Kähler manifolds and transcendental techniques in algebraic geometry

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Complex manifolds / (p, q)-forms

- Goal : study the geometric / topological / cohomological properties of compact Kähler manifolds
- A complex n-dimensional manifold is given by coordinate charts equipped with local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$.
- A differential form u of type (p, q) can be written as a sum

$$u(z) = \sum_{|J|=p,|K|=q} u_{JK}(z)\, dz_J \wedge d\overline{z}_K$$

where $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$,

$$dz_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}, \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_q}.$$

Complex manifolds / Currents

- A current is a differential form with distribution coefficients

$$T(z) = i^{pq} \sum_{|J|=p,|K|=q} T_{JK}(z)\, dz_J \wedge d\overline{z}_K$$

- The current T is said to be positive if the distribution $\sum \lambda_j \overline{\lambda}_k T_{JK}$ is a positive real measure for all $(\lambda_J) \in \mathbb{C}^N$ (so that $T_{KJ} = \overline{T}_{JK}$, hence $\overline{T} = T$).
- The coefficients T_{JK} are then complex measures – and the diagonal ones T_{JJ} are positive real measures.
- T is said to be closed if $dT = 0$ in the sense of distributions.

Complex manifolds / Basic examples of Currents

- The current of integration over a codimension p analytic cycle $A = \sum c_j A_j$ is defined by duality as $[A] = \sum c_j [A_j]$ with

$$\langle [A_j], u \rangle = \int_{A_j} u_{|A_j}$$

for every $(n-p, n-p)$ test form u on X.

- Hessian forms of plurisubharmonic functions :

$$\varphi \ \text{ plurisubharmonic} \Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right) \geq 0$$

then

$$T = i\partial\overline{\partial}\varphi \qquad \text{is a closed positive } (1,1)\text{-current.}$$

Complex manifolds / Kähler metrics

- A Kähler metric is a smooth positive definite $(1,1)$-form

$$\omega(z) = i \sum_{1\le j,k\le n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k \qquad \text{such that } d\omega = 0.$$

- The manifold X is said to be Kähler (or of Kähler type) if it possesses at least one Kähler metric ω [Kähler 1933]
- Every complex analytic and locally closed submanifold $X \subset \mathbb{P}^N_{\mathbb{C}}$ in projective space is Kähler when equipped with the restriction of the Fubini-Study metric

$$\omega_{FS} = \frac{i}{2\pi} \log(|z_0|^2 + |z_1|^2 + \ldots + |z_N|^2).$$

- Especially projective algebraic varieties are Kähler.

Sheaf / De Rham / Dolbeault / cohomology

- Sheaf cohomology $H^q(X,\mathcal{F})$
  especially when $\mathcal{F}$ is a coherent analytic sheaf.
- Special case : cohomology groups $H^q(X,R)$ with values in constant coefficient sheaves $R=\mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C},\ \ldots$
  $=$ De Rham cohomology groups.
- $\Omega^p_X = \mathcal{O}(\Lambda^p T^*_X) =$ sheaf of holomorphic p-forms on X.
- Cohomology classes [forms / currents yield same groups]

$$\alpha\ d\text{-closed } k\text{-form/current to } \mathbb{C} \longmapsto \{\alpha\}\in H^k(X,\mathbb{C})$$
$$\alpha\ \overline{\partial}\text{-closed } (p,q)\text{-form/current to } F \longmapsto \{\alpha\}\in H^{p,q}(X,F)$$

Dolbeault isomorphism (Dolbeault - Grothendieck 1953)

$$H^{0,q}(X,F)\simeq H^q(X,\mathcal{O}(F)),$$
$$H^{p,q}(X,F)\simeq H^q(X,\Omega^p_X\otimes\mathcal{O}(F))$$

Hodge decomposition theorem

- **Theorem.** *If (X,ω) is compact Kähler, then*
$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}).$$

- *Each group $H^{p,q}(X,\mathbb{C})$ is isomorphic to the space of (p,q) harmonic forms α with respect to ω, i.e. $\Delta_\omega \alpha = 0$.*
- **Hodge Conjecture** [a millenium problem!]. *If X is a projective algebraic manifold, Hodge (p,p)-classes $= H^{p,p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{Q})$ are generated by classes of algebraic cycles of codimension p with $\mathbb{Q}$-coefficients.*
- **(Claire Voisin, 2001)** $\exists$ *4-dimensional complex torus X possessing a non trivial Hodge class of type $(2,2)$, such that every coherent analytic sheaf $\mathcal{F}$ on X satisfies $c_2(\mathcal{F}) = 0$.*

Idea of proof of Claire Voisin's counterexample

The idea is to show the existence of a 4-dimensional complex torus $X = \mathbb{C}^4/\Lambda$ which does not contain any analytic subset of positive dimension, and such that the Hodge classes of degree 4 are perpendicular to ω^{n-2} for a suitable choice of the Kähler metric ω.

