Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials

Jean-Pierre Demailly
Université de Grenoble I, Institut Fourier

Lecture Notes of a Course given at Santa Cruz (CA, USA)
at the AMS Summer Research Institute
in Algebraic Geometry (July 1995)

Abstract. These notes are an expanded version of lectures delivered at the AMS Summer School on Algebraic Geometry, held at Santa Cruz in July 1995. The main goal of the notes is to study complex varieties (mostly compact or projective algebraic ones), through a few geometric questions related to hyperbolicity in the sense of Kobayashi. A convenient framework for this is the category of "directed manifolds", that is, the category of pairs (X, V) where X is a complex manifold and V a holomorphic subbundle of T_X. If X is compact, the pair (X, V) is hyperbolic if and only if there are no nonconstant entire holomorphic curves $f : \mathbb{C} \to X$ tangent to V (Brody's criterion). We describe a construction of projectivized k-jet bundles P_kV, which generalizes a construction made by Semple in 1954 and allows to analyze hyperbolicity in terms of negativity properties of the curvature. More precisely, $\pi_k : P_kV \to X$ is a tower of projective bundles over X and carries a canonical line bundle $\mathcal{O}_{P_kV}(1)$; the hyperbolicity of X is then conjecturally equivalent to the existence of suitable singular hermitian metrics of negative curvature on $\mathcal{O}_{P_kV}(-1)$ for k large enough. The direct images $(\pi_k)_\star\mathcal{O}_{P_kV}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization. Following an approach initiated by Green and Griffiths, we establish a basic Ahlfors-Schwarz lemma in the situation when $\mathcal{O}_{P_kV}(-1)$ has a (possibly singular) metric of negative curvature, and we infer that every nonconstant entire curve $f : \mathbb{C} \to V$ tangent to V must be contained in the base locus of the metric. This basic result is then used to obtain a proof of the Bloch theorem, according to which the Zariski closure of an entire curve in a complex torus is a translate of a subtorus. Our hope, supported by explicit Riemann-Roch calculations and other geometric considerations, is that the Semple bundle construction should be an efficient tool to calculate the base locus. Necessary or sufficient algebraic criteria for hyperbolicity are then obtained in terms of inequalities relating genera of algebraic curves drawn on the variety, and singularities of such curves. We finally describe some techniques introduced by Siu in value distribution theory, based on a use of meromorphic connections. These techniques have been developped later by Nadel to produce elegant examples of hyperbolic surfaces of low degree in projective 3-space; thanks to a suitable concept of "partial projective projection" and the associated Wronskian operators, substantial improvements on Nadel's degree estimate will be achieved here.

Key words: Kobayashi hyperbolic variety, directed manifold, genus of curves, jet bundle, jet differential, jet metric, Chern connection and curvature, negativity of jet curvature, variety of general type.

A.M.S. Classification (1991): 32H20, 32L10, 53C55, 14J40

Contents

§0. Introduction

In these notes, we investigate some geometric questions related to the concept of hyperbolic variety in the sense of Kobayashi [Kob70]. Hyperbolic algebraic varieties have attracted considerable attention, in part because of their conjectured diophantine properties. For instance, [Lang86] has conjectured (among other things) that any hyperbolic complex projective variety over a number field K can contain only finitely many rational points over K; this conjecture, which seems at present far beyond reach, may be regarded as a higher dimensional analogue of the Mordell conjecture. The reader can consult P. Vojta [Voj87] for aspects connected to diophantine problems.

We will be concerned here only with the geometric aspects of the theory which, although a priori more tractable than the diophantine aspects, are still conjectural for a major part; in fact very few satisfactory general purpose theorems are available. We hope that some of the ideas presented here will prove useful to achieve substantial progress. The reader is referred to S. Lang's survey [Lang86] and book [Lang87] for an overview of the theory until the mid 80's, and to J. Noguchi-T. Ochiai [NoOc90], P.M. Wong [Wong93] and M. Zaidenberg [Zai93] for a good exposition of more recent problems. Our goal here is not to provide an exhaustive compilation of known results, but rather to emphasize two or three important ideas around the concepts of jet bundles and jet metrics. Similar ideas have been applied successfully in a somewhat special situation in the recent work [SiYe96a] by Siu and Yeung, where the authors prove the hyperbolicity of the complement of an irreducible generic curve of high degree $d \geqslant 10^{13}$ in $\mathbb{P}^2$. Let us fix here our terminology: the word "generic" will refer to a property which holds true in the complement of a global algebraic or analytic subset in the parameter space, and the expression "very generic" will be used when the exceptional set of parameters is a countable union of algebraic or analytic subsets. As we will see in several instances, the geometry of jets conveys many natural interesting problems concerning the relationship between hyperbolicity and jet curvature negativity.

We now give a short outline of the contents. Recall that a complex variety is *hyperbolic in the sense of Kobayashi* if the family of holomorphic maps $f : \Delta \to X$ from the unit disk into X is a normal family. If X is compact (e.g. projective algebraic), it is well known that X is Kobayashi hyperbolic if and only if it is *Brody hyperbolic*, that is, if there are no nonconstant entire holomorphic curve $f : \mathbb{C} \to X$. In particular X has no rational or elliptic curves, and more generally every holomorphic map $f : Z \to X$ from an abelian variety (or complex torus) to X must be constant. Conversely, it has been suggested by Kobayashi [Kob70] and [Lang86] that these algebraic properties could be equivalent to hyperbolicity. To prove this, one would have to construct a torus Z and a nontrivial holomorphic map $f : Z \to X$ whenever X is non hyperbolic. A hint that this should be true is given by the following observation: if X is hyperbolic, there is an absolute constant $\varepsilon > 0$ such that the genus of any compact curve of X is bounded below by ε times the degree; conversely, this property fails to be true in many examples of nonhyperbolic projective varieties. Our belief, supported by some heuristic arguments, is that any sequence of compact curves (C_ℓ) with $\mathrm{genus}(C_\ell)/\mathrm{degree}(C_\ell) \to 0$ should have a cluster set swept out by the image of a map $f : Z \to X$ from a complex torus Z, such that the limit of some subsequence of the sequence of universal covering maps $\Delta \to C_\ell \to X$ (suitably reparametrized) coincides with the image of a (non necessarily compact) straight line of Z into X. A related conjecture of [Lang86] states that a projective variety is hyperbolic if and only if all its irreducible algebraic varieties are of general type. The most elementary step would be to exclude the case of manifolds with $c_1 \equiv 0$ (say, Calabi-Yau manifolds and symplectic manifolds), by showing for instance that they do admit a sequence of compact curves (C_ℓ) with $\mathrm{genus}(C_\ell)/\mathrm{degree}(C_\ell) \to 0$.

We next introduce jet bundles and jet differentials, extending some ideas of Green and Griffiths [GrGr80] (actually, the idea of using jet differentials can be traced back to the work of A. Bloch [Blo26, 26′], H. Cartan [Car28], L. Ahlfors [Ahl41] and T. Ochiai [Och77]). The basic idea is to introduce a bundle $E_{k,m}$ of algebraic differential operators $Q(f', f'', \ldots, f^{(k)})$ of weighted degree m, acting on germs of holomorphic curves ("*jet differentials of order k and degree m*"). When no other restrictions are made on Q, one obtains a "huge" bundle which will be denoted $E_{k,m}^{\mathrm{GG}}$ in reference to Green-Griffiths' work. In our case, the main goal is to determine the conformal type of entire curves drawn on the variety. Therefore, the way curves are parametrized is irrelevant. For this reason, one is led to introduce a subbundle $E_{k,m} \subset E_{k,m}^{\mathrm{GG}}$ of "special jet differential operators" $Q(f', f'', \ldots, f^{(k)})$, namely operators which have the property of being invariant by reparametrization, i.e.

$$Q((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}) = (\varphi')^m Q(f', f'', \ldots, f^{(k)})$$

for any local reparametrization $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the curve. For instance, all Wronskian determinants appearing in the wedge products f', $f' \wedge f''$, $\ldots$, $f' \wedge f'' \wedge \cdots \wedge f^{(\ell)}$ (computed in any coordinate system), and all polynomial combinations of these give rise to local sections of $E_{k,m}$. The subbundle $E_{k,m}$ turns out to have better positivity properties than $E^{\rm GG}_{k,m}$ (see section § 12). The bundles $E^{\rm GG}_{k,m}$ and $E_{k,m}$ have natural filtrations for which the graded pieces split in irreducible tensor product representations of $T^\star_X$ (Schur fonctor representations). However, in the case of $E_{k,m}$, it seems to be a highly non trivial question to decide which representations are actually involved, except for the simpler cases of jet differentials of order $k \leqslant 2$.

A basic observation is that any entire curve $f : \mathbb{C} \to X$ must automatically satisfy all algebraic differential equations $Q(f', f'', \ldots, f^{(k)}) = 0$ arising from global jet differential operators $Q \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ which vanish on some ample divisor A. Our proof is based on a strong pointwise version of the Ahlfors-Schwarz lemma (Lemma 3.2 and Theorem 7.8), and actually completes the scheme of proof suggested in [GrGr80] in the case of invariant jet differentials; the general pointwise case of the Ahlfors-Schwarz lemma for non necessarily invariant jet differentials, however, seems to be still unsettled. Let us mention, although we will not need it here, that the above vanishing theorem is still true with sections of $E^{\rm GG}_{k,m} \otimes \mathcal{O}(-A)$ in place of $E_{k,m} \otimes \mathcal{O}(-A)$ (see Remark 7.11 and [SiYe96c]). The vanishing theorem provides a way of investigating the hyperbolicity of X by trying to compute the base locus of sections in $H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ for large k and m (we will call this set the "*Green-Griffiths locus*" of X, although [GrGr80] deals rather with sections of $E^{\rm GG}_{k,m} \otimes \mathcal{O}(-A)$). We believe that the use of $E_{k,m}$ in place of $E^{\rm GG}_{k,m}$ should make easier to understand the structure of the base locus (especially in the case of higher values of k and $n = \dim X$), since the dimension of the projectivized jet bundles under consideration is smaller and $E_{k,m}$ is "more positive" than $E^{\rm GG}_{k,m}$. At least in the case of surfaces of general type, the existence of sections in $E_{k,m}$ and $E^{\rm GG}_{k,m}$ can be obtained by Riemann-Roch computations and suitable vanishing theorems proved by Bogomolov [Bog79] (here again, the conditions involved for the existence of sections in $E_{k,m}$ are better than those for $E^{\rm GG}_{k,m}$). It is reasonable to hope that suitable refinements of these ideas could lead in the future to a complete proof of the conjecture that every surface of general type only has finitely many rational and elliptic curves. Such a result is indeed obtained in [Bog77] for the class of surfaces X satisfying $c_1(X)^2 > c_2(X)$, thanks to a finiteness theorem for integral curves of foliations on surfaces (cf. Jouanolou [Jou78]). When $c_1(X)^2 > 2\,c_2(X)$, Lu-Yau [LuYa90] proved the additional result that the transcendental curves are algebraically degenerate; the argument is based on a result of Miyaoka asserting that the base locus of the first order jet differentials in $H^0(X, S^k T^\star_X \otimes \mathcal{O}(-A))$ is at most 1-dimensional for $c_1(X)^2 > 2\,c_2(X)$ and $k \gg 0$. Recently, S. Lu [Lu96] also obtained a proof for the limit case $c_1(X)^2 = 2\,c_2(X)$. By a different method, Lu-Miyaoka [LuMi95] investigate the case of arbitrary surfaces of general type, and get the desired finiteness results under suitable restrictions on the curve singularities.

In a similar manner, for varieties of general type of arbitrary dimension, it is conjectured that all entire curves $f : \mathbb{C} \to X$ are contained in a proper algebraic subset. One of the most celebrated result in this direction is the proof of the "Bloch theorem", i.e. the special case of the above conjecture when X has irregularity $q = h^0(X, \Omega^1_X) > \dim X$; the subject was revived by Ochiai [0ch77], who considerably clarified the ideas introduced in [Blo26], and formulated a technical result that would yield what he termed the "Bloch conjecture". The Bloch theorem was finally settled in the affirmative, by means of various techniques, in fundamental papers by Noguchi [Nog77, 81, 84], Kawamata [Kaw80] and Green-Griffiths [GrGr80] (to avoid the slight technical difficulty with the proof of the Ahlfors-Schwarz lemma in [GrGr80], we will prove here Bloch's theorem by replacing the Green-Griffiths jet differentials $E^{\mathrm{GG}}_{k,m}$ with the invariant jet differentials $E_{k,m}$). A very interesting related result is the proof by Siu-Yeung [SiYe96b] of the hyperbolicity of complements of ample divisors in abelian varieties (see also Noguchi [Nog96a] for an extension to the semi-abelian case). In § 10, we give a brief overview of these results, following an alternative approach suggested by Dethloff and Lu [DLu96], which makes use of a "logarithmic version" of our jet bundles. An earlier interesting result is the construction by Nadel [Nad89] of explicit hyperbolic algebraic surfaces in $\mathbb{P}^3$. Nadel's method is based on a use of meromorphic connections with low pole order, according to ideas introduced by Siu [Siu87]. In our setting, Nadel's technique is just the very special case when the jet differential operators under consideration are Wronskian operators associated with a meromorphic connection. Unfortunately, the method seems to work only for a restricted class of hypersurfaces or complete intersections defined by polynomials with few monomials. Thanks to a new flexible concept of *partial projective connection*, we have been able to improve Nadel's bounds and show the existence of hyperbolic algebraic surfaces in $\mathbb{P}^3$ of any degree $\geqslant 11$ (the conjectured optimal bound should be 5); see §11 and [EG96, 97], [DeEG97]*.

Contrary to most prior methods, including J. Noguchi's "jet projection method" (see [NoOc90], [Nog96]), our method does not use any Nevanlinna theory at all. Our approach is to construct suitable jet metrics of negative curvature, following original ideas of Grauert-Reckziegel [GRec65], [Gra89], Kobayashi [Kob75], Cowen-Griffiths [CoGr76] and Green-Griffiths [GrGr80]. In fact, the sheaves $\mathcal{O}(E_{k,m})$ of jet differentials defined above are the direct image sheaves of some canonical invertibles sheaves $\mathcal{O}_{X_k}(m)$ defined over suitable "projectivized k-jet bundles" $X_k \to X$. The k-jet bundle X_k is a tower of projective bundles, and can be obtained by iterating a natural fonctorial construction $(X, V) \rightsquigarrow (X_1, V_1)$ in the category of "*directed manifolds*". By definition, objects of this category are pairs (X, V), where X is a complex manifold and V a holomorphic subbundle of T_X, and the arrows are holomorphic morphisms preserving the V subbundles. We show in § 6 that the bundle X_k is a canonical smooth compactification of the bundle of "geometric" k-jets of regular curves (by "geometric jets", we mean that one does not pay attention to the way curves are parametrized). Such bundles X_k appear to be a natural generalization of a construction introduced by Semple [Sem54] in 1954, which has been used recently as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [ASS92], [CoKe94]).

* After these notes were completed, similar results have been announced independently in [SiYe96c].

In fact, almost all concepts pertaining to hyperbolicity can be extended in the general framework of directed manifolds (X, V), which we may think of as a "relative" situation (in fact, it is not necessary to assume that V is an integrable subbundle of T_X, but the case when $V = T_{X/S}$ is the relative tangent bundle of a smooth map $X \to S$ is of special interest). For instance, (X, V) is said to be Brody hyperbolic if there are no global holomorphic curve $f : \mathbb{C} \to X$ tangent to V. In the inductive definition of the k-jet spaces (X_k, V_k), X_k is simply the projectivization $P(V_{k-1})$ of $V_{k-1} \to X_{k-1}$, and $\mathcal{O}_{X_{k-1}}(-1)$ is the tautological line subbundle of the inverse image of V_{k-1} over X_k. Now, we say that X has k-jet negative curvature if $\mathcal{O}_{X_k}(-1)$ can be equipped with a hermitian metric (for which some type of singularities have to be allowed), such that the $(1,1)$ curvature form of the metric is negative along V_k. The negativity property of the k-jet curvature is closely related to the existence of sections of large degree in $H^0(X, E_{k,m})$, exactly in the same way positivity and ampleness are related. A variant of the Ahlfors-Schwarz lemma shows that the negativity of k-jet curvature implies hyperbolicity. Conversely, Kobayashi [Kob70] and [Lang86] raised the question whether hyperbolicity is equivalent to 1-jet negativity (in our terminology). We show that this optimistic picture is unfortunately wrong. In fact the k-jet negativity property yields the following necessary algebraic condition: there exists a constant $\varepsilon > 0$ such that every algebraic curve $C \subset X$ satisfies

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg(C) + \sum_{x \in C_{k-1}} (m_{C_{k-1}}(x) - 1),$$

where $\overline{C}$ is the normalization of C and $m_{C_{k-1}}(x)$ are the multiplicities of the singular points in the $(k-1)$-st jet lifting of C. Using this criterion, we construct for every integer k_0 an hyperbolic algebraic surface which cannot have any k-jet metric of negative curvature when $k \leqslant k_0$. It is nevertheless reasonable to expect that hyperbolicity is equivalent to the existence of a sufficiently large integer k_1 such that X has k-jet negative curvature for $k \geqslant k_1$.

We want to stress that many important questions have been left out in these notes, especially Nevanlinna theory and its applications to hyperbolicity theory [CaGr72], [Nog83], [Siu87], [Wong89], [RuSt91], [Nog91]. Especially noticeable in this respect is the work of Dethloff-Wong-Schumacher [DSW92, 94] on the hyperbolicity of complements of 3 or more generic curves in the projective plane, and the construction by Masuda-Noguchi [MaNo93] of hyperbolic hypersurfaces of large degree in $\mathbb{P}^n$. Also, in a more algebraic setting, there is an extensive literature dealing with the question of computing genus of curves in algebraic surfaces, bearing an intimate connection with hyperbolicity ([Bog77], [Cle86], [CKM88], [LuYa90], [Lu91], [LuMi95], [Lu96] [Xu94]). Last but not least, there are several important questions of Number Theory which either depend on Nevanlinna theory or suggest new tools for the study of differential geometric problems. The reader may profitably consult McQuillan's paper [McQu96], in which the method of Vojta-Faltings is adapted to give a completely new proof of the Bloch theorem.

I wish to express our gratitude to the organizers of the AMS Summer Institute held at Santa Cruz in July 1995 for giving me the opportunity of making a series of lectures on hyperbolicity theory. I warmly thank Gerd Dethloff, Siegmund

Kosarew, Steven Lu, Bernard Shiffman, Yum-Tong Siu and Mikhail Zaidenberg for formal or informal discussions which got me started in the subject and helped me to improve these notes.

§1. Hyperbolicity concepts and directed manifolds

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob70, Kob76]. Let X be a complex n-dimensional manifold. We denote by $f : \Delta \to X$ an arbitrary holomorphic map from the unit disk $\Delta \subset \mathbb{C}$ to X. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\,\exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}$$

(see H. Royden [Roy71], [Roy74]). In the terminology of Kobayashi [Kob75], a *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) positive function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\,N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E.$$

A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi pseudodistance* $d_K(x, y)$ is the geodesic pseudodistance obtained by integrating the Kobayashi-Royden infinitesimal metric. The manifold X is said to be *hyperbolic* (in the sense of Kobayashi) if d_K is actually a distance, namely if $d_K(x, y) > 0$ for all pairs of distinct points (x, y) in X. In this context, we have the following well-known results of Brody [Bro78].

