From the Ohsawa-Takegoshi theorem to asymptotic cohomology estimates

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dedicated to Prof. Takeo Ohsawa and Tetsuo Ueda on the occasion of their sixtieth birthday

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The Ohsawa-Takegoshi extension theorem

Let $(X, \omega)$ be a Kähler manifold, which is compact or weakly pseudoconvex, $n = \dim_{\mathbb{C}} X$, $L \to X$ a hermitian line bundle, $E$ a hermitian holomorphic vector bundle, and $s \in H^0(X, E)$ s.t. $Y = \{ x \in X ; s(x) = 0, \Lambda^r ds(x) \neq 0 \}$ is dense in $\overline{Y} = \{ s(x) = 0 \}$, so that $p = \dim \overline{Y} = n - r$. Assume that $\exists \alpha(x) \geq 1$ continuous s.t.

(i) $i \Theta_L + r i \partial \overline{\partial} \log |s|^2 \geq \max \left( 0, \alpha^{-1} \left\{ i \Theta_E s, s \right\} \right)$

(ii) $|s| \leq e^{-\alpha}$. Then

$\forall f \in H^0(Y, (K_X \otimes L)|_Y)$ s.t. $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_{Y, \omega} < +\infty$, $\exists F \in H^0(X, K_X \otimes L)$ s.t. $F|_Y = f$ and

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log |s|)^2} dV_{X, \omega} \leq C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y, \omega}.$$
The Ohsawa-Takegoshi extension theorem (II)

**Theorem (Ohsawa-Takegoshi 1987)**

Let $X = \Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex set, $\varphi$ a plurisubharmonic function on $\Omega$ and $Y = \Omega \cap S$ where $S$ is an affine linear subspace of $\mathbb{C}^n$ of any codimension $r$.

For every $f \in H^0(Y, \mathcal{O}_Y)$ such that $\int_Y |f|^2 e^{-\varphi} dV_Y < +\infty$, there exists $F \in H^0(\Omega, \mathcal{O}_\Omega)$ s.t. $F|_Y = f$ and

$$
\int_{\Omega} |F|^2 e^{-\varphi} dV_\Omega \leq C_r (\text{diam } \Omega)^{2r} \int_Y |f|^2 e^{-\varphi} dV_Y.
$$

Even the case when $Y = \{z_0\}$ is highly non trivial, thanks to the $L^2$ estimate: $\forall z_0 \in \Omega$, $\exists F \in H^0(\Omega, \mathcal{O}_\Omega)$, such that $F(z_0) = C_n^{-1/2} (\text{diam } \Omega)^{-n} e^{\varphi(z_0)/2}$ and

$$
\|F\|^2 = \int_{\Omega} |F|^2 e^{-\varphi} dV_\Omega \leq 1.
$$
Let $\Omega \subset \subset \mathbb{C}^n$ be a bounded pseudoconvex set, $\varphi$ a plurisubharmonic function on $\Omega$. Consider the Hilbert space

$$\mathcal{H}(\Omega, m\varphi) = \{ f \in \mathcal{O}(\Omega) ; \int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty \}.$$

One defines an “approximating sequence” of $\varphi$ by putting

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{j \in \mathbb{N}} |g_{j,m}(z)|^2$$

where $(g_{j,m})$ is a Hilbert basis of $\mathcal{H}(\Omega, m\varphi)$ (Bergman kernel procedure).

If $\text{ev}_z : \mathcal{H}(\Omega, m\varphi) \rightarrow \mathbb{C}$ is the evaluation linear form, one also has

$$\varphi_m(z) = \frac{1}{m} \log \| \text{ev}_z \| = \frac{1}{m} \sup_{f \in \mathcal{H}(\Omega,m\varphi), \|f\| \leq 1} \log |f(z)|.$$
The Ohsawa-Takegoshi approximation theorem implies

$$\varphi_m(z) \geq \varphi(z) - \frac{C_1}{m}$$

In the other direction, the mean value inequality gives

$$\varphi_m(z) \leq \sup_{B(z, r)} \varphi + \frac{n}{m} \log \frac{C_2}{r}, \quad \forall B(z, r) \subset \Omega$$