The lattice Λ is explicitly found via a number theoretic construction of Weil based on the number field $\mathbb{Q}[i]$, also considered by S. Zucker.

The theorem of existence of Hermitian Yang-Mills connections for stable bundles combined with Lübke's inequality then implies $c_2(\mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on the torus.

Kodaira embedding theorem

Theorem. *X a compact complex n-dimensional manifold. Then the following properties are equivalent.*

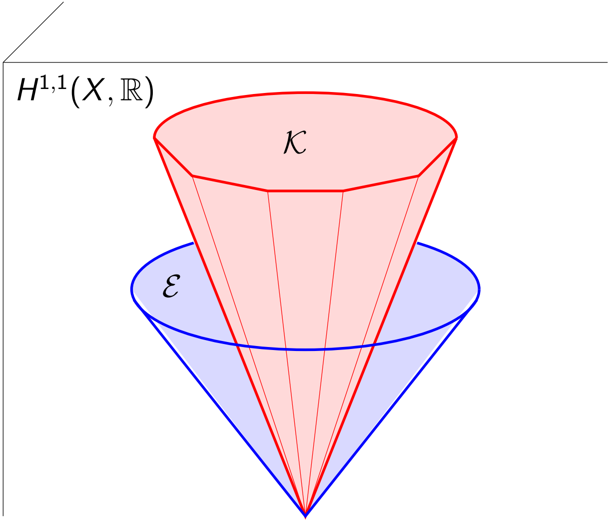
- *X can be embedded in some projective space $\mathbb{P}^N_{\mathbb{C}}$ as a closed analytic submanifold (and such a submanifold is automatically algebraic by Chow's thorem).*
- *X carries a hermitian holomorphic line bundle (L, h) with positive definite smooth curvature form $i\Theta_{L,h} > 0$.*
  For $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$,
  $$i\Theta_{L,h} = i\partial\overline{\partial}\varphi = -i\partial\overline{\partial}\log h,$$
  $$c_1(L) = \left\{\frac{i}{2\pi}\Theta_{L,h}\right\}.$$
- *X possesses a Hodge metric, i.e., a Kähler metric ω such that $\{\omega\} \in H^2(X, \mathbb{Z})$.*

Positive cones

Definition. Let X be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone.
  (by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.
- One can have: $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
  if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Kähler (red) cone and pseudoeffective (blue) cone

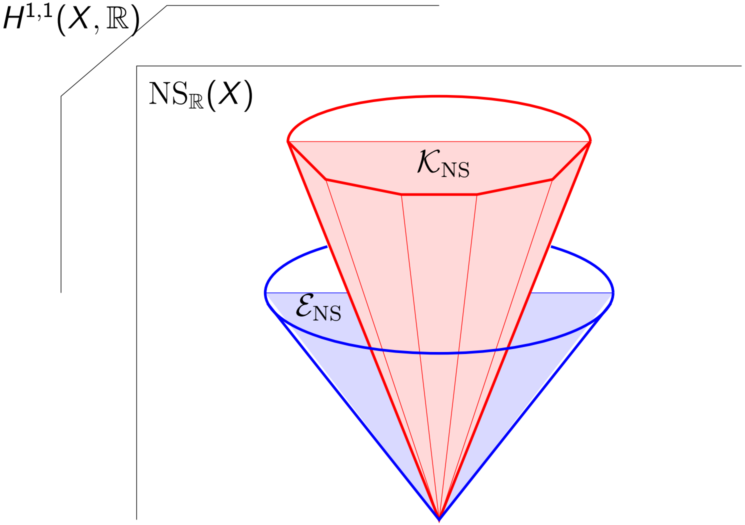


Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X, \mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$\begin{aligned}
\mathrm{NS}(X) &:= H^{1,1}(X, \mathbb{R}) \cap \big(H^2(X, \mathbb{Z})/\{\text{torsion}\}\big), \\
\mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \\
\mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \\
\mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).
\end{aligned}$$

