 + \zeta^2 + \zeta^3 + \zeta^5)/3,
\]

\[
\alpha_2 = (((1 + i)\sqrt{3} + 3(1 - i))/6 = (1 + \bar{\zeta} + \bar{\zeta}^3 + \bar{\zeta}^5)/3.
\]

More specifically, the \((2,1)\)-entry of the matrix \((R_3R_2R_4)^2(R_4R_2R_3)^{-2} - \text{Id} \) gives a polynomial of degree 4 in \( \alpha \) (here we write \( \bar{\alpha} = 2/3\alpha \) and keep only the numerator of the
corresponding rational fraction), which turns out to be a square, and with two roots given by the values $\alpha_1, \alpha_2$ given above. One then checks by painful computation that these values actually imply $(R_3R_2R_4)^2(R_2R_4R_3)^{-2} = Id$.

We denote by $\Gamma_j$ the group generated by $R_1, \ldots, R_4$ when $\alpha = \alpha_j$. Note that $\alpha_1$ and $\alpha_2$ are Galois conjugates, the homomorphism induced by $\zeta \mapsto \zeta^5$ maps $\alpha_1$ to $\alpha_2$.

Note that the upper left corner $k \times k$ submatrices of $H$ have positive determinant for $k = 1, 2, 3$, so the signature of $H$ is either $(3,1)$ or $(4,0)$, depending on the sign of

$$\det H = \frac{\alpha + \overline{\alpha}}{3} - \frac{5}{9} = \frac{-2 \pm \sqrt{3}}{9}.$$  

Since the last expression is negative for both choices of $\pm$, the signature $(3,1)$ for both values $\alpha_1$ and $\alpha_2$.

**Proposition 9.** The group $\Gamma_2$ is not discrete.

**Proof:** In this case, the lower right $3 \times 3$ submatrix of $H$ gives a positive definite Hermitian form, so the subgroup $\Gamma_2^{34}$ of $\Gamma_2$ generated by $R_2, R_3, R_4$ has a fixed point inside the ball.

One easily checks that $R_2R_3R_4$ is elliptic (the 1-eigenvector is negative), but has infinite order. The easiest way to check this is to consider its Galois conjugate (obtained by $\zeta \mapsto \zeta^5$) is loxodromic. □

Since the lattice giving the holonomy group of the complex hyperbolic structure of the CHL lattice $C(G_{29}, 3)$ must be isomorphic to either $\Gamma_1$ or $\Gamma_2$, we get

**Proposition 10.** The group $\Gamma_1$ is conjugate to $C(G_{29}, 3)$.

In this section, from this point on, we denote by $\Gamma$ either $\Gamma_1$ or $C(G_{29}, 3)$.

The generators $R_1, \ldots, R_4$ for $\Gamma$ have entries in $\mathbb{Q}(\zeta)$, whose maximal totally real subfield is $\mathbb{Q}(\sqrt{3})$, so $\text{tr}\text{Ad} \Gamma \subset \mathbb{Q}(\sqrt{3})$.

One checks

$$\text{tr}(R_2R_3R_4) = 2 + \zeta - i,$$

which implies

$$|\text{tr}(R_2R_3R_4)|^2 = 5 + 2\sqrt{3}.$$  

So we have the other inclusion $\text{tr}\text{Ad} \Gamma \subset \mathbb{Q}(\sqrt{3})$.

**Proposition 11.** The group $C(G_{29}, 3)$ has adjoint trace field given by $\mathbb{Q}(\sqrt{3})$, and it is not arithmetic.

**Proof:** The matrices described above have algebraic integer entries, so we can apply Theorem 4.

One then considers the matrix in equation (4). The non-trivial Galois conjugate can be represented by $\zeta_{12} \mapsto \zeta_{5}^{12}$, and this gives a Hermitian form with signature $(3,1)$. The criterion in Theorem 4 shows that the group is non-arithmetic. □
5.2.2. The case \( p = 4 \). In this case we have \(|\langle v_j, v_{j+1} \rangle| = 1/\sqrt{2}\), and we may assume \( \langle v_j, v_{j+1} \rangle = 1/\sqrt{2} \). The condition that \( br_4(R_2, R_4) \) implies \( |\alpha| = 1 \), and \( br_4(R_3, R_2R_4) \) implies \( \alpha = 1 \) or \( i \).

In other words, in suitable coordinates, the group \( \Gamma_4 \) preserves one of the following two Hermitian forms:

\[
\begin{pmatrix}
1 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & 1 \\
0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\
0 & 1 & \frac{1}{\sqrt{2}} & 1
\end{pmatrix},
\]

or

\[
\begin{pmatrix}
1 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & i \\
0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\
0 & -i & \frac{1}{\sqrt{2}} & 1
\end{pmatrix}.
\]

Considering the values of \( \kappa_L \) given in section 11 for \( \mathcal{L} = m_2 \cap m_3 \cap m_4 \), we see that the CHL group actually corresponds to taking \( \alpha = i \). Indeed, for \( \alpha = 1 \), the lower right 3 \( \times \) 3 submatrix of the Hermitian form gives a degenerate form, but \( \kappa_{L234} = 2 > 1 \), so we should be getting signature \((2, 1)\).

Now using the diagonal matrix \( K = \text{diag}(1, \sqrt{2}, 1, \sqrt{2}) \), we see that the four matrices \( K^{-1}R_jK \) have entries in the ring \( \mathbb{Z}[i] \) of Gaussian integers. This implies:

**Proposition 12.** The lattice \( \mathcal{C}(G_{29}, 4) \) has adjoint trace field \( \mathbb{Q} \), and it is arithmetic.

5.3. Lattices derived from the group \( G_{30} \). Since there is no loop in the Coxeter diagram, we may assume that all inner products \( \langle v_j, v_k \rangle \) are real. Because of the braid relations

\[
br_2(R_1, R_3), br_2(R_1, R_4), br_2(R_2, R_4), br_3(R_1, R_2), br_3(R_2, R_3), br_5(R_3, R_4)
\]

we may take the invariant Hermitian form to be

\[
\begin{pmatrix}
1 & a & 0 & 0 \\
a & 1 & a & 0 \\
0 & a & 1 & b \\
0 & 0 & b & 1
\end{pmatrix},
\]

where

\[
a = \frac{1}{2 \sin \frac{\pi}{p}}, \quad b = \frac{\sqrt{5} \pm 1}{2} \frac{1}{2 \sin \frac{\pi}{p}}.
\]

**Proposition 13.** The group \( \mathcal{C}(G_{30}, p) \) is a lattice if \( p = 3, 5 \), and in both cases it is cocompact. Both groups are arithmetic, with \( \mathbb{Q}(\text{tr} \text{Ad} \Gamma) = \mathbb{Q}(\sqrt{5}) \).

**Proof:**

5.3.1. The case \( p = 3 \). The fact that these groups are lattices, and the cocompactness is proved in [5].

Using the above Hermitian form, one writes

\[
R_1 = \begin{pmatrix}
\quad u & a(u - 1) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad R_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & u & a(u - 1) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
\[ R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a(u-1) & u & b(u-1) \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b(u-1) & \omega \end{pmatrix}. \]

where \( u = e^{2\pi i/p} \).

Only one of the two possible values of \( b \) give a Hermitian form with signature (3,1), namely \( b = \frac{1+\sqrt{5}}{2\sqrt{3}} \). We change coordinates by \( \text{diag}(\sqrt{3}, 1, \sqrt{3}, 1) \), to get

\[ \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 & 1+\sqrt{5} & 1 \end{pmatrix}. \]

In these coordinates, the reflection matrices have entries in \( \mathbb{Q}(\sqrt{5}, \omega) \), hence the adjoint trace field is contained in \( \mathbb{Q}(\sqrt{5}) \).

In order to prove that the trace field is indeed \( \mathbb{Q}(\sqrt{5}) \), it is enough to exhibit a matrix \( \gamma \in \Gamma \) such that \(|\text{tr}\gamma|^2 \notin \mathbb{Q}\). For instance, one verifies

\[ |\text{tr}(R_1 R_2 R_3 R_4)|^2 = \frac{3 - \sqrt{5}}{2}. \]

In order to check integrality, we consider the element \( J = R_3 R_4 R_2 R_3 R_4 \), and the basis given by the vectors \( f_1 = e_1/(\omega-1), f_2 = e_2, f_3 = Je_2, f_4 = J^{-1} e_2 \). In that basis, the four reflections have algebraic integer entries.

Hence we can apply the arithmeticity criterion of Theorem 4. The nontrivial Galois conjugate matrix is positive definite, so the group \( \mathcal{C}(G_{30}, 3) \) is arithmetic.

**5.3.2. The case \( p = 5 \).** In this case, in the above notation, there are two possibilities, namely \( a = \frac{1}{2\sin \frac{\pi}{5}}, b = \frac{\sqrt{5}+1}{4\sin \frac{\pi}{5}} \). The value \( b = \frac{\sqrt{5}-1}{4\sin \frac{\pi}{5}} \) is not relevant, since it would give a cusp, fixed by \( R_2, R_3 \) and \( R_4 \) (the restriction of the Hermitian form to the last three standard basis vectors is degenerate), but the group \( \mathcal{C}(G_{30}, 5) \) is cocompact.

Hence we take \( b = \frac{\sqrt{5}+1}{4\sin \frac{\pi}{5}} \). Then changing coordinates with \( \text{diag}(1, \sqrt{\frac{10}{5+\sqrt{5}}}, 1, \sqrt{\frac{5}{5+2\sqrt{5}}}) \), we get a Hermitian form

\[ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & \sqrt{5} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 5-2\sqrt{5} \end{pmatrix}. \]

In these coordinates, the reflection matrices have entries in \( \mathbb{Q}(\zeta_5) \), hence the adjoint trace field is contained in \( \mathbb{Q}(\sqrt{5}) \).

In order to prove that the trace field is indeed \( \mathbb{Q}(\sqrt{5}) \), it is enough to exhibit a matrix \( \gamma \in \Gamma \) such that \(|\text{tr}\gamma|^2 \notin \mathbb{Q}\). For instance, one verifies

\[ |\text{tr}(R_1)|^2 = \frac{17 + 3\sqrt{5}}{2}. \]
The integrality check is almost the same as in the case $p = 3$, namely we consider the element $J = R_3 R_4 R_2 R_3 R_4$, and the basis given by the vectors $f_1 = \mu e_1, f_2 = e_2, f_3 = J e_2, f_4 = J^{-1} e_2$, where $\mu = (\zeta_{45}^{21} + \zeta_{45}^{18} + 2 \zeta_{45}^9 + \zeta_{45}^6 - 2)/5$. In that basis, the four reflections have algebraic integer entries.

One checks that the Galois conjugate matrix obtained by replacing $\sqrt{5}$ by $-\sqrt{5}$ is positive definite, so the group $C(G_{30}, 5)$ is arithmetic.

5.4. Lattices derived from the group $G_{31}$. There are two groups in the CHL list, corresponding to $p = 3$ and $p = 5$. These are a bit more difficult computationally, but not conceptually.

The initial difficulty is that the corresponding Shephard-Todd is not well-generated, i.e. it requires five generators (and not four as one may expect from the dimension). We will parametrize quadruples of reflections with extra parameters, and then solve in the parameter space the equations that express the existence of a 5-th reflection that satisfies the appropriate relations with the first 4 reflections.

According to [3], the group is generated by reflections $R_1, \ldots, R_5$ that satisfy

(5) \[ br_3(R_1, R_2), br_3(R_2, R_5), br_3(R_5, R_3), br_3(R_3, R_4), \]
(6) \[ br_2(R_2, R_4), br_2(R_1, R_3), br_2(R_2, R_3), \]
(7) \[ R_5 R_4 R_1 = R_4 R_1 R_3 = R_1 R_5 R_4. \]

We denote by $v_j$ a vector polar to the mirror of $R_j$. Note that the last relation implies that the polar vectors $v_1, v_4, v_5$ are linearly dependent.

Since the action of the group generated by all the $R_j$ must be irreducible on $\mathbb{C}^4$, the vectors $v_1, v_2, v_3, v_4$ must be linearly independent. We write the Hermitian form in the corresponding basis. The right angles coming from the above commutation relations imply that we may assume the corresponding Hermitian matrix has the form

\[
H = \begin{pmatrix}
1 & \alpha & 0 & \beta \\
\alpha & 1 & 0 & 0 \\
0 & 0 & 1 & \alpha \\
\beta & \alpha & 1 & 0
\end{pmatrix},
\]

where $\alpha = \frac{1}{2 \sin \frac{\pi}{p}}$ and $\beta \in \mathbb{R}^+$. We then write $w$ for a vector polar to the mirror of $R_5$. Because of the linear dependence between $v_1, v_4$ and $w$, we can write it as $(x_1, 0, 0, x_4)$. If $R_1$ and $R_5$ have the same mirror, then they coincide, and this would imply that $R_1, R_4$ commute, in which case the action cannot be irreducible on $\mathbb{C}^4$.

Hence we must have $x_1 \neq 0$, hence we can take $w = (1, 0, 0, z)$ for some $z \in \mathbb{C}$ (in fact $z \neq 0$ by a reasoning similar to the previous one, but we will not need this).

One then writes equations on $b, z$ expressing the relations from (5) that involve $R_5$.

5.4.1. The group CHL($G_{31}, 3$). Note that all the polynomial systems mentioned above turn out to be 0-dimensional, i.e. they have finitely many solutions. Their solutions can
be listed, with an explicit field of definition, using the rational univariate representation (see [19]).