**Corollary 1** ("strong psh approximation")

One has \(\lim \varphi_m = \varphi\) and the Lelong-numbers satisfy

$$\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z).$$

**Corollary 2** (new proof of Siu’s Theorem, 1974)

The Lelong-number sublevel sets

$$E_c(\varphi) = \{z \in \Omega; \nu(\varphi, z) \geq c\}, \ c > 0$$

are analytic subsets.
Approximation of global closed \((1,1)\) currents

Let \((X, \omega)\) be a compact Kähler manifold and \(\{\alpha\} \in H^{1,1}(X, \mathbb{R})\) a cohomology class given by a smooth representative \(\alpha\).

Let \(T \in \{\alpha\}\) be an almost positive current, i.e. a closed \((1,1)\)-current such that

\[
T = \alpha + i\partial\bar{\partial}\varphi, \quad T \geq \gamma
\]

where \(\gamma\) is a continuous \((1,1)\)-form (e.g. \(\gamma = 0\) in case \(T \geq 0\)).

One can write \(T = \alpha + i\partial\bar{\partial}\varphi\) for some quasi-psh potential \(\varphi\) on \(X\), with \(i\partial\bar{\partial}\varphi \geq \gamma - \alpha\). Then use a finite covering \((B_j)_{1 \leq j \leq N}\) of \(X\) by coordinate balls, a partition of unity \((\theta_j)\), and set

\[
\varphi_m(z) = \sum_{j=1}^{N} \theta_j(z) \left( \psi_{j,m} + \sum_{k=1}^{n} \lambda_{j,k} |z_k^{(j)}|^2 \right)
\]

where \(\psi_{j,m}\) are Bergman approximations of

\[
\psi_j(z) := \varphi(z) - \sum \lambda_{j,k} |z_k^{(j)}|^2 \quad \text{(coordinates } z^{(j)} \text{ and coefficients } \lambda_{j,k} \text{ are chosen so that } \psi_j \text{ is psh on } B_j)\).
Let \((X, \omega)\) be a compact Kähler manifold and \(T = \alpha + i\partial\bar{\partial}\varphi \geq \gamma\) a quasi-positive closed \((1,1)\)-currents.

Then \(T = \lim T_m\) weakly where

(i) \(T_m = \alpha + i\partial\bar{\partial}\varphi_m \geq \gamma - \varepsilon_m \omega, \quad \varepsilon_m \to 0\)

(ii) \(\nu(T, z) - \frac{n}{m} \leq \nu(T_m, z) \leq \nu(T, z)\)

(iii) the potentials \(\varphi_m\) have only analytic singularities of the form \(\frac{1}{2m} \log \sum |g_{j,m}|^2 + C\infty\)

(iv) The local coherent ideal sheaves \((g_{j,m})\) glue together into a global ideal \(I_m = \text{multiplier ideal sheaf } I(m\varphi)\).

The OT theorem implies that \((\varphi_{2m})\) is decreasing, i.e. that the singularities of \(\varphi_{2m}\) increase to those of \(\varphi\) by “subadditivity”: 

\[I(\varphi + \psi) \subset I(\varphi) + I(\psi) \quad \Rightarrow \quad I(2^{m+1}\varphi) \subset (I(2^m\varphi))^2.\]
Kähler (red) cone and pseudoeffective (blue) cone

\[ H^{1,1}(X, \mathbb{R}) \]

Kähler classes:
\[ \mathcal{K} = \left\{ \{\alpha\} \ni \omega \right\} \]
(open convex cone)

pseudoeffective classes:
\[ \mathcal{E} = \left\{ \{\alpha\} \ni T \geq 0 \right\} \]
(closed convex cone)
In case $X$ is projective, it is interesting to consider the “algebraic part” of our “transcendental cones” $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X, \mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

\[
\begin{align*}
\text{NS}(X) & := H^{1,1}(X, \mathbb{R}) \cap (H^2(X, \mathbb{Z})/\{\text{torsion}\}), \\
\text{NS}_\mathbb{R}(X) & := \text{NS}(X) \otimes \mathbb{R}, \\
\mathcal{K}_{\text{NS}} & := \mathcal{K} \cap \text{NS}_\mathbb{R}(X) = \text{cone of ample divisors}, \\
\mathcal{E}_{\text{NS}} & := \mathcal{E} \cap \text{NS}_\mathbb{R}(X) = \text{cone of effective divisors}.
\end{align*}
\]