Neron Severi parts of the cones



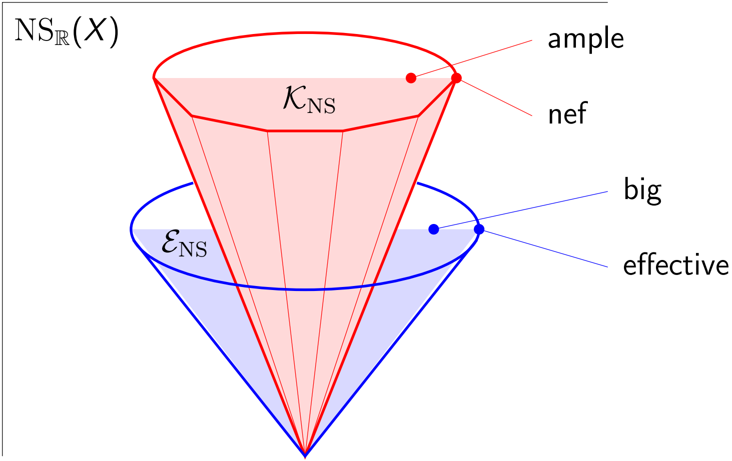
ample / nef / effective / big divisors

Theorem (Kodaira+successors, D90). *Assume X projective.*

- $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X,\mathcal{O}(A))$ provides an embedding of X in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D\cdot C \geq 0$ for every curve C.*
- $\mathcal{E}_{\mathrm{NS}}$ *is the closure of the cone of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*
- *The interior $\mathcal{E}^{\circ}_{\mathrm{NS}}$ is the cone of big divisors, namely divisors D such that $h^0(X,\mathcal{O}(kD)) \geq c\, k^{\dim X}$ for k large.*

Proof: L^2 estimates for $\overline{\partial}$ / Bochner-Kodaira technique

ample / nef / effective / big divisors



Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*
- **Theorem.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha = \{T\}$, $T =$ *a Kähler current.*

  We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.
- **Theorem** (D92). *Any Kähler current T can be written*

  $$T = \lim T_m$$

  where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e. $\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that

  $$\mu_m^\star T_m = [E_m] + \beta_m$$

  where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Idea of proof of analytic Zariski decomposition (1)

Locally one can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ on X. The approximating potentials φ_m of φ are defined as

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty\}.$$

The Ohsawa-Takegoshi L^2 extension theorem (applied to extension from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$. On the other hand $\varphi = \lim_{m\to+\infty} \varphi_m$ by a Bergman kernel trick and by the mean value inequality.

Idea of proof of analytic Zariski decomposition (2)

The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. The modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, with

$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective $\mathbb{Q}$-divisor E_m with normal crossings on $\widetilde{X}_m$. Now, we set $T_m = i\partial\overline{\partial}\varphi_m$ and $\beta_m = \mu_m^* T_m - [E_m]$. Then $\beta_m = i\partial\overline{\partial}\psi_m$ where

$$\psi_m = \frac{1}{2m} \log \sum_\ell |g_{\ell,m} \circ \mu_m / h|^2 \quad \text{locally on } \widetilde{X}_m$$

and h is a generator of $\mathcal{O}(-mE_m)$, and we see that β_m is a smooth semi-positive form on $\widetilde{X}_m$. The construction can be made global by using a gluing technique, e.g. via partitions of unity, and β_m can be made Kähler by a perturbation argument.

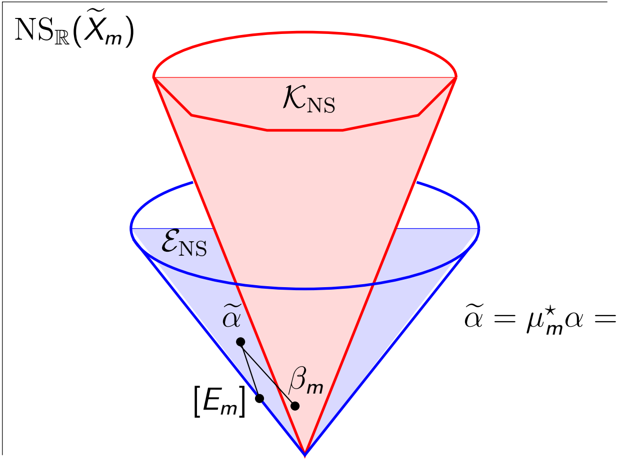
Algebraic analogue

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \;\; \text{effective}, \quad D_m \;\; \text{free}.$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.