It turns out the system has four solutions, that come in pairs with opposite values of $\beta$ (and $w$ adjusted accordingly by changing the sign of $x_4$). Here we give only two of the four solutions, namely

\begin{itemize}
  \item $\beta = 1$, $w = (1, 0, 0, -\frac{1+i\sqrt{3}}{2})$,
  \item $\beta = \frac{1}{\sqrt{3}}$, $w = (1, 0, 0, -\frac{\sqrt{3}+i}{2})$.
\end{itemize}

We rule out the second solution, since the corresponding Hermitian form is positive definite. Hence we take $\beta = 1$. Using $Q = \text{diag}(\sqrt{3}, 1, 1, \sqrt{3})$, one gets a Hermitian form with entries in $\mathbb{Q}$, so $R_1, \ldots, R_4$ can be written with entries in $\mathbb{Q}(\omega)$. One easily checks that in that case $R_5$ also has entries in $\mathbb{Q}(\omega)$, so $\mathbb{Q}(\text{tr}Ad\Gamma) = \mathbb{Q}$.

Since the generators have algebraic integer entries, we get:

**Proposition 14.** The group $\text{CHL}(G_{31}, 3)$ has adjoint trace field $\mathbb{Q}$ and it is arithmetic.

5.4.2. The group $\text{CHL}(G_{31}, 5)$. There are four solutions given up to obvious sign change by

\begin{itemize}
  \item $\beta = 1+\sqrt{5}, w = (1, 0, 0, -\zeta_5)$,
  \item $\beta = \sqrt{\frac{4-\sqrt{5}}{10}}, w = (1, 0, 0, -\zeta_{20})$.
\end{itemize}

We rule out the second possibility, since the lattice $\text{CHL}(G_{31}, 5)$ is cocompact. Indeed, in that case, one considers the $3 \times 3$ submatrix obtained from $H$ by removing the third row and column, which gives a degenerate Hermitian form, hence a cusp group.

For the first value, using $Q = \text{diag}(\sqrt{\frac{10}{5+\sqrt{5}}}, 1, 1, \sqrt{\frac{5+\sqrt{5}}{2}})$ we get $Q^*HQ$ with entries in $\mathbb{Q}(\sqrt{5})$, namely

\[
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & \frac{5+\sqrt{5}}{2} & 0 & 0 \\
0 & 0 & 1 & \frac{1-\sqrt{5}}{2} \\
1 & 0 & \frac{1-\sqrt{5}}{2} & \frac{5-\sqrt{5}}{2}
\end{pmatrix}.
\]

One can also check that $Q^{-1}R_5Q$ then has entries that are in $\mathbb{Q}(\zeta_5)$, hence the group can be generated by matrices with entries in $\mathbb{Q}(\zeta_5)$.

**Proposition 15.** The lattice $\mathcal{C}(G_{31}, 5)$ has adjoint trace field $\mathbb{Q}(\sqrt{5})$, and it is arithmetic.

**Proof:** One checks that, using the above coordinates, the five generators have algebraic integer entries. Replacing $\sqrt{5}$ by $-\sqrt{5}$ in the above matrix gives a positive definite matrix, hence $\text{CHL}(G_{31}, 5)$ is arithmetic by Theorem 4. \qed

5.5. Lattices derived from the group $G_{33}$. The group $C(G_{33}, 3)$ is generated by 5 reflections $R_1, \ldots, R_5$ of order 3. The Hermitian form in the basis given by suitably chosen
vectors polar to the mirrors is given

\[
\begin{pmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & 1
\end{pmatrix}
\]

According to Bessis and Michel, we must have \((R_2R_4R_3)^2 = (R_3R_4R_2)^2 = (R_3R_2R_4)^2\).

Writing out matrices for the \(R_j\) in terms of \(\alpha\), we find that there are two values of \(\alpha\) such that this happens, namely \(\alpha = i\) and \(\alpha = \frac{\sqrt{3} - i}{2}\). The latter gives a degenerate Hermitian form, so only \(\alpha = i\) is allowed.

For that value, and \(Q = \text{diag}(1, \sqrt{3}, 1, \sqrt{3}, 1)\), we find \(Q^*HQ\) has entries in \(\mathbb{Q}(\omega)\), so the group can be generated by matrices with entries in \(\mathbb{Q}(\omega)\). This implies \(\mathbb{Q}(tr.\text{Ad}\Gamma) = \mathbb{Q}\).

We now explain how to get integral entries for the generators. In order to do this, we choose a basis that exhibits the cyclic symmetry of order three between the mirrors of \(R_2\), \(R_3\) and \(R_4\). Specifically, since \(i = \zeta_{12}^3\), we can use the Hermitian form

\[
\begin{pmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & 1
\end{pmatrix}
\]

then adjust by using the diagonal matrix \(\text{diag}(\zeta_{12}, 1, 1, 1, \bar{\zeta}_{12})\).

This gives the following (after multiplying the Hermitian matrix by 3)

\[
\begin{pmatrix}
3 & 1 - \omega & 0 & 0 & 0 \\
1 - \bar{\omega} & 3 & 1 - \omega & 1 - \bar{\omega} & 0 \\
0 & 1 - \bar{\omega} & 3 & 1 - \omega & 0 \\
0 & 1 - \omega & 1 - \bar{\omega} & 3 & 1 - \omega \\
0 & 0 & 0 & 1 - \bar{\omega} & 3
\end{pmatrix}
\]

and the corresponding reflections have integral entries.

We now have the following.

**Proposition 16.** The lattice \(\mathcal{C}(G_{33}, 3)\) is arithmetic with adjoint trace field \(\mathbb{Q}\).

5.6. The group \(G_{34}\). The group \(\mathcal{C}(G_{34}, 3)\) is generated by 6 reflections \(R_1, \ldots, R_6\), the braid group is the same as the previous one, with one extra generator that commutes with the first four, and braids with length 3 with the fifth.
The Hermitian form in the basis given by suitably chosen vectors polar to the mirrors is given
\[
\begin{pmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & \alpha & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 1 & \sqrt{3} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 1
\end{pmatrix}.
\]

We must still have \((R_2R_4R_3)^2 = (R_4R_3R_2)^2 = (R_3R_2R_4)^2\), and this once again implies \(\alpha = i\) or \(\frac{\sqrt{3} - i}{2}\), but now both Hermitian forms have signature \((5, 1)\), so a priori it is not clear which group corresponds to \(C(G_{34}, 3)\).

Actually, according to Couwenberg-Heckman-Looijenga, the group \(C(G_{34}, 3)\) is non-cocompact, so there must be a cusp.

One easily checks that for \(\alpha = \frac{\sqrt{3} - i}{2}\), the group generated by \(R_2, R_3, R_4\) acts as a finite group on the span of \(v_2, v_3, v_4\) (in fact it gives the group \(G_{25}\), of order 648, with projeotiveization of order 216). In fact, the corresponding group would be cocompact, since all triples of reflections would then generate a finite group.

In particular, we must have \(\alpha = i\). In that case, \(Q = \text{diag}(1, \sqrt{3}, 1, \sqrt{3}, 1, \sqrt{3})\) gives a matrix with entries in \(\mathbb{Q}(\omega)\), so \(\mathbb{Q}(\text{tr Ad} \Gamma) = \mathbb{Q}\).

In order to write integral matrix generators, one simply extends the Hermitian matrix from the case \(G_{33}\), namely one can take
\[
\begin{pmatrix}
3 & 1 - \omega & 0 & 0 & 0 & 0 \\
1 - \bar{\omega} & 3 & 1 - \omega & 1 - \bar{\omega} & 0 & 0 \\
0 & 1 - \bar{\omega} & 3 & 1 - \omega & 0 & 0 \\
0 & 1 - \omega & 1 - \bar{\omega} & 3 & 1 - \omega & 0 \\
0 & 0 & 0 & 1 - \bar{\omega} & 3 & 1 - \omega \\
0 & 0 & 0 & 0 & 1 - \bar{\omega} & 3
\end{pmatrix},
\]
and the corresponding reflections have integral entries.

We have the following.

**Proposition 17.** The lattice \(C(G_{34}, 3)\) is arithmetic with adjoint trace field \(\mathbb{Q}\).

5.7. **Lattices derived from the group** \(G_{35}\). We can write the Hermitian matrix as
\[
\begin{pmatrix}
1 & r & 0 & 0 & 0 & 0 \\
r & 1 & r & 0 & 0 & 0 \\
0 & r & 1 & r & 0 & r \\
0 & 0 & r & 1 & r & 0 \\
0 & 0 & 0 & r & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]
where \(r = \frac{1}{\zeta - 1}, \zeta = e^{2\pi i/p}\). The corresponding reflections have entries in \(\mathbb{Z}[\zeta]\), hence we have the following.
Proposition 18. The lattices $\mathcal{C}(G_{35}, 3)$ and $\mathcal{C}(G_{35}, 4)$ are both arithmetic with adjoint trace field $\mathbb{Q}$.

5.8. The group $G_{36}$. We can write the Hermitian matrix as

$$
\begin{pmatrix}
1 & r & 0 & 0 & 0 & 0 \\
\bar{r} & 1 & r & 0 & 0 & 0 \\
0 & \bar{r} & 1 & r & 0 & 0 \\
0 & 0 & \bar{r} & 1 & r & 0 \\
0 & 0 & 0 & \bar{r} & 1 & 0 \\
0 & 0 & 0 & 0 & \bar{r} & 0 \\
\end{pmatrix}
$$

where $r = \frac{1}{\omega-1}$. This gives matrices with entries in $\mathbb{Z}[\omega]$, so we have:

Proposition 19. The lattices $\mathcal{C}(G_{36}, 3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.

5.9. Lattices derived from the group $G_{37}$. We can write the Hermitian matrix as

$$
\begin{pmatrix}
1 & r & 0 & 0 & 0 & 0 & 0 \\
\bar{r} & 1 & r & 0 & 0 & 0 & 0 \\
0 & \bar{r} & 1 & r & 0 & 0 & 0 \\
0 & 0 & \bar{r} & 1 & r & 0 & 0 \\
0 & 0 & 0 & \bar{r} & 1 & r & 0 \\
0 & 0 & 0 & 0 & \bar{r} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \bar{r} & 1 \\
\end{pmatrix}
$$

where $r = \frac{1}{\omega-1}$. This gives matrices with entries in $\mathbb{Z}[\omega]$, so we have:

Proposition 20. The lattices $\mathcal{C}(G_{37}, 3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.

6. The proof of Theorem 3

The fact that $\mathcal{C}(G_{29}, 3)$ is not cocompact is mentioned in the tables in [5]. The adjoint trace field was determined in section 5. The only thing that is left to prove is that the fact that it is not commensurable to the Deligne-Mostow group $\Gamma_\mu$ with $\mu = (3, 3, 3, 3, 5, 7)/12$.

This is not obvious, since both groups have the same rough commensurability invariants (both are non-uniform, have non-arithmeticity index one, and they have the same adjoint trace field).

We will argue by comparing the cusps in both groups. It is known that both groups have a single orbit of cusps, but we will show that the corresponding cusps are not commensurable.

6.1. The cusp of $\mathcal{C}(G_{29}, 3)$. It follows from the analysis in [5] that the lattice $\mathcal{C}(G_{29}, 3)$ has a single (conjugacy class) of cusps. In order to make the structure of this cusp explicit, we write the cusp generators in suitable form, see [18] for instance.
First note that the reflections $R_1$, $R_2$ and $R_4$ fix a point in the ideal boundary $\partial_\infty H_3^\mathbb{C}$, given in the basis used in section 5.2 by the vector

$$v = (1, \zeta(\omega - 1), 0, -\zeta^2(1 + i)).$$

We use this as the first vector, and (a suitable multiple of) $R_3v$ as the last basis vector. As the second vector, we use (a suitable multiple of) the polar vector to the mirror of $R_1$, and as the third one we take one that is orthogonal to both $v$ and $R_3v$, so that the matrices in that basis are simple enough.

Concretely, we take

$$Q = \begin{pmatrix}
1 & \zeta^2 - 2 & -\zeta^3 + \zeta^2 - \zeta + 1 & -2\zeta^2 + 3\zeta - 2 \\
\zeta^3 - 2\zeta & 0 & -\zeta^3 + 3\zeta^2 - \zeta & -3\zeta^2 + 6\zeta - 3 \\
-\zeta^3 - \zeta^2 + \zeta & 0 & \zeta^3 - 3\zeta^2 + \zeta & -\zeta^3 + 4\zeta^2 - 4\zeta + 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

Writing $S_j = Q^{-1} R_j Q$ (and denoting by $\ast$ entries that are irrelevant for us), we get

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
S_2 = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & -\omega & \omega & \zeta^3 - \zeta^2 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
S_4 = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & \frac{1}{2}\zeta^3 - \frac{1}{2}\zeta^2 - \frac{1}{2} \zeta + \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which preserve the Hermitian form

$$Q^* H Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & \bar{\omega} & -1 \\ 0 & \omega - 1 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Keeping the lower right 3x3 submatrices, we get

$$(8) A_1 = \begin{pmatrix} \omega & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\omega & \omega & \zeta^3 - \zeta^2 + 1 \\ 0 & 0 & 1 \end{pmatrix},
A_4 = \begin{pmatrix} 1 & -1 & \zeta^3 - \frac{1}{2}\zeta^2 - \frac{1}{2}\zeta + \frac{1}{2} \\ 0 & \omega & \frac{3}{2}\zeta^3 - \zeta^2 + \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives an explicit description of the cross section of the cusp as the quotient of $\mathbb{C}^2$ by an affine crystallographic group, i.e. by a subgroup of $U(2) \ltimes \mathbb{C}^2$ (note that the unitary group $U(2)$ as we write it is the group of a non-standard positive definite Hermitian form). Indeed, the linear part of the generators are given by the upper left 2x2 corner of the matrices $A_j$, whereas their translation parts are given by the first two entries of their third columns. In other words, writing $A_j$ in block form as follows

$$A_j = \begin{pmatrix} B_j & v_j \\ 0 & 1 \end{pmatrix},$$

the linear part of $A_j$ is given by $B_j$, and the translation part by $v_j$.