1.1. Brody reparametrization lemma. *Let ω be a hermitian metric on X and let $f : \Delta \to X$ be a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geqslant (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0, R)$ onto $(1-\varepsilon)\Delta$ such that*

$$\|(f\circ\psi)'(0)\|_\omega = 1, \qquad \|(f\circ\psi)'(t)\|_\omega \leqslant \frac{1}{1-|t|^2/R^2} \quad \text{for every } t \in D(0, R).$$

In particular, if X is compact, given any sequence of holomorphic mappings $f_\nu : \Delta \to X$ such that $\lim \|f'_\nu(0)\|_\omega = +\infty$, one can find a sequence of homographic transformations $\psi_\nu : D(0, R_\nu) \to (1-1/\nu)\Delta$ with $\lim R_\nu = +\infty$, such that, after passing possibly to a subsequence, $(f_\nu\circ\psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a non constant holomorphic map $g : \mathbb{C} \to X$ with $\|g'(0)\|_\omega = 1$ and $\sup_{t\in\mathbb{C}} \|g'(t)\|_\omega \leqslant 1$.

Proof. The first assertion of Brody's lemma is obtained by selecting $t_0 \in \Delta$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t = t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t) : T_\Delta \to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on T_Δ, which is conformally invariant under $\mathrm{Aut}(\Delta)$. One then adjusts R and ψ so that

$\psi(0) = (1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega = 1$. As $|\psi'(0)| = \frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R = (1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega \geqslant (1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. □

1.2. Corollary (Brody's theorem)**.** *A compact complex manifold X is hyperbolic if and only if there are no non constant entire holomorphic maps $g : \mathbb{C} \to X$.*

Proof. The arguments are rather standard and will be developped in more detail in the proof of Prop. 1.5 below. □

Now, more generally, let (X, V) be a complex manifold equipped with a *holomorphic subbundle* $V \subset T_X$. We will refer to such a pair as being a *complex directed manifold.* A morphism $\Phi : (X, V) \to (Y, W)$ in the category of complex directed manifolds is a holomorphic map such that $\Phi_\star(V) \subset W$. Our philosophy is that directed manifolds are also useful to study the "absolute case", i.e. the case $V = T_X$, because there are fonctorial constructions which work better in the category of directed manifolds (see e.g. § 4, 5, 6). We think of directed manifolds as a kind of "relative situation", covering e.g. the case when V is the relative tangent sheaf to a smooth map $X \to S$. We want to stress here that no assumption need be made on the Lie bracket tensor $[\ ,\ ] : V \times V \to T_X/V$, and the rank $r = \operatorname{rank} V$ may be an arbitrary integer in the range $1 \leqslant r \leqslant n := \dim_{\mathbb{C}} X$. For the sake of generality, one might also wish to allow singularities in the subbundle V: for this, one can take V to be given by an arbitrary coherent subsheaf $\mathcal{V} \subset \mathcal{O}(T_X)$ such that $\mathcal{O}(T_X)/\mathcal{V}$ has no torsion; then V is a subbundle outside an analytic subset of codimension at least 2 (it is however somewhat safer to view $\mathcal{V}^\star$ as given by a quotient sheaf morphism $\Omega^1_X \to \mathcal{V}^\star$ and let $V^\star$ be the associated linear space, see Remark 3.10 below). For the sake of simplicity, we will assume most of the time that V is actually a subbundle of T_X. In this situation, we generalize the notion of hyperbolicity as follows.

1.3. Definition. *Let (X, V) be a complex directed manifold.*

i) *The Kobayashi-Royden infinitesimal metric of (X, V) is the Finsler metric on V defined for any $x \in X$ and $\xi \in V_x$ by*

$$\mathbf{k}_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\ \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi,\ f'(\Delta) \subset V\big\}.$$

Here $\Delta \subset \mathbb{C}$ is the unit disk and the map f is an arbitrary holomorphic map which is tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all $t \in \Delta$. We say that (X, V) is infinitesimally hyperbolic if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon\|\xi\|_\omega$ in terms of any smooth hermitian metric ω on X, when x describes a compact subset of X.

ii) *More generally, the Kobayashi-Eisenman infinitesimal pseudometric of (X, V) is the pseudometric defined on all decomposable p-vectors $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V_x$, $1 \leqslant p \leqslant r = \operatorname{rank} V$, by*

$$\mathbf{e}^p_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\ \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_\star(\tau_0) = \xi,\ f_\star(T_{\mathbb{B}_p}) \subset V\big\}$$

where $\mathbb{B}_p$ is the unit ball in $\mathbb{C}^p$ and $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$ is the unit p-vector of $\mathbb{C}^p$ at the origin. We say that (X, V) is infinitesimally p-measure hyperbolic if $\mathbf{e}^p_{(X,V)}$ is positive definite on every fiber $\Lambda^p V_x$ and satisfies a locally uniform lower bound in terms of any smooth metric.

If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed manifolds, it is immediate to check that we have the monotonicity property

$$(1.4) \qquad \mathbf{k}_{(Y,W)}(\Phi_\star \xi) \leqslant \mathbf{k}_{(X,V)}(\xi), \qquad \forall \xi \in V,$$

$$(1.4^p) \qquad \mathbf{e}^p_{(Y,W)}(\Phi_\star \xi) \leqslant \mathbf{e}^p_{(X,V)}(\xi), \qquad \forall \xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V.$$

The following proposition shows that virtually all reasonable definitions of the hyperbolicity property are equivalent if X is compact (in particular, the additional assumption that there is locally uniform lower bound for $\mathbf{k}_{(X,V)}$ is not needed). We merely say in that case that (X, V) is *hyperbolic.*

1.5. Proposition. *For an arbitrary directed manifold (X, V), the Kobayashi-Royden infinitesimal metric $\mathbf{k}_{(X,V)}$ is upper semicontinuous on the total space of V. If X is compact, (X, V) is infinitesimally hyperbolic if and only if there are no non constant entire curves $g : \mathbb{C} \to X$ tangent to V. In that case, $\mathbf{k}_{(X,V)}$ is a continuous* (*and positive definite*) *Finsler metric on V.*

Proof. The proof is almost identical to the standard proof for $\mathbf{k}_X$, so we only give a brief outline of the ideas. In order to prove the upper semicontinuity, let $\xi_0 \in V_{x_0}$ and $\varepsilon > 0$ be given. Then there is a curve $f : \Delta \to X$ tangent to V such that $f(0) = x_0$ and $\lambda\, f'(0) = \xi_0$ with $0 < \lambda < \mathbf{k}_X(\xi_0) + \varepsilon$. Take $\lambda = 1$ for simplicity, and replace ξ_0 by $\lambda^{-1}\xi_0$. We may assume that f is a proper embedding, otherwise we replace (X, V) by $(X', V') = (X \times \Delta, \mathrm{pr}_1^\star V \oplus \mathrm{pr}_2^\star T_\Delta)$, f by $f \times \mathrm{Id}_\Delta$, ξ_0 by $\xi_0 \oplus 1$, and use a monotonicity argument for the projection $\mathrm{pr}_1 : X' \to X$. If f is an embedding, then $f(\Delta)$ is a Stein submanifold of X, and thus $f(\Delta)$ has a Stein neighborhood Ω. As Ω is Stein, there exists a section $\theta \in H^0(\Omega, \mathcal{O}(V))$ extending $f' \in H^0(f(\Delta), \mathcal{O}(V))$. The map f can be viewed as the solution of the differential equation $f' = \theta(f)$ with initial value $f(0) = x_0$. Take a small perturbation $g' = \theta_\eta(g)$ with initial value $g(0) = x$, where $\theta_\eta = \theta + \sum \eta_j s_j$ and $s_1, \ldots, s_N$ are finitely many sections of $H^0(\Omega, \mathcal{O}(V))$ which generate V in a neighborhood of x_0. We can achieve that $g'(0) = \theta_\eta(x)$ is equal to any prescribed vector $\xi \in V_x$ close to $\xi_0 = \theta(x_0)$, and the solution g exists on $(1 - \varepsilon)\Delta$ if the perturbation is small enough. We conclude that $\mathbf{k}_{(X,V)}$ is upper semicontinuous by considering $t \mapsto g((1 - \varepsilon)t)$.

If there exists a non constant entire curve $g : \mathbb{C} \to X$ tangent to V, it is clear that $\mathbf{k}_{(X,V)}(g'(t)) \equiv 0$, hence (X, V) cannot be hyperbolic. Conversely, if X is compact and if there are no non constant entire curves $g : \mathbb{C} \to X$ tangent to V, Brody's lemma implies that there is an absolute bound $\|f'(0)\|_\omega \leqslant C$ for all holomorphic maps $f : \Delta \to X$ tangent to V ; hence $\mathbf{k}_{(X,V)}(\xi) \geqslant C^{-1}\|\xi\|_\omega$ and (X, V) is infinitesimally hyperbolic. By reparametrizing f with an arbitrary automorphism of Δ, we find $\|f'(t)\|_\omega \leqslant C/(1-|t|^2)$. The space of maps $f : \Delta \to X$ tangent to V is therefore compact for the topology of uniform convergence on compact subsets of Δ, thanks to Ascoli's theorem. We easily infer from this that $\mathbf{k}_{(X,V)}$ is lower semicontinuous on V. $\square$

We conclude this section by showing that hyperbolicity is an open property.

1.6. Proposition. *Let $(\mathcal{X},\mathcal{V}) \to S$ be a holomorphic family of compact directed manifolds (by this, we mean a proper holomorphic map $\mathcal{X} \to S$ together with a holomorphic subbundle $\mathcal{V} \subset T_{\mathcal{X}/S}$ of the relative tangent bundle, defining a deformation $(X_t, V_t)_{t\in S}$ of the fibers). Then the set of $t \in S$ such that the fiber (X_t, V_t) is hyperbolic is open in S with respect to the euclidean topology.*

Proof. Take a sequence of non hyperbolic fibers (X_{t_ν}, V_{t_ν}) with $t_\nu \to t$ and fix a hermitian metric ω on $\mathcal{X}$. By Brody's lemma, there is a sequence of entire holomorphic maps $g_\nu : \mathbb{C} \to X_{t_\nu}$ tangent to V_{t_ν}, such that $\|g'_\nu(0)\|_\omega = 1$ and $\|g'_\nu\| \leqslant 1$. Ascoli's theorem shows that there is a subsequence of (g_ν) converging uniformly to a limit $g : \mathbb{C} \to X_t$, tangent to V_t, with $\|g'(0)\|_\omega = 1$. Hence (X_t, V_t) is not hyperbolic, and the collection of non hyperbolic fibers is closed in S. $\square$

§2. Hyperbolicity and bounds for the genus of curves

In the case of projective algebraic varieties, hyperbolicity is expected to be related to other properties of a more algebraic nature. Theorem 2.1 below is a first step in this direction.

2.1. Theorem. *Let (X, V) be a compact complex directed manifold and let $\sum \omega_{jk} dz_j \otimes d\overline{z}_k$ be a hermitian metric on X, with associated positive $(1,1)$-form $\omega = \frac{i}{2}\sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Consider the following three properties, which may or not be satisfied by (X, V) :*

i) *(X, V) is hyperbolic.*

ii) *There exists $\varepsilon > 0$ such that every compact irreducible curve $C \subset X$ tangent to V satisfies*
$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C)$$
where $g(\overline{C})$ is the genus of the normalization $\overline{C}$ of C, $\chi(\overline{C})$ its Euler characteristic and $\deg_\omega(C) = \int_C \omega$. (This property is of course independent of ω.)

iii) *There does not exist any non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_\star(T_Z) \subset V$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii).

Proof. i) $\Rightarrow$ ii). If (X, V) is hyperbolic, there is a constant $\varepsilon_0 > 0$ such that $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon_0 \|\xi\|_\omega$ for all $\xi \in V$. Now, let $C \subset X$ be a compact irreducible curve tangent to V and let $\nu : \overline{C} \to C$ be its normalization. As (X, V) is hyperbolic, $\overline{C}$ cannot be a rational or elliptic curve, hence $\overline{C}$ admits the disk as its universal covering $\rho : \Delta \to \overline{C}$.

The Kobayashi-Royden metric $\mathbf{k}_\Delta$ is the Finsler metric $|dz|/(1 - |z|^2)$ associated with the Poincaré metric $|dz|^2/(1 - |z|^2)^2$ on Δ, and $\mathbf{k}_{\overline{C}}$ is such that

$\rho^\star \mathbf{k}_{\overline{C}} = \mathbf{k}_\Delta$. In other words, the metric $\mathbf{k}_{\overline{C}}$ is induced by the unique hermitian metric on $\overline{C}$ of constant Gaussian curvature -4. If $\sigma_\Delta = \frac{i}{2} dz \wedge d\overline{z}/(1-|z|^2)^2$ and $\sigma_{\overline{C}}$ are the corresponding area measures, the Gauss-Bonnet formula (integral of the curvature $= 2\pi\,\chi(\overline{C})$) yields

$$\int_{\overline{C}} d\sigma_{\overline{C}} = -\frac{1}{4}\int_{\overline{C}} \mathrm{curv}(\mathbf{k}_{\overline{C}}) = -\frac{\pi}{2}\chi(\overline{C})$$

On the other hand, if $j : C \to X$ is the inclusion, the monotonicity property (1.4) applied to the holomorphic map $j \circ \nu : \overline{C} \to X$ shows that

$$\mathbf{k}_{\overline{C}}(t) \geqslant \mathbf{k}_{(X,V)}\big((j\circ\nu)_\star t\big) \geqslant \varepsilon_0 \big\|(j\circ\nu)_\star t\big\|_\omega, \qquad \forall t \in T_{\overline{C}}.$$

From this, we infer $d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 (j\circ\nu)^\star\omega$, thus

$$-\frac{\pi}{2}\chi(\overline{C}) = \int_{\overline{C}} d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 \int_{\overline{C}} (j\circ\nu)^\star\omega = \varepsilon_0^2 \int_C \omega.$$

Property ii) follows with $\varepsilon = 2\varepsilon_0^2/\pi$.

ii) $\Rightarrow$ iii). First observe that ii) excludes the existence of elliptic and rational curves tangent to V. Assume that there is a non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_\star(T_Z) \subset V$. We must have $\dim \Phi(Z) \geqslant 2$, otherwise $\Phi(Z)$ would be a curve covered by images of holomorphic maps $\mathbb{C} \to \Phi(Z)$, and so $\Phi(Z)$ would be elliptic or rational, contradiction. Select a sufficiently general curve Γ in Z (e.g., a curve obtained as an intersection of very generic divisors in a given very ample linear system $|L|$ in Z). Then all isogenies $u_m : Z \to Z$, $s \mapsto ms$ map Γ in a $1:1$ way to curves $u_m(\Gamma) \subset Z$, except maybe for finitely many double points of $u_m(\Gamma)$ (if $\dim Z = 2$). It follows that the normalization of $u_m(\Gamma)$ is isomorphic to Γ. If Γ is general enough, similar arguments show that the images

$$C_m := \Phi(u_m(\Gamma)) \subset X$$

are also generically $1:1$ images of Γ, thus $\overline{C}_m \simeq \Gamma$ and $g(\overline{C}_m) = g(\Gamma)$. We would like to show that C_m has degree $\geqslant \mathrm{Const}\, m^2$. This is indeed rather easy to check if ω is Kähler, but the general case is slightly more involved. We write

$$\int_{C_m} \omega = \int_\Gamma (\Phi\circ u_m)^\star\omega = \int_Z [\Gamma] \wedge u_m^\star(\Phi^\star\omega),$$

where Γ denotes the current of integration over Γ. Let us replace Γ by an arbitrary translate $\Gamma + s$, $s \in Z$, and accordingly, replace C_m by $C_{m,s} = \Phi\circ u_m(\Gamma+s)$. For $s \in Z$ in a Zariski open set, $C_{m,s}$ is again a generically $1:1$ image of $\Gamma + s$. Let us take the average of the last integral identity with respect to the unitary Haar measure $d\mu$ on Z. We find

$$\int_{s\in Z}\left(\int_{C_{m,s}} \omega\right) d\mu(s) = \int_Z \left(\int_{s\in Z} [\Gamma+s]\, d\mu(s)\right) \wedge u_m^\star(\Phi^\star\omega).$$

Now, $\gamma := \int_{s\in Z}[\Gamma + s]\, d\mu(s)$ is a translation invariant positive definite form of type $(p-1,p-1)$ on Z, where $p = \dim Z$, and γ represents the same cohomology class as $[\Gamma]$, i.e. $\gamma \equiv c_1(L)^{p-1}$. Because of the invariance by translation, γ has constant coefficients and so $(u_m)_\star\gamma = m^2\gamma$. Therefore we get

$$\int_{s\in Z} d\mu(s) \int_{C_{m,s}} \omega = m^2 \int_Z \gamma \wedge \Phi^\star\omega.$$

In the integral, we can exclude the algebraic set of values z such that $C_{m,s}$ is not a generically $1:1$ image of $\Gamma + s$, since this set has measure zero. For each m, our integral identity implies that there exists an element $s_m \in Z$ such that $g(\overline{C}_{m,s_m}) = g(\Gamma)$ and

$$\deg_\omega(C_{m,s_m}) = \int_{C_{m,s_m}} \omega \geqslant m^2 \int_Z \gamma \wedge \Phi^\star\omega.$$

As $\int_Z \gamma \wedge \Phi^\star\omega > 0$, the curves C_{m,s_m} have bounded genus and their degree is growing quadratically with m, contradiction to property ii). □

2.2. Definition. *We say that a projective directed manifold (X,V) is "algebraically hyperbolic" if it satisfies property* 2.1 ii)*, namely, if there exists $\varepsilon > 0$ such that every algebraic curve $C \subset X$ tangent to V satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C).$$

A nice feature of algebraic hyperbolicity is that it satisfies an algebraic analogue of the openness property.

2.3. Proposition. *Let $(\mathcal{X},\mathcal{V}) \to S$ be an algebraic family of projective algebraic directed manifolds (given by a projective morphism $\mathcal{X} \to S$). Then the set of $t \in S$ such that the fiber (X_t, V_t) is algebraically hyperbolic is open with respect to the "countable Zariski topology" of S (by definition, this is the topology for which closed sets are countable unions of algebraic sets).*

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathcal{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling back the Fubini-Study metric via an embedding in a projective space. If integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ tangent to V_t with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semicontinuous). Now, the set of non algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0} \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof (of course, one has to know that the countable Zariski topology is actually a topology, namely that the class of countable unions of algebraic sets is stable under arbitrary intersections; this can be easily checked by an induction on dimension). □

2.4. Remark. More explicit versions of the openness property have been dealt with in the literature. H. Clemens ([Cle86] and [CKL88]) has shown that on a very generic surface of degree $d \geqslant 5$ in $\mathbb{P}^3$, the curves of type (d,k) are of genus $g > kd(d-5)/2$ (recall that a very generic surface $X \subset \mathbb{P}^3$ of degree $\geqslant 4$ has Picard group generated by $\mathcal{O}_X(1)$ thanks to the Noether-Lefschetz theorem, thus any curve on the surface is a complete intersection with another hypersurface of degree k; such a curve is said to be of type (d,k); genericity is taken here in the sense of the countable Zariski topology). Improving on this result of Clemens, Geng Xu [Xu94] has shown that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2-2$. This actually shows that a very generic surface of degree $d \geqslant 6$ is algebraically hyperbolic. Although a very generic quintic surface has no rational or elliptic curves, it seems to be unknown whether a (very) generic quintic surface is algebraically hyperbolic in the sense of Definition 2.2.