The interior $\mathcal{E}^\circ$ is by definition the cone of big classes.
Neron Severi parts of the cones

\[ H^{1,1}(X, \mathbb{R}) \]

\[ \text{NS}_\mathbb{R}(X) \]

\[ \mathcal{K}_{\text{NS}} \]

\[ \mathcal{E}_{\text{NS}} \]
Definition

On $X$ compact Kähler, a \textbf{Kähler current} $T$ is a closed positive $(1, 1)$-current $T$ such that $T \geq \delta \omega$ for some smooth hermitian metric $\omega$ and a constant $\delta \ll 1$.

Observation

$\alpha \in \mathcal{E}^\circ \iff \alpha = \{T\}$, $T = \text{a Kähler current}$.

Consequence of approximation theorem

Any Kähler current $T$ can be written $T = \lim T_m$ where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e.

$\exists$ a modification $\mu_m : \tilde{X}_m \to X$ such that $\mu_m^* T_m = [E_m] + \beta_m$

where $E_m$ effective $\mathbb{Q}$-divisor and $\beta_m$ Kähler form on $\tilde{X}_m$.  

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Proof of the consequence

Since $T \geq \delta \omega$, the main approximation theorem implies

$$T_m = i\partial\bar{\partial} \frac{1}{2m} \log \sum_j |g_{j,m}|^2 (\text{mod } C^\infty) \geq \frac{\delta}{2} \omega, \quad m \geq m_0$$

and $\mathcal{I}_m = \mathcal{I}(m\varphi)$ is a global coherent sheaf. The modification $\mu_m : \tilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, so that

$$\mu_m^* \mathcal{J}_m = \mathcal{O}(-mE_m)$$

for some effective $\mathbb{Q}$-divisor $E_m$ with normal crossings on $\tilde{X}_m$. If $h$ is a generator of $\mathcal{O}(-mE_m)$, and we see that

$$\beta_m = \mu_m^* T_m - [E_m] = \frac{1}{2m} \log \sum_j |g_{j,m} \circ \mu_m / h|^2 \quad \text{locally on } \tilde{X}_m$$

hence $\beta_m$ is a smooth semi-positive form on $\tilde{X}_m$ which is $> 0$ on $\tilde{X}_m \setminus \text{Supp } E_m$. By a perturbation argument using transverse negativity of exceptional divisors, $\beta_m$ can easily be made Kähler.
**Analytic Zariski decomposition**

**Theorem**

*For every class \( \{ \alpha \} \in \mathcal{E} \), there exists a positive current \( T_{\min} \in \{ \alpha \} \) with minimal singularities.*

**Proof.** Take \( T = \alpha + i \partial \overline{\partial} \varphi_{\min} \) where
\[
\varphi_{\min}(x) = \max\{ \varphi(x) ; \varphi \leq 0 \text{ and } \alpha + i \partial \overline{\partial} \varphi \geq 0 \}.
\]

**Theorem**

*Let \( X \) be compact Kähler and let \( \{ \alpha \} \in \mathcal{E}^\circ \) be a big class and \( T_{\min} \geq 0 \) be a current with minimal singularities. Then \( T_{\min} = \lim T_m \) where \( T_m \) are Kähler currents such that
(i) \( \exists \) modification \( \mu_m : \tilde{X}_m \to X \) with \( \mu_m^* T_m = [E_m] + \beta_m \), where \( E_m \) is a \( \mathbb{Q} \)-divisor and \( \beta_m \) a Kähler form on \( \tilde{X}_m \).
(ii) \( \int_{\tilde{X}_m} \beta_m^n \) is an increasing sequence converging to
\[
\Vol(X, \{ \alpha \}) := \int_X (T_{\min})_{ac}^n = \sup_{T \in \{ \alpha \}, \text{anal.sing}} \int_X T^n.
\]
Orthogonality estimate

Theorem (Boucksom-Demailly-Păun-Peternell 2004)

Assume $X$ projective and $\{\alpha\} \in E_{\text{NS}}^\circ$. Then $\beta_m = [D_m]$ is an ample $\mathbb{Q}$-divisor such that

$$(D_m^{n-1} \cdot E_m)^2 \leq 20 (C\omega)^n (\text{Vol}(\alpha) - D_m^n)$$

where $\omega = c_1(H)$ is a fixed polarization and $C \geq 0$ is a constant such that $\pm \alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ nef).