One checks that the matrices $B_1, B_2, B_4$ generate a group isomorphic to the Shephard-Todd group $G_5$. Indeed, the matrices $B_2$ and $B_4$ are reflections of order 3 and $br(B_2, B_4) =$
4, so they generate a copy of $G_5$ (see [4] for instance). One then checks that $B_1$ is in the group generated by $B_2$ and $B_4$, for instance $B_1 = B_2B_4^{-1}B_2B_4^{-1}$.

This proves the following.

**Proposition 21.** The cusp of $C(G_{29}, 3)$ is a central extension of a 2-dimensional affine crystallographic group generated by reflections, with linear part $G_5$.

We can also describe the corresponding subgroup of pure translations of this complex affine crystallographic group. Denote by $T_v$ the matrix

$$
\begin{pmatrix}
1 & 0 & v_1 \\
0 & 1 & v_2 \\
0 & 0 & 1
\end{pmatrix}
$$

**Proposition 22.** The subgroup of pure translations of $G = \langle A_1, A_2, A_4 \rangle$ is generated by

$$
\begin{align*}
T_{1,1} &= A_1A_2A_1^{-1}A_4^{-1}A_2^{-1}A_4 \\
T_{1,0} &= A_1A_2A_4^{-1}A_2^{-1}A_4A_1^{-1} \\
T_{2,\bar{\omega}} &= A_1^{-1}A_2^{-1}A_4A_1A_2A_1^{-1} \\
T_{0,\omega} &= A_1A_4A_2A_1^{-1}A_4^{-1}A_2^{-1}
\end{align*}
$$

**Proof:** Denote by $\Lambda$ the kernel of the map $\pi : G \to G_5$, and by $\Lambda_0$ the subgroup of $G$ generated by the four elements in the statement of the proposition. The fact that these elements are indeed translations follows from explicit computation using (8).

In particular $\Lambda_0 \subset \Lambda$. In order to show that this inclusion is an equality, we observe that the group $\langle x_1, x_2, x_4|x_3^3, x_3^3, x_4^3, \text{br}_3(x_1, x_2), \text{br}_4(x_2, x_4), \text{br}_2(x_1, x_4), w_1, w_2, w_3, w_4 \rangle$ has order 72, where $w_1 = x_1x_2^{-1}x_1^{-1}x_4^{-1}x_2^{-1}x_4$, $w_2 = x_1x_2x_4^{-1}x_2^{-1}x_4x_2^{-1} \ldots$ are the words from the statement of the proposition.

It is easy to see that the vectors $(1, 1), (1, 0), (\bar{\omega}, \bar{\omega}), (0, \omega)$ that appear in Proposition 22 generate the lattice $\mathbb{Z}[\omega] \times \mathbb{Z}[\omega]$, in other words they generate the same lattice in $\mathbb{C}^2$ as $(1, 0), (\omega, 0), (0, 1)$ and $(0, \omega)$. We summarize this as follows.

**Proposition 23.** The pure translation group of the cusp of $C(G_{29}, 3)$ is given by $\Lambda = \mathbb{Z}[\omega] \times \mathbb{Z}[\omega]$.

6.2. The cusp of the Deligne-Mostow non-arithmetic lattice in $PU(3, 1)$. We now review some facts about the cusp of the group $\Gamma_\mu$ where $\mu = (3, 3, 3, 3, 5, 7)/12$, see §15 of [8] (for notational convenience in the computations with braid groups, we have reordered the exponents). We write $\alpha_j = e^{2\pi i \mu_j}$.

We will write explicit generators for $\Gamma_\mu$. The starting point is the observation that $\Gamma_\mu$ is a homomorphic image of a subgroup of the braid group $B_6$, namely $\phi^{-1}(\Sigma)$, where $\phi : B_6 \to S_6$ is the natural homomorphism to the symmetric group, and $\Sigma \simeq S_3$ is generated by (12), (23) and (34). Recall that $B_6$ has a presentation of the form

$$\langle \sigma_1, \ldots, \sigma_5 | \text{br}_3(\sigma_j, \sigma_{j+1}), j = 1, 2, 3, 4, \text{br}_2(\sigma_j, \sigma_k) \text{whenever} |j - k| \geq 2 \rangle,$$

where the $\sigma_j$ correspond to a half-twist between the strands $j$ and $j + 1$. 
For notational convenience, we assume that $\sigma_1$ is a half-twist between $x_6$ and $x_1$, and $\sigma_j, j = 2, \ldots 5$ is a half-twist between $x_{j-1}$ and $x_j$. We write

$$H = \phi^{-1}(\Sigma).$$

It is clear that $\sigma_1^2, \sigma_2, \sigma_3, \sigma_4, \sigma_5^2$ are in $H$, and one checks (using the Reidemeister-Schreier method) that these generate a subgroup of index 30 in $\mathcal{B}_6$. It follows that these five elements generate $H$.

The monodromy transformations corresponding to these braids can be computed as in section §12 of [7]. We pick a tree $T$ joining all 6 vertices, with 5 edges and no branch point, and denote by $T_1$ the tree obtained from $T$ by deleting the last vertex, and $T_2$ the tree obtained by deleting the first 5 vertices (hence $T_2$ has a single vertex and no edge).

We denote by $w_1, \ldots, w_5$ for the elements of $H_1^f(P_0, \hat{L})$ obtained from the tree $T$. Since $H_1^f(P_0, \hat{L})$ has dimension 4, there must be a relation between these vectors, which can be computed by writing $w_5$ in two ways, using parallel transport.

This gives

$$w_5 = \overline{\alpha_6} \{ w_5 + w_4 + w_3 + w_2 + w_1 - \alpha_5 w_1 - \alpha_5 \overline{\alpha_5} w_2 - \alpha_5 \overline{\alpha_5} \overline{w_3} - \alpha_5 \overline{\alpha_5} \overline{w_4} \},$$

where we have written $\alpha$ for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Solving this equation for $w_5$ allows one to express it as a linear combination of $w_1, w_2, w_3, w_4$.

We write $\rho = \rho_{\mu, \Sigma}$ for the monodromy representation. One then computes the matrices $M_1 = \rho(\sigma_1^2), M_2 = \rho(\sigma_2), M_3 = \rho(\sigma_3), M_4 = \rho(\sigma_4), M_5 = \rho(\sigma_5^2)$ as in [7] (p.70-72).

Explicitly, we have

$$M_1 = \begin{pmatrix} \alpha \alpha_5 & \alpha(1 - \alpha_5) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -\alpha & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -\alpha & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\alpha \end{pmatrix}, M_5 = \begin{pmatrix} 1 & 0 & 0 & \overline{\alpha_5} - 1 \\ 0 & 1 & 0 & \overline{\alpha_5} \alpha - 1 \\ 0 & 0 & 1 & \overline{\alpha_5} \overline{\alpha_5} - 1 \\ 0 & 0 & 0 & \overline{\alpha_5} \overline{\alpha_5} \overline{\alpha_5} \overline{\alpha_5} - 1 \end{pmatrix},$$

where $\alpha = \zeta^3 = i$ and $\alpha_5 = \zeta^5$. Up to scaling, there is a unique (nonzero) Hermitian form preserved by the group $\Gamma_{\mu}$. Writing out a generic Hermitian matrix $H$, the conditions $M_j^* H M_j = H$ give equations for the entries of $H$. Explicitly, we have

$$H = \begin{pmatrix} 1 & \beta & 0 & 0 \\ \bar{\beta} & \gamma & \beta & 0 \\ 0 & \bar{\beta} & \gamma & \beta \\ 0 & 0 & \bar{\beta} & \gamma \end{pmatrix},$$

where $\beta = - (\zeta^3 + \zeta^2 + \zeta + 1)/3, \gamma = 1 + \frac{1}{\sqrt{3}}.$
We now study the cusp for this group, which is generated by $M_2, M_3, M_4$. We consider the span $V$ of $e_2, e_3, e_4$. The restriction of the Hermitian form is degenerate, with 1-dimensional kernel corresponding to the common fixed point of $M_2, M_3, M_4$. This gives $v = (0, i, 1 + i, 1)$.

As in the previous section, we build a basis of $\mathbb{C}^4$ by using $v, R_1 v$ and suitable vectors orthogonal to both of these. Explicitly, we take

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 - i \\ i & 0 & 0 & -2 - \zeta - \zeta^2 - \zeta^3 \\ 1 + i & 2 & -1 - i & -2 + 2\zeta \\ 1 & 1 - i & 0 & \zeta - \zeta^2 + \zeta^3 \end{pmatrix}$$

and get the following conjugates $S_j = Q^{-1} R_j Q$:

$$S_2 = \begin{pmatrix} 1 & 2 & -1 - i & -1 + i \\ 0 & -i & i & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ We keep only the lower-right 3x3 blocks:

$$A_2 = \begin{pmatrix} -i & i & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & -i & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whose upper left 2x2 gives the linear part in the description of the cusp cross section as a quotient of $\mathbb{C}^2$ by a subgroup of $U(2) \ltimes \mathbb{C}^2$; the top two components of the last column give the translation part (see the notation in section 6.1).

One easily checks that the linear parts $B_2, B_3, B_4$ generate a copy of the Shephard-Todd group $G_8$, which has order 96 (and center of order 4). This gives the first part of the following proposition.

**Proposition 24.** The cusp cross section is a quotient of $\mathbb{C}^2$ by a subgroup of $U(2) \ltimes \mathbb{C}^2$ with linear part $G_8$, and translation subgroup given by $\mathbb{Z}[i] \times \mathbb{Z}[i]$.

**Proof:** It is clear that the translation subgroup is contained in $\mathbb{Z}[i] \times \mathbb{Z}[i]$. We claim that it is precisely equal to it, which can be checked by computing,

(9) \hspace{1cm} T_{1,0} = A_2 A_3^2 A_4^{-1} A_2^2 \\
(10) \hspace{1cm} T_{i,0} = A_3^{-2} A_4^{-1} A_3^{-1} A_2 \\
(11) \hspace{1cm} T_{0,1} = A_2^{-1} A_3 A_4^{-1} A_2 A_3 A_2^{-1} \\
(12) \hspace{1cm} T_{0,i} = A_2 A_3^{-1} A_4^{-1} A_2^{-1} A_3$

and checking that the group

$$\langle x_2, x_3, x_4, x_2^3, x_3^3, x_4^3, br_3(x_2, x_3), br_3(x_3, x_4), br_2(x_2, x_4), w_1, w_2, w_3, w_4 \rangle$$

has order 96 (where $w_1 = x_2 x_3 x_2^{-1} x_3^4, \ldots$ are the words that appear on the right hand sides of (9)). \qed
6.3. Incommensurability of the cusps. We denote by $\Lambda$ the lattice corresponding to the cusp of $C(G_{29}, 3)$, which is the $\mathbb{Z}$-span of $v_1 = (1, \omega), v_2 = (1, -\omega), v_3 = (\bar{\omega}, -1), v_4 = (-\bar{\omega}, -1)$.

The following should be well known to experts, but since we could not find it in the literature, we give a detailed elementary proof.

**Proposition 25.** There is no $M \in \text{GL}(2, \mathbb{C})$ that maps a sublattice of $\mathbb{Z}[i] \times \mathbb{Z}[i]$ to a sublattice of $\Lambda$.

**Proof:** We argue by contradiction. Assume such a matrix exists, and write

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + i \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},$$

for real matrices $A$ and $B$. Viewing $\mathbb{C}^2$ as $\mathbb{R}^4$ and using real coordinates $(x_1, x_2, y_1, y_2)$, the action of $M$ is given by the $4 \times 4$ matrix

$$\begin{pmatrix} a_1 & a_2 & -b_1 & -b_2 \\ a_3 & a_4 & -b_3 & -b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix}.$$

Let us write $(p_j, q_j, r_j, s_j) \in \mathbb{Z}^4, j = 1, 2, 3, 4$ for a basis for the sublattice of $\mathbb{Z}[i] \times \mathbb{Z}[i]$, and assume it is sent to a basis of $\Lambda$. We may write the image of the $q$-th basis vector as $j_q v_1 + k_q v_2 + l_q v_3 + m_q v_4$, for some $j_q, k_q, l_q, m_q \in \mathbb{Z}$.

Then

$$\begin{pmatrix} a_1 & a_2 & -b_1 & -b_2 \\ a_3 & a_4 & -b_3 & -b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} p_t \\ q_t \\ r_t \\ s_t \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} j_t \\ k_t \\ l_t \\ m_t \end{pmatrix}$$

for all $t = 1, 2, 3, 4$.

Since the $(p_t, q_t, r_t, s_t)$ form a basis of $\mathbb{Q}^4$, and so do the $(j_t, k_t, l_t, m_t)$, the matrix

(13)

$$U = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & a_2 & -b_1 & -b_2 \\ a_3 & a_4 & -b_3 & -b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 + \frac{b_1}{\sqrt{3}} & a_2 + \frac{b_2}{\sqrt{3}} & -b_1 + \frac{a_1}{\sqrt{3}} & -b_2 + \frac{a_2}{\sqrt{3}} \\ a_3 + \frac{b_3}{\sqrt{3}} & a_4 + \frac{b_4}{\sqrt{3}} & -b_3 + \frac{a_3}{\sqrt{3}} & -b_4 + \frac{a_4}{\sqrt{3}} \\ 2a_1 & 2a_2 & 2b_1 & 2b_2 \\ 2a_3 & 2a_4 & 2b_3 & 2b_4 \end{pmatrix}$$

must be have rational entries. The second and fourth row show that $M$ must be $\sqrt{3}$ times a rational matrix. If that is the case, then the first and third equation show that $M$ must be also be rational.

Since $\mathbb{Q} \cap \sqrt{3} \mathbb{Q} = \{0\}$, this implies $M = 0$, which contradicts the fact that it is invertible. □
7. Proof of Theorem 3.