2.5. Remark. It would be interesting to know whether algebraic hyperbolicity is open with respect to the euclidean topology; still more interesting would be to know whether Kobayashi hyperbolicity is open for the countable Zariski topology (of course, both properties would follow immediately if one knew that Zariski and Kobayashi hyperbolicity coincide, but they seem otherwise highly non trivial to establish). The latter openness property has raised an important amount of work around the following more particular question: is a (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d large enough (say $d \geqslant 2n+1$) Kobayashi hyperbolic? Again, "very generic" is to be taken here in the sense of the countable Zariski topology. Brody-Green [BrGr77] and Nadel [Nad89] produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for all degrees $d \geqslant 50$, and Masuda-Noguchi [MaNo93] recently gave examples of such hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geqslant 2$, of degree $d \geqslant d_0(n)$ large enough. The question of studying the hyperbolicity of complements $\mathbb{P}^n \smallsetminus D$ of generic divisors is in principle closely related to this; in fact if $D = \{P(z_0,\ldots,z_n)=0\}$ is a smooth generic divisor of degree d, one may look at the hypersurface

$$X = \big\{z_{n+1}^d = P(z_0,\ldots,z_n)\big\} \subset \mathbb{P}^{n+1}$$

which is a cyclic $d\!:\!1$ covering of $\mathbb{P}^n$. Since any holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \smallsetminus D$ can be lifted to X, it is clear that the hyperbolicity of X would imply the hyperbolicity of $\mathbb{P}^n \smallsetminus D$. The hyperbolicity of complements of divisors in $\mathbb{P}^n$ has been investigated by many authors. M. Green [Green77] proved the hyperbolicity of the complement of $(2n+1)$ generic hyperplanes in $\mathbb{P}^n$. Zaidenberg [Zai89] showed the existence of curves with hyperbolic complement for every degree $d \geqslant 5$. In response to a conjecture of [Zai89], [DSW92, 94] showed that the complement of the union of at least 3 generic curves is hyperbolic, when the sum of degrees is at least 5. More recently, Siu and Yeung [SiYe96a] proved the harder fact that the complement of a generic irreducible curve of high degree in $\mathbb{P}^2$ is hyperbolic. Their approach uses jet bundle techniques, and it is one of our goals to explain some of the underlying geometric ideas. □

In the "absolute case" $V = T_X$, it seems reasonable to expect that all three properties 2.1 i), ii), iii) are equivalent, in particular that Kobayashi and algebraic hyperbolicity coincide. In fact, S. Lang made the following conjecture.

2.6. Conjecture ([Lang 86, 87]). *A compact complex manifold X is hyperbolic if and only if there are no nontrivial holomorphic maps $Z \to X$ where $Z = \mathbb{C}^p/\Lambda$ is a compact complex torus.*

The "only if" part of the conjecture is of course clearly true. On the other hand, if X is projective algebraic, every holomorphic map $Z \to X$ of a compact complex torus Z to X admits a factorization $Z \to Z' \to X$ where Z' is an abelian variety (see e.g. A. Weil [We57]). Thus, for X projective algebraic and $V = T_X$, a positive solution to Conjecture 2.5 would imply the equivalence of properties i), ii), iii) in Theorem 2.1.

2.7. Hint of heuristic proof of Lang's conjecture. Although Lang's conjecture seems at present far beyond reach, we would like to present here a heuristic argument showing how things could possibly work. These ideas arose through discussions with S. Kosarew. Let $\mathcal{B}_X$ be the space of all Brody curves, that is, the set of all entire holomorphic curves $f : \mathbb{C} \to X$ with $\|f'\|_\omega \leqslant 1$ for some given hermitian metric ω on X. For each $\varepsilon \geqslant 0$, we define a distance δ_ε on $\mathcal{B}_X$ by putting

$$\delta_\varepsilon(f, g) = \sup_{t\in\mathbb{C}} d_\omega(f(t), g(t))e^{-\varepsilon|t|}$$

where d_ω is the geodesic distance on X. For all $\varepsilon > 0$, d_ε defines the topology of uniform convergence of compact sets, whilst d_0 is the topology of uniform convergence up to infinity. Hence $(\mathcal{B}_X, d_\varepsilon)$ is a compact metric space for $\varepsilon > 0$. Assume that this is still true for $\varepsilon = 0$ and assume moreover that X is not hyperbolic, i.e. $\mathcal{B}_X \neq \emptyset$. We then consider the compact topological group $G_{\mathcal{B}}$ of isometries of $(\mathcal{B}_X, d_0)$ and look at the group homomorphism

$$\Psi : (\mathbb{C}, +) \to (G_{\mathcal{B}}, \circ), \qquad a \mapsto f_a \quad \text{where } f_a(t) = f(t - a).$$

We claim that $G_{\mathcal{B}}$ should be a finite dimensional Lie group and $Z = \overline{\Psi(\mathbb{C})} \subset G_{\mathcal{B}}$ a compact commutative complex subgroup, thus a complex torus. In fact, a compact Banach Lie group is finite dimensional, and the "Lie algebra" of $G_{\mathcal{B}}$ seems to be interpretable as a closed subspace of the Banach space of bounded holomorphic sections in $H^0(\mathcal{B}_X \times \mathbb{C}, \mathrm{ev}^\star T_X)$ where $\mathrm{ev} : \mathcal{B}_X \times \mathbb{C} \to X$ is the evaluation map $(f, t) \mapsto f(t)$. The complex structure on Z should arise from the complex structure on that Banach space. Now, we obtain a non trivial holomorphic map $\Phi : Z \to X$ by selecting an $f \in \mathcal{B}_X$ which is not a fixed point of Z and putting $\Phi(\gamma) = \gamma(f)(0)$, $\gamma \in Z$. □

In the general context of directed manifolds, algebraic hyperbolicity can be strictly weaker than Kobayashi hyperbolicity. The simplest example is provided by a 2-dimensional abelian variety $X=\mathbb{C}^2/\Lambda$ equipped with a constant subbundle $V\subset T_X$ given by a complex line $V_0\subset\mathbb{C}^2$ such that $V_0\cap\Lambda=\{0\}$. Then all leaves of V are isomorphic to complex lines, in particular there are no compact curves tangent to V, and thus 2.1 ii) and 2.1 iii) are satisfied (as void conditions). This pathology can somehow be corrected by observing that the hyperbolicity of (X,V) implies a statement analogue to 2.1 ii) but substantially stronger, namely a lower bound of $-\chi(\overline{C})$ for curves C which are almost tangent to V, in the sense that their "deviation with respect to V" is small.

2.8. Definition. *Let X be equipped with a hermitian $(1,1)$-form ω, and let $C\subset X$ be a compact curve in X. We define the L^2-deviation of C with respect to V by*

$$\mathrm{dev}^2_\omega(C/V)=\int_C\omega_{V^\perp}$$

where $\omega=\omega_V\oplus\omega_{V^\perp}$ is the orthogonal decomposition of ω on $V\oplus V^\perp$. Similarly, if $\nu:\overline{C}\to X$ is the normalization map and $\overline{C}\not\simeq\mathbb{P}^1$, we define the L^∞-deviation (resp. the L^p-deviation) to be

$$\mathrm{dev}^\infty_\omega(C/V)=\sup_{t\in\overline{C}}\frac{\nu^\star\omega_{V^\perp}(t)}{d\widetilde{\sigma}(t)}=\sup_{t\in\overline{C}}\|\nu'(t)^\perp\|^2_{\widetilde{\sigma},\omega_{V^\perp}},$$

$$\mathrm{dev}^p_\omega(C/V)=\Big[\int_{t\in\overline{C}}\Big(\frac{\nu^\star\omega_{V^\perp}(t)}{d\widetilde{\sigma}(t)}\Big)^{p/2}d\widetilde{\sigma}(t)\Big]^{2/p}=\Big[\int_{t\in\overline{C}}\|\nu'(t)^\perp\|^p_{\widetilde{\sigma},\omega_{V^\perp}}\,d\widetilde{\sigma}(t)\Big]^{2/p},$$

where $d\widetilde{\sigma}$ is the normalized Poincaré metric on $\overline{C}$ (hermitian metric of constant curvature with $\int_{\overline{C}}d\widetilde{\sigma}=1$), and $\nu'(t)^\perp$ is the projection of the tangent vector $\nu'(t)$ on $V^\perp$. If $\overline{C}\simeq\mathbb{P}^1$, we set instead

$$\mathrm{dev}^\infty_\omega(C/V)=\inf_{\gamma\in\mathrm{PGL}_2(\mathbb{C})}\ \sup_{t\in\overline{C}}\ \frac{\nu^\star\omega_{V^\perp}(t)}{\gamma^\star d\widetilde{\sigma}(t)}$$

$$\mathrm{dev}^p_\omega(C/V)=\inf_{\gamma\in\mathrm{PGL}_2(\mathbb{C})}\ \Big[\int_{t\in\overline{C}}\|\nu'(t)^\perp\|^p_{\gamma^\star\widetilde{\sigma},\omega_{V^\perp}}\,d\gamma^\star\widetilde{\sigma}(t)\Big]^{2/p}.$$

2.9. Proposition. *Let (X,V) be a compact directed manifold equipped with a hermitian metric ω. If (X,V) is hyperbolic, there exists a constant $\varepsilon>0$ such that*

$$\max\big(-\chi(\overline{C}),\mathrm{dev}^\infty_\omega(C/V)\big)\geqslant\varepsilon\,\mathrm{deg}_\omega(C),$$

for every compact curve $C\subset X$.

Proof. Otherwise, there would exist a sequence of curves (C_ℓ) and a sequence of positive numbers ε_ℓ converging to 0, such that

$$-\chi(\overline{C}_\ell)\leqslant\varepsilon_\ell\,\mathrm{deg}_\omega(C_\ell),\qquad \mathrm{dev}^\infty_\omega(C_\ell/V)\big)\leqslant\varepsilon_\ell\,\mathrm{deg}_\omega(C_\ell).$$

First assume that all curves C_ℓ have geometric genus $g(\overline{C}_\ell)\geqslant 2$. Let $\nu_\ell:\overline{C}_\ell\to X$ be the normalization map of C_ℓ, and let $d\sigma_\ell$ be the area measure associated with the Poincaré metric on $\overline{C}_\ell$ and $d\widetilde{\sigma}_\ell = \lambda_\ell^{-1} d\sigma_\ell$ the normalized Poincaré metric with $\lambda_\ell = \int_{\overline{C}_\ell} d\sigma_\ell = \frac{\pi}{2}(-\chi(C_\ell))$. Select a point $x_\ell \in \overline{C}_\ell$ where the ratio $\nu_\ell^\star\omega/d\sigma_\ell$ is maximum. Since $\int_{\overline{C}_\ell} \nu_\ell^\star\omega = \deg_\omega(C_\ell)$, we have

$$R_\ell^2 := \frac{\nu_\ell^\star\omega(x_\ell)}{d\sigma_\ell(x_\ell)} \geqslant \frac{\int_{\overline{C}_\ell} \nu_\ell^\star\omega}{\int_{\overline{C}_\ell} d\sigma_\ell} = \frac{\deg_\omega(C_\ell)}{\lambda_\ell} \geqslant \frac{2}{\pi\varepsilon_\ell} \to +\infty.$$

Let $\rho_\ell : \Delta \to \overline{C}_\ell$ be the universal covering map of $\overline{C}_\ell$, chosen such that $\rho_\ell(0) = x_\ell$. We get a holomorphic map $f_\ell = \nu_\ell \circ \rho_\ell : \Delta \to X$ such that $\|f_\ell'(0)\|_\omega = R_\ell \to +\infty$. By Brody's reparametrization lemma, we can reparametrize f_ℓ as $g_\ell(t) = f_\ell(t/R_\ell)$ so that some subsequence of (g_ℓ) converges uniformly on every compact set to a limit $g : \mathbb{C} \to X$ with $\|g'(0)\|_\omega = 1$. We claim that g must be tangent to V. In fact, by definition of the L^∞ deviation, we get $\nu_\ell^\star\omega_{V^\perp} \leqslant \varepsilon_\ell \deg_\omega(C_\ell) d\widetilde{\sigma}_\ell$, thus $f_\ell^\star\omega_{V^\perp} = \rho_\ell^\star(\nu_\ell^\star\omega_{V^\perp})$ and $g_\ell^\star\omega_{V^\perp}$ satisfy

$$\begin{aligned} f_\ell^\star\omega_{V^\perp} &\leqslant \frac{1}{\lambda_\ell}\varepsilon_\ell \deg_\omega(C_\ell)\frac{|dt|^2}{(1-|t|^2)^2}, \\ g_\ell^\star\omega_{V^\perp} &\leqslant \frac{\deg_\omega(C_\ell)}{\lambda_\ell}\varepsilon_\ell \frac{R_\ell^{-2}|dt|^2}{(1-|t|^2/R_\ell^2)^2} \leqslant \varepsilon_\ell \frac{|dt|^2}{(1-|t|^2/R_\ell^2)^2}. \end{aligned}$$

From this we conclude that $g^\star\omega_{V^\perp} = 0$, hence g is tangent to V, contradiction. If the curves C_ℓ are of genus 0 or 1, the arguments are similar and will be left to the reader. □

§3. The Ahlfors-Schwarz lemma for metrics of negative curvature

One of the most basic ideas is that hyperbolicity should somehow be related with suitable negativity properties of the curvature. For instance, it is a standard fact already observed in Kobayashi [Kob70] that the negativity of T_X (or the ampleness of $T_X^\star$) implies the hyperbolicity of X. There are many ways of improving or generalizing this result. We present here a few simple examples of such generalizations. If (V, h) is a holomorphic vector bundle equipped with a smooth hermitian metric, we denote by $\nabla_h = \nabla_h' + \nabla_h''$ the associated Chern connection and by $\Theta_h(V) = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor.

3.1. Proposition. *Let (X, V) be a compact directed manifold. Assume that $V^\star$ is ample. Then (X, V) is hyperbolic.*

Proof (from an original idea of [Kob75]). Recall that a vector bundle E is said to be ample if $S^m E$ has enough global sections $\sigma_1, \ldots, \sigma_N$ so as to generate 1-jets of sections at any point, when m is large. One obtains a Finsler metric N on $E^\star$ by putting

$$N(\xi) = \Big(\sum_{1\leqslant j\leqslant N} |\sigma_j(x)\cdot\xi^m|^2\Big)^{1/2m}, \qquad \xi \in E_x^\star,$$

and N is then a strictly plurisubharmonic function on the total space of $E^\star$ minus the zero section (in other words, the line bundle $\mathcal{O}_{P(E^\star)}(1)$ has a metric of positive curvature). By the ampleness assumption on $V^\star$, we thus have a Finsler metric N on V which is strictly plurisubharmonic outside the zero section. By Brody's lemma, if (X,V) is not hyperbolic, there is a non constant entire curve $g:\mathbb{C}\to X$ tangent to V such that $\sup_{\mathbb{C}}\|g'\|_\omega\leqslant 1$ for some given hermitian metric ω on X. Then $N(g')$ is a bounded subharmonic function on $\mathbb{C}$ which is strictly subharmonic on $\{g'\neq 0\}$. This is a contradiction, for any bounded subharmonic function on $\mathbb{C}$ must be constant. □

This result can be generalized a little bit further by means of the Ahlfors-Schwarz lemma (see e.g. [Lang87]).

3.2. Ahlfors-Schwarz lemma. *Let $\gamma(t)=\gamma_0(t)\,i\,dt\wedge d\overline{t}$ be a hermitian metric on Δ_R where $\log\gamma_0$ is a subharmonic function such that $i\,\partial\overline{\partial}\log\gamma_0(t)\geqslant A\,\gamma(t)$ in the sense of currents, for some positive constant A. Then γ can be compared with the Poincaré metric of Δ_R as follows:*

$$\gamma(t)\leqslant\frac{2}{A}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2}.$$

More generally, let $\gamma=i\sum\gamma_{jk}dt_j\wedge d\overline{t}_k$ be an almost everywhere positive hermitian form on the ball $B(0,R)\subset\mathbb{C}^p$, such that $-\mathrm{Ricci}(\gamma):=i\,\partial\overline{\partial}\log\det\gamma\geqslant A\gamma$ in the sense of currents, for some constant $A>0$ (this means in particular that $\det\gamma=\det(\gamma_{jk})$ is such that $\log\det\gamma$ is plurisubharmonic). Then

$$\det(\gamma)\leqslant\Big(\frac{p+1}{AR^2}\Big)^p\frac{1}{(1-|t|^2/R^2)^{p+1}}.$$

Proof. It is of course sufficient to deal with the more general case of a ball in $\mathbb{C}^p$. First assume that γ is smooth and positive definite on $\overline{B}(0,R)$. Take a point $t_0\in B(0,R)$ at which $(1-|t|^2/R^2)^{p+1}\det(\gamma(t))$ is maximum. The logarithmic $i\,\partial\overline{\partial}$-derivative of this function at t_0 must be $\leqslant 0$, hence

$$i\,\partial\overline{\partial}\log\det\gamma(t)_{t=t_0}-(p+1)\,i\,\partial\overline{\partial}\log(1-|t|^2/R^2)^{-1}_{t=t_0}\leqslant 0.$$

The hypothesis on the Ricci curvature implies

$$A^p\,\gamma(t_0)^p\leqslant\big(i\,\partial\overline{\partial}\log\det\gamma(t)_{t=t_0}\big)^p\leqslant(p+1)^p\,\big(i\,\partial\overline{\partial}\log(1-|t|^2/R^2)^{-1}_{t=t_0}\big)^p.$$

An easy computation shows that the determinant of $i\,\partial\overline{\partial}\log(1-|t|^2/R^2)^{-1}$ is equal to $R^{-2p}(1-|t|^2/R^2)^{-p-1}$. From this, we conclude that

$$(1-|t|^2/R^2)^{p+1}\det\gamma(t)\leqslant(1-|t_0|^2/R^2)^{p+1}\det\gamma(t_0)\leqslant\Big(\frac{p+1}{AR^2}\Big)^p.$$

If γ is not smooth, we use a regularization argument. Namely, we shrink R a little bit and look at the maximum of the function

$$u(t)=(1-|t|^2/R^2)^{p+1}\exp\big(\rho_\varepsilon\star\log\det\gamma(t)\big)$$

where (ρ_ε) is a family of regularizing kernels. The argument goes through because

$$i\,\partial\overline{\partial}(\rho_\varepsilon\star\log\det\gamma)\geqslant A\,\rho_\varepsilon\star\gamma$$

and $\log\det(\rho_\varepsilon\star\gamma)\geqslant\rho_\varepsilon\star\log\det\gamma$ by concavity of the log det function. □

3.3. Proposition. *Let (X,V) be a compact directed manifold. Assume that $V^\star$ is "very big" in the following sense: there exists an ample line bundle L and a sufficiently large integer m such that the global sections in $H^0(X,S^mV^\star\otimes L^{-1})$ generate all fibers over $X\smallsetminus Y$, for some analytic subset $Y\subsetneq X$. Then all entire curves $f:\mathbb{C}\to X$ tangent to V satisfy $f(\mathbb{C})\subset Y$ [under our assumptions, X is a projective algebraic manifold and Y is an algebraic subvariety, thus it is legitimate to say that the entire curves are "algebraically degenerate"].*

Proof. Let $\sigma_1,\ldots,\sigma_N\in H^0(X,S^mV^\star\otimes L^{-1})$ be a basis of sections generating $S^mV^\star\otimes L^{-1}$ over $X\smallsetminus Y$. If $f:\mathbb{C}\to X$ is tangent to V, we define a semipositive hermitian form $\gamma(t)=\gamma_0(t)\,|dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t)=\sum\|\sigma_j(f(t))\cdot f'(t)^m\|_{L^{-1}}^{2/m}$$

where $\|\ \|_L$ denotes a hermitian metric with positive curvature on L. If $f(\mathbb{C})\not\subset Y$, the form γ is not identically 0 and we then find

$$i\,\partial\overline{\partial}\log\gamma_0\geqslant\frac{2\pi}{m}f^\star\Theta(L)$$

where $\Theta(L)$ is the curvature form. The positivity assumption combined with an obvious homogeneity argument yield

$$\frac{2\pi}{m}f^\star\Theta(L)\geqslant\varepsilon\|f'(t)\|_\omega^2\,|dt|^2\geqslant\varepsilon'\,\gamma(t)$$

for any given hermitian metric ω on X. Now, for any t_0 with $\gamma_0(t_0)>0$, the Ahlfors-Schwarz lemma shows that f can only exist on a disk $D(t_0,R)$ such that $\gamma_0(t_0)\leqslant\frac{2}{\varepsilon'}R^{-2}$, contradiction. □

There are similar results for p-measure hyperbolicity, e.g.