Proof similar to projection of a point onto a convex set, using elementary case of Morse inequalities:

$$\text{Vol}(\beta - \gamma) \geq \beta^n - n\beta^{n-1} \cdot \gamma$$

$\forall \beta, \gamma$ ample classes
Duality between $\mathcal{E}_{NS}$ and $\mathcal{M}_{NS}$

**Theorem (BDPP, 2004)**

*For $X$ projective, a class $\alpha$ is in $\mathcal{E}_{NS}$ (pseudo-effective) if and only if $\alpha \cdot C_t \geq 0$ for all mobile curves, i.e. algebraic curves which can be deformed to fill the whole of $X$. In other words, $\mathcal{E}_{NS}$ is the dual cone of the cone $\mathcal{M}_{NS}$ of mobile curves with respect to Serre duality.*

**Proof.** We want to show that $\mathcal{E}_{NS} = \mathcal{M}_{NS}^\vee$. By obvious positivity of the integral pairing, one has in any case

$$\mathcal{E}_{NS} \subset (\mathcal{M}_{NS})^\vee.$$ 

If the inclusion is strict, there is an element $\alpha \in \partial \mathcal{E}_{NS}$ on the boundary of $\mathcal{E}_{NS}$ which is in the interior of $\mathcal{N}_{NS}^\vee$. Hence

$$\alpha \cdot \Gamma \geq \varepsilon \omega \cdot \Gamma$$

for every moving curve $\Gamma$, while $\langle \alpha^n \rangle = \text{Vol}(\alpha) = 0$. 
Then use approximate Zariski decomposition of \( \{ \alpha + \delta \omega \} \) and orthogonality relation to contradict (\( \ast \)) with \( \Gamma = \langle \alpha^{n-1} \rangle \).
Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1$.  

**Theorem** (Boucksom-Demailly-Paun-Peternell 2004) 

A projective manifold $X$ has its canonical bundle $K_X$ pseudo-effective, i.e. $K_X \in \mathcal{E}_{NS}$, if and only if $X$ is not uniruled.

**Proof** (of the non trivial implication). If $K_X \notin \mathcal{E}_{NS}$, the duality pairing shows that there is a moving curve $C_t$ such that $K_X \cdot C_t < 0$. The standard “bend-and-break” lemma of Mori then implies that there is family $\Gamma_t$ of rational curves with $K_X \cdot \Gamma_t < 0$, so $X$ is uniruled.

**Note:** Mori’s proof uses characteristic $p$, so it is hard to extend to the Kähler case!
### Definition

Let $X$ be a compact complex manifold and let $L \to X$ be a holomorphic line bundle.

(i) \( \hat{h}^q(X, L) := \limsup_{k \to +\infty} \frac{n!}{m^n} h^q(X, L^\otimes m) \)

(ii) (asymptotic Morse partial sums)
\[
\hat{h}^\leq q(X, L) := \limsup_{m \to +\infty} \frac{n!}{m^n} \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, L^\otimes m).
\]

### Conjecture

\( \hat{h}^q(X, L) \) and \( \hat{h}^\leq q(X, L) \) depend only on \( c_1(L) \in H^{1,1}_{\text{BC}}(X, \mathbb{R}) \).

### Theorem (Küronya, 2005), (D, 2010)

This is true if \( c_1(L) \) belongs to the “divisorial Neron-Severi group” \( \text{DNS}_\mathbb{R}(X) \) generated by divisors.
Holomorphic Morse inequalities

Theorem (D, 1985)

Let $L \rightarrow X$ be a holomorphic line bundle on a compact complex manifold. Then

(i) $\hat{h}^q(X, L) \leq \inf_{u \in c_1(L)} \int_X (u, q) (-1)^q u^n$

(ii) $\hat{h}^{\leq q}(X, L) \leq \inf_{u \in c_1(L)} \int_X (u, \leq q) (-1)^q u^n$

where $X(u, q)$ is the $q$-index set of the $(1, 1)$-form $u$ and $X(u, \leq q) = \bigcup_{0 \leq j \leq q} X(u, j)$.