Analytic Zariski decomposition



$$\widetilde{\alpha} = \mu_m^\star \alpha = [E_m] + \beta_m$$

Characterization of the Fujiki class $\mathcal{C}$

Theorem (Demailly-Păun 2004). *A compact complex manifold X is bimeromorphic to a Kähler manifold $\widetilde{X}$ (or equivalently, dominated by a Kähler manifold $\widetilde{X}$) if and only if it carries a Kähler current T.*

Proof. If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$, then $T = \mu_\star \widetilde{\omega}$ is a Kähler current on X.

Conversely, if T is a Kähler current, we take $\widetilde{X} = \widetilde{X}_m$ and $\widetilde{\omega} = \beta_m$ for m large enough.

Definition. *The class of compact complex manifolds X bimeromorphic to some Kähler manifold $\widetilde{X}$ is called the Fujiki class $\mathcal{C}$.*
Hodge decomposition still holds true in $\mathcal{C}$.

Numerical characterization of the Kähler cone

Theorem (Demailly-Păun 2004).
Let X be a compact Kähler manifold. Let

$$\mathcal{P} = \{\alpha \in H^{1,1}(X, \mathbb{R})\,;\ \int_Y \alpha^p > 0,\ \forall Y \subset X,\ \dim Y = p\}.$$

"cone of numerically positive classes".
Then the Kähler cone $\mathcal{K}$ is
one of the connected components of $\mathcal{P}$.

Corollary (Projective case).
If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.

Note: this is a "transcendental version" of the Nakai-Moishezon criterion.

Example (non projective) for which $\mathcal{K} \subsetneq \mathcal{P}$.

Take $X =$ generic complex torus $X = \mathbb{C}^n/\Lambda$.

Then X does not possess any analytic subset except finite subsets and X itself.

Hence $\mathcal{P} = \{\alpha \in H^{1,1}(X,\mathbb{R})\,;\ \int_X \alpha^n > 0\big\}$.

Since $H^{1,1}(X,\mathbb{R})$ is in one-to-one correspondence with constant hermitian forms, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, i.e.
possessing an even number of negative eigenvalues.

$\mathcal{K}$ is the component with all eigenvalues > 0.

Proof of the theorem : use Monge-Ampère

Fix $\alpha \in \overline{\mathcal{K}}$ so that $\int_X \alpha^n > 0$.

If ω is Kähler, then $\{\alpha + \varepsilon\omega\}$ is a Kähler class $\forall\varepsilon > 0$.

Use the Calabi-Yau theorem (Yau 1978) to solve the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = f_\varepsilon$$

where $f_\varepsilon > 0$ is a suitably chosen volume form.

Necessary and sufficient condition :

$$\int_X f_\varepsilon = (\alpha + \varepsilon\omega)^n \quad \text{in } H^{n,n}(X, \mathbb{R}).$$

Otherwise, the volume form of the Kähler metric $\alpha_\varepsilon = \alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon$ can be spread randomly.

Proof of the theorem : concentration of mass

In particular, the mass of the right hand side f_ε can be spread in an ε-neighborhood U_ε of any given subvariety $Y \subset X$.

If $\operatorname{codim} Y = p$, on can show that

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^p \to \Theta \quad \text{weakly}$$

where Θ positive (p,p)-current and $\Theta \geq \delta[Y]$ for some $\delta > 0$.

Now, "diagonal trick": apply the above result to

$$\widetilde{X} = X \times X, \qquad \widetilde{Y} = \text{diagonal} \subset \widetilde{X}, \qquad \widetilde{\alpha} = \mathrm{pr}_1^*\,\alpha + \mathrm{pr}_2^*\,\alpha.$$