3.4. Proposition. *Let (X,V) be a compact directed manifold. Assume that $\Lambda^pV^\star$ is ample. Then (X,V) is infinitesimally p-measure hyperbolic. More generally, assume that $\Lambda^pV^\star$ is very big with base locus contained in $Y\subsetneq X$* (see 3.3). *Then $\mathbf{e}^p$ is non degenerate over $X\smallsetminus Y$.*

Proof. By the ampleness assumption, there is a smooth Finsler metric N on Λ^pV which is strictly plurisubharmonic outside the zero section. We select also a hermitian metric ω on X. For any holomorphic map $f:\mathbb{B}_p\to X$ we define a semipositive hermitian metric $\widetilde{\gamma}$ on $\mathbb{B}_p$ by putting $\widetilde{\gamma}=f^\star\omega$. Since ω need not have any good curvature estimate, we introduce the function $\delta(t)=N_{f(t)}(\Lambda^pf'(t)\cdot\tau_0)$, where $\tau_0=\partial/\partial t_1\wedge\cdots\wedge\partial/\partial t_p$, and select a metric $\gamma=\lambda\widetilde{\gamma}$ conformal to $\widetilde{\gamma}$ such that $\det\gamma=\delta$. Then λ^p is equal to the ratio $N/\Lambda^p\omega$ on the element

$\Lambda^p f'(t) \cdot \tau_0 \in \Lambda^p V_{f(t)}$. Since X is compact, it is clear that the conformal factor λ is bounded by an absolute constant independent of f. From the curvature assumption we then get

$$i\,\partial\overline{\partial}\log\det\gamma = i\,\partial\overline{\partial}\log\delta \geqslant (f, \Lambda^p f')^\star(i\,\partial\overline{\partial}\log N) \geqslant \varepsilon f^\star\omega \geqslant \varepsilon'\,\gamma.$$

By the Ahlfors-Schwarz lemma we infer that $\det\gamma(0) \leqslant C$ for some constant C, i.e., $N_{f(0)}(\Lambda^p f'(0)\cdot\tau_0) \leqslant C'$. This means that the Kobayashi-Eisenman pseudometric $\mathbf{e}^p_{(X,V)}$ is positive definite everywhere and uniformly bounded from below. In the case $\Lambda^p V^\star$ is very big with base locus Y, we use essentially the same arguments, but we then only have N being positive definite on $X \smallsetminus Y$. □

3.5. Corollary ([Gri71], KobO71]). *If X is a projective variety of general type, the Kobayashi-Eisenmann volume form $\mathbf{e}^n$, $n = \dim X$, can degenerate only along a proper algebraic set $Y \subsetneq X$.*

The converse of Corollary 3.5 is expected to be true, namely, the generic non degeneracy of $\mathbf{e}^n$ should imply that X is of general type, but this is only known for surfaces (see [GrGr80] and [MoMu82]):

3.6. Conjecture (Green-Griffiths [GrGr80]). *A projective algebraic variety X is almost measure hyperbolic (i.e. $\mathbf{e}^n$ degenerates only along a proper algebraic subvariety) if and only if X is of general type.*

In the same vein, Green-Griffiths and Lang proposed the following conjectures.

3.7. Conjecture (Green-Griffiths [GrGr80]). *If X is a variety of general type, there exists a proper algebraic set $Y \subsetneq X$ such that every entire holomorphic curve $f : \mathbb{C} \to X$ is contained in Y.*

The most outstanding result in the direction of Conjecture 3.7 is the proof of the Bloch theorem, as proposed by Bloch [Blo26] and Ochiai [Och77]. The Bloch theorem is the special case of 3.7 when the irregularity of X satisfies $q = h^0(X, \Omega^1_X) > \dim X$. Various solutions have then been obtained in fundamental papers of Noguchi [Nog77, 81, 84], Kawamata [Kaw80] and Green-Griffiths [GrGr80], by means of different techniques. See section § 9 for a proof based on jet bundle techniques.

3.8. Conjecture ([Lang86, 87]). *A projective algebraic variety X is hyperbolic if and only if all its algebraic subvarieties (including X itself) are of general type.*

An essential step in the proof of the necessity of having general type subvarieties would be to show that manifolds of Kodaira dimension 0 (say, Calabi-Yau manifolds and symplectic manifolds, all of which have $c_1(X) = 0$) are not hyperbolic, e.g. by exhibiting a sequence of curves C_ℓ such that $(2g(\overline{C}_\ell)-2)/\deg(C_\ell) \to 0$. In fact, it is even expected that there are covering families of such curves, whereby proving that such manifolds are not measure hyperbolic. An analogous conjecture in the relative situation might stand as follows.

3.9. Conjecture. *Let (X,V) be a projective directed manifold. Assume that* $\det V^\star$ *is big and that V is semistable in some sense (e.g. with respect to* $\det V^\star$*, if* $\det V^\star$ *is ample). Then there is a proper algebraic subset $Y \subsetneq X$ such that every entire curve $f:\mathbb{C}\to X$ tangent to V satisfies $f(\mathbb{C})\subset Y$.*

Recall that a line bundle L is said to be *big* if it has maximal Kodaira dimension, in other words, if $H^0(X,L^{\otimes m})\geqslant c\,m^{\dim X}$ for some constant $c>0$, when $m\geqslant m_0$ is sufficiently large. Some sort of semistability condition is clearly required, otherwise one might take X to be an abelian variety of dimension $\geqslant 3$ and $V=L+L'$ where $L\subset T_X$ is a constant line subbundle with dense trajectories, and $L'\subset T_X$ a "generic" sufficiently negative line bundle (also take a blow-up $\widetilde X\to X$ to resolve the singularities of V, so as to obtain a subbundle $\widetilde V\subset T_{\widetilde X}$). In the absolute case $V=T_X$, the semistability condition is probably not needed since T_X tends to be always semistable in some sense (if K_X is ample, there is always a Kähler-Einstein metric, hence T_X is K_X-semistable).

3.10. Remark. One should take care of the fact that Propositions 3.1, 3.3 and 3.4 cannot be extended without modifications to the case when V admits singularities. For instance, take $X=\mathbb{P}^n$ and let $\ell=\mathbb{P}^1\subset X$ be a line. Take a section of $T_\ell\otimes\mathcal{O}_\ell(d)\simeq\mathcal{O}_\ell(d+2)$ admitting only one zero z_0 of multiplicity $d+2$, and extend it as a section σ of $T_{\mathbb{P}^n}\otimes\mathcal{O}(d)$ admitting only isolated zeroes (this is always possible for arbitrary $d>0$). Then σ defines a sheaf injection $\sigma:\mathcal{O}(-d)\hookrightarrow\mathcal{O}(T_{\mathbb{P}^n})$ with $\mathcal{V}=\sigma(\mathcal{O}(-d))\simeq\mathcal{O}(-d)$ negative, nevertheless $\mathcal{V}$ admits a complex line $\ell\smallsetminus\{z_0\}\simeq\mathbb{C}$ as one of its integral curves. The correct assumption guaranteeing the hyperbolicity of (X,V) is that $\mathcal{W}=\mathrm{Im}(\Omega^1_X\to\mathcal{V}^\star)$ should be ample.

§4. Projectivization of a directed manifold

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde X,\widetilde V)$ from a given one (X,V). The new structure $(\widetilde X,\widetilde V)$ plays the role of a space of 1-jets over X. We let

$$\widetilde X=P(V),\qquad \widetilde V\subset T_{\widetilde X}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde V$ of $T_{\widetilde X}$ defined as follows: for every point $(x,[v])\in\widetilde X$ associated with a vector $v\in V_x\smallsetminus\{0\}$,

$$\widetilde V_{(x,[v])}=\big\{\xi\in T_{\widetilde X,\,(x,[v])}\,;\,\pi_\star\xi\in\mathbb{C}\,v\big\},\qquad \mathbb{C}\,v\subset V_x\subset T_{X,x},\tag{4.1}$$

where $\pi:\widetilde X=P(V)\to X$ is the natural projection and $\pi_\star:T_{\widetilde X}\to\pi^\star T_X$ is its differential. On $\widetilde X=P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde X}(-1)\subset\pi^\star V$ such that $\mathcal{O}_{\widetilde X}(-1)_{(x,[v])}=\mathbb{C}\,v$. The bundle $\widetilde V$ is characterized by the two exact sequences

$$0\longrightarrow T_{\widetilde X/X}\longrightarrow\widetilde V\xrightarrow{\ \pi_\star\ }\mathcal{O}_{\widetilde X}(-1)\longrightarrow 0,\tag{4.2}$$

$$0\longrightarrow\mathcal{O}_{\widetilde X}\longrightarrow\pi^\star V\otimes\mathcal{O}_{\widetilde X}(1)\longrightarrow T_{\widetilde X/X}\longrightarrow 0,\tag{4.2$'$}$$

where $T_{\widetilde X/X}$ denotes the relative tangent bundle of the fibration $\pi:\widetilde X\to X$. The first sequence is a direct consequence of the definition of $\widetilde V$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \qquad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{4.3}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^\star \det V \otimes \mathcal{O}_{\widetilde{X}}(r)$, thus

$$\det \widetilde{V} = \pi^\star \det V \otimes \mathcal{O}_{\widetilde{X}}(r-1). \tag{4.4}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is fonctorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{V}) & \xrightarrow{\pi} & (X, V) \\ \widetilde{\Phi} \downarrow & & \downarrow \Phi \\ (\widetilde{Y}, \widetilde{W}) & \xrightarrow{\pi} & (Y, W) \end{array} \tag{4.5}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_\star : V \to \Phi^\star W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_\star : V \to \Phi^\star W$ is injective).

Now, suppose that we are given a holomorphic curve $f : \Delta_R \to X$ parametrized by the disk Δ_R of centre 0 and radius R in the complex plane, and that f is a *tangent trajectory* of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in \Delta_R$. If f is non constant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde{f} : \Delta_R \to \widetilde{X}, \qquad t \mapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{4.6}$$

is holomorphic (at a stationary point t_0, we just write $f'(t) = (t - t_0)^s u(t)$ with $s \in \mathbb{N}^\star$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\, u(t)$, hence the derivative f' defines a section

$$f' : T_{\Delta_R} \to \widetilde{f}^\star \mathcal{O}_{\widetilde{X}}(-1). \tag{4.7}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_\star \widetilde{f}'(t) = f'(t) \in \mathbb{C}\, u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t), u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z\in\Omega}$ can be defined by linear equations

$$V_z = \Big\{\xi = \sum_{1\leqslant j\leqslant n} \xi_j \frac{\partial}{\partial z_j}\,;\ \xi_j = \sum_{1\leqslant k\leqslant r} a_{jk}(z)\xi_k \text{ for } j = r+1, \ldots, n\Big\}, \tag{4.8}$$

where (a_{jk}) is a holomorphic $(n-r)\times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1, \ldots, \xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction\Omega}$ can be described by the coordinate system

$$\Big(z_1, \ldots, z_n; \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_r}{\xi_j}\Big). \tag{4.9}$$

Let $f \simeq (f_1, \ldots, f_n)$ be the components of f in the coordinates $(z_1, \ldots, z_n)$ (we suppose here R so small that $f(\Delta_R) \subset \Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1, \ldots, f_r)$. Indeed, as $f'(t) \in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f'_j(t) = \sum_{1\leqslant k\leqslant r} a_{jk}(f(t)) f'_k(t), \qquad j > r, \tag{4.10}$$

on a neighborhood of 0, with initial data $f(0) = x$. We denote by $m = m(f, t_0)$ the *multiplicity* of f at any point $t_0 \in \Delta_R$, that is, $m(f, t_0)$ is the smallest integer $m \in \mathbb{N}^\star$ such that $f_j^{(m)}(t_0) \neq 0$ for some j. By (4.10), we can always suppose $j \in \{1, \ldots, r\}$, for example $f_r^{(m)}(t_0) \neq 0$. Then $f'(t) = (t - t_0)^{m-1}u(t)$ with $u_r(t_0) \neq 0$, and the lifting $\widetilde{f}$ is described in the coordinates of the affine chart $\xi_r \neq 0$ of $P(V)_{\restriction\Omega}$ by

$$\widetilde{f} \simeq \Big(f_1, \ldots, f_n; \frac{f'_1}{f'_r}, \ldots, \frac{f'_{r-1}}{f'_r}\Big). \tag{4.11}$$

We end this section with a few curvature computations. Assume that V is equipped with a smooth hermitian metric h. Denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_h(V) = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor. For every point $x_0 \in X$, there exists a "normalized" holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ on a neighborhood of x_0, such that

$$\langle e_\lambda, e_\mu\rangle_h = \delta_{\lambda\mu} - \sum_{1\leqslant j,k\leqslant n} c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3), \tag{4.12}$$

with respect to any holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0. A computation of $d'\langle e_\lambda, e_\mu\rangle_h = \langle\nabla'_h e_\lambda, e_\mu\rangle_h$ and $\nabla_h^2 e_\lambda = d''\nabla'_h e_\lambda$ then gives

$$\nabla'_h e_\lambda = -\sum_{j,k,\mu} c_{jk\lambda\mu} \overline{z}_k\, dz_j \otimes e_\mu + O(|z|^2),$$

$$\Theta_h(V)_{x_0} = \frac{i}{2\pi} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu. \tag{4.13}$$

The above curvature tensor can also be viewed as a hermitian form on $T_X\otimes V$. In fact, one associates with $\Theta_h(V)$ the hermitian form $\langle\Theta_h(V)\rangle$ on $T_X\otimes V$ defined for all $(\zeta,v)\in T_X\times_X V$ by

$$\langle\Theta_h(V)\rangle(\zeta\otimes v)=\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{jk\lambda\mu}\zeta_j\overline{\zeta}_k v_\lambda\overline{v}_\mu. \tag{4.14}$$

Let h_1 be the hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)\subset\pi^\star V$ induced by the metric h of V. We compute the curvature $(1,1)$-form $\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))$ at an arbitrary point $(x_0,[v_0])\in P(V)$, in terms of $\Theta_h(V)$. For simplicity, we suppose that the frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ has been chosen in such a way that $[e_r(x_0)]=[v_0]\in P(V)$ and $|v_0|_h=1$. We get holomorphic local coordinates $(z_1,\ldots,z_n\,;\,\xi_1,\ldots,\xi_{r-1})$ on a neighborhood of $(x_0,[v_0])$ in $P(V)$ by assigning

$$(z_1,\ldots,z_n\,;\,\xi_1,\ldots,\xi_{r-1})\longmapsto(z,[\xi_1e_1(z)+\cdots+\xi_{r-1}e_{r-1}(z)+e_r(z)])\in P(V).$$

Then the function

$$\eta(z,\xi)=\xi_1e_1(z)+\cdots+\xi_{r-1}e_{r-1}(z)+e_r(z)$$

defines a holomorphic section of $\mathcal{O}_{P(V)}(-1)$ in a neighborhood of $(x_0,[v_0])$. By using the expansion (4.12) for h, we find

$$\begin{aligned}
|\eta|^2_{h_1}&=|\eta|^2_h=1+|\xi|^2-\sum_{1\leqslant j,k\leqslant n}c_{jkrr}z_j\overline{z}_k+O((|z|+|\xi|)^3),\\
\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))_{(x_0,[v_0])}&=-\frac{i}{2\pi}d'd''\log|\eta|^2_{h_1}\\
&=\frac{i}{2\pi}\Big(\sum_{1\leqslant j,k\leqslant n}c_{jkrr}dz_j\wedge d\overline{z}_k-\sum_{1\leqslant\lambda\leqslant r-1}d\xi_\lambda\wedge d\overline{\xi}_\lambda\Big).
\end{aligned}\tag{4.15}$$

Now, the connection ∇_h on V defines on $\widetilde{X}=P(V)$ a C^∞ decomposition

$$T_{\widetilde{X}}={}^HT_{\widetilde{X}}\oplus{}^VT_{\widetilde{X}},\qquad {}^HT_{\widetilde{X},(x,[v])}\simeq T_{X,x},\quad {}^VT_{\widetilde{X},(x,[v])}\simeq T_{P(V_x),[v]},$$

in horizontal and vertical components. With respect to this decomposition, (4.15) can be rewritten as

$$\langle\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))\rangle_{(x_0,[v_0])}(\tau)=\langle\Theta_h(V)\rangle_{x_0}({}^H\tau\otimes v_0)-|{}^V\tau|^2_{\mathrm{FS}} \tag{4.16}$$

where $|\ \ |_{\mathrm{FS}}$ is the Fubini-Study metric along the fibers $T_{P(V_x)}$. By definition of $\widetilde{V}$, we have $\widetilde{V}_{(x,[v])}\subset V_x\oplus T_{P(V_x),[v]}$ with respect to the decomposition. By this observation, if we equip $P(V)$ with the Fubini-Study metric rescaled by $\rho^2>0$, the metric h on V induces a canonical hermitian metric $\widetilde{h}_\rho$ on $\widetilde{V}$ such that

$$|w|^2_{\widetilde{h}_\rho}=|{}^Hw|^2_h+\rho^2|{}^Vw|^2_h\qquad\text{for } w\in\widetilde{V}_{(x_0,[v_0])},$$

where ${}^Hw\in\mathbb{C}\,v_0\subset V_{x_0}$ and ${}^Vw\in T_{P(V_{x_0}),[v_0]}$ is viewed as an element of $v_0^\perp\subset V_{x_0}$. A computation (left to the reader) gives the formula

$$\begin{aligned}
\langle\Theta_{\widetilde{h}_\rho}(\widetilde{V})\rangle_{(x_0,[v_0])}(\tau\otimes w)&=\langle\Theta_h(V)\rangle_{x_0}({}^H\tau\otimes v_0)\ (|{}^Hw|^2_h-\rho^2|{}^Vw|^2_h)\\
&\quad+\rho^2\langle\Theta_h(V)\rangle_{x_0}({}^H\tau\otimes{}^Vw)\\
&\quad+\rho^2\big(|\langle{}^V\tau,{}^Vw\rangle_h|^2+|{}^V\tau|^2_h|{}^Vw|^2_h\big)-|{}^V\tau|^2_h|{}^Hw|^2_h\\
&\quad+O(\rho)|\tau|^2_\omega|w|^2_{\widetilde{h}_\rho},\qquad \tau\in T_{\widetilde{X}},\ w\in\widetilde{V},
\end{aligned}\tag{4.17}$$

where $|\tau|^2_\omega$ is computed from a fixed hermitian metric ω on T_X.

§5. Jets of curves and Semple jet bundles

Let X be a complex n-dimensional manifold. Following ideas of Green-Griffiths [GrGr80], we let $J_k \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_k \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber $J_{k,x}$, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors $(f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_k is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_k is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes; see formula (6.2) in § 6).

According to the philosophy developed throughout this paper, we describe the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V \subset T_X$, we associate to V a k-jet bundle $J_k V$ as follows.

5.1. Definition. *Let (X, V) be a complex directed manifold. We define $J_k V \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that $J_k V$ is actually a subbundle of J_k. In fact, by using (4.8) and (4.10), we see that the fibers $J_k V_x$ are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_k. □

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles $J_k V$ (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS92] is also concerned with such generalizations of jet bundles*.

* Very recently, a preprint [LaTh96] by Laksov and Thorup has also appeared, dealing in depth with algebraic-theoretic properties of jet differentials. The formalism of "higher order" differentials has been part of the mathematical folklore during the 18th and 19th centuries (without too much concern, in those times, on the existence of precise definitions!). During the 20th century, this formalism almost disappeared, before getting revived in several ways. See e.g. the interesting article by P.A. Meyer [Mey89], which was originally motivated by applications to probability theory.