Proof: If the lattices were commensurable, their cusps should be commensurable. This is impossible given the above description of their respective cusps, and Proposition 25. □

8. Presentations

From the above results, one can easily obtain explicit presentations for the CHL lattices. Indeed, recall that we denote $V = \mathbb{C}^{n+1}$, $V^0 \subset V$ the complement of the arrangement (given by the union of the mirrors of reflections in $G$). According to Theorem 7.1 in [5], a presentation is given by adjoining to a presentation of the braid group $\pi_1(G\setminus V^0)$ specific relations corresponding to the (irreducible) strata in the arrangement. More specifically, for each irreducible stratum $L$, consider the set of mirrors $\mathcal{H}_L$ that contain $L$, and the braid group $G_L$ generated by the reflections in the elements in $\mathcal{H}_L$, which has infinite cyclic center, generated by an element $\alpha_L$. If we denote the monodromy representation by $\rho: \pi_1(G\setminus V^0) \to \Gamma$, the CHL relations correspond to imposing the order of $\rho(\alpha_L)$.

Presentations $\pi_1(\mathbb{P}(V^0/G))$ are given in [3] (some of the results given there were conjectural at the time, but the proof of their validity was given by Bessis in [2]). It is easy to determine conjugacy classes of loops corresponding to the conjugacy classes described in section 7.1 of [5], by determining the conjugacy classes of (irreducible) mirror intersections in $G$, and then taking a generator of the center of each stabilizer.

We list the results for 3-dimensional groups in Tables 9 through 12, where we give the order of the monodromy image of central elements for each irreducible stratum. Strictly speaking, not all these relations are needed in order to get a presentation (see the precise statement in section 7.1 of [5]).

For example for groups derived from $G_{28}$ (see Table 9), the first three columns correspond to irreducible codimension 2 strata, whereas the last two correspond to codimension 3.

In codimension 2, we list $(R_1 R_2)^3$ which generates the center of the group generated by $R_1$ and $R_2$ in the braid group they generate (in that case $\text{br}(R_1, R_2) = 3$). We also list $(R_2 R_3)^2$, since $\text{br}(R_2, R_3) = 4$ (see section 2.2 [17] for a discussion of braid groups generated by two elements).

In codimension 3, generators for the center are a bit more complicated to obtain, but they are listed in [4], for instance. One can also check their result by using the explicit matrices described in our paper. For example, we list $(R_1 R_2 R_3)^3$ which generates the center of the braid group generated by $R_1$, $R_2$ and $R_3$. Indeed, these generate a braid group of type $G_{26}$, and a generator for the center is given in the fifth column of Table 1 in [4].

For some strata, the central element is a bit trickier to determine. For instance, for the lattices derived from $G_{29}$, the braid group generated by $R_2$, $R_3$ and $R_4$ has type $G(4, 4, 3)$, i.e. $G(e, e, r)$ with $e = 4$, $r = 3$ on page 186 in [4]. One needs to be a bit careful with the order of the generators, since $(R_4 R_3 R_2)^8$ generates the center, but $(R_2 R_3 R_4)^8$ is not even central in that group.
Table 7. Central elements in the stabilizer of mirror intersections and their orders, for groups derived from $A_4$; boldface indicates that the corresponding relation is needed in the presentation.

<table>
<thead>
<tr>
<th>$(p, q)$</th>
<th>$(R_1R_2)^3$</th>
<th>$(R_1R_2R_3)^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 4)$</td>
<td>4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$(2, 5)$</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>$(2, 6)$</td>
<td>6</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$(2, 8)$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$(3, 3)$</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$(3, 4)$</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>$(3, 6)$</td>
<td>$\infty$</td>
<td>6</td>
</tr>
<tr>
<td>$(4, 3)$</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>$(4, 4)$</td>
<td>$\infty$</td>
<td>4</td>
</tr>
<tr>
<td>$(4, 8)$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$(6, 3)$</td>
<td>$\infty$</td>
<td>2</td>
</tr>
<tr>
<td>$(6, 4)$</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>$(6, 6)$</td>
<td>6</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$(10, 5)$</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>$(12, 3)$</td>
<td>12</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 8. Central elements in the stabilizer of mirror intersections and their orders, for groups derived from $B_4$; boldface indicates that the corresponding relation is needed in the presentation.

Note also that for $G_{31}$, the stabilizer of the statum $L_{235}$ corresponding to the intersection of the mirrors of $R_2$, $R_3$ and $R_5$ is not always the same braid group, namely it has type $G(2p, 2p, 3)$.

9. Volumes

It follows from the results in [5] that the quotient of a CHL lattice derived from the Shephard-Todd group $G$ is given as a normal analytic space by $\hat{X}/G$, where $\hat{X}$ is a fairly explicit birational modification of $\mathbb{P}^n$. The information is contained in the proof of Theorem 6.2 in [5].

For simplicity, in the discussion below, we assume $\kappa_L$ is never equal to 1 (if $\kappa_L = 1$, the developing map sends the corresponding strata to the boundary of complex hyperbolic space, and they do not contribute to volume). As mentioned in [5], the developing map
of the Couwenberg-Heckman-Looijenga geometric structures are holomorphic on the space obtained by blowing up all mirror intersections $L$ in the arrangement with $\kappa_L > 1$, but it is in general not a local biholomorphism, the developing map actually contracts some of the exceptional divisors to curves. More precisely, the developing map maps strata of dimension $d$ to subspaces of codimension $d$, see Proposition 6.9 in [5]. Here the dimension refers to the dimension of the linear subspaces in $\mathbb{C}^{n+1}$.

We will only handle the 3-dimensional case (the 2-dimensional case is trivial, but in higher dimensions, it is probably much more complicated). We denote by $\hat{X}$ the space

<table>
<thead>
<tr>
<th>$(p, q)$</th>
<th>$(R_1 R_2)^3$</th>
<th>$(R_2 R_3)^2$</th>
<th>$(R_3 R_4)^3$</th>
<th>$(R_1 R_2 R_3)^3$</th>
<th>$(R_2 R_3 R_4)^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 4)</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>1</td>
<td>6</td>
<td>$\infty$</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>(2, 8)</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>(2, 12)</td>
<td>1</td>
<td>12</td>
<td>4</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>2</td>
<td>12</td>
<td>4</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>2</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>(3, 12)</td>
<td>2</td>
<td>12</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>2</td>
<td>4</td>
<td>$\infty$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(6, 6)</td>
<td>2</td>
<td>6</td>
<td>$\infty$</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 9. Central elements in the stabilizer of mirror intersections and their orders, for groups derived from $G_{28}$; boldface indicates that the corresponding relation is needed in the presentation.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$(R_1 R_2)^3$</th>
<th>$(R_2 R_4)^2$</th>
<th>$(R_1 R_2 R_3)^4$</th>
<th>$(R_1 R_2 R_4)^3$</th>
<th>$(R_1 R_3 R_2 R_4)^8$</th>
<th>$(R_1 R_2 R_3^{-1} R_4 R_3)^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>$\infty$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>4</td>
<td>1</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table 10. Groups derived from $G_{29}$; boldface required in presentation.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$(R_1 R_2)^3$</th>
<th>$(R_3 R_4)^5$</th>
<th>$(R_1 R_2 R_3)^4$</th>
<th>$(R_2 R_3 R_4)^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11. Groups derived from $G_{30}$; boldface required in presentation.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$(R_1 R_2)^3$</th>
<th>$(R_1 R_4)^p$</th>
<th>$(R_5 R_2 R_1)^{4p/(2p \wedge 3)}$</th>
<th>$(R_2 R_3 R_5)^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>$\infty$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 12. Groups derived from $G_{31}$. Note $4p/(2p \wedge 3)$ is 4 for $p = 3$, and 20 for $p = 5$. Boldface required in presentation.
obtained by blowing up $X = \mathbb{P}^3$ at orbits of strata $L$ in the mirror arrangement of $G$ (the blow-up should be performed along strata of increasing dimensions). The exceptional divisors corresponding to lines that get blown-up are then isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which we then contract in the other direction (these correspond to linear subspaces of $\mathbb{C}^4$ of dimension 2, so the developing map sends them to complex curves). Of course, in general, this contraction may produce singularities.

We denote by $\pi : \hat{X} \to X$ the blow-up, and by $f : \hat{X} \to Y$ the corresponding contraction, see diagram (14).

\begin{equation}
\begin{array}{ccc}
M, D, E \subset \hat{X} & \xrightarrow{\pi} & f \\
\pi_*M \subset X & \xrightarrow{\pi} & f_*M, f_*D \subset Y
\end{array}
\end{equation}

Here $D$ corresponds to the exceptional locus above points, $E$ is above lines in the arrangement, and $M$ corresponds to the proper transform in $\hat{X}$ of the arrangement in $\mathbb{P}^3$.

For simplicity, we start by assuming that every divisor in the above diagram is a $G$-orbit of a single irreducible divisor, in which case the relevant orbifold pair is given by $(Y, \Delta)$ where

$$\Delta = (1 - \frac{2}{p})f_*M + (1 - \frac{1}{m})f_*D.$$ 

We then intend to compute

$$c_1(Y, \Delta)^3 = 16c_3(Y, \Delta),$$

which will give us the orbifold Euler characteristic of the ball quotient.

Indeed, the corresponding ball quotient is given by $Y/G$, and since the quotient by $G$ has degree $|\mathbb{P} G|$, we have

$$c_3(Y/G, \mathcal{D}) = c_3(Y, \Delta)/|\mathbb{P} G|.$$

where $\Delta = \varphi^*\mathcal{D}$, and $\varphi : Y \to Y/G$ denotes the quotient map. Beware that the orbifold weights need not be the same on the level of $Y$ or $Y/G$, since the quotient map ramifies.

In order to compute $c_1(Y, \Delta)^3$, note that $c_1$ is the opposite of the log-canonical, and compute $(K_Y + \Delta)^3 = (f^*(K_Y + \Delta))^3$ on the level of $\hat{X}$. This is most readily done by using the map $\pi : \hat{X} \to \mathbb{P}^3$.

When $M$ consists of several $G$-orbits of irreducible divisors, the corresponding orbits usually pick up different orbifold weights, so one should replace $(1 - \frac{1}{p})f_*M$ by a sum $\sum_j (1 - \frac{1}{p_j})M_j$ (it turns out in all CHL examples, there are at most two such orbits, i.e. the sum has at most two terms). A similar remark is of course in order for $f_*D$ and $E$, since in general we may have to blow-up several $G$-orbits of strata.

The results of the volume computations are given in Table 14 on p. 48, see also Table 13 on p. 45 for groups commensurable with Deligne-Mostow groups.
9.1. Cases where no blow-up is needed. The above computation is extremely easy to perform in the few cases where no blow-up is needed, i.e. when \( X = \hat{X} = Y \) and the orbifold locus is simply supported by the hyperplane arrangement.

For instance, in the \( A_4 \) case and \( p = 4 \), the log-canonical divisor is numerically equivalent to \((-4 + 10(1 - \frac{2}{p}))H\), where \( H \) denotes the class of a hyperplane, so \((K_X + D)^3 = 1\). We have \(|\mathbb{P}G| = |G| = 120\), and 3-dimensional ball quotients satisfy \( c_1(X)^3 = 16c_3(X) \), so the Euler characteristic is given by

\[
\frac{1}{120 \cdot 16} = -\frac{1}{1920}.
\]

This is the orbifold Euler characteristic of the Deligne-Mostow lattice for \( \mu = (1, 1, 1, 1, 1, 3)/4 \), see Table 3 in [16].

The are a few more cases where the computation is as simple as the one we just performed. For instance, consider \( C(G_{28}, 2, 4) \). For \( G = G_{28} \), \(|\mathbb{P}G| = 1152/2 = 576\) and \( K_X + D \) is numerically equivalent to \((-4 + 12(1 - \frac{2}{2}) + 12(1 - \frac{2}{4}))H\), so that \((K_X + D)^3 = 8\),

\[
c^\text{orb}_1(C(G_{28}, 2, 4))^3 = \frac{8}{576} = -\frac{1}{72},
\]

and

\[
\chi^\text{orb}(C(G_{28}, 2, 4)) = -\frac{1}{72 \cdot 16} = -\frac{1}{1152}.
\]

For \( C(G_{28}, 3, 3) \), the same computation gives

\[
\chi^\text{orb}(C(G_{28}, 3, 3)) = -\frac{1}{144}.
\]

For all other 3-dimensional lattices, some blowing-up is needed, so the computations are a bit more intricate.

9.2. Cases where we only blow-up orbits of points.

9.3. The \( A_4 \) case. For the \( A_4 \) arrangement and the case \( p = 5 \), we have \( \kappa_{L_{12}} = 1 - \frac{1}{10} \) and \( \kappa_{L_{123}} = 1 + \frac{1}{5} > 1 \), so we need to blow up the five points in the orbit of \( L_{123} \). Let \( \pi : \hat{X} \to X \) denote that blow up.

We denote by \( M = M_1 + \cdots + M_{10} \) the proper transform in \( \hat{X} \) of the arrangement. Since there are mirrors through \( L_{123} \), we have

\[
\pi^*\pi_* M = M + 6D, \quad K_{\hat{X}} = \pi^* K_X + 2D.
\]
In the last formula, the factor 2 comes from the codimension minus one for the locus blown-up in \( \mathbb{P}^3 \). We then compute
\[
\left( K_X + (1 - \frac{2}{p})M + (1 - \frac{1}{m})D \right)^3 = \left( \pi^* K_X + (1 - \frac{2}{p})M + (3 - \frac{1}{m})D \right)^3 = (\pi^*(K_X + A) + \alpha D)^3 = \lambda^3 + 5\alpha^3, \]
where
\[
\lambda = -4 + 10(1 - \frac{2}{p}) \\
\alpha = 3 - \frac{1}{m} - 6(1 - \frac{2}{p}),
\]
and we have written
\[A = (1 - \frac{2}{p})\pi_* M.\]

Note that \( m \) is the number given by the Schwarz condition for the stratum \( L_{123} \), i.e. \( m = 5 \) for \( p = 5 \) (see the tables in section 11). More specifically, it is defined by the relation
\[
\kappa_{L_{123}} = 1 + \frac{1}{m}.
\]
This gives \((c^\text{orb}_1)^3 = 136/25\), hence
\[
\chi^\text{orb} = c^\text{orb}_3 = -\frac{136}{25 \cdot 120 \cdot 16} = -\frac{17}{6000}.
\]
This agrees with the Euler characteristic of the Deligne-Mostow lattice for with \( \mu = (3, 3, 3, 3, 3, 5)/10 \).