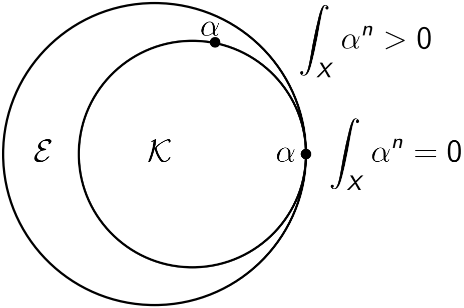
As $\widetilde{\alpha}$ is nef on $\widetilde{X}$ and $\int_{\widetilde{X}}(\widetilde{\alpha})^{2n} > 0$, it follows by the above that the class $\{\widetilde{\alpha}\}^n$ contains a Kähler current Θ such that $\Theta \geq \delta[\widetilde{Y}]$ for some $\delta > 0$. Therefore

$$T := (\mathrm{pr}_1)_*(\Theta \wedge \mathrm{pr}_2^*\,\omega)$$

is numerically equivalent to a multiple of α and dominates $\delta\omega$, and we see that T is a Kähler current.

Generalized Grauert-Riemenschneider result

Main conclusion (Demailly-Păun 2004).
Let X be a compact Kähler manifold and let $\{\alpha\} \in \overline{\mathcal{K}}$ such that $\int_X \alpha^n > 0$.
Then $\{\alpha\}$ contains a Kähler current T, i.e. $\{\alpha\} \in \mathcal{E}^\circ$.



Final step of proof

Clearly the open cone $\mathcal{K}$ is contained in $\mathcal{P}$, hence in order to show that $\mathcal{K}$ is one of the connected components of $\mathcal{P}$, we need only show that $\mathcal{K}$ is closed in $\mathcal{P}$, i.e. that $\overline{\mathcal{K}} \cap \mathcal{P} \subset \mathcal{K}$. Pick a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X \alpha^n > 0$. Hence $\{\alpha\}$ contains a Kähler current T.

Now, an induction on dimension using the assumption $\int_Y \alpha^p > 0$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on Y.

We conclude that $\{\alpha\}$ is a Kähler class by results of Paun (PhD 1997), therefore $\{\alpha\} \in \mathcal{K}$.

Variants of the main theorem

Corollary 1 (DP2004). *Let X be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is Kähler} \Leftrightarrow \exists\omega \text{ Kähler s.t.} \int_Y \alpha^k \wedge \omega^{p-k} > 0$$

for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.

Proof. Argue with $(1-t)\alpha + t\omega$, $t \in [0,1]$.

Corollary 2 (DP2004). *Let X be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is nef } (\alpha \in \overline{\mathcal{K}}) \Leftrightarrow \forall\omega \text{ Kähler} \int_Y \alpha \wedge \omega^{p-1} \geq 0$$

for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.

Consequence. *the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X,\mathbb{R})$.*

Deformations of compact Kähler manifolds

A deformation of compact complex manifolds is a proper holomorphic map

$$\pi : \mathcal{X} \to S \qquad \text{with smooth fibers } X_t = \pi^{-1}(t).$$

Basic question (Kodaira $\sim$ 1960). Is every compact Kähler manifold X a limit of projective manifolds :

$$X \simeq X_0 = \lim X_{t_\nu}, \quad t_\nu \to 0, \quad X_{t_\nu} \text{ projective ?}$$

Recent results by Claire Voisin (2004)

- *In any dimension ≥ 4, $\exists X$ compact Kähler manifold which does not have the homotopy type (or even the homology ring) of a complex projective manifold.*
- *In any dimension ≥ 8, $\exists X$ compact Kähler manifold such that no compact bimeromorphic model X' of X has the homotopy type of a projective manifold.*

Conjecture on deformation stability of the Kähler property

Theorem (Kodaira and Spencer 1960).
The Kähler property is open with respect to deformation :
if X_{t_0} is Kähler for some $t_0 \in S$, then the nearby fibers X_t are also Kähler (where "nearby" is in metric topology).

We expect much more.

Conjecture. *Let $\mathcal{X} \to S$ be a deformation with irreducible base space S such that some fiber X_{t_0} is Kähler. Then there should exist a countable union of analytic strata $S_\nu \subset S$, $S_\nu \neq S$, such that*

- *X_t is Kähler for $t \in S \smallsetminus \bigcup S_\nu$.*
- *X_t is bimeromorphic to a Kähler manifold (i.e. has a Kähler current) for $t \in \bigcup S_\nu$.*

Theorem on deformation stability of Kähler cones

Theorem (Demailly-Păun 2004). *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are $\nabla^{1,1}$-invariant over $S \smallsetminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇ in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