We define inductively the *projectivized k-jet bundle* $P_kV = X_k$ (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(X_0, V_0) = (X, V), \qquad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}). \tag{5.2}$$

In other words, $(P_kV, V_k) = (X_k, V_k)$ is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in § 4. By (4.2–4.7), we find

$$\dim P_kV = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{5.3}$$

together with exact sequences

$$0 \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{P_kV}(-1) \longrightarrow 0, \tag{5.4}$$

$$0 \longrightarrow \mathcal{O}_{P_kV} \longrightarrow \pi_k^\star V_{k-1} \otimes \mathcal{O}_{P_kV}(1) \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow 0. \tag{5.4$'$}$$

where π_k is the natural projection $\pi_k : P_kV \to P_{k-1}V$ and $(\pi_k)_\star$ its differential. Formula (4.4) yields

$$\det V_k = \pi_k^\star \det V_{k-1} \otimes \mathcal{O}_{P_kV}(r-1). \tag{5.5}$$

Every non constant tangent trajectory $f : \Delta_R \to X$ of (X, V) lifts to a well defined and unique tangent trajectory $f_{[k]} : \Delta_R \to P_kV$ of (P_kV, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^\star \mathcal{O}_{P_kV}(-1). \tag{5.6}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (4.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{5.7}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $P_kV \to P_{k-1}V$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^\star V_{k-1}$, and a composition with the projection $(\pi_{k-1})_\star$ (analogue for order $k-1$ of the arrow $(\pi_k)_\star$ in sequence (5.4)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^\star V_{k-1} \xrightarrow{(\pi_k)^\star(\pi_{k-1})_\star} \pi_k^\star \mathcal{O}_{P_{k-1}V}(-1), \tag{5.8}$$

which admits precisely $D_k = P(T_{P_{k-1}V/P_{k-2}V}) \subset P(V_{k-1}) = P_kV$ as its zero divisor (clearly, D_k is a hyperplane subbundle of P_kV). Hence we find

$$\mathcal{O}_{P_kV}(1) = \pi_k^\star \mathcal{O}_{P_{k-1}V}(1) \otimes \mathcal{O}(D_k). \tag{5.9}$$

Now, we consider the composition of projections

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : P_k V \longrightarrow P_j V. \tag{5.10}$$

Then $\pi_{0,k} : P_k V \to X = P_0 V$ is a locally trivial holomorphic fiber bundle over X, and the fibers $P_k V_x = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X, V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" nonsingular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 6.8 below). The following Proposition will help us to understand a little bit more about the geometric structure of $P_k V$. As usual, we define the *multiplicity* $m(f, t_0)$ of a curve $f : \Delta_R \to X$ at a point $t \in \Delta_R$ to be the smallest integer $s \in \mathbb{N}^\star$ such that $f^{(s)}(t_0) \neq 0$, i.e., the largest s such that $\delta(f(t), f(t_0)) = O(|t - t_0|^s)$ for any hermitian or riemannian geodesic distance δ on X. As $f_{[k-1]} = \pi_k \circ f_{[k]}$, it is clear that the sequence $m(f_{[k]}, t)$ is non increasing with k.

5.11. Proposition. *Let $f : (\mathbb{C}, 0) \to X$ be a non constant germ of curve tangent to V. Then for all $j \geqslant 2$ we have $m(f_{[j-2]}, 0) \geqslant m(f_{[j-1]}, 0)$ and the inequality is strict if and only if $f_{[j]}(0) \in D_j$. Conversely, if $w \in P_k V$ is an arbitrary element and $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_{k-1} \geqslant 1$ is a sequence of integers with the property that*

$$\forall j \in \{2, \ldots, k\}, \qquad m_{j-2} > m_{j-1} \quad \textit{if and only if } \pi_{j,k}(w) \in D_j,$$

there exists a germ of curve $f : (\mathbb{C}, 0) \to X$ tangent to V such that $f_{[k]}(0) = w$ and $m(f_{[j]}, 0) = m_j$ for all $j \in \{0, \ldots, k-1\}$.

Proof. i) Suppose first that f is given and put $m_j = m(f_{[j]}, 0)$. By definition, we have $f_{[j]} = (f_{[j-1]}, [u_{j-1}])$ where $f'_{[j-1]}(t) = t^{m_{j-1}-1} u_{j-1}(t) \in V_{j-1}$, $u_{j-1}(0) \neq 0$. By composing with the differential of the projection $\pi_{j-1} : P_{j-1}V \to P_{j-2}V$, we find $f'_{[j-2]}(t) = t^{m_{j-1}-1} (\pi_{j-1})_\star u_{j-1}(t)$. Therefore

$$m_{j-2} = m_{j-1} + \mathrm{ord}_{t=0} (\pi_{j-1})_\star u_{j-1}(t),$$

and so $m_{j-2} > m_{j-1}$ if and only if $(\pi_{j-1})_\star u_{j-1}(0) = 0$, that is, if and only if $u_{j-1}(0) \in T_{P_{j-1}V/P_{j-2}V}$, or equivalently $f_{[j]}(0) = (f_{[j-1]}(0), [u_{j-1}(0)]) \in D_j$.

ii) Suppose now that $w \in P_k V$ and $m_0, \ldots, m_{k-1}$ are given. We denote by $w_{j+1} = (w_j, [\eta_j])$, $w_j \in P_j V$, $\eta_j \in V_j$, the projection of w to $P_{j+1}V$. Fix coordinates $(z_1, \ldots, z_n)$ on X centered at w_0 such that the r-th component $\eta_{0,r}$ of η_0 is non zero. We prove the existence of the germ f by induction on k, in the form of a Taylor expansion

$$f(t) = a_0 + t\, a_1 + \cdots + t^{d_k} a_{d_k} + O(t^{d_k+1}), \qquad d_k = m_0 + m_1 + \cdots + m_{k-1}.$$

If $k = 1$ and $w = (w_0, [\eta_0]) \in P_1 V_x$, we simply take $f(t) = w_0 + t^{m_0}\eta_0 + O(t^{m_0+1})$. In general, the induction hypothesis applied to $P_k V = P_{k-1}(V_1)$ over $X_1 = P_1 V$

yields a curve $g : (\mathbb{C},0) \to X_1$ such that $g_{[k-1]} = w$ and $m(g_{[j]},0) = m_{j+1}$ for $0 \leqslant j \leqslant k-2$. If $w_2 \notin D_2$, then $[g'_{[1]}(0)] = [\eta_1]$ is not vertical, thus $f = \pi_1 \circ g$ satisfies $m(f,0) = m(g,0) = m_1 = m_0$ and we are done.

If $w_2 \in D_2$, we express $g = (G_1,\ldots,G_n;G_{n+1},\ldots,G_{n+r-1})$ as a Taylor expansion of order $m_1+\cdots+m_{k-1}$ in the coordinates (4.9) of the affine chart $\xi_r \neq 0$. As $\eta_1 = \lim_{t\to 0} g'(t)/t^{m_1-1}$ is vertical, we must have $m(G_s,0) > m_1$ for $1 \leqslant j \leqslant n$. It follows from (5.7) that $G_1,\ldots,G_n$ are never involved in the calculation of the liftings $g_{[j]}$. We can therefore replace g by $f \simeq (f_1,\ldots,f_n)$ where $f_r(t) = t^{m_0}$ and $f_1,\ldots,f_{r-1}$ are obtained by integrating the equations $f'_j(t)/f'_r(t) = G_{n+j}(t)$, i.e., $f'_j(t) = m_0t^{m_0-1}G_{n+j}(t)$, while $f_{r+1},\ldots,f_n$ are obtained by integrating (4.10). We then get the desired Taylor expansion of order d_k for f. □

Since we can always take $m_{k-1} = 1$ without restriction, we get in particular:

5.12. Corollary. *Let $w \in P_kV$ be an arbitrary element. Then there is a germ of curve $f : (\mathbb{C},0) \to X$ such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$ (thus the liftings $f_{[k-1]}$ and $f_{[k]}$ are regular germs of curve). Moreover, if $w_0 \in P_kV$ and w is taken in a sufficiently small neighborhood of w_0, then the germ $f = f_w$ can be taken to depend holomorphically on w.*

Proof. Only the holomorphic dependence of f_w with respect to w has to be guaranteed. If f_{w_0} is a solution for $w = w_0$, we observe that $(f_{w_0})'_{[k]}$ is a non vanishing section of V_k along the regular curve defined by $(f_{w_0})_{[k]}$ in P_kV. We can thus find a non vanishing section ξ of V_k on a neighborhood of w_0 in P_kV such that $\xi = (f_{w_0})'_{[k]}$ along that curve. We define $t \mapsto F_w(t)$ to be the trajectory of ξ with initial point w, and we put $f_w = \pi_{0,k} \circ F_w$. Then f_w is the required family of germs. □

Now, we can take $f : (\mathbb{C},0) \to X$ to be regular at the origin (by this, we mean $f'(0) \neq 0$) if and only if $m_0 = m_1 = \cdots = m_{k-1} = 1$, which is possible by Proposition 5.11 if and only if $w \in P_kV$ is such that $\pi_{j,k}(w) \notin D_j$ for all $j \in \{2,\ldots,k\}$. For this reason, we define

$$(5.13)\qquad \begin{aligned} P_kV^{\mathrm{reg}} &= \bigcap_{2\leqslant j\leqslant k} \pi_{j,k}^{-1}(P_jV \smallsetminus D_j),\\ P_kV^{\mathrm{sing}} &= \bigcup_{2\leqslant j\leqslant k} \pi_{j,k}^{-1}(D_j) = P_kV \smallsetminus P_kV^{\mathrm{reg}}, \end{aligned}$$

in other words, P_kV^{reg} is the set of values $f_{[k]}(0)$ reached by all regular germs of curves f. One should take care however that there are singular germs which reach the same points $f_{[k]}(0) \in P_kV^{\mathrm{reg}}$, e.g., any s-sheeted covering $t \mapsto f(t^s)$. On the other hand, if $w \in P_kV^{\mathrm{sing}}$, we can reach w by a germ f with $m_0 = m(f,0)$ as large as we want.

5.14. Corollary. *Let $w \in P_kV^{\mathrm{sing}}$ be given, and let $m_0 \in \mathbb{N}$ be an arbitrary integer larger than the number of components D_j such that $\pi_{j,k}(w) \in D_j$. Then there is a germ of curve $f : (\mathbb{C},0) \to X$ with multiplicity $m(f,0) = m_0$ at the origin, such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$.*

§6. Jet differentials

Following Green-Griffiths [GrGr80], we now introduce the concept of jet differential. This concept gives an intrinsic way of describing holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^\star,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k V$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^\star \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^\star$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^\star$ of homotheties $\varphi(t) = \lambda t$ is a (non normal) subgroup of $\mathbb{G}_k$, and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr80], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m} V^\star \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of $J_k V$, of weighted degree m with respect to the $\mathbb{C}^\star$ action defined by H, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{6.1}$$

for all $\lambda \in \mathbb{C}^\star$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$\big((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \ldots; (f^{(k)}_1, \ldots, f^{(k)}_r)\big) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of $J_k V$ makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of $J_k V$, given by a formula

$$(\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1 + j_2 + \cdots + j_s = j} c_{j_1 \ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) \tag{6.2}$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). In the "absolute case" $V = T_X$, we simply write $E^{\rm GG}_{k,m}T^\star_X = E^{\rm GG}_{k,m}$. If $V \subset W \subset T_X$ are holomorphic subbundles, there are natural inclusions

$$J_kV \subset J_kW \subset J_k, \qquad P_kV \subset P_kW \subset P_k.$$

The restriction morphisms induce surjective arrows

$$E^{\rm GG}_{k,m} \to E^{\rm GG}_{k,m}W^\star \to E^{\rm GG}_{k,m}V^\star,$$

in particular $E^{\rm GG}_{k,m}V^\star$ can be seen as a quotient of $E^{\rm GG}_{k,m}$. (The notation $V^\star$ is used here to make the contravariance property implicit from the notation).

If $Q \in E^{\rm GG}_{k,m}V^\star$ is decomposed into multihomogeneous components of multidegree $(\ell_1, \ell_2, \ldots, \ell_k)$ in $f', f'', \ldots, f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E^{\rm GG}_{k,m}V^\star$ will be called the *bundle of jet differentials of order k and weighted degree m*. It is clear from (6.2) that a coordinate change $f \mapsto \Psi \circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \leqslant s \leqslant k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(k)})$ which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1, \ldots, k$, we get a well defined (i.e., coordinate invariant) decreasing filtration $F^\bullet_s$ on $E^{\rm GG}_{k,m}V^\star$ as follows:

$$(6.3) \qquad F^p_s(E^{\rm GG}_{k,m}V^\star) = \left\{ \begin{matrix} Q(f', f'', \ldots, f^{(k)}) \in E^{\rm GG}_{k,m}V^\star \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix} \right\}, \qquad \forall p \in \mathbb{N}.$$

The graded terms $\mathrm{Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^\star)$ associated with the filtration $F^p_{k-1}(E^{\rm GG}_{k,m}V^\star)$ are precisely the homogeneous polynomials $Q(f', \ldots, f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1} = p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$, and $\mathrm{Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^\star) = 0$ unless $k|m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f \mapsto \Psi\circ f$, and they are described by substituting the arguments of $Q(f', \ldots, f^{(k)})$ according to formula (6.2), namely $f^{(j)} \mapsto (\Psi\circ f)^{(j)}$ for $j < k$, and $f^{(k)} \mapsto \Psi'(f)\circ f^{(k)}$ for $j = k$ (when $j = k$, the other terms fall in the next stage F^{p+1}_{k-1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V \subset T_X$ under coordinate changes. We thus find

$$(6.4) \qquad G^{m-k\ell_k}_{k-1}(E^{\rm GG}_{k,m}V^\star) = E^{\rm GG}_{k-1,m-k\ell_k}V^\star \otimes S^{\ell_k}V^\star.$$

Combining all filtrations $F^\bullet_s$ together, we find inductively a filtration $F^\bullet$ on $E^{\rm GG}_{k,m}V^\star$ such that the graded terms are

$$(6.5) \qquad \mathrm{Gr}^\ell(E^{\rm GG}_{k,m}V^\star) = S^{\ell_1}V^\star \otimes S^{\ell_2}V^\star \otimes \cdots \otimes S^{\ell_k}V^\star, \qquad \ell \in \mathbb{N}^k, \quad |\ell|_k = m.$$

The bundles $E_{k,m}^{\mathrm{GG}}V^\star$ have other interesting properties. In fact,

$$E_{k,\bullet}^{\mathrm{GG}}V^\star := \bigoplus_{m\geqslant 0} E_{k,m}^{\mathrm{GG}}V^\star$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\mathrm{GG}}V^\star \subset E_{k+1,\bullet}^{\mathrm{GG}}V^\star$ of algebras, hence $E_{\infty,\bullet}^{\mathrm{GG}}V^\star = \bigcup_{k\geqslant 0} E_{k,\bullet}^{\mathrm{GG}}V^\star$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E_{\infty,\bullet}^{\mathrm{GG}}V^\star)$ admits a canonical derivation ∇ given by a collection of $\mathbb{C}$-linear maps

$$\nabla : \mathcal{O}(E_{k,m}^{\mathrm{GG}}V^\star) \to \mathcal{O}(E_{k+1,m+1}^{\mathrm{GG}}V^\star), \tag{6.6}$$

constructed in the following way. A holomorphic section of $E_{k,m}^{\mathrm{GG}}V^\star$ on a coordinate open set $\Omega \subset X$ can be seen as a differential operator on the space of germs $f : (\mathbb{C},0) \to \Omega$ of the form

$$Q(f) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$$

in which the coefficients $a_{\alpha_1\ldots\alpha_k}$ are holomorphic functions on Ω. Then ∇Q is given by the formal derivative $(\nabla Q)(f)(t) = d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q \in H^0(\Omega, \mathcal{O}(E_{2,4}^{\mathrm{GG}}))$ is the section of weighted degree 4

$$Q(f) = a(f_1,f_2)\,f_1'^3 f_2' + b(f_1,f_2)\,f_1''^2,$$

we find that $\nabla Q \in H^0(\Omega, \mathcal{O}(E_{3,5}^{\mathrm{GG}}))$ is given by

$$\begin{aligned}(\nabla Q)(f) = {}& \frac{\partial a}{\partial z_1}(f_1,f_2)\,f_1'^4 f_2' + \frac{\partial a}{\partial z_2}(f_1,f_2)\,f_1'^3 f_2'^2 + \frac{\partial b}{\partial z_1}(f_1,f_2)\,f_1' f_1''^2\\ &+ \frac{\partial b}{\partial z_2}(f_1,f_2)\,f_2' f_1''^2 + a(f_1,f_2)\,\big(3f_1'^2 f_1'' f_2' + f_1'^3 f_2''\big) + b(f_1,f_2)\,2f_1'' f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E_{k,\bullet}^{\mathrm{GG}}V^\star$, we have an analytic fiber bundle $\mathrm{Proj}(E_{k,\bullet}^{\mathrm{GG}}V^\star) = J_kV^{\mathrm{nc}}/\mathbb{C}^\star$ over X, which has weighted projective spaces $\mathbb{P}(r,\ldots,r\,;1,2,\ldots,k)$ as fibers (these weighted projective spaces are singular for $k>1$, but they only have quotient singularities, see [Dol81] ; here J_kV^{nc} denotes the set of non constant jets of order k ; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj fonctor). However, we are not really interested in the bundles $J_kV^{\mathrm{nc}}/\mathbb{C}^\star$ themselves, but rather on their quotients $J_kV^{\mathrm{nc}}/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle P_kV constructed in § 5 plays the role of such a quotient. First we introduce a canonical bundle subalgebra of $E_{k,\bullet}^{\mathrm{GG}}V^\star$.

6.7. Definition. *We introduce a subbundle $E_{k,m}V^\star \subset E_{k,m}^{\mathrm{GG}}V^\star$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^\star$ is* *the set of polynomial differential operators* $Q(f', f'', \ldots, f^{(k)})$ *which are invariant under arbitrary changes of parametrization, i.e., for every* $\varphi \in \mathbb{G}_k$

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^\star = (E^{\rm GG}_{k,m}V^\star)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\rm GG}_{k,m}V^\star$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^\star = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^\star$ is a subalgebra of $E^{\rm GG}_{\infty,\bullet}V^\star = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E^{\rm GG}_{k,m}V^\star$ (observe however that the algebra $E_{\infty,\bullet}V^\star$ is not invariant under the derivation ∇, since e.g. $f''_j = \nabla f_j$ is not an invariant polynomial). In addition to this, there are natural induced filtrations $F^p_s(E_{k,m}V^\star) = E_{k,m}V^\star \cap F^p_s(E^{\rm GG}_{k,m}V^\star)$ (all locally trivial over X). These induced filtrations will play an important role in Section 12.

6.8. Theorem. *Suppose that* V *has rank* $r \geqslant 2$. *Let* $\pi_{0,k} : P_kV \longrightarrow X$ *be the Semple jet bundles constructed in section* 5, *and let* $J_kV^{\rm reg}$ *be the bundle of regular* k*-jets of maps* $f : (\mathbb{C}, 0) \to X$, *that is, jets* f *such that* $f'(0) \neq 0$.

i) *The quotient* $J_kV^{\rm reg}/\mathbb{G}_k$ *has the structure of a locally trivial bundle over* X, *and there is a holomorphic embedding* $J_kV^{\rm reg}/\mathbb{G}_k \hookrightarrow P_kV$ *over* X, *which identifies* $J_kV^{\rm reg}/\mathbb{G}_k$ *with* $P_kV^{\rm reg}$ (*thus* P_kV *is a relative compactification of* $J_kV^{\rm reg}/\mathbb{G}_k$ *over* X).

ii) *The direct image sheaf*

$$(\pi_{0,k})_\star \mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^\star)$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^\star$.

iii) *For every* $m > 0$, *the relative base locus of the linear system* $|\mathcal{O}_{P_kV}(m)|$ *is equal to the set* $P_kV^{\rm sing}$ *of singular* k*-jets. Moreover,* $\mathcal{O}_{P_kV}(1)$ *is relatively big over* X.