The same formula also works for \( p = 6 \), if we take \( m = 3 \), where we get
\[
\chi^\text{orb} = -\frac{1}{270}.
\]
This is coherent with the formula in [16], note that this lattice has index \( 6 = 6!/5! \) in the corresponding Deligne-Mostow lattice, i.e. the one with \( \mu = (1, 1, 1, 1, 1, 1)/3 \), and
\[
-\frac{1}{6} \cdot \frac{1}{270} = -\frac{1}{1620}.
\]

9.3.1. Groups derived from \( G_{28} \). We treat the cases \((p_1, p_2) = (2, 5)\) (where we need to blow-up points in the orbit of \( L_{234} \)) and \((3, 4)\) (where we need to blow-up two orbits of points, namely \( L_{234} \) and \( L_{123} \)).

We do the computations for the case where we blow-up two orbits, the other one is similar (or simply remove the term coming from the corresponding exceptional divisor). Denote by \( \pi : \widehat{X} \to X \) the corresponding blow-up, and by \( M_1 \) and \( M_2 \) the proper transform of the two orbits of mirrors in \( G_{28} \).
Also write $D_1$ (resp. $D_2$) for the exceptional above the orbit of $L_{123}$ (resp. $L_{234}$). We have

\begin{align*}
K_X = \pi^*K_X + 2D_1 + 2D_2 \\
\pi^*\pi_*M_1 = 6D_1 + 3D_2 \\
\pi^*\pi_*M_2 = 3D_1 + 6D_2
\end{align*}

The relevant divisor for the orbifold pair is

$$\Delta = (1 - \frac{2}{p_1})M_1 + (1 - \frac{2}{p_2})M_2 + (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2,$$

and we need to compute

$$\begin{align*}
(K_X + \Delta)^3 &= \left(\pi^*K_X + (1 - \frac{2}{p_1})M_1 + (1 - \frac{2}{p_2})M_2 + (3 - \frac{1}{m_1})D_1 + (3 - \frac{1}{m_2})D_2 \right) \\
&= (\lambda\pi^*H + \alpha_1D_1 + \alpha_2D_2)^3 \\
&= \lambda^3 + 12\alpha_1^3 + 12\alpha_2^3
\end{align*}$$

where

$$\begin{align*}
\lambda &= -4 + 12(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha_1 &= 3 - \frac{1}{m_1} - 6(1 - \frac{2}{p_1}) - 3(1 - \frac{2}{p_2}) \\
\alpha_2 &= 3 - \frac{1}{m_2} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}).
\end{align*}$$

The factors 12 in the last formula come from the fact that each $D_j, j = 1, 2$ has 12 components, note also that $D_j^3 = 1$.

Once again, the values for $m_1$ and $m_2$ can be computed from the Schwarz condition using the formulas in [5], they are also tabulated in section 11. More specifically, they are given by

$$\kappa_{L_{123}} = 1 + \frac{1}{m_1}, \quad \kappa_{L_{234}} = 1 + \frac{1}{m_2}.$$ 

For $(p_1, p_2) = (3, 4)$, we take $m_1 = 3$ and $m_2 = 6$, and get

$$\chi_{\text{orb}}(C(G_{28}, 3, 4)) = -\frac{23}{1152}.$$ 

For $(p_1, p_2) = (2, 5)$, we remove the term corresponding to $D_1$ (i.e. the exceptional above the orbit of $L_{123}$, which is not supposed to get blown up). In other words, with the same notation for $\alpha$ and $\lambda$, we compute

$$(K_X + \Delta)^3 = \lambda^3 + 12\alpha_2^3.$$ 

Taking $p_1 = 2, p_2 = 5, m = 5$, we get

$$\chi_{\text{orb}}(C(G_{28}, 2, 5)) = -\frac{13}{4500}.$$
9.3.2. Groups derived from $G_{29}$. For $p = 3$, the same method as the one in the previous section gives

$$(K_{\hat{X}} + \Delta)^3 = \lambda^3 + 20\alpha^3,$$

where

$$\begin{align*}
\lambda &= -4 + 40(1 - \frac{2}{p}) \\
\alpha &= 3 - \frac{1}{m} - 12(1 - \frac{2}{p}).
\end{align*}$$

Taking $m = 3$ (see the values in section 11), we get

$$\chi_{\text{orb}}(\mathcal{C}(G_{29}, 3)) = -\frac{323}{12960}.$$ 

For $p = 4$, we need to blow up more points, but the formula is similar, we get

$$(K_{\hat{X}} + \Delta)^3 = \lambda^3 + 60\alpha^3,$$

where

$$\begin{align*}
\lambda &= -4 + 60(1 - \frac{2}{p}) \\
\alpha_1 &= 3 - \frac{1}{m_1} - 12(1 - \frac{2}{p}) \\
\alpha_2 &= 3 - \frac{1}{m_2} - 9(1 - \frac{2}{p}).
\end{align*}$$

Taking $m_1 = 1$ and $m_2 = 2$, we get

$$\chi_{\text{orb}}(\mathcal{C}(G_{29}, 4)) = -\frac{13}{160}.$$ 

9.3.3. Groups derived from $G_{30}$. The same method as the one in the previous section gives

$$(K_{\hat{X}} + \Delta)^3 = \lambda^3 + 60\alpha^3,$$

where

$$\begin{align*}
\lambda &= -4 + 60(1 - \frac{2}{p}) \\
\alpha &= 3 - \frac{1}{m} - 15(1 - \frac{2}{p}).
\end{align*}$$

Taking $p = 3$, $m = 3/2$, we get

$$\chi_{\text{orb}}(\mathcal{C}(G_{30}, 3)) = -\frac{52}{2025}.$$ 

9.3.4. Groups derived from $G_{31}$. The same method as the one in the previous section gives

$$(K_{\hat{X}} + \Delta)^3 = \lambda^3 + 60\alpha^3,$$

where

$$\begin{align*}
\lambda &= -4 + 60(1 - \frac{2}{p}) \\
\alpha &= 3 - \frac{1}{m} - 15(1 - \frac{2}{p}).
\end{align*}$$
Taking $p = 3$, $m = 3/2$, we get
\[ \chi_{\text{orb}}(C(G_{31}, 3)) = -\frac{13}{810}. \]

9.4. Cases where we blow-up orbits of points and lines.

9.4.1. Preliminary calculations. We start with some preliminary computations that will be used in all volume computations that follow. Let $\pi = \pi_2 \circ \pi_1 : \hat{X} \to \mathbb{P}^3$ be obtained by blowing up $n$ points on a projective line $L$, then blowing up the strict transform of $L$ (we will always assume $n \geq 2$). Denote by $D_1, \ldots, D_n$ the exceptional locus over the points that were blown up in $\pi_1$, and by $E$ the exceptional divisor over $L$.

Note that $E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, in particular $\text{Pic}(E) \simeq \mathbb{Z}l_1 \oplus \mathbb{Z}l_2$, where we assume $l_1$ projects to $L$ in $\mathbb{P}^3$. We then have
\[ E|_E = -l_1 - (n - 1)l_2, \quad E^3 = (-l_1 - (n - 1)l_2)^2 = 2n - 2. \]
We will be interested in the space obtained from $\hat{X}$ by contracting $E$ to a $\mathbb{P}^1$, by contracting the factor given by $l_1$. Note that the space $Y$ is singular (unless $n = 2$). We denote by $f : \hat{X} \to Y$ the corresponding map.

One verifies that
\[ K_{\hat{X}} = f^*K_Y - \frac{n - 3}{n - 1}E. \]

We will also need to study $f^*f_*Z$ for various divisors $Z$. The first one is the proper transform of a plane $H$ in $\mathbb{P}^3$.

**Proposition 26.** Let $Z \subset \hat{X}$ be the proper transform of a plane in $H \subset \mathbb{P}^3$.

1. If $H \cap L$ is one of the points blown-up in $\pi_1$, or if $H$ contains $L$, then $f^*f_*Z = Z$.
2. If $H \cap L$ is a point which is not one of the points blown-up in $\pi_1$, then
\[ f^*f_*Z = Z + \frac{1}{n - 1}E. \]

The first case follows from elementary properties of blow-ups, the second follows from the fact that $Z|_E$ is equivalent to $l_2$. Indeed, if we write $f^*f_*Z = Z + \alpha E$ for some $\alpha \in \mathbb{Q}$, intersecting both sides the fiber $l_1$ that gets contracted in $f$, we get
\[ 0 = l_1 \cdot l_2 + \alpha l_1 (l_1 - (n - 1)l_2), \]
which gives the desired formula (note that the assumption on $H$ implies that the proper transform restricts to $l_2$ in $E$).

Also, for any $D_j$, $D_j|_E$ is equivalent to $l_2$, so we also have a similar formula
\[ f^*f_*D_j = D_j + \frac{1}{n - 1}E. \]
9.4.2. The case $C(A_4, 8)$. We treat the group derived from $A_4$ with $p = 8$ in detail, which corresponds to the Deligne-Mostow group for $\mu = (1, 3, 3, 3, 3, 3)/8$.

In this case the mirrors can be thought of as the 3-dimensional analogue of the complete quadrilateral, i.e. we take 5 points in general position and take as mirrors the planes through triples of points (there are 10 of these).

For the details of the combinatorics of the configuration, see Figure 1. The strata $L$ with $\kappa_L > 1$ consist of the original five points, and the 10 lines that join them.

We denote by $\pi_1 : \tilde{X} \to X$ the blow-up of $X = \mathbb{P}^3$ at five points, by $\pi_2 : \tilde{X} \to \tilde{X}$ the further blow-up of the proper transform of the 10 lines joining the 5 points, and $\pi = \pi_2 \circ \pi_1$.

Let $E$ be the exceptional divisor of $\pi_2$, and let $E_j, j = 1, \ldots, 10$ denote the components of $E$ (so that $E_j$ is the exceptional locus above one of the lines joining two of the five points).

The space $Y$ is obtained by contracting the fibers these copies of $\mathbb{P}^1 \times \mathbb{P}^1$ in the other direction than $\pi_2$, which yields a smooth space (this operation is in fact a flop). We then use the formulas discussed in section 9.4.1. If $f : \tilde{X} \to Y$ denotes the contraction, we have

$$K_{\tilde{X}} = f^* K_Y + E.$$ 

It is easy to check that $f^* f_* M = M + E$ (one needs to count the planes in the arrangement that intersect each given line $\pi_*(L_j)$ away from the points blown-up, it turns out this number is one). Also $f^* f_* D = D + 2E$, since there each line blown-up intersects precisely two $D_k$ (see section 11).

Now we get

$$\left(K_Y + (1 - \frac{2}{p}) f_* M + (1 - \frac{1}{m}) f_* D\right)^3 =$$

$$\left(K_{\tilde{X}} - E + (1 - \frac{2}{p})(M + E) + (1 - \frac{1}{m})(D + 2E)\right)^3 =$$

$$\left(\pi^* K_X + 2D + E - E + (1 - \frac{2}{p})(M + E) + (1 - \frac{1}{m})(D + 2E)\right)^3 =$$

$$\left(\pi^* K_X + (1 - \frac{2}{p}) M + (3 - \frac{1}{m}) D(-2(1 - \frac{2}{p}) + 2(1 - \frac{1}{m})) E\right)^3.$$ 

Note that

$$\pi^* \pi_* M = M + 6D + 3E,$$

since there are six mirrors through each point blown-up, and three mirrors containing each line blown-up.

Setting $A = (1 - \frac{2}{p}) \pi_* M$, we have

$$(K_Y + \Delta)^3 = (\lambda \pi^* H + \alpha D + \beta E)^3$$

where $H$ denotes the class of a plane in $\mathbb{P}^3$, and

$$\lambda = -4 + 10(1 - \frac{2}{p})$$

$$\alpha = 3 - \frac{1}{m} - 6(1 - \frac{2}{p})$$

$$\beta = -2(1 - \frac{2}{p}) + 2(1 - \frac{1}{m}).$$
Finally, we get
\[
(K_Y + \Delta)^3 = 
\lambda^3 + 5\alpha^3 + 10\beta^3 \cdot E^3 + 3 \cdot 10 \cdot \beta^2 (\lambda \pi^*H \cdot E^2 + \alpha D \cdot E^2) = 
\lambda^3 + 5\alpha^3 + 20\beta^3 - 30\beta^2 (\lambda + 2\alpha).
\]

To explain the last two equalities, the key point is that for each irreducible component $E_j$ of $E$,
\[
E_j|_{E_j} = -l_1 - l_2,
\]
where $l_1$ and $l_2$ are the respective fibers in $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that when developing the cube, most crossed terms disappear because
\[
(\pi^*H)^2 \cdot D = (\pi^*H)^2 \cdot E = \pi^*H \cdot D^2 = \pi^*H \cdot D \cdot E = 0.
\]

Moreover, whenever $D_k$ intersects $E_j$, $D_k|_{E_j} = l_1$. Also, we can represent $H$ by a plane that is transverse to $\pi(E_j)$, so that $\pi^*H|_{E_j} = l_1$, so we have
\[
D_k \cdot E_j^2 = -1, \quad \pi^*H \cdot E_j^2 = -1.
\]

For more details on this, see section 9.4.1.