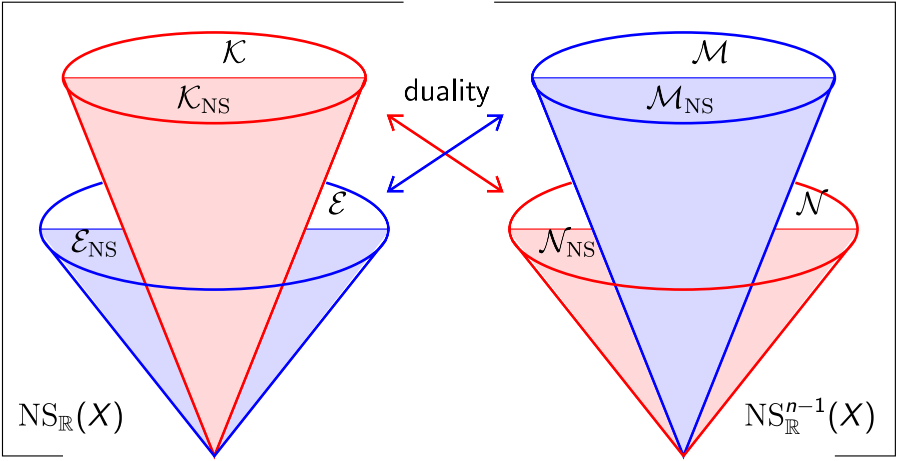
on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Positive cones in $H^{n-1,n-1}(X)$ and Serre duality

Definition. *Let X be a compact Kähler manifold.*

- *Cone of $(n-1,n-1)$ positive currents*
  $\mathcal{N} = \overline{\mathit{cone}}\big\{\{T\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\big\}$.
- *Cone of effective curves*
  $$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap \mathrm{NS}_{\mathbb{R}}^{n-1,n-1}(X),$$
  $$= \overline{\mathit{cone}}\big\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ C \text{ effective curve}\big\}.$$
- *Cone of movable curves : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M}_{\mathrm{NS}} = \overline{\mathit{cone}}\big\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ [C] = \mu_\star(H_1 \cdots H_{n-1})\big\}$
  where H_j = ample hyperplane section of $\widetilde{X}$.
- *Cone of movable currents : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M} = \overline{\mathit{cone}}\big\{\{T\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ T = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})\big\}$
  where $\widetilde{\omega}_j$ = Kähler metric on $\widetilde{X}$.

Main duality theorem



$H^{1,1}(X,\mathbb{R}) \longleftarrow$ Serre duality $\longrightarrow \quad H^{n-1,n-1}(X,\mathbb{R})$

Precise duality statement

Recall that the Serre duality pairing is

$$(\alpha^{(p,q)}, \beta^{(n-p,n-q)}) \longmapsto \int_X \alpha \wedge \beta.$$

Theorem (Demailly-Păun 2001)
If X is compact Kähler, then
$\mathcal{K}$ and $\mathcal{N}$ are dual cones.
(*well known since a long time : $\mathcal{K}_{\rm NS}$ and $\mathcal{N}_{\rm NS}$ are dual*)

Theorem (Boucksom-Demailly-Paun-Peternell 2004)
If X is projective algebraic, then
$\mathcal{E}_{\rm NS}$ and $\mathcal{M}_{\rm NS}$ are dual cones.

Conjecture (Boucksom-Demailly-Paun-Peternell 2004)
If X is Kähler, then
$\mathcal{E}$ and $\mathcal{M}$ should be dual cones.

Concept of volume (very important !)

Definition (Boucksom 2002).
The volume (movable self-intersection) of a big class $\alpha \in \mathcal{E}^\circ$ is

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^\star T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Theorem. (Boucksom 2002). *If L is a big line bundle and*
$\mu_m^*(mL) = [E_m] + [D_m]$
(*where $E_m =$ fixed part, $D_m =$ moving part*), *then*

$$\mathrm{Vol}(c_1(L)) = \lim_{m \to +\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m \to +\infty} D_m^n.$$

Approximate Zariski decomposition

In other words, the volume measures the amount of sections and the growth of the degree of the images of the rational maps

$$\Phi_{|mL|} : X \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$$

By Fujita 1994 and Demailly-Ein-Lazarsfeld 2000, one has

Theorem. *Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \leq \mathrm{Vol}(\beta) \leq \mathrm{Vol}(L).$$