Proof. i) For $f \in J_kV^{\rm reg}$, the lifting $\widetilde{f}$ is obtained by taking the derivative $(f, [f'])$ without any cancellation of zeroes in f', hence we get a uniquely defined $(k-1)$-jet $\widetilde{f} : (\mathbb{C}, 0) \to \widetilde{X}$. Inductively, we get a well defined $(k-j)$-jet $f_{[j]}$ in P_jV, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f\circ\varphi)^\sim = \widetilde{f}\circ\varphi$ and more generally $(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi$, we conclude that there is a well defined set-theoretic map

$$J_kV^{\rm reg}/\mathbb{G}_k \to P_kV^{\rm reg}, \qquad f \mod \mathbb{G}_k \mapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1, \ldots, z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1, \ldots, \partial/\partial z_r)$. Let $f = (f_1, \ldots, f_n)$ be a regular k-jet tangent to V. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $f'_i(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f\circ\varphi = g = (g_1, g_2, \ldots, g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space P_kV is a k-stage tower of

$\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$\big((g_1'(0),\ldots,g_{r-1}'(0));(g_1''(0),\ldots,g_{r-1}''(0));\ldots;(g_1^{(k)}(0),\ldots,g_{r-1}^{(k)}(0))\big).$$

[Recall that the other components $(g_{r+1},\ldots,g_n)$ can be recovered from $(g_1,\ldots,g_r)$ by integrating the differential system (4.10)]. Thus the map $J_kV^{\rm reg}/\mathbb{G}_k\to P_kV$ is a bijection onto $P_kV^{\rm reg}$, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq(\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau}=\frac{1}{f_r'(t)}\frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau)=d^jg_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t)=d^jf_i/dt^j$

$$\begin{aligned}
&(g_1',\ldots,g_{r-1}')=\Big(\frac{f_1'}{f_r'},\ldots,\frac{f_{r-1}'}{f_r'}\Big);\\
(6.9)\quad&(g_1'',\ldots,g_{r-1}'')=\Big(\frac{f_1''f_r'-f_r''f_1'}{f_r'^3},\ldots,\frac{f_{r-1}''f_r'-f_r''f_{r-1}'}{f_r'^3}\Big);\ \ldots\ ;\\
&(g_1^{(k)},\ldots,g_{r-1}^{(k)})=\Big(\frac{f_1^{(k)}f_r'-f_r^{(k)}f_1'}{f_r'^{k+1}},\ldots,\frac{f_{r-1}^{(k)}f_r'-f_r^{(k)}f_{r-1}'}{f_r'^{k+1}}\Big)+(\text{order}<k).
\end{aligned}$$

Also, it is easy to check that $f_r'^{2k-1}g_i^{(k)}$ is an invariant polynomial in $f',f'',\ldots,f^{(k)}$ of total degree $2k-1$, i.e., a section of $E_{k,2k-1}V^*$.

ii) Since the bundles P_kV and $E_{k,m}V^\star$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{P_kV}(m)$ over a fiber $P_kV_x=\pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V_x^\star$, at any point $x\in X$. Let $f\in J_kV_x^{\rm reg}$ be a regular k-jet at x. By (5.6), the derivative $f_{[k-1]}'(0)$ defines an element of the fiber of $\mathcal{O}_{P_kV}(-1)$ at $f_{[k]}(0)\in P_kV$. Hence we get a well defined complex valued operator

$$Q(f',f'',\ldots,f^{(k)})=\sigma(f_{[k]}(0))\cdot(f_{[k-1]}'(0))^m. \tag{6.10}$$

Clearly, Q is holomorphic on $J_kV_x^{\rm reg}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Def. 6.7 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and $(f\circ\varphi)_{[k-1]}'(0)=f_{[k-1]}'(0)\varphi'(0)$. Now, $J_kV_x^{\rm reg}$ is the complement of a linear subspace of codimension n in J_kV_x, hence Q extends holomorphically to all of $J_kV_x\simeq(\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r\geqslant 2$; if $r=1$, the situation is anyway not interesting since $P_kV=X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f',f'',\ldots,f^{(k)})=\sum_{\alpha_1,\alpha_2,\ldots,\alpha_k\in\mathbb{N}^r}a_{\alpha_1\ldots\alpha_k}\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance (6.7) implies in particular that Q must be multihomogeneous in the sense of (6.1), and thus Q must be a polynomial. We conclude that $Q\in E_{k,m}V_x^\star$, as desired.

Conversely, Corollary 5.12 implies that there is a holomorphic family of germs $f_w:(\mathbb{C},0)\to X$ such that $(f_w)_{[k]}(0)=w$ and $(f_w)_{[k-1]}'(0)\neq 0$, for all w in a neighborhood of any given point $w_0 \in P_kV_x$. Then every $Q \in E_{k,m}V_x^\star$ yields a holomorphic section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x by putting

$$\sigma(w) = Q(f'_w, f''_w, \ldots, f_w^{(k)})(0)\,\big((f_w)'_{[k-1]}(0)\big)^{-m}. \tag{6.11}$$

iii) By what we saw in i-ii), every section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x is given by a polynomial $Q \in E_{k,m}V_x^\star$, and this polynomial can be expressed on the Zariski open chart $f'_r \neq 0$ of $P_kV_x^{\mathrm{reg}}$ as

$$Q(f', f'', \ldots, f^{(k)}) = f_r'^m \widehat{Q}(g', g'', \ldots, g^{(k)}), \tag{6.12}$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau) = \tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f'_r = 1$ and $f_r^{(j)} = 0$ for $j \geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (6.9).

In this context, the jet differentials $f \mapsto f'_1, \ldots, f \mapsto f'_r$ can be viewed as sections of $\mathcal{O}_{P_kV}(1)$ on a neighborhood of the fiber P_kV_x. Since these sections vanish exactly on P_kV^{sing}, the relative base locus of $\mathcal{O}_{P_kV}(m)$ is contained in P_kV^{sing} for every $m > 0$. We see that $\mathcal{O}_{P_kV}(1)$ is big by considering the sections of $\mathcal{O}_{P_kV}(2k-1)$ associated with the polynomials $Q(f', \ldots, f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f'_r \neq 0$ of $P_kV_x^{\mathrm{reg}}$.

Now, we check that every section σ of $\mathcal{O}_{P_kV}(m)$ over P_kV_x must vanish on $P_kV_x^{\mathrm{sing}}$. Pick an arbitrary element $w \in P_kV^{\mathrm{sing}}$ and a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$, $f'_{[k-1]}(0) \neq 0$ and $s = m(f, 0) \gg 0$ (such an f exists by Corollary 5.14). There are local coordinates $(z_1, \ldots, z_n)$ on X such that $f(t) = (f_1(t), \ldots, f_n(t))$ where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau), \ldots, g_{r-1}(\tau), \tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$ if $j \geqslant 2$. Hence the Puiseux expansion of $\widehat{Q}(g', g'', \ldots, g^{(k)})$ can only involve powers of τ of exponent $\geqslant -\max_\ell((1-\frac{1}{s})\ell_2 + \cdots + (k-1-\frac{1}{s})\ell_k)$. Finally $f'_r(t) = st^{s-1} = s\tau^{1-1/s}$, thus the lowest exponent of τ in $Q(f', \ldots, f^{(k)})$ is at least equal to

$$\begin{aligned}\Big(1-\frac{1}{s}\Big)m - \max_\ell\Big(\Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(k-1-\frac{1}{s}\Big)\ell_k\Big)&\\ \geqslant \min_\ell \Big(1-\frac{1}{s}\Big)\ell_1 + \Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(1-\frac{k-1}{s}\Big)\ell_k&\end{aligned}$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$, $|\alpha_j| = \ell_j$, occurring in Q. Choosing $s \geqslant k$, we already find that the minimal exponent is positive, hence $Q(f', \ldots, f^{(k)})(0) = 0$ and $\sigma(w) = 0$ by (6.11). □

Theorem (6.8 iii) shows that $\mathcal{O}_{P_kV}(1)$ is never relatively ample over X for $k \geqslant 2$. In order to overcome this difficulty, we define for every $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ a line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ on P_kV such that

$$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \pi_{1,k}^\star \mathcal{O}_{P_1V}(a_1) \otimes \pi_{2,k}^\star \mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k). \tag{6.13}$$

By (5.9), we have $\pi_{j,k}^\star \mathcal{O}_{P_jV}(1) = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\pi_{j+1,k}^\star D_{j+1} - \cdots - D_k)$, thus by putting $D_j^\star = \pi_{j+1,k}^\star D_{j+1}$ for $1 \leqslant j \leqslant k-1$ and $D_k^\star = 0$, we find an identity

$$\begin{aligned}
&\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star), \qquad \text{where}\\
&\boldsymbol{b} = (b_1,\ldots,b_k) \in \mathbb{Z}^k, \quad b_j = a_1+\cdots+a_j,\\
&\boldsymbol{b}\cdot D^\star = \sum_{1\leqslant j\leqslant k-1} b_j\, \pi_{j+1,k}^\star D_{j+1}.
\end{aligned} \tag{6.14}$$

In particular, if $\boldsymbol{b} \in \mathbb{N}^k$, i.e., $a_1+\cdots+a_j \geqslant 0$, we get a morphism

$$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star) \to \mathcal{O}_{P_kV}(b_k). \tag{6.15}$$

6.16. Proposition. *Let $\boldsymbol{a} = (a_1,\ldots,a_k) \in \mathbb{Z}^k$ and $m = a_1+\cdots+a_k$.*

i) *We have the direct image formula*

$$(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \simeq \mathcal{O}(\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star) \subset \mathcal{O}(E_{k,m}V^\star)$$

where $\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star$ is the subbundle of polynomials $Q(f',f'',\ldots,f^{(k)}) \in E_{k,m}V^\star$ involving only monomials $(f^{(\bullet)})^\ell$ such that

$$\ell_{s+1} + 2\ell_{s+2} + \cdots + (k-s)\ell_k \leqslant a_{s+1} + \cdots + a_k$$

for all $s = 0,\ldots,k-1$.

ii) *If $a_1 \geqslant 3a_2, \ldots, a_{k-2} \geqslant 3a_{k-1}$ and $a_{k-1} \geqslant 2a_k \geqslant 0$, the line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ is relatively nef over X.*

iii) *If $a_1 \geqslant 3a_2, \ldots, a_{k-2} \geqslant 3a_{k-1}$ and $a_{k-1} > 2a_k > 0$, the line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ is relatively ample over X.*

Proof. i) By (6.15), we find a sheaf injection

$$(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \hookrightarrow (\pi_{0,k})_\star \mathcal{O}_{P_kV}(m) = \mathcal{O}(E_{k,m}V^\star).$$

Given a section σ of $\mathcal{O}_{P_kV}(\boldsymbol{a})$ over a fiber P_kV_x, the associated polynomial $Q(f',f'',\ldots,f^{(k)}) \in E_{k,m}V_x^\star$ is given by the identity

$$Q(f',f'',\ldots,f^{(k)}) = \sigma(f_{[k]}(0)) \cdot (f'(0))^{a_1} \cdot (f'_{[1]}(0))^{a_2} \cdots (f'_{[k-1]}(0))^{a_k}.$$

Indeed, we see this from (6.10) and from the fact that $f'_{[k-1]}(0)$ is mapped to $f'_{[j-1]}(0)$ by the projection morphism

$$(\pi_{j-1,k-1})_\star : \mathcal{O}_{P_kV}(-1) \to \pi_{j,k}^\star \mathcal{O}_{P_jV}(-1)$$

(cf. (5.8)), which is dual to the corresponding morphism (6.15). Now, we prove the inclusion $(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \subset \mathcal{O}(\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star)$ by induction on k. For $s = 0$, the desired inequality comes from the weighted homogeneity condition, hence we may assume $s \geqslant 1$. Let f run over all regular germs having their first derivative $f'(0)$ fixed. This means that σ is viewed as a section of $\pi_{2,k}^\star \mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k)$ on the fibers of the projection $P_kV = P_{k-1}V_1 \to X_1 = P_1V$. Then we get a polynomial $Q_1 \in E_{k-1,m-a_1}V_1^\star$ such that

$$Q_1(f'_{[1]}, f''_{[1]}, \ldots, f^{(k-1)}_{[1]}) = Q(f', f'', \ldots, f^{(k)}).$$

In the affine chart $f'_r \neq 0$, the map $f_{[1]}$ is defined in coordinates by $f_{[1]} \simeq (f_1, \ldots, f_n\,;\, f'_1/f'_r, \ldots, f'_{r-1}/f'_r)$. Its derivative $f'_{[1]} \in V_1$ can thus be described by $f'_{[1]} \simeq \big((f'_1/f'_r)', \ldots, (f'_{r-1}/f'_r)', f'_r\big)$, by taking $r-1$ vertical components and a horizontal one. All this becomes much simpler if we replace f by $g = f \circ f_r^{-1}$, since $g_r(t) = t$ and $g'_r(t) = 1$. Then we get

$$(g', g'', \ldots, g^{(k)}) \simeq \big((g'_1, \ldots, g'_{r-1}, 1), (g''_1, \ldots, g''_{r-1}, 0), \ldots, (g^{(k)}_1, \ldots, g^{(k)}_{r-1}, 0)\big),$$
$$(g'_{[1]}, g''_{[1]}, \ldots, g^{(k)}_{[1]}) \simeq \big((g''_1, \ldots, g''_{r-1}, 1), (g'''_1, \ldots, g'''_{r-1}, 0), \ldots, (g^{(k)}_1, \ldots, g^{(k)}_{r-1}, 0)\big)$$

in the corresponding charts of J_kV and $J_{k-1}V_1$. The inequality (6.16 i) for the monomials $(g^{(\bullet)})^\ell$ of $Q(g', g'', \ldots, g^{(k)})$ follows clearly from the corresponding inequality on the monomials $(g^{(\bullet)}_{[1]})^\ell$ of Q_1, when (k,s) is replaced by $(k-1, s-1)$. Now, thanks to (6.9), we get $Q(f', f'', \ldots, f^{(k)}) = (f'_r)^m Q(g', g'', \ldots, g^{(k)})$, and the desired inequality (6.16 i) for the monomials $(f^{(\bullet)})^\ell$ follows easily. In the opposite direction, if we are given a section $Q(f', f'', \ldots, f^{(k)}) \in \mathcal{O}(\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star)$, we see by induction on k that Q defines a section of

$$\mathcal{O}_{P_1V}(a_1) \otimes (\pi_{1,k})_\star\big(\pi_{2,k}^\star \mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k)\big)$$

on P_1V, and we conclude that we get a section of $(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a})$ by taking the direct image by $(\pi_1)_\star$.

ii-iii) By induction on k, we construct a relatively ample line bundle L_{k-1} on $P_{k-1}V$ such that $\mathcal{O}_{P_kV}(1) \otimes \pi_k^\star L_{k-1}$ is relatively nef; by definition, this is equivalent to saying that the vector bundle $V_{k-1}^\star \otimes L_{k-1}$ is relatively nef (for the notion of a nef vector bundle, see e.g. [DPS94]). Since $\mathcal{O}_{P_1V}(1)$ is relatively ample, we can start with $L_0 = \mathcal{O}_X$. Suppose that L_{k-1} has been constructed. The dual of (5.4) yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{P_kV}(1) \longrightarrow V_k^\star \longrightarrow T^\star_{P_kV/P_{k-1}V} \longrightarrow 0.$$

As an extension of nef vector bundles is nef, it is enough to select L_k in such a way that

$$\mathcal{O}_{P_kV}(1) \otimes L_k \quad \text{and} \quad T^\star_{P_kV/P_{k-1}V} \otimes L_k \quad \text{are relatively nef.} \tag{6.17}$$

By taking the second wedge power of the central term in (5.4′), we get an injection

$$0 \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow \Lambda^2\big(\pi_k^\star V_{k-1} \otimes \mathcal{O}_{P_kV}(1)\big).$$

By dualizing and twisting with $\mathcal{O}_{P_{k-1}V}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$, we find a surjection

$$\pi_k^\star \Lambda^2(V_{k-1}^\star \otimes L_{k-1}) \longrightarrow T^\star_{P_kV/P_{k-1}V} \otimes \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2} \longrightarrow 0.$$

As $V_{k-1}^\star \otimes L_{k-1}$ is relatively nef by the induction hypothesis, we obtain that its quotient $T^\star_{P_kV/P_{k-1}V} \otimes \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ is also relatively nef. Hence Condition (6.17) is achieved if we take $L_k \geqslant \pi_k^\star L_{k-1}$ and $L_k \geqslant \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ (the ordering relation $\geqslant$ is the one given by the cone of relatively nef line bundles). We need only define L_k inductively by

$$L_k = \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 3}.$$

The relative ampleness of L_k is then clear by induction, since $\mathcal{O}_{P_kV}(1) \otimes \pi_k^\star L_{k-1}$ is relatively nef over X and relatively ample over $P_{k-1}V$. The resulting formula for L_k is

$$L_k = \mathcal{O}_{P_kV}\big((2\cdot 3^{k-1}, 2\cdot 3^{k-2}, \ldots, 6, 2)\big).$$

By definition, we then find

$$\mathcal{O}_{P_kV}(1) \otimes \pi_k^\star L_{k-1} = \mathcal{O}_{P_kV}\big((2\cdot 3^{k-2}, 2\cdot 3^{k-3}, \ldots, 6, 2, 1)\big) \qquad \text{relatively nef.}$$

These properties imply ii) and iii) by taking suitable convex combinations. □

6.18. Remark. As in Green-Griffiths [GrGr80], Riemann's extension theorem shows that for every meromorphic map $\Phi : X \dashrightarrow Y$ there are well-defined pullback morphisms

$$\Phi^\star : H^0(Y, E^{\mathrm{GG}}_{k,m}) \to H^0(X, E^{\mathrm{GG}}_{k,m}), \qquad \Phi^\star : H^0(Y, E_{k,m}) \to H^0(X, E_{k,m}).$$

In particular the dimensions $h^0(X, E^{\mathrm{GG}}_{k,m})$ and $h^0(X, E^{\mathrm{GG}}_{k,m})$ are bimeromorphic invariants of X. The same is true for spaces of sections of any subbundle of $E^{\mathrm{GG}}_{k,m}$ or $E_{k,m}$ constructed by means of the canonical filtrations $F_s^\bullet$.

§7. *k*-jet metrics with negative curvature

The goal of this section is to show that hyperbolicity is closely related to the existence of k-jet metrics with suitable negativity properties of the curvature. The connection between these properties is in fact a simple consequence of the Ahlfors-Schwarz lemma. Such ideas have been already developed long ago by Grauert-Reckziegel [GRec65], Kobayashi [Kob75] for 1-jet metrics (i.e., Finsler metrics on T_X) and by Cowen-Griffiths [CoGr76], Green-Griffiths [GrGr80] and Grauert [Gra89] for higher order jet metrics. However, even in the standard case $V = T_X$, the definition given below differs from that of [GrGr80], in which the k-jet metrics are not supposed to be $\mathbb{G}'_k$-invariant. We prefer to deal here with $\mathbb{G}'_k$-invariant objects, because they reflect better the intrinsic geometry. Grauert [Gra89] actually deals with $\mathbb{G}'_k$-invariant metrics, but he apparently does not take care of the way the quotient space $J_k^{\mathrm{reg}}V/\mathbb{G}_k$ can be compactified; also, his metrics are always induced by the Poincaré metric, and it is not at all clear whether these metrics have the expected curvature properties (see 7.14 below). In the present situation, it is important to allow also hermitian metrics possessing some singularities ("singular hermitian metrics" in the sense of [Dem90]).