Finally we have $D_j^2 = 1$ and
\[
E_j^3 = E_j|_{E_j} \cdot E_j|_{E_j} = (-l_1 - l_2)^2 = 2.
\]

For $p = 8$ and $m = 2$ this gives
\[
(K_Y + \Delta)^3 = \frac{33}{8},
\]
and this gives
\[
\chi^{orb}(C(A_4, 8)) = -\frac{11}{5120}.
\]

Note that this agrees with the formula in [16].

9.4.3. The cases $C(B_4, (p_1, p_2))$ with $(p_1, p_2) = (6, 4)$ or $(12, 3)$. Here we have two orbits of planes in the group, we write $M_1, M_2$ for the corresponding divisors in $\hat{X}$. We need to compute
\[
\left( K_Y + \left(1 - \frac{2}{p_1}\right)f_*M_1 + \left(1 - \frac{2}{p_2}\right)f_*M_2 + \left(1 - \frac{1}{m}\right)D \right)^3.
\]

As in the $A_4$ case, $K_{\hat{X}} = f^*K_Y + E$, and one checks using the combinatorics of the arrangement that
\[
f^*f_*M_1 = M_1, \quad f^*f_*M_2 = M_2 + 2E, \quad f^*f_*D = D + 2E.
\]

Note also that
\[
\pi^*\pi_*M_1 = M_1 + 3D + 2E, \quad \pi^*\pi_*M_2 = M_2 + 6D + 2E,
\]
because for each $j$, $\pi_*D_j$ is on 3 mirrors in the first orbit, and 6 mirrors in the second orbit; and $\pi^*E_j$ lies on 2 mirrors from each orbit.
This gives
\[(K_Y + \Delta)^3 = (\lambda \pi^* H + \alpha D + \beta E)^3\]
where
\[
\lambda = -4 + 4(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha = 3 - \frac{1}{m_1} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}) \\
\beta = -2(1 - \frac{2}{p_1}) + 2(1 - \frac{1}{m_1}).
\]
Finally we get
\[
(K_Y + \Delta)^3 = \lambda^3 + 4\alpha^3 + 6\beta^3 \cdot 2 + 3\lambda \beta^2 \pi^* H \cdot E^2 + 3\alpha \beta^2 D \cdot E^2 \\
= \lambda^3 + 4\alpha^3 + 6\beta^3 \cdot 2 - 3 \cdot 6 \cdot \lambda \beta^2 - 3 \cdot 6 \cdot 2 \cdot \alpha \beta^2.
\]
For \(p_1 = 6, p_2 = 4, m = 3/2\), this gives
\[
\chi_{\text{orb}}(C(B_4, 6, 4)) = \frac{31}{3456},
\]
and for \(p_1 = 12, p_2 = 3, m = 2\), this gives
\[
\chi_{\text{orb}}(C(B_4, 12, 3)) = -\frac{23}{10368},
\]
as it should in comparison with the values expected from [16].

9.4.4. The cases \(C(B_4, (p_1, p_2))\) with \((p_1, p_2) = (6, 6)\) or \((10, 5)\). Here the situation is almost the same as in the previous two cases, except that there is one more orbit of points to blow-up, but these points do not intersect the lines that need to be blow-up.

In other words, one gets the same formula as before with \(D\) replaced by \(D_1\) and \(D_2\), but \(D_2\) has no interaction with either \(D_1\) or \(E\).
\[
\lambda = -4 + 4(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha_1 = 3 - \frac{1}{m_1} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}), \quad \alpha_2 = 3 - \frac{1}{m_2} - 6(1 - \frac{2}{p_2}) \\
\beta = -2(1 - \frac{2}{p_1}) + 2(1 - \frac{1}{m_1}).
\]
This gives
\[
(K_Y + \Delta)^3 = \lambda^3 + 4\alpha^3 + 8\alpha_2^3 + 6\beta^3 \cdot 2 - 3 \cdot 6 \cdot \lambda \beta^2 - 3 \cdot 6 \cdot 2 \cdot \alpha_1 \beta^2,
\]
For \(p_1 = 6, p_2 = 6, m_1 = 1, m_2 = 3\), we get
\[
\chi_{\text{orb}}(C(B_4, 6, 6)) = -\frac{5}{432},
\]
and for \(p_1 = 10, p_2 = 5, m_1 = 1, m_2 = 5\), we get
\[
\chi_{\text{orb}}(C(B_4, 10, 5)) = -\frac{13}{1500},
\]
as expected.
9.4.5. The case $C(B_4(2, 8))$. This case is similar to the previous one. We now wish to compute

$$
\left( K_Y + \left(1 - \frac{2}{p_1}\right)f_\ast M_1 + \left(1 - \frac{2}{p_2}\right)f_\ast M_2 + \left(1 - \frac{1}{m_1}\right)D_1 + \left(1 - \frac{1}{m_2}\right)D_2 \right)^3.
$$

Note that

$$
K_\hat{X} = K_Y
$$

$$
f_\ast f_\ast M_1 = M_1 + \frac{1}{2}E,
$$

$$
f_\ast f_\ast M_2 = M_2
$$

$$
f_\ast f_\ast D_1 = D_1 + \frac{1}{2}E,
$$

$$
f_\ast f_\ast D_2 = D_2 + E.
$$

Indeed, each line in $\mathbb{P}^3$ below a component of $E$ contains three of the points that get blown-up (one in the orbit of $L_{123}$, two in the orbit of $L_{234}$), and it has a single transverse intersection with a mirror in the first orbit of mirrors.

Using the blow-up map and the combinatorics of the arrangement, we have

$$
\pi_\ast f_\ast M_1 = M_1 + 3D_1
$$

$$
\pi_\ast f_\ast M_2 = M_2 + 6D_1 + 6D_2 + 3E,
$$

and computations similar to the ones in the previous sections show that $(K_Y + \Delta)^3$ is given by

$$
(\lambda \pi_\ast H + \alpha_1 D_1 + \alpha_2 D_2 + \beta E)^3
$$

where

$$
\lambda = -4 + 4(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2})
$$

$$
\alpha_1 = 3 - \frac{1}{m_1} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}),
$$

$$
\alpha_2 = 3 - \frac{1}{m_2} - 6(1 - \frac{2}{p_2})
$$

$$
\beta = 1 + \frac{1}{2}(1 - \frac{2}{p_1}) - 3(1 - \frac{2}{p_2}) + \frac{1}{2}(1 - \frac{1}{m_1}) + (1 - \frac{1}{m_2}).
$$

Finally, developing the cube, we get

$$
\lambda^3 + 4\alpha_1^3 + 8\alpha_2^3 + 16\beta^4 \cdot 4 + 3\alpha_1\beta^2 D_1 \cdot E^2 + 3\alpha_2\beta^2 D_2 \cdot E^2 + 3\lambda\beta^2\pi_\ast H \cdot E^2.
$$

Using the combinatorics and the above description for $E_j|_{E_j}$ (see section 9.4.1), we get

$$
\lambda^3 + 4\alpha_1^3 + 8\alpha_2^3 + 16\beta^4 \cdot 4 - 3 \cdot 16 \cdot \alpha_1\beta^2 - 3 \cdot 16 \cdot 2 \cdot \alpha_2\beta^2 - 3 \cdot 16 \cdot \lambda\beta^2.
$$

This gives

$$
\chi^{orb}(C(B_4, 2, 8)) = -\frac{11}{1024} = -\frac{11}{5120} \cdot 5,
$$

as it should since it has index 5 in the corresponding Deligne-Mostow group (see Figure 13).
9.4.6. The case \( C(B_4(4,8)) \). This case is the most painful case to handle, but it simply combines the difficulties we have encountered before. Here we blow up the orbits of \( L_{123} \) (4 copies), \( L_{234} \) (8 copies), \( L_{12} \) (6 copies) and \( L_{23} \) (16 copies). Accordingly we have 4 exceptions \( D_1, D_2, E_1, E_2 \) in \( \hat{X} \), and still wish to compute
\[
\left( K_Y + (1 - \frac{2}{p_1}) f_* M_1 + (1 - \frac{2}{p_2}) f_* M_2 + (1 - \frac{1}{m_1}) D_1 + (1 - \frac{1}{m_2}) D_2 \right)^3.
\]
Note that
\[
K_{\hat{X}} = f^* K_Y + E_1
\]
\[
f^* f_* M_1 = M_1 + \frac{1}{2} E_2, \quad f^* f_* M_2 = M_2 + 2 E_1
\]
\[
f^* f_* D_1 = D_1 + 2 E_1 + \frac{1}{2} E_2, \quad f^* f_* D_2 = D_2 + E_2
\]
\[
\pi^* \pi_* M_1 = M_1 + 3 D_1 + 2 E_1, \quad \pi^* \pi_* M_2 = M_2 + 6 D_1 + 6 D_2 + 2 E_1 + 3 E_2
\]
The same computations as before now give
\[
(K_X + \Delta)^3 = (\lambda \pi^* H + \alpha_1 D_1 + \alpha_2 D_2 + \beta_1 E_1 + \beta_2 E_2)^3
\]
where
\[
\lambda = -4 + 4(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2})
\]
\[
\alpha_1 = 3 - \frac{1}{m_1} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}) - 3 \cdot 16 \alpha_1 \beta_1 - 3 \cdot 6 \alpha_1 \beta_2 - 3 \cdot 6 \alpha_2 \beta_2 - 3 \cdot 16 \alpha_2 \beta_2 - 3 \cdot 16 \alpha_1 \beta_2 - 3 \cdot 16 \alpha_2 \beta_2.
\]
Inspecting the combinatorics of the arrangement and using \( E_1^{(j)} |_{E_1^{(j)}} = -l_1 - 2 l_2, E_2^{(j)} |_{E_2^{(j)}} = -l_1 - l_2 \), we then get
\[
\lambda^3 + 4 \alpha_1^3 + 8 \alpha_2^3 + 16 \cdot 4 \cdot \beta_1^3 + 6 \cdot 2 \cdot \beta_2^3 - 3 \cdot 6 \lambda \beta_1^2 - 3 \cdot 16 \lambda \beta_2^2 - 3 \cdot 6 \cdot 2 \alpha_1 \beta_1^2 - 3 \cdot 16 \alpha_1 \beta_2^2 - 3 \cdot 16 \cdot 2 \alpha_2 \beta_2^2.
\]
This gives
\[
\chi^{orb}(C(B_4, 4,8)) = -\frac{11}{1024},
\]
which is again the expected value.

9.4.7. The case \( C(G_{28}, p_1, p_2) \) for \( (p_1, p_2) = (2, 8) \) or \( (2, 12) \). In this case we blow up (12 copies of) \( L_{234} \) and (16 copies of) \( L_{34} \). Since each copy of \( L_{34} \) contains 3 copies of \( L_{234} \), \( f : \hat{X} \to Y \) is crepant (see section 9.4.1). On \( \hat{X} \) we have \( D, E, M_1, M_2 \).
\[
K_{\hat{X}} = f^* K_Y
\]
\[
f^* f_* M_1 = M_1 + \frac{3}{2} E, \quad f^* f_* M_2 = M_2
\]
\[
f^* f_* D = D + \frac{3}{2} E
\]
\[
\pi^* \pi_* M_1 = M_1 + 3 D, \quad \pi^* \pi_* M_2 = M_2 + 6 D + 3 E
\]
The same computations as before now give
\[(K_X + \Delta)^3 = (\lambda \pi^* H + \alpha D + \beta E)^3\]
where
\[
\begin{align*}
\lambda &= -4 + 12(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha &= 3 - \frac{1}{m} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}) \\
\beta &= 1 + \frac{4}{3}(1 - \frac{2}{p_1}) - 3(1 - \frac{2}{p_2}) + \frac{2}{3}(1 - \frac{1}{m})
\end{align*}
\]

We then get
\[
\lambda^3 + 12\alpha^3 + 16 \cdot 4 \cdot \beta^3 - 3 \cdot 16 \cdot \lambda\beta^2 - 3 \cdot 16 \cdot 3 \alpha \beta^2,
\]
which for \(p_1 = 2, p_2 = 8, m = 2\) gives
\[
\chi_{orb}(C(G_{28}, 2, 8)) = -\frac{11}{3072},
\]
and for \(p_1 = 2, p_2 = 12, m = 3/2\) gives
\[
\chi_{orb}(C(G_{28}, 2, 12)) = -\frac{23}{10368}.
\]

9.4.8. The case \(C(G_{28}, 6, 6))\). In this case we blow up (12 copies of) \(L_{123}\), (12 copies of) \(L_{234}\) and (16 copies of) \(L_{34}\). Since each copy of \(L_{23}\) contains 2 copies of \(L_{234}\) and 2 copies of \(L_{123}\), we have
\[
K_{\hat{X}} = f^*K_Y - \frac{n - 3}{n - 1} E
\]
with \(n = 2 + 2 = 4\), i.e. \(f^*K_Y = K_{\hat{X}} + \frac{1}{3} E\).