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\}.$$

- $\forall k = 1, 2, \ldots, n$, $\exists$ *canonical "movable intersection product"*

  $$\mathcal{E}\times\cdots\times\mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle\alpha_1{\cdot}\alpha_2\cdots\alpha_{k-1}{\cdot}\alpha_k\rangle$$

  such that $\mathrm{Vol}(\alpha) = \langle\alpha^n\rangle$ *whenever* α *is a big class.*
- *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

  $$\langle\alpha_1\cdots(\alpha'_j{+}\alpha''_j)\cdots\alpha_k\rangle \geq \langle\alpha_1\cdots\alpha'_j\cdots\alpha_k\rangle{+}\langle\alpha_1\cdots\alpha''_j\cdots\alpha_k\rangle.$$

  It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ *are nef classes.*

Movable intersection theory (continued)

- *For $k = 1$, one gets a "divisorial Zariski decomposition"*

$$\alpha = \{N(\alpha)\} + \langle\alpha\rangle$$

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  where :

  - *$N(\alpha)$ is a uniquely defined effective divisor which is called the "negative divisorial part" of α. The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive ;*
  - *$\langle\alpha\rangle$ is "nef outside codimension 2".*

Construction of the movable intersection product

First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous more and more accurate log-resolutions $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \beta_{j,m}.$$

We define

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m \to +\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the limit is unique in cohomology ; this is based on "monotonicity properties" of the Zariski decomposition.

Generalized abundance conjecture

Definition. *For a class* $\alpha \in H^{1,1}(X, \mathbb{R})$*, the numerical dimension* $\nu(\alpha)$ *is*

- $\nu(\alpha) = -\infty$ *if* α *is not pseudo-effective,*
- $\nu(\alpha) = \max\{p \in \mathbb{N}\ ;\ \langle\alpha^p\rangle \neq 0\} \quad \in \{0, 1, \ldots, n\}$ *if* α *is pseudo-effective.*

Conjecture ("generalized abundance conjecture"). *For an arbitrary compact Kähler manifold* X*, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(c_1(K_X)).$$

Remark. The generalized abundance conjecture holds true when $\nu(c_1(K_X)) = -\infty$, 0, n (cases $-\infty$ and n being easy).

Orthogonality estimate

Theorem. *Let X be a projective manifold. Let $\alpha = \{T\} \in \mathcal{E}^{\circ}_{\mathrm{NS}}$ be a big class represented by a Kähler current T, and consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \le 20\,(C\omega)^n \big(\,\mathrm{Vol}(\alpha) - D_m^n\big)$$

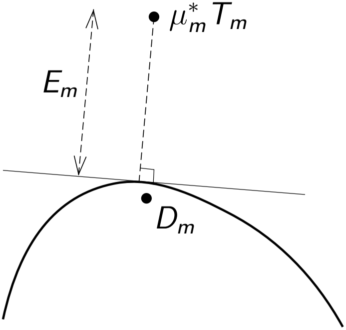
where $\omega = c_1(H)$ is a Kähler form and $C \ge 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

By going to the limit, one gets

Corollary. $\alpha \cdot \langle \alpha^{n-1} \rangle - \langle \alpha^n \rangle = 0.$

Schematic picture of orthogonality estimate

The proof is similar to the case of projecting a point onto a convex set, where the segment to closest point is orthogonal to tangent plane.



Proof of duality between $\mathcal{E}_{\rm NS}$ and $\mathcal{M}_{\rm NS}$

Theorem (Boucksom-Demailly-Păun-Peternell 2004).
For X projective, a class α is in $\mathcal{E}_{\rm NS}$ (pseudo-effective) if and only if it is dual to the cone $\mathcal{M}_{\rm NS}$ of moving curves.

Proof of the theorem. We want to show that $\mathcal{E}_{\rm NS} = \mathcal{M}_{\rm NS}^\vee$. By obvious positivity of the integral pairing, one has in any case