Question (or Conjecture !)

Are these inequalities always equalities ?

If the answer is yes, then $\hat{h}^q(X, L)$ and $\hat{h}^{\leq q}(X, L)$ actually only depend only on $c_1(L)$ and can be extended to $H^{1,1}_{BC}(X, \mathbb{R})$, e.g.

$$h^{\leq q}_{tr}(X, \alpha) := \inf_{u \in \alpha} \int_X (u, \leq q) (-1)^q u^n, \quad \forall \alpha \in H^{1,1}_{BC}(X, \mathbb{R})$$
Converse of Andreotti-Grauert theorem

**Theorem (D, 2010) / related result S.-I. Matsumura, 2011**

Let $X$ be a projective variety. Then

(i) the conjectures are true for $q = 0$:
\[
\hat{h}^0(X, L) = \text{Vol}(X, c_1(L)) = \inf_{u \in c_1(L)} \int_{X(u, 0)} u^n
\]

(ii) The conjectures are true for $\dim X \leq 2$

The limsup’s are *limits* in all of these cases.

**Observation 1.** The question is invariant by Serre duality:
\[
\hat{h}^q(X, L) = \hat{h}^{n-q}(X, -L)
\]

**Observation 2.** (Birational invariance). If $\mu : \tilde{X} \to X$ is a modification, then $\hat{h}^q(X, L) = \hat{h}^q(\tilde{X}, \mu^* L)$ by the Leray spectral sequence and
\[
\inf_{u \in \alpha} \int_{X(u, \leq q)} (-1)^q u^n = \inf_{\tilde{u} \in \mu^* \alpha} \int_{\tilde{X}(\tilde{u}, \leq q)} (-1)^q \tilde{u}^n.
\]
Main idea of the proof

It is enough to consider the case of a big line bundle $L$. Then use approximate Zariski decomposition:

$$\forall \delta > 0, \exists \mu = \mu_\delta : \tilde{X} \to X, \quad \mu^* L = E + A$$

where $E$ is $\mathbb{Q}$-effective and $A$ $\mathbb{Q}$-ample, and

$$\text{Vol}(X, L) - \delta \leq A^n \leq \text{Vol}(X, L), \quad E \cdot A^{n-1} \leq C\delta^{1/2},$$

the latter inequality by the orthogonality estimate.

Take $\omega \in c_1(A)$ a Kähler form and a metric $h$ on $O(E)$ such that

$$\Theta_{O(E), h} \wedge \omega^{n-1} = c_\delta \omega^n, \quad c_\delta = O(\delta^{1/2}).$$

The last line is obtained simply by solving a Laplace equation, thanks to the orthogonality estimate.
End of the proof

\[ \mu^* L = E + A \implies \tilde{u} = \Theta_{O(E),h} + \omega \in c_1(\mu^* L). \]

If \( \lambda_1 \leq \ldots \leq \lambda_n \) are the eigenvalues of \( \Theta_{O(E),h} \) with respect to \( \omega \), then \( \sum \lambda_i = \text{trace} \leq C\delta^{1/2} \). We have

\[ \tilde{u}^n = \prod (1 + \lambda_i)\omega^n \leq \left(1 + \frac{1}{n} \sum \lambda_i\right)^n \omega^n \leq (1 + O(\delta^{1/2})\omega^n, \]

therefore

\[ \int_{\tilde{X}(u,0)} \tilde{u}^n \leq (1 + O(\delta^{1/2}) \int_X \omega^n \leq (1 + O(\delta^{1/2}) \text{Vol}(X, L). \]

As \( \delta \to 0 \) we find

\[ \inf_{u \in c_1(L)} \int_{X(u,0)} u^n = \inf_{\mu} \inf_{\tilde{u} \in c_1(\mu^* L)} \int_{\tilde{X}(u,0)} \tilde{u}^n \leq \text{Vol}(X, L). \quad \text{QED} \]