On \(\hat{X}\) we have \(D_1, D_2, E, M_1, M_2\).
\[
K_{\hat{X}} = f^*K_Y + \frac{1}{3} E_1
\]
\[
f^*f_*, M_1 = M_1, \quad f^*f_*, M_2 = M_2
\]
\[
f^*f_*, D_1 = D_1 + \frac{2}{3} E, \quad f^*f_*, D_2 = D_2 + \frac{2}{3} E
\]
\[
K_{\hat{X}} = \pi^*K_X + 2D_1 + 2D_2 + E
\]
\[
\pi^*\pi_*, M_1 = M_1 + 6D_1 + 3D_2 + 2E, \quad \pi^*\pi_*, M_2 = M_2 + 3D_1 + 6D_2 + 2E
\]
The same computations as before now give
\[(K_X + \Delta)^3 = (\lambda \pi^* H + \alpha D + \beta E)^3\]
where
\[
\begin{align*}
\lambda &= -4 + 12(1 - \frac{2}{p_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha_1 &= 3 - \frac{1}{m_1} - 6(1 - \frac{2}{p_1}) - 3(1 - \frac{2}{p_2}), \quad \alpha_2 = 3 - \frac{1}{m_2} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}) \\
\beta &= \frac{4}{3} - 2(1 - \frac{2}{p_1}) - 2(1 - \frac{2}{p_2}) + \frac{2}{3}(1 - \frac{1}{m_1}) + \frac{2}{3}(1 - \frac{1}{m_2})
\end{align*}
\]
Using the combinatorics and the self intersection of $E$ we get

\[ \lambda^3 + 12\alpha_1^3 + 12\alpha_2^3 + 18 \cdot 6 \cdot \beta^3 - 3 \cdot 18 \cdot \lambda \beta^2 - 3 \cdot 18 \cdot 2 \cdot \alpha_1 \beta^2 - 3 \cdot 18 \cdot 2 \cdot \alpha_2 \beta^2 \]

which for $p_1 = 6$, $p_2 = 6$, $m_1 = m_2 = 1$ gives

\[ \chi^{\text{orb}}(C(G_{28}, 6, 6)) = -\frac{5}{144}. \]

9.4.9. The case $C(G_{28}, 3, 12)$). In this case we blow up (12 copies of) $L_{123}$, (12 copies of) $L_{234}$, (18 copies of) $L_{23}$ and (16 copies of $L_{34}$. On $\hat{X}$ we have $D_1, D_2, E_1, E_2, M_1, M_2$.

\[
\begin{align*}
K_{\hat{X}} &= f^*K_Y - \frac{1}{2}E_1 \\
 f^*f_*M_1 &= M_1 + \frac{2}{3}E_2, \quad f^*f_*M_2 = M_2 \\
 f^*f_*D_1 &= D_1 + \frac{2}{3}E_1, \quad f^*f_*D_2 = D_2 + \frac{2}{3}E_1 + \frac{2}{3}E_2 \\
 K_{\hat{X}} &= \pi^*K_X + 2D_1 + 2D_2 + E_1 + E_2 \\
\pi^*\pi_*M_1 &= M_1 + 6D_1 + 3D_2 + 2E_1, \quad \pi^*\pi_*M_2 = M_2 + 3D_1 + 6D_2 + 2E_1 + 3E_2 
\end{align*}
\]

The same computations as before now give

\[ (K_X + \Delta)^3 = (\lambda \pi^*H + \alpha D + \beta E)^3 \]

where

\[
\begin{align*}
\lambda &= -4 + 12(1 - \frac{2}{m_1}) + 12(1 - \frac{2}{p_2}) \\
\alpha_1 &= 3 - \frac{1}{m_1} - 6(1 - \frac{2}{p_1}) - 3(1 - \frac{2}{p_1}), \quad \alpha_2 = 3 - \frac{1}{m_2} - 3(1 - \frac{2}{p_1}) - 6(1 - \frac{2}{p_2}) \\
\beta_1 &= \frac{4}{3} - 2(1 - \frac{2}{p_1}) - 2(1 - \frac{2}{p_2}) + \frac{\alpha}{3}(1 - \frac{1}{m_1}) + \frac{\beta}{3}(1 - \frac{1}{m_2}) \\
\beta_2 &= 1 + \frac{\alpha}{3}(1 - \frac{1}{m_1}) - 3(1 - \frac{2}{p_2}) + \frac{\beta}{2}(1 - \frac{1}{m_1})
\end{align*}
\]

Using the combinatorics and the self intersection of $E_1$ and $E_2$ (see the previous sections), we get

\[ \lambda^3 + 12\alpha_1^3 + 12\alpha_2^3 + 18\beta_1^3 + 18\beta_2^3 + 4 - 3 \cdot 18 \cdot \lambda \beta^2 - 3 \cdot 16 \cdot \lambda \beta^2 - 3 \cdot 18 \cdot 2 \cdot \alpha_1 \beta^2 - 3 \cdot 18 \cdot 2 \cdot \alpha_2 \beta^2 - 3 \cdot 16 \cdot 3 \cdot \alpha_2 \beta^2 \]

which for $p_1 = 3$, $p_2 = 12$, $m_1 = 2$, $m_2 = 1$ gives

\[ \chi^{\text{orb}}(C(G_{28}, 3, 12)) = -\frac{23}{1152}. \]

9.4.10. The case $C(G_{30}, 5))$. In this case we blow up 300 copies of $L_{123}$, 60 copies of $L_{234}$, and 72 copies of $L_{34}$. We denote the corresponding exceptions by $D_1, D_2, E$, and note

\[
\begin{align*}
K_{\hat{X}} &= f^*K_Y - \frac{1}{2}E \\
f^*f_*L &= L + \frac{5}{4}E \\
f^*f_*D_1 &= D_1, \quad f^*f_*D_2 = D_2 + \frac{5}{4}E \\
K_{\hat{X}} &= \pi^*K_X + 2D_1 + 2D_2 + E \\
\pi^*\pi_*L &= L + 6D_1 + 15D_2 + 5E.
\end{align*}
\]
The same computations as before now give

\[(K_X + \Delta)^3 = (\lambda \pi^* H + \alpha_1 D_1 + \alpha_2 D_2 + \beta E)^3\]

where

\[
\lambda = -4 + 60(1 - \frac{2}{p})
\]

\[
\alpha_1 = 3 - \frac{1}{m_1} - 6(1 - \frac{2}{p}), \quad \alpha_2 = 3 - \frac{1}{m_2} - 15(1 - \frac{2}{p})
\]

\[
\beta = \frac{3}{2} - \frac{15}{4}(1 - \frac{2}{p}) + \frac{5}{4}(1 - \frac{1}{m_2}).
\]

Using the combinatorics and the self intersection of \(E_1\) and \(E_2\) (see the previous sections), we get

\[\lambda^3 + 300\alpha_1^3 + 60\alpha_2^3 + 72 \cdot 8 \cdot \beta^3 - 3 \cdot 72 \cdot \lambda \beta^2 - 3 \cdot 72 \cdot 5 \cdot \alpha_2 \beta^2\]

which for \(p = 5, m_1 = 5, m_2 = 1/2\) gives

\[\chi_{\text{orb}}(C(G_{30}, 5)) = -\frac{41}{1125}\]

9.4.11. The case \(C(G_{31}, 5)\). In this case we blow up 60 copies of \(L_{125}\), 480 copies of \(L_{235}\), and 30 copies of \(L_{14}\). We denote the corresponding exceptionals by \(D_1, D_2, E\), and note

\[K_{\hat{X}} = f^* K_Y - \frac{3}{5} E\]

\[f^* f_* L = L\]

\[f^* f_* D_1 = D_1 + \frac{6}{5} E, \quad f^* f_* D_2 = D_2\]

\[K_{\hat{X}} = \pi^* K_X + 2D_1 + 2D_2 + E\]

\[\pi^* \pi_* L = L + 15D_1 + 6D_2 + 6E.\]

The same computations as before now give

\[(K_X + \Delta)^3 = (\lambda \pi^* H + \alpha_1 D_1 + \alpha_2 D_2 + \beta E)^3\]

where

\[
\lambda = -4 + 60(1 - \frac{2}{p})
\]

\[
\alpha_1 = 3 - \frac{1}{m_1} - 15(1 - \frac{2}{p})
\]

\[
\alpha_2 = 3 - \frac{1}{m_2} - 6(1 - \frac{2}{p})
\]

\[
\beta = \frac{8}{5} - 6(1 - \frac{2}{p}) + \frac{6}{5}(1 - \frac{1}{m_2}).
\]

Using the combinatorics and the self intersection of \(E_1\) and \(E_2\) (see the previous sections), we get

\[\lambda^3 + 60\alpha_1^3 + 480\alpha_2^3 + 30 \cdot 10 \cdot \beta^3 - 3 \cdot 30 \cdot \lambda \beta^2 - 3 \cdot 30 \cdot 6 \cdot \alpha_1 \beta^2\]

which for \(p = 5, m_1 = 1/2, m_2 = 5\) gives

\[\chi_{\text{orb}}(C(G_{31}, 5)) = -\frac{41}{1125}\]
10. Relation with Deligne-Mostow groups

As mentioned in [5], their construction applies to reflection groups of type $A_n$ and $B_n$ give lattices commensurable to the Deligne-Mostow examples. We give some details of that relationship in the case of lattices in $PU(3,1)$, in the form of a table (see Figure 13). The basic point is that Deligne-Mostow lattices in $PU(n,1)$ are representations of spherical braid groups on $N=n+3$ strands (which can be thought of as the corresponding plane braid group modulo its center).

More precisely, the group that gets represented is $\phi^{-1}(\Sigma)$ for some subgroup $\Sigma \subset S_N$ ($\Sigma$ acts as a symmetry group of the $N$-tuple of weights for the corresponding hypergeometric functions), where $\phi : B_N \to S_N$ corresponds to remembering only the permutation effected by the braid.

For simplicity, when describing Deligne-Mostow groups, we will take $\Sigma$ to be the full symmetry group of the $N$-tuple $\mu = (\mu_1, \ldots, \mu_N)$ of weights, but the corresponding CHL subgroups will be obtained by taking a subgroup $\Sigma_0 \subset \Sigma$.

One then observes that standard generators of $B_N$ (either commute or) satisfy the usual braid relation, $aba = bab$, and moreover $a^2ba^2b = ba^2ba^2$. This allows us to produce several subgroups in Deligne-Mostow groups $\Gamma_{\mu, \Sigma}$ of type $A_4$ when $\mu$ has 5 equal weights, and of type $B_4$ when $\mu$ has 4 equal weights. The index of these subgroups can be worked out using the index of $\Sigma_0$ in $\Sigma$. The relationship between these groups and Deligne-Mostow is given in Table 13, on p. 45.

11. Combinatorial data

In Figures 5 through 9, we list combinatorial data that allow to check the Schwarz conditions (see section 4 of [5]) and to compute volumes (see section 9). For the group $G_{28}$, there are two orbits of mirrors, which can be assigned independent weights. Accordingly, we give the number of mirrors containing a given $L$ in the form $j + k$, where $j$ (resp. $k$) is the number of mirrors from the first (resp. second) orbit.

For each group orbit of irreducible mirror intersections (see p. 88 of [5]), we list the corresponding weight $\kappa_L$, which is the ratio

\begin{equation}
\kappa_L = \frac{\#(H_L)}{\text{codim}L},
\end{equation}

where $H_L$ is the set of hyperplanes in the mirror arrangement that contain $L$.

We also list the order of the center $Z(G_L)$ of the Schwarz symmetry group $G_L$. Recall that $G_L$ is obtained as the fixed point stabilizer of $L$, and it is a reflection group (although the stabilizer of $L$ need not be). The Schwarz condition amounts to requiring that, for every irreducible $L$ such that $\kappa_L > 1$,

\[ \kappa_L - 1 = \frac{|Z(G_L)|}{n_L} \]

for some integer $n_L \geq 2$.

In order to describe strata in the arrangement, we label them with an index that indicates the mirrors of reflections that define a given intersection using the numbering of
the reflection generators. For instance, $L_j$ denotes the mirror of the $j$-th reflection $R_j$, $L_{ijk}$ denotes the intersection of the mirrors of the reflections $R_j$ and $R_k$, $L_{ijk}$ denotes the intersection of the three mirrors of $R_i$, $R_j$ and $R_k$, etc. We extend this notation slightly to include conjugates of the generators, for instance $L_{12343}$ denotes the intersection of the mirrors of $R_1$, $R_2$ and $R_3R_4R_3$.