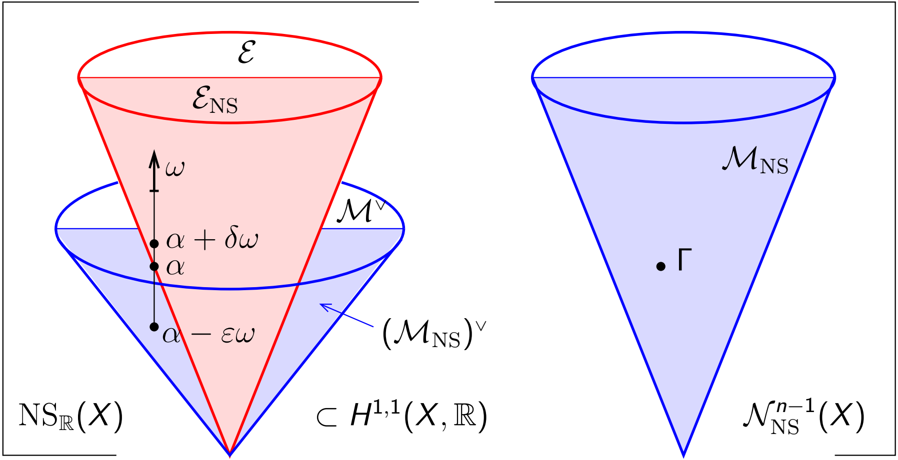
$$\mathcal{E}_{\rm NS} \subset (\mathcal{M}_{\rm NS})^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\rm NS}$ on the boundary of $\mathcal{E}_{\rm NS}$ which is in the interior of $\mathcal{N}_{\rm NS}^\vee$. Hence

$$(*) \qquad \alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve Γ, while $\langle\alpha^n\langle = \mathrm{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)
A projective manifold X is not uniruled if and only if K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.

Proof (of the non trivial implication). If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Plurigenera and the Minimal Model Program

Fundamental question. Prove that every birational class of non uniruled algebraic varieties contains a "minimal" member X with mild singularities, where "minimal" is taken in the sense of avoiding unnecessary blow-ups; minimality actually means that K_X is nef and not just pseudo-effective (pseudo-effectivity is known by the above results).

This requires performing certain birational transforms known as flips, and one would like to know whether
a) flips are indeed possible ("existence of flips"),
b) the process terminates ("termination of flips").
Thanks to Kawamata 1992 and Shokurov (1987, 1996), this has been proved in dimension 3 at the end of the 80's and more recently in dimension 4 (C. Hacon and J. McKernan also introduced in 2005 a new induction procedure).

Finiteness of the canonical ring

Basic questions.

- *Finiteness of the canonical ring:*
  Is the canonical ring $R = \bigoplus H^0(X, mK_X)$ *of a variety of general type always finitely generated ?*

  If true, $Proj(R)$ *of this graded ring* R *yields of course a "canonical model" in the birational class of* X*.*
- *Boundedness of pluricanonical embeddings:*
  Is there a bound r_n *depending only on dimension* $\dim X = n$*, such that the pluricanonical map* Φ_{mK_X} *of a variety of general type yields a birational embedding in projective space for* $m \geq r_n$ *?*
- *Invariance of plurigenera:*
  Are plurigenera $p_m = h^0(X, mK_X)$ *always invariant under deformation ?*

Recent results on extension of sections

The following is a very slight extension of results by M. Păun (2005) and B. Claudon (2006), which are themselves based on the ideas of Y.T. Siu 2000 and S. Takayama 2005.

Theorem. *Let $\pi : \mathcal{X} \to \Delta$ be a family of projective manifolds over the unit disk, and let $(L_j, h_j)_{0 \le j \le m-1}$ be (singular) hermitian line bundles with semipositive curvature currents $i\Theta_{L_j,h_j} \ge 0$ on $\mathcal{X}$. Assume that*

- *the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*
- *additionally the multiplier ideal sheaf $\mathcal{I}(h_{j|X_0})$ is trivial for $1 \le j \le m-1$.*

Then any section σ of $\mathcal{O}(mK_{\mathcal{X}} + \sum L_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ over the central fiber X_0 extends to $\mathcal{X}$.

Proof / invariance of plurigenera

The proof relies on a clever iteration procedure based on the Ohsawa-Takegoshi L^2 extension theorem, and a convergence process of an analytic nature (no algebraic proof at present !)

The special case of the theorem obtained by taking all bundles L_j trivial tells us in particular that any pluricanonical section σ of $mK_{\mathcal{X}}$ over X_0 extends to $\mathcal{X}$. By the upper semi-continuity of $t \mapsto h^0(X_t, mK_{X_t})$, this implies

Corollary (Siu 2000). *For any projective family $t \mapsto X_t$ of algebraic varieties, the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ do not depend on t.*