7.1. Definition. *Let $L \to X$ be a holomorphic line bundle over a complex manifold X. We say that h is a singular metric on L if for any trivialization $L_{\restriction U} \simeq U \times \mathbb{C}$ of L, the metric is given by $|\xi|_h^2 = |\xi|^2 e^{-\varphi}$ for some real valued weight function $\varphi \in L^1_{\mathrm{loc}}(U)$. The curvature current of L is then defined to be the closed $(1,1)$-current $\Theta_h(L) = \frac{i}{2\pi}\partial\overline{\partial}\varphi$, computed in the sense of distributions. We say that h admits a closed subset $\Sigma \subset X$ as its degeneration set if φ is locally bounded on $X \smallsetminus \Sigma$ and is unbounded on a neighborhood of any point of Σ.*

An especially useful situation is the case when the curvature of h is positive definite. By this, we mean that there exists a smooth positive definite hermitian metric ω and a continuous positive function ε on X such that $\Theta_h(L) \geqslant \varepsilon\omega$ in the sense of currents, and we write in this case $\Theta_h(L) \gg 0$. We need the following basic fact (quite standard when X is projective algebraic; however we want to avoid any algebraicity assumption here, so as to be able the case of general complex tori in § 9).

7.2. Proposition. *Let L be a holomorphic line bundle on a compact complex manifold X.*

i) *L admits a singular hermitian metric h with positive definite curvature current $\Theta_h(L) \gg 0$ if and only if L is big.*

Now, define B_m to be the base locus of the linear system $|H^0(X, L^{\otimes m})|$ and let

$$\Phi_m : X \smallsetminus B_m \to \mathbb{P}^N$$

be the corresponding meromorphic map. Let Σ_m be the closed analytic set equal to the union of B_m and of the set of points $x \in X \smallsetminus B_m$ such that the fiber $\Phi_m^{-1}(\Phi_m(x))$ is positive dimensional.

ii) *If $\Sigma_m \neq X$ and G is any line bundle, the base locus of $L^{\otimes k} \otimes G^{-1}$ is contained in Σ_m for k large. As a consequence, L admits a singular hermitian metric h with degeneration set Σ_m and with $\Theta_h(L)$ positive definite on X.*

iii) *Conversely, if L admits a hermitian metric h with degeneration set Σ and positive definite curvature current $\Theta_h(L)$, there exists an integer $m > 0$ such that the base locus B_m is contained in Σ and $\Phi_m : X \smallsetminus \Sigma \to \mathbb{P}_m$ is an embedding.*

iv) *Assume that L admits a singular hermitian metric h with positive definite curvature current, such that the degeneration set Σ is an analytic subset of X. Assume moreover that for each irreducible component Σ_j of Σ, $L_{|\Sigma_j}$ admits a singular hermitian metric h_j with positive definite curvature current on Σ_j and degeneration set $\Sigma_{j,k} \subset \Sigma_j$. Then L admits a singular hermitian metric $\widetilde{h}$ of positive curvature current on X, with degeneration set $\widetilde{\Sigma} = \bigcup_{j,k} \Sigma_{j,k}$.*

Proof. i) is proved e.g. in [Dem90, 92], so we will only briefly sketch the details. If L is big, then X is Moishezon and we can even assume that X is projective algebraic after taking a suitable modification $\widetilde{X}$ (apply Hironaka [Hir64]; observe moreover that the direct image of a strictly positive current is strictly positive). So, assume that X is projective algebraic. Then it is well-known that some large multiple of L can be written as $L^{\otimes m} = \mathcal{O}_X(D+A)$ with divisors D, A such that D is effective and A ample. The invertible sheaf $\mathcal{O}_X(D)$ can be viewed as a subsheaf of the sheaf of meromorphic functions. We get a singular metric $|s|^2$ on sections of $\mathcal{O}_X(D)$ by just taking the square of the modulus of s viewed as a complex valued (meromorphic) function. By the Lelong-Poincaré equation, the curvature current of that metric is equal to the current of integration $[D] \geqslant 0$ over the divisor D. We thus get $\Theta(L) = \frac{1}{m}([D] + \Theta(A)) \geqslant \frac{1}{m}\Theta(A) \gg 0$ for a suitable choice of the metric on $\mathcal{O}_X(A)$. In the other direction, if $\Theta_h(L)$ is positive, one can construct a "lot of" sections in $H^0(X, L^{\otimes m})$, $m \gg 0$, by using Hörmander's L^2 estimates; the Hörmander-Bombieri-Skoda technique implies that these sections can be taken to have arbitrary jets at all points in a given finite subset of $X \smallsetminus \Sigma$, if Σ is the degeneration set of h. This also proves property iii).

ii) The assumption $\Sigma_m \neq X$ shows that there is a generically finite meromorphic map from X to an algebraic variety, and this implies again that X is Moishezon. By blowing-up the ideal

$$\mathcal{J}_m = \mathrm{Im}\left(H^0(X, L^{\otimes m}) \otimes \mathcal{O}_X(L^{\otimes -m}) \to \mathcal{O}_X\right) \subset \mathcal{O}_X$$

and resolving the singularities, we obtain a smooth modification $\mu : \widetilde{X} \to X$ and a line bundle $\widetilde{L} = \mu^\star(L^{\otimes m}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ (where E is a μ-exceptional divisor with support in $\mu^{-1}(\Sigma_m)$, such that $\widetilde{L}$ is base point free; after possibly blowing-up again, we may assume furthermore that $\widetilde{X}$ is projective algebraic. Clearly, it is enough to prove the result for $\widetilde{L}$, and we are thus reduced to the case when L is base point free and X is projective algebraic. We may finally assume that G is very ample (other we add a large ample divisor to G to make it very ample). In this situation, we have a holomorphic map $\Phi_m : X \to \mathbb{P}^N$ such that $L^{\otimes m} = \Phi_m^{-1}(\mathcal{O}(1))$, and Φ_m is finite-to-one outside Σ_m. Hence, if $x \in X \smallsetminus \Sigma_m$, the set $\Phi_m^{-1}(\Phi_m(x))$ is finite, and we can take a smooth divisor $D \in |G|$ such that $D \cap \Phi_m^{-1}(\Phi_m(x)) = \emptyset$. Thus $\Phi_m(D) \not\ni \varphi_m(x)$ in $\mathbb{P}^N$. It follows that there exists a hypersurface $H = \sigma^{-1}(0) \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ of sufficiently large degree k, such that H contains $\Phi_m(D)$ but does not pass through $\Phi_m(x)$. Then $\Phi_m^\star\sigma$ can be viewed as a section of $\Phi_m^\star\mathcal{O}_{\mathbb{P}^N}(k) \otimes \mathcal{O}_X(-D) = L^{\otimes km} \otimes G^{-1}$, and $\Phi_m^\star\sigma$ does not vanish at x. By the Noetherian property, there exists k_0 such that the base locus of $L^{\otimes km} \otimes G^{-1}$ is contained in Σ_m for $k \geqslant k_0$ large. Claim ii) follows.

iv) is obtained by extending the metric h_j to a metric $\widetilde{h}_j$ on a neighborhood of Σ_j (it is maybe necessary to modify $\widetilde{h}_j$ slightly by adding some "transversally convex terms" in the weight, so as to obtain positive curvature in all directions of T_X, on a suitable neighborhood of Σ_j), and then taking $\widetilde{h} = \min(h, \varepsilon\widetilde{h}_j)$ with $\varepsilon > 0$ small enough. □

We now come to the main definitions. By (5.6), every regular k-jet $f \in J_kV$

gives rise to an element $f'_{[k-1]}(0) \in \mathcal{O}_{P_kV}(-1)$. Thus, measuring the "norm of k-jets" is the same as taking a hermitian metric on $\mathcal{O}_{P_kV}(-1)$.

7.3. Definition. *A smooth, (resp. continuous, resp. singular) k-jet metric on a complex directed manifold (X,V) is a hermitian metric h_k on the line bundle $\mathcal{O}_{P_kV}(-1)$ over P_kV (i.e. a Finsler metric on the vector bundle V_{k-1} over $P_{k-1}V$), such that the weight functions φ representing the metric are smooth (resp. continuous, $L^1_{\rm loc}$). We let $\Sigma_{h_k} \subset P_kV$ be the singularity set of the metric, i.e., the closed subset of points in a neighborhood of which the weight φ is not locally bounded.*

We will always assume here that the weight function φ is quasi psh. Recall that a function φ is said to be quasi psh if φ is locally the sum of a plurisubharmonic function and of a smooth function (so that in particular $\varphi \in L^1_{\rm loc}$). Then the curvature current

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

is well defined as a current and is locally bounded from below by a negative $(1,1)$-form with constant coefficients.

7.4. Definition. *Let h_k be a k-jet metric on (X,V). We say that h_k has negative jet curvature (resp. negative total jet curvature) if $\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))$ is negative definite along the subbundle $V_k \subset T_{P_kV}$ (resp. on all of T_{P_kV}), i.e., if there is $\varepsilon > 0$ and a smooth hermitian metric ω_k on T_{P_kV} such that*

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k}, \qquad \forall\xi \in V_k \subset T_{P_kV} \quad (\textit{resp.} \quad \forall\xi \in T_{P_kV}).$$

(If the metric h_k is not smooth, we suppose that its weights φ are quasi psh, and the curvature inequality is taken in the sense of distributions.)

It is important to observe that for $k \geqslant 2$ there cannot exist any smooth hermitian metric h_k on $\mathcal{O}_{P_kV}(1)$ with positive definite curvature along $T_{X_k/X}$, since $\mathcal{O}_{P_kV}(1)$ is not relatively ample over X. However, it is relatively big, and Prop. 7.2 i) shows that $\mathcal{O}_{P_kV}(-1)$ admits a singular hermitian metric with negative total jet curvature (whatever the singularities of the metric are) if and only if $\mathcal{O}_{P_kV}(1)$ is big over P_kV. It is therefore crucial to allow singularities in the metrics in Def. 7.4.

7.5. Special case of 1-jet metrics. A 1-jet metric h_1 on $\mathcal{O}_{P_1V}(-1)$ is the same as a Finsler metric $N = \sqrt{h_1}$ on $V \subset T_X$. Assume until the end of this paragraph that h_1 is smooth. By the well known Kodaira embedding theorem, the existence of a smooth metric h_1 such that $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive on all of T_{P_1V} is equivalent to $\mathcal{O}_{P_1V}(1)$ being ample, that is, $V^\star$ ample. In the absolute case $V = T_X$, there are only few examples of varieties X such that $T^\star_X$ is ample, mainly quotients of the ball $\mathbb{B}_n \subset \mathbb{C}^n$ by a discrete cocompact group of automorphisms. The 1-jet negativity condition considered in Definition 7.4 is much weaker. For example, if the hermitian metric h_1 comes from a (smooth) hermitian metric h on V, then formula (4.16) implies that h_1 has negative total jet curvature (i.e. $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive) if and only if $\langle\Theta_h(V)\rangle(\zeta\otimes v) < 0$ for all $\zeta \in T_X \smallsetminus \{0\}$, $v \in V \smallsetminus \{0\}$, that is, if (V,h) is *negative in the sense of Griffiths.* On the other hand, $V_1 \subset T_{P_1V}$ consists by definition of tangent vectors $\tau \in T_{P_1V,(x,[v])}$ whose horizontal projection ${}^H\tau$ is proportional to v, thus $\Theta_{h_1}(\mathcal{O}_{P_1V}(-1))$ is negative definite on V_1 if and only if $\Theta_h(V)$ satisfies the much weaker condition that the *holomorphic sectional curvature* $\langle\Theta_h(V)\rangle(v\otimes v)$ is negative on every complex line. □

We now come back to the general situation of jets of arbitrary order k. Our first observation is the fact that the k-jet negativity property of the curvature becomes actually weaker and weaker as k increases.

7.6. Lemma. *Let (X,V) be a compact complex directed manifold. If (X,V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there is a k-jet metric h_k with negative jet curvature such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$. (The same holds true for negative total jet curvature).*

Proof. Let ω_{k-1}, ω_k be given smooth hermitian metrics on $T_{P_{k-1}V}$ and T_{P_kV}. The hypothesis implies

$$\langle\Theta_{h_{k-1}^{-1}}(\mathcal{O}_{P_{k-1}V}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_{k-1}}, \qquad \forall\xi\in V_{k-1}$$

for some constant $\varepsilon>0$. On the other hand, as $\mathcal{O}_{P_kV}(D_k)$ is relatively ample over $P_{k-1}V$ (D_k is a hyperplane section bundle), there exists a smooth metric $\widetilde{h}$ on $\mathcal{O}_{P_kV}(D_k)$ such that

$$\langle\Theta_{\widetilde{h}}(\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant \delta|\xi|^2_{\omega_k} - C|(\pi_k)_\star\xi|^2_{\omega_{k-1}}, \qquad \forall\xi\in T_{P_kV}$$

for some constants $\delta, C>0$. Combining both inequalities (the second one being applied to $\xi\in V_k$ and the first one to $(\pi_k)_\star\xi\in V_{k-1}$), we get

$$\begin{aligned}\langle\Theta_{(\pi_k^\star h_{k-1})^{-p}\widetilde{h}}(\pi_k^\star\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k))\rangle(\xi) &\geqslant \\ \geqslant \delta|\xi|^2_{\omega_k} + (p\varepsilon - C)|(\pi_k)_\star\xi|^2_{\omega_{k-1}}, &\qquad \forall\xi\in V_k.\end{aligned}$$

Hence, for p large enough, $(\pi_k^\star h_{k-1})^{-p}\widetilde{h}$ has positive definite curvature along V_k. Now, by (5.9), there is a sheaf injection

$$\mathcal{O}_{P_kV}(-p) = \pi_k^\star\mathcal{O}_{P_{k-1}V}(-p)\otimes\mathcal{O}_{P_kV}(-pD_k) \hookrightarrow \big(\pi_k^\star\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k)\big)^{-1}$$

obtained by twisting with $\mathcal{O}_{P_kV}((p-1)D_k)$. Therefore $h_k := ((\pi_k^\star h_{k-1})^{-p}\widetilde{h})^{-1/p} = (\pi_k^\star h_{k-1})\widetilde{h}^{-1/p}$ induces a singular metric on $\mathcal{O}_{P_kV}(1)$ in which an additional degeneration divisor $p^{-1}(p-1)D_k$ appears. Hence we get $\Sigma_{h_k} = \pi_k^{-1}\Sigma_{h_{k-1}} \cup D_k$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{p}\Theta_{(\pi_k^\star h_{k-1})^{-p}\widetilde{h}} + \frac{p-1}{p}[D_k]$$

is positive definite along V_k. The same proof works in the case of negative total jet curvature. □

One of the main motivations for the introduction of k-jets metrics is the following list of algebraic sufficient conditions.

7.7. Algebraic sufficient conditions. We suppose here that X is projective algebraic, and we make one of the additional assumptions i), ii) or iii) below.

i) Assume that there exist integers $k, m > 0$ and $\boldsymbol{b} \in \mathbb{N}^k$ such that the line bundle $\mathcal{O}_{P_kV}(m) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^\star)$ is ample over P_kV. Set $A = \mathcal{O}_{P_kV}(m) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^\star)$. Then there is a smooth hermitian metric h_A on A with positive definite curvature on P_kV. By means of the morphism $\mu : \mathcal{O}_{P_kV}(-m) \to A^{-1}$, we get an induced metric $h_k = (\mu^\star h_A^{-1})^{1/m}$ on $\mathcal{O}_{P_kV}(-1)$ which is degenerate on the support of the zero divisor $\mathrm{div}(\mu) = \boldsymbol{b} \cdot D^\star$. Hence $\Sigma_{h_k} = \mathrm{Supp}(\boldsymbol{b} \cdot D^\star) \subset P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{m}\Theta_{h_A}(A) + \frac{1}{m}[\boldsymbol{b} \cdot D^\star] \geqslant \frac{1}{m}\Theta_{h_A}(A) > 0.$$

In particular h_k has negative total jet curvature.

ii) Assume more generally that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_kV, \mathcal{O}_{P_kV}(m) \otimes \pi_{0,k}^\star L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_kV$ be the base locus of these sections; necessarily $Z \supset P_kV^{\mathrm{sing}}$ by 6.8 iii). By taking a smooth metric h_L with positive curvature on L, we get a singular metric h'_k on $\mathcal{O}_{P_kV}(-1)$ such that

$$h'_k(\xi) = \Big(\sum_{1 \leqslant j \leqslant N} |\sigma_j(w) \cdot \xi^m|^2_{h_L^{-1}} \Big)^{1/m}, \qquad w \in P_kV, \quad \xi \in \mathcal{O}_{P_kV}(-1)_w.$$

Then $\Sigma_{h'_k} = Z$, and by computing $\frac{i}{2\pi}\partial\overline{\partial}\log h'_k(\xi)$ we obtain

$$\Theta_{h_k'^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{m}\pi_{0,k}^\star\Theta(L).$$

By (6.15) and 6.16 iii), there exists $\boldsymbol{b} \in \mathbb{Q}_+^k$ such that $\mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^\star)$ is relatively ample over X. Hence $A = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^\star) \otimes \pi_{0,k}^\star L^{\otimes p}$ is ample on X for $p \gg 0$. The arguments used in i) show that there is a k-jet metric h''_k on $\mathcal{O}_{P_kV}(-1)$ with $\Sigma_{h''_k} = \mathrm{Supp}(\boldsymbol{b} \cdot D^\star) = P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k''^{-1}}(\mathcal{O}_{P_kV}(1)) = \Theta(A) + [\boldsymbol{b} \cdot D^\star] - p\,\pi_{0,k}^\star\Theta(L),$$

where $\Theta(A)$ is positive definite on P_kV. The metric $h_k = (h_k'^{mp} h''_k)^{1/(mp+1)}$ then satisfies $\Sigma_{h_k} = \Sigma_{h'_k} = Z$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{mp+1}\Theta(A) > 0.$$

iii) If $E_{k,m}V^\star$ is ample, there is an ample line bundle L and a sufficiently high symmetric power such that $S^p(E_{k,m}V^\star) \otimes L^{-1}$ is generated by sections. These sections can be viewed as sections of $\mathcal{O}_{P_kV}(mp) \otimes \pi_{0,k}^\star L^{-1}$ over P_kV, and their base locus is exactly $Z = P_kV^{\mathrm{sing}}$ by 6.8 iii). Hence the k-jet metric h_k constructed in ii) has negative total jet curvature and satisfies $\Sigma_{h_k} = P_kV^{\mathrm{sing}}$. $\square$

An important fact, first observed by [GRe65] for 1-jet metrics and by [GrGr80] in the higher order case, is that k-jet negativity implies hyperbolicity. In particular, the existence of enough global jet differentials implies hyperbolicity.