### References


<table>
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<th>ST group</th>
<th>Order(s)</th>
<th>DM group</th>
<th>Index</th>
<th>C/NC</th>
<th>A/NA</th>
<th>trAd*</th>
<th>$\chi^{or\beta}$</th>
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<td>A</td>
<td>$\mathbb{Q}$</td>
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<td>A</td>
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</tr>
</tbody>
</table>

**Table 13.** CHL groups for $A_4$ and $B_4$ are subgroups of specific Deligne-Mostow lattices.
| $G$  | $|G|$ | $|Z(G)|$ | Mirror orbit | $|\text{orbit}|$ | Weight |
|------|------|--------|--------------|----------------|--------|
| $A_4, S_5$ | 120 | 1 | $m_1$ | 10 | $1 - \frac{2}{p}$ |

| $L$ | #(Orb) | #(mirrors) | $|Z(G_L)|$ | $\kappa_L$ | Vertices |
|-----|--------|------------|---------|-------------|----------|
| $L_{12}$ | 10 | 3 | 1 | $\frac{3}{2}(1 - \frac{2}{p})$ | $2 \times L_{123}, L_{134}$ |
| $L_{13}$ | 15 | 2 | 1 | (reducible) | $L_{123}, 2 \times L_{134}$ |

<table>
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<tr>
<th>$p$</th>
<th>$A_3$</th>
<th>$G_{25}$</th>
<th>Cusp</th>
<th>$\Gamma(5, \frac{4}{5})$</th>
<th>$\Gamma(6, \frac{1}{3})$</th>
<th>$\Gamma(8, \frac{1}{5})$</th>
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<td>0</td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>3</td>
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<td></td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
<td>$1 + \frac{1}{5}$</td>
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</table>

**Figure 1.** Combinatorial data for $A_4$
| $G$          | $|G|$ | $|Z(G)|$ | Mirror orbit | orbit | Weight       |
|--------------|------|---------|--------------|-------|--------------|
| $B_4, G(2, 1, 4)$ | 384  | 2       | $m_1$        | 4     | $1 - \frac{2}{p_1}$ |
|              |      |         | $m_2 (m_3, m_4)$ | 12    | $1 - \frac{2}{p_2}$ |

<table>
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<th>Finite</th>
<th>Parabolic</th>
<th>Hyperbolic</th>
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<td>$(p_1, p_2) = (n, 2)(G(n, 1, 4))$</td>
<td>$(p_1, p_2) = (2, 3)$</td>
<td>$(p_1, p_2) = (2, 4), (2, 5), (2, 6), (2, 8), (3, 3), (3, 4), (3, 6)$</td>
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<td>$(4, 3), (4, 4), (4, 8), (6, 3), (6, 4), (6, 6), (10, 5), (12, 3)$</td>
<td></td>
</tr>
</tbody>
</table>

**Remark 1.** The group derived from $B_4$ and orders $(p_1, p_2) = (5, 5)$ is the Deligne-Mostow group $(3, 3, 3, 3, 3)/10$, so it is a lattice; however, it does not satisfy the Schwarz condition in [5], since in that case $\kappa_{L_{123}} - 1 = 4/5$, but $|Z(G_{L_{123}})| = 2$ only allows numerator $1$ or $2$, not $4$. This group can also be described as $C(A_4, 5)$, where the Schwarz condition does hold.

| $L$ | $\#$(Orb) | $\#$(mirrors) | $|Z(G_L)|$ | $\kappa_L$ | Vertices            |
|-----|-----------|---------------|-------------|------------|---------------------|
| $L_{123}$ | 4        | 3+6           | 2           | $1 - \frac{2}{p_1} + 2(1 - \frac{2}{p_2})$ | $2 \times L_{123}, 4 \times L_{123}$ |
| $L_{124}$ | 12       | 2+6           | (reducible) | (reducible) | $2 \times L_{124}$, $L_{123}, L_{124}, 2 \times L_{134}$ |
| $L_{134}$ | 16       | 1+3           | (reducible) | (reducible) | $L_{123}, L_{134}, 2 \times L_{234}$ |
| $L_{234}$ | 8        | 0+6           | 1           | $2(1 - \frac{2}{p_2})$ | $2 \times L_{124}, 2 \times L_{234}$ |

**Figure 3.** Combinatorial data for $B_4$
Figure 4. Point stabilizers for $B_4$

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<th>$(p_1, p_2)$</th>
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<th>Group</th>
<th>$\kappa_{L_{234}}$</th>
<th>Group</th>
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<td>$A_3$</td>
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<td>$G_{26}$</td>
<td>$1 - \frac{1}{3}$</td>
<td>$G_{25}$</td>
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<td>Cusp</td>
<td>1</td>
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<td>$1 + \frac{1}{10}$</td>
<td>$\Gamma(5, \frac{1}{10})$</td>
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<td>$\Gamma(6, \frac{1}{2})$</td>
<td>$1 + \frac{1}{2}$</td>
<td>$\Gamma(6, \frac{1}{2})$</td>
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<tr>
<td>$(2, 8)$</td>
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<td>$\Gamma(8, \frac{1}{4})$</td>
<td>$1 + \frac{1}{4}$</td>
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</tr>
<tr>
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<td>Cusp</td>
<td>$1 - \frac{1}{3}$</td>
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<td>$\Gamma(3, \frac{1}{3})$</td>
<td>$1 - \frac{1}{3}$</td>
<td>$G_{25}$</td>
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Table 14. Rough commensurability invariants and orbifold Euler characteristics, for groups in $PU(3, 1)$. 
| $G$ | $|G|$ | $|Z(G)|$ | Mirror orbit | $|\text{orbit}|$ | Weight |
|-----|------|--------|---------------|---------|--------|
| $G_{28}, F_4$ | 1152 | 2      | $m_1(m_2, m_3, m_4)$ | 12      | $1 - \frac{1}{p_1}$ $1 - \frac{1}{p_2}$ |

| $L$ | $\#(\text{Orb})$ | $\#(\text{mirrors})$ | $|Z(G_L)|$ | $\kappa_L$ | Vertices |
|-----|----------------|----------------|--------|---------|---------|
| $L_{12}$ | 16 | 3+0 | 1 | $3(\frac{1}{2} - \frac{1}{p_1})$ | $(2,12), 3 \times L_{12}$ |
| $L_{14}$ | 72 | 1+1 | (reducible) | (reducible) | $L_{12}, L_{134}, 2 \times L_{12}$ |
| $L_{23}$ | 18 | 2+2 | 2 | $2(1 - \frac{1}{p_1} - \frac{1}{p_2})$ | $2 \times L_{12}, 2 \times L_{234}$ |
| $L_{34}$ | 16 | 0+3 | 1 | $3(\frac{1}{2} - \frac{1}{p_2})$ | $3 \times L_{234}, 3 \times L_{134}$ |

<table>
<thead>
<tr>
<th>$(p_1, p_2)$</th>
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<th>(2,3)</th>
<th>(2,4)</th>
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| $L$ | $\#(\text{Orb})$ | $\#(\text{mirrors})$ | $|Z(G_L)|$ | $\kappa_L$ | Adj. lines |
|-----|----------------|----------------|--------|---------|----------|
| $L_{123}$ | 12 | 6+3 | 2 | $2(1 - \frac{1}{p_1} + (1 - \frac{1}{p_2})$ | $4 \times L_{12}, 6 \times L_{14}, 3 \times L_{23}$ |
| $L_{234}$ | 12 | 3+6 | 2 | $(1 - \frac{1}{p_1} + (1 - \frac{1}{p_2})$ | $6 \times L_{14}, 3 \times L_{23}, 4 \times L_{34}$ |
| $L_{134}$ | 48 | 1+3 | (reducible) | (reducible) | $3 \times L_{14}, 1 \times L_{34}$ |
| $L_{124}$ | 48 | 3+1 | (reducible) | (reducible) | $1 \times L_{12}, 3 \times L_{14}$ |

<table>
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<th>(2,3)</th>
<th>(2,4)</th>
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<th>$(p_1, p_2)$</th>
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<th>(3,6)</th>
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<td>$1 + \frac{1}{3}$</td>
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*Figure 5.* Combinatorial data for $G_{28}$.
\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
\((p_1, p_2)\) & \(\kappa_{L,123}\) & \text{Group} & \(\kappa_{L,234}\) & \text{Group} \\
\hline
(2, 2) & 0 & \(G(2, 1, 3)\) & 0 & \(G(2, 1, 3)\) \\
(2, 3) & \frac{1}{2} & \(G(3, 1, 3)\) & \frac{2}{3} & \(G_{26}\) \\
(2, 4) & \frac{1}{2} & \(G(4, 1, 3)\) & 1 & \text{Cusp} \\
(2, 5) & \frac{1}{2} & \(G(5, 1, 3)\) & 1 + \frac{1}{3} & \(\Gamma(5, \frac{1}{10})\) \\
(2, 6) & \frac{1}{2} & \(G(6, 1, 3)\) & 1 + \frac{1}{2} & \(\Gamma(6, \frac{2}{3})\) \\
(2, 8) & \frac{1}{2} & \(G(8, 1, 3)\) & 1 + \frac{1}{4} & \(\Gamma(8, \frac{2}{3})\) \\
(2, 12) & \frac{1}{2} & \(G(12, 1, 3)\) & 1 + \frac{1}{6} & \(\Gamma(12, \frac{1}{2})\) \\
(3, 3) & 1 & \text{Cusp} & 1 & \text{Cusp} \\
(3, 4) & 1 + \frac{1}{3} & \(\Gamma(3, \frac{1}{3})\) & 1 + \frac{1}{3} & \(\Gamma(4, \frac{2}{3})\) \\
(3, 6) & 1 + \frac{1}{4} & \(\Gamma(3, \frac{1}{4})\) & 1 + \frac{1}{4} & \(\Gamma(6, \frac{2}{3})\) \\
(3, 12) & 1 + \frac{1}{6} & \(\Gamma(3, 0)\) & 1 + \frac{1}{6} & \(\Gamma(12, \frac{1}{2})\) \\
(4, 4) & 1 + \frac{1}{2} & \(\Gamma(4, \frac{1}{2})\) & 1 + \frac{1}{2} & \(\Gamma(4, \frac{3}{2})\) \\
(6, 6) & 1 + \frac{1}{3} & \(\Gamma(6, 0)\) & 1 + \frac{1}{3} & \(\Gamma(6, 0)\) \\
\hline
\end{tabular}
\end{table}

\textbf{Figure 6.} Point stabilizers for \(G_{28}\)


| $G$ | $|G|$ | $|Z(G)|$ | $(\text{mirrors})$ |
|----|-----|------|----------------|
| $G_{29}$ | 7680 | 4 | 40 |

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<tr>
<th>Finite</th>
<th>Parabolic</th>
<th>Hyperbolic</th>
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</thead>
<tbody>
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<td>$p = 3, 4$</td>
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</tbody>
</table>

| $L$ | $(\text{Orb})$ | $(\text{mirrors})$ | $|Z(G_L)|$ | $\kappa_L$ | Vertices |
|-----|----------------|-------------------|---------|----------|----------|
| $L_{12}$ | 160 | 3 | 1 | $\frac{2}{3}(1 - \frac{1}{p})$ | $(\text{reducible})$ | $2 \times L_{123}, 2 \times L_{12343}, L_{124}, L_{134}, 2 \times L_{234}$ |
| $L_{13}$ | 120 | 2 | 2 | $2(1 - \frac{1}{p})$ | | $2 \times L_{123}, 2 \times L_{12343}, 2 \times L_{124}, 4 \times L_{134}$ |
| $L_{24}$ | 30 | 4 | 2 | | | $4 \times L_{12343}, 2 \times L_{234}$ |

<table>
<thead>
<tr>
<th>$p$</th>
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<th>4</th>
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<tr>
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</table>

| $L$ | $(\text{Orb})$ | $(\text{mirrors})$ | $|Z(G_L)|$ | $\kappa_L$ | Vertices |
|-----|----------------|-------------------|---------|----------|----------|
| $L_{123}$ | 80 | 6 | 1 | $2(1 - \frac{1}{p})$ | | $4 \times L_{12}, 3 \times L_{13}$ |
| $L_{12343}$ | 80 | 6 | 1 | $2(1 - \frac{1}{p})$ | | $4 \times L_{12}, 3 \times L_{13}$ |
| $L_{124}$ | 40 | 9 | 2 | $3(1 - \frac{1}{p})$ | | $4 \times L_{12}, 6 \times L_{13}, 3 \times L_{24}$ |
| $L_{134}$ | 160 | 4 | $(\text{reducible})$ | $(\text{reducible})$ | | $1 \times L_{12}, 3 \times L_{13}$ |
| $L_{234}$ | 20 | 12 | 1 | $4(1 - \frac{2}{p})$ | | $16 \times L_{12}, 3 \times L_{24}$ |

<table>
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<tr>
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<tbody>
<tr>
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</tr>
<tr>
<td>$\kappa_{L_{124}}$</td>
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<td>$1 + \frac{1}{2}$</td>
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<td>$\kappa_{L_{234}}$</td>
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<td>$1 + 1$</td>
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<table>
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<th>Group</th>
<th>$\kappa_{L_{124}}$</th>
<th>Group</th>
<th>$\kappa_{L_{234}}$</th>
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<tr>
<td>3</td>
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<td>$G_{25}$</td>
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<td>$\Gamma(4, \frac{1}{4})$</td>
<td>$1 + \frac{1}{2}$</td>
<td>$\Gamma(3, \frac{1}{3})$</td>
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<tr>
<td>4</td>
<td>1</td>
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</table>

Figure 7. Combinatorial data for $G_{29}$
| $L$   | #(Orb) | #(mirrors) | $|Z(G_L)|$ | $\kappa_L$ | Vertices                                 |
|-------|--------|------------|-----------|------------|------------------------------------------|
| $L_{12}$ | 200 | 3 | 1 | 4/5(1 − 1/2) | 6 × $L_{123}$, 3 × $L_{124}$, 3 × $L_{234}$ |
| $L_{13}$ | 450 | 2 | (reducible) | (reducible) | 2 × $L_{123}$, 4 × $L_{124}$, 4 × $L_{134}$, 2 × $L_{234}$ |
| $L_{34}$ | 72  | 5 | 1 | 5/2(1 − 1/6) | 5 × $L_{134}$, 5 × $L_{234}$ |

| $p$ | 2 | 3 | 5 |
| $\kappa_{L_{12}}$ | 0 | 3/5 | 7/10 |
| $\kappa_{L_{34}}$ | 0 | 1 + 1/5 |

| $L$   | #(Orb) | #(mirrors) | $|Z(G_L)|$ | $\kappa_L$ | Adj. lines |
|-------|--------|------------|-----------|------------|------------|
| $L_{123}$ | 300 | 6 | 1 | 3(1 − 2/5) | 4 × $L_{12}$, 3 × $L_{13}$ |
| $L_{124}$ | 600 | 4 | (reducible) | (reducible) | 1 × $L_{12}$, 3 × $L_{13}$ |
| $L_{134}$ | 360 | 6 | (reducible) | (reducible) | 5 × $L_{13}$, 1 × $L_{34}$ |
| $L_{234}$ | 60  | 15 | 2 | 5(1 − 2/5) | 10 × $L_{12}$, 15 × $L_{13}$, 6 × $L_{34}$ |

| $p$ | 3 | 5 |
| $\kappa_{L_{123}}$ | 3/5 | 1 + 1/
| $\kappa_{L_{234}}$ | 1 + 1/3 |

<table>
<thead>
<tr>
<th>$p$</th>
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<th>Group</th>
<th>$\kappa_{L_{234}}$</th>
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<td>$\Gamma(5, 7/10)$</td>
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Figure 8. Combinatorial data for $G_{30}$
Fig. 9. Combinatorial data for $G_{31}$.