7.8. Theorem. *Let (X,V) be a compact complex directed manifold. If (X,V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f:\mathbb{C}\to X$ tangent to V is such that $f_{[k]}(\mathbb{C})\subset\Sigma_{h_k}$. In particular, if $\Sigma_{h_k}\subset P_kV^{\rm sing}$, then (X,V) is hyperbolic.*

Proof. The main idea is to use the Ahlfors-Schwarz lemma, following the approach of [GrGr80]. However we will give here all necessary details because our setting is slightly different. Assume that there is a k-jet metric h_k as in the hypotheses of Theorem 7.8. Let ω_k be a smooth hermitian metric on T_{P_kV}. By hypothesis, there exists $\varepsilon>0$ such that

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi)\geqslant\varepsilon|\xi|^2_{\omega_k}\qquad\forall\xi\in V_k.$$

Moreover, by (5.4), $(\pi_k)_\star$ maps V_k continuously to $\mathcal{O}_{P_kV}(-1)$ and the weight e^φ of h_k is locally bounded from above. Hence there is a constant $C>0$ such that

$$|(\pi_k)_\star\xi|^2_{h_k}\leqslant C|\xi|^2_{\omega_k},\qquad\forall\xi\in V_k.$$

Combining these inequalities, we find

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi)\geqslant\frac{\varepsilon}{C}|(\pi_k)_\star\xi|^2_{h_k},\qquad\forall\xi\in V_k.$$

Now, let $f:\Delta_R\to X$ be a non constant holomorphic map tangent to V on the disk Δ_R. We use the line bundle morphism (5.6)

$$F=f'_{[k-1]}:T_{\Delta_R}\to f^\star_{[k]}\mathcal{O}_{P_kV}(-1)$$

to obtain a pullback metric

$$\gamma=\gamma_0(t)\,dt\otimes d\bar t=F^\star h_k\qquad\text{on }T_{\Delta_R}.$$

If $f_{[k]}(\Delta_R)\subset\Sigma_{h_k}$ then $\gamma\equiv0$. Otherwise, $F(t)$ has isolated zeroes at all singular points of $f_{[k-1]}$ and so $\gamma(t)$ vanishes only at these points and at points of the degeneration set $(f_{[k]})^{-1}(\Sigma_{h_k})$ which is a polar set in Δ_R. At other points, the Gaussian curvature of γ satisfies

$$\frac{i\,\partial\overline\partial\log\gamma_0(t)}{\gamma(t)}=\frac{-2\pi\,(f_{[k]})^\star\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))}{F^\star h_k}=\frac{\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(f'_{[k]}(t))}{|f'_{[k-1]}(t)|^2_{h_k}}\geqslant\frac{\varepsilon}{C},$$

since $f'_{[k-1]}(t)=(\pi_k)_\star f'_{[k]}(t)$. The Ahlfors-Schwarz lemma 3.2 implies that γ can be compared with the Poincaré metric as follows:

$$\gamma(t)\leqslant\frac{2C}{\varepsilon}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2}\qquad\Longrightarrow\qquad|f'_{[k-1]}(t)|^2_{h_k}\leqslant\frac{2C}{\varepsilon}\frac{R^{-2}}{(1-|t|^2/R^2)^2}.$$

If $f : \mathbb{C} \to X$ is an entire curve tangent to V such that $f_{[k]}(\mathbb{C}) \not\subset \Sigma_{h_k}$, the above estimate implies as $R \to +\infty$ that $f_{[k-1]}$ must be a constant, hence also f. Now, if $\Sigma_{h_k} \subset P_k V^{\mathrm{sing}}$, the inclusion $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$ implies $f'(t) = 0$ at every point, hence f is a constant and (X, V) is hyperbolic. $\square$

Combining Theorem 7.8 with 7.7 ii) and iii), we get the following consequences.

7.9. Corollary. *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_k V, \mathcal{O}_{P_k V}(m) \otimes \pi_{0,k}^\star L^{-1}) \simeq H^0(X, E_{k,m}(V^\star) \otimes L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_k V$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve f must satisfy the algebraic differential equation $P(f) = 0$.* $\square$

7.10. Corollary. *Let (X, V) be a compact complex directed manifold. If $E_{k,m} V^\star$ is ample for some positive integers k, m, then (X, V) is hyperbolic.* $\square$

7.11. Remark. Green and Griffiths [GrGr80] stated that Corollary 7.9 is even true with sections $\sigma_j \in H^0(X, E_{k,m}^{\mathrm{GG}}(V^\star) \otimes L^{-1})$, in the special case $V = T_X$ they consider. We refer to the recent preprint [SiYe96c] by Siu and Yeung for a detailed proof of this fact, based on a use of the well-known logarithmic derivative lemma in Nevanlinna theory (the original proof given in [GrGr80] does not seem to be complete, as it relies on an unsettled pointwise version of the Ahlfors-Schwarz lemma for general jet differentials); other proofs seem to have been circulating in the literature in the last years. We give here a very short proof for the case when f is supposed to have a bounded derivative (thanks to Brody's theorem, this is enough if one is merely interested in proving hyperbolicity, thus Corollary 7.10 will be valid with $E_{k,m}^{\mathrm{GG}} V^\star$ in place of $E_{k,m} V^\star$). In fact, if f' is bounded, one can apply the Cauchy inequalities to all components f_j of f with respect to a finite collection of coordinate patches covering X. As f' is bounded, we can do this on sufficiently small discs $D(t, \delta) \subset \mathbb{C}$ of constant radius $\delta > 0$. Therefore all derivatives f', f'', $\ldots$ $f^{(k)}$ are bounded. From this we conclude that $\sigma_j(f)$ is a bounded section of $f^\star L^{-1}$. Its norm $|\sigma_j(f)|_{L^{-1}}$ (with respect to any positively curved metric $|\ \ |_L$ on L) is a bounded subharmonic function, which is moreover strictly subharmonic at all points where $f' \neq 0$ and $\sigma_j(f) \neq 0$. This is a contradiction unless f is constant or $\sigma_j(f) \equiv 0$. $\square$

The above results justify the following definition and problems.

7.12. Definition. *We say that X, resp. (X, V), has non degenerate negative k-jet curvature if there exists a k-jet metric h_k on $\mathcal{O}_{P_k V}(-1)$ with negative jet curvature such that $\Sigma_{h_k} \subset P_k V^{\mathrm{sing}}$.*

7.13. Conjecture. *Let (X, V) be a compact directed manifold. Then (X, V) is hyperbolic if and only if (X, V) has nondegenerate negative k-jet curvature for k large enough.*

This is probably a hard problem. In fact, we will see in the next section that the smallest admissible integer k must depend on the geometry of X and need not be uniformly bounded as soon as $\dim X \geqslant 2$ (even in the absolute case $V = T_X$). On the other hand, if (X, V) is hyperbolic, we get for each integer $k \geqslant 1$ a generalized Kobayashi-Royden metric $\mathbf{k}_{(P_{k-1}V, V_{k-1})}$ on V_{k-1} (see Definition 1.3), which can be also viewed as a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$; we will call it the *Grauert k-jet metric* of (X, V), although it formally differs from the jet metric considered in [Gra89] (see also [DGr91]). By looking at the projection $\pi_k : (P_kV, V_k) \to (P_{k-1}V, V_{k-1})$, we see that the sequence h_k is monotonic, namely $\pi_k^\star h_k \leqslant h_{k+1}$ for every k. If (X, V) is hyperbolic, then h_1 is nondegenerate and therefore by monotonicity $\Sigma_{h_k} \subset P_kV^{\rm sing}$ for $k \geqslant 1$. Conversely, if the Grauert metric satisfies $\Sigma_{h_k} \subset P_kV^{\rm sing}$, it is easy to see that (X, V) is hyperbolic. The following problem is thus especially meaningful.

7.14. Problem. *Estimate the k-jet curvature $\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))$ of the Grauert metric h_k on (P_kV, V_k) as k tends to $+\infty$.*

§8. Algebraic criterion for the negativity of jet curvature

Our goal is to show that the negativity of k-jet curvature implies strong restrictions of an algebraic nature, similar to property 2.1 ii). Using this we give examples, for any prescribed integer k, of hyperbolic projective surfaces which do not admit any k-jet metric of negative jet curvature.

8.1. Theorem. *Let (X, V) be a compact complex directed manifold and let ω be a hermitian metric on X. If (X, V) has negative k-jet curvature, there exists a constant $\varepsilon > 0$ such that every closed irreducible curve $C \subset X$ tangent to V satisfies*

$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C) + \sum_{t \in \overline{C}} (m_{k-1}(t) - 1) > 0$$

where $g(\overline{C})$ is the genus of the normalization $\nu : \overline{C} \to C \subset X$, and $m_k(t)$ is the multiplicity at point t of the k-th lifting $\nu_{[k]} : \overline{C} \to P_kV$ of ν.

Proof. By (5.6), we get a lifting $\nu_{[k]} : \overline{C} \to P_kV$ of the normalization map ν, and there is a canonical map

$$\nu'_{[k-1]} : T_{\overline{C}} \to \nu_{[k]}^\star \mathcal{O}_{P_kV}(-1).$$

Let $t_j \in \overline{C}$ be the singular points of $\nu_{[k-1]}$, and let $m_j = m_{k-1}(t_j)$ be the corresponding multiplicity. Then $\nu'_{[k-1]}$ vanishes at order $m_j - 1$ at t_j and thus we find

$$T_{\overline{C}} \simeq \nu_{[k]}^\star \mathcal{O}_{P_kV}(-1) \otimes \mathcal{O}_{\overline{C}}\Big(- \sum (m_j - 1) p_j \Big).$$

Taking any k-jet metric h_k with negative jet curvature on $\mathcal{O}_{P_kV}(-1)$, the Gauss-Bonnet formula yields

$$2g(\overline{C}) - 2 = \int_{\overline{C}} \Theta(T_{\overline{C}}^\star) = \sum (m_j - 1) + \int_{\overline{C}} \nu_{[k]}^\star \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)).$$

Now, the curvature hypothesis implies

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon'|\xi|^2_{\omega_k} \geqslant \varepsilon''|(\pi_{0,k})_\star\xi|^2_\omega \qquad \forall\xi\in V_k,$$

for some $\varepsilon', \varepsilon'' > 0$ and some smooth hermitian metric ω_k on P_kV. As $\pi_{0,k}\circ\nu_{[k]}=\nu$, we infer from this $\nu_{[k]}^\star\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\geqslant\nu^\star\omega$, hence

$$\int_{\overline{C}}\nu_{[k]}^\star\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{\varepsilon''}{2\pi}\int_{\overline{C}}\nu^\star\omega = \varepsilon\,\deg_\omega(C)$$

with $\varepsilon=\varepsilon''/2\pi$. Theorem 8.1 follows. □

8.2. Theorem. *Let $k\geqslant 1$ be any positive integer. Then there is a nonsingular algebraic surface X (depending on k) which is hyperbolic, but does not carry any nondegenerate k-jet metric with negative jet curvature. In fact, given any two curves Γ,Γ' of genus at least 2, the surface X may be constructed as a fibration $X\to\Gamma$ in which one of the fibers C_0 is singular and has Γ' as its normalization.*

Proof. The idea is to construct X in such a way that the singular fiber C which is normalized by Γ' violates the inequality obtained in Theorem 8.1. For this we need only having a singular point t_0 such that $m_{k-1}(t_0)-1>2g(\overline{C})-2$, i.e., $m_{k-1}(t_0)\geqslant 2g(\Gamma')$. Moreover, as Γ is hyperbolic, X will be hyperbolic if and only if all fibers of $X\to\Gamma$ have geometric genus at least 2.

We first construct from Γ' a singular curve Γ'' with normalization $\overline{\Gamma}''=\Gamma'$, simply by modifying the structure sheaf $\mathcal{O}_{\Gamma'}$ at one given point $w_0\in\Gamma'$. Let t be a holomorphic coordinate on Γ' at w_0. We replace $\mathcal{O}_{\Gamma',w_0}=\mathbb{C}\{t\}$ by $\mathcal{O}_{\Gamma'',w_0}=\mathbb{C}\{t^a,t^b\}$, where $a<b$ are relatively prime integers. The corresponding singularity is described by the germ of embedding $t\mapsto f(t)=(t^a,t^b)$ in $(\mathbb{C}^2,0)$. Now, $f'(t)=(at^{a-1},bt^{b-1})$, thus $[f'(t)]\in\mathbb{P}^1\simeq\mathbb{C}\cup\{\infty\}$ is given by $[f'(t)]=\frac{b}{a}t^{b-a}$. By induction, we see that the singularity of the j-th lifting $f_{[j]}$ is described by the embedding

$$t\mapsto(t^a,t^b,c_1t^{b-a},\ldots,c_jt^{b-ja})\in\mathbb{C}^{j+2},\qquad c_j=a^{-j}b(b-a)\cdots(b-(j-1)a)$$

if $b>ja$. Then we have $m(f_{[j]},0)=\min(a,b-ja)$. If we take for instance $a=2g(\Gamma')$ and $b=ka+1$, then $m(f_{[k-1]},0)=a$. We embed Γ'' in some projective space $\mathbb{P}^n$ and let $C=p(\Gamma'')$ to be a generic projection to a plane $\mathbb{P}^2\subset\mathbb{P}^n$ in such a way that C has only $x_0=p(w_0)$ and some nodes (ordinary double points) as its singular points. By construction, the Zariski tangent space to Γ'' at w_0 is 2-dimensional, so we may assume that p projects that plane injectively into $T_{\mathbb{P}^2}$. Then we get a curve $C\subset\mathbb{P}^2$ with $\overline{C}=\Gamma'$, such that $m(\nu_{[k-1]},w_0)=a=2g(\overline{C})$, if $\nu:\overline{C}\to\mathbb{P}^2$ is the normalization.

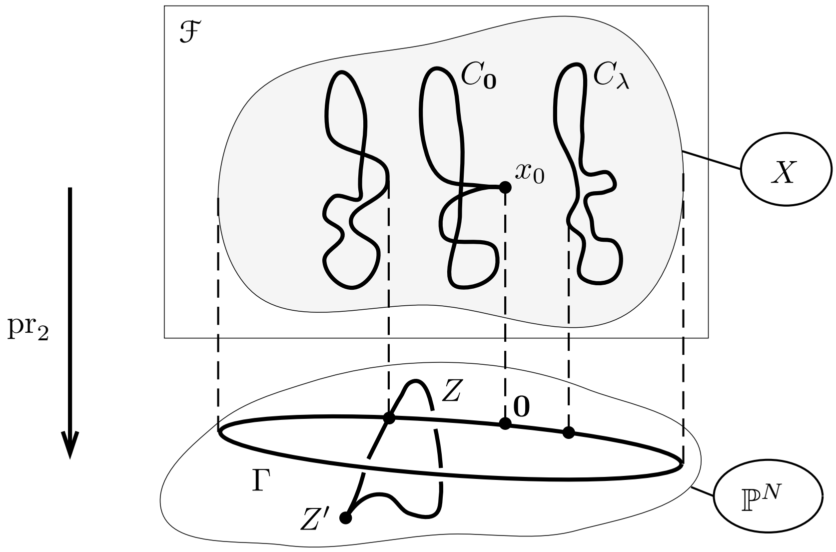

Figure 1. Construction of the surface X

Let $P_0(z_0, z_1, z_2) = 0$ be an equation of C in $\mathbb{P}^2$. Since C has geometric genus at least 2, we have $d = \deg P_0 \geqslant 4$. We complete P_0 into a basis $(P_0, \ldots, P_N)$ of the space of homogeneous polynomials of degree d, and consider the universal family

$$\mathcal{F} = \Big\{([z_0 : z_1 : z_2], [\lambda_0, \lambda_1, \ldots, \lambda_N]) \in \mathbb{P}^2 \times \mathbb{P}^N \,;\, \sum \lambda_j P_j(z) = 0\Big\} \subset \mathbb{P}^2 \times \mathbb{P}^N$$

of curves $C_\lambda = \{\sum \lambda_j P_j(z) = 0\}$ of degree d in $\mathbb{P}^2$. As is well known, the set Z of points $\lambda \in \mathbb{P}^N$ such that C_λ is a singular curve is an algebraic hypersurface, and the set $Z' \subset Z$ of points λ such that C_λ has not just a node in its singularity set satisfies $\operatorname{codim} Z' \geqslant 2$. The curve $C = C_{\mathbf{0}}$ itself corresponds to the point $\mathbf{0} = [1 : 0 : \cdots : 0] \in Z'$. Since $\operatorname{codim} Z' \geqslant 2$, we can embed Γ in $\mathbb{P}^N$ in such a way that $\Gamma \cap Z' = \{\mathbf{0}\}$. We then take $X \to \Gamma$ to be the family of curves $(C_\lambda)_{\lambda\in\Gamma}$. If X is singular, we move Γ by a generic automorphism of $\mathbb{P}^N$ leaving $\mathbf{0}$ fixed. Then, since $\mathcal{F}$ is smooth (it is a smooth $\mathbb{P}^{N-1}$ subbundle of $\mathbb{P}^2 \times \mathbb{P}^N$ over $\mathbb{P}^2$), Bertini's theorem implies that $X \smallsetminus C_{\mathbf{0}}$ will become nonsingular. That X will be also nonsingular near $C_{\mathbf{0}}$ depends only on the following first order condition: if $[1 : \alpha\lambda_1^0 : \cdots : \alpha\lambda_N^0]$, $\alpha \in \mathbb{C}$, is the tangent line to Γ at $\mathbf{0}$, then $\sum_{j\geqslant 1} \lambda_j^0 P_j(z)$ does not vanish at any of the singular points of $C_{\mathbf{0}}$. Now, all nonsingular fibers C_λ of the fibration $X \to \Gamma$ have genus $(d-1)(d-2)/2 \geqslant 3$, and the singular ones other than $C_{\mathbf{0}}$ only have one node, so their genus is $(d-1)(d-2)/2 - 1 \geqslant 2$. $\square$

If we make an assumption on the total jet curvature (as is the case with the algebraic sufficient conditions 7.7), Theorem 8.1 can be strengthened to curves which are not necessarily tangent to V, again by introducing the concept of deviation. We start with a general purpose statement.

8.3. Proposition. *Let (X,V) be a compact complex directed manifold and let L be a holomorphic line bundle over X. Assume that L is equipped with a singular hermitian metric h of degeneration set Σ_h, such that the curvature (computed in the sense of distributions) satisfies*

$$\Theta_h(L) \geqslant \alpha, \qquad \alpha_{\restriction V} \geqslant \delta\omega_{\restriction V}$$

where δ is a positive constant, ω a smooth hermitian metric and α is a continuous real $(1,1)$-form on X. Then for every compact irreducible curve $C \subset X$ not contained in Σ_h, there exists a constant $\varepsilon > 0$ such that the following a priori inequality holds

$$\max\big(L\cdot C, \mathrm{dev}^2_\omega(C/V)\big) \geqslant \varepsilon\ \deg_\omega(C).$$

Proof. By the continuity of α and the compactness of X, our assumption $\alpha_{\restriction V} \geqslant \delta\omega$ implies that there is a constant $M > 0$ such that

$$\alpha + M\,\omega^{V^\perp} \geqslant \frac{\delta}{2}\omega$$

(to get this, one merely needs to apply the Cauchy-Schwarz inequality to mixed terms $V^\star \otimes (V^\perp)^\star$ in a hermitian form on V). In particular, we find

$$\Theta_h(L) + M\,\omega^{V^\perp} \geqslant \frac{\delta}{2}\omega$$

This inequality gives rise to a corresponding numerical inequality on every irreducible curve $C \not\subset \Sigma_h$, for the difference has a well defined and nonnegative restriction to C (we use here the fact that the weight of h is quasi-psh and locally bounded at some point of C, hence locally integrable along C). From this we infer

$$L\cdot C + M\,\mathrm{dev}^2_\omega(C/V) \geqslant \frac{\delta}{2}\deg_\omega(C),$$

and the left hand side is at most equal to $(M+1)\max\big(L\cdot C, \mathrm{dev}^2_\omega(C/V)\big)$. $\square$

8.4. Proposition. *Let (X,V) be a compact complex directed manifold. Assume that there are integers $k,m > 0$ and $\boldsymbol{b} \in \mathbb{N}^k$ such that $\mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)$ is an ample line bundle over P_kV. Then (X,V) is hyperbolic and there exists $\varepsilon > 0$ such that every closed curve $C \subset X$ satisfies*

$$\max\Big(-\chi(\overline{C}) - \sum_{t\in\overline{C}}(m_{k-1}(t)-1), \mathrm{dev}^\infty_\omega(C/V)\Big) \geqslant \varepsilon\ \deg_\omega(C).$$

Proposition 8.4 is likely to be true also if we assume more generally that (X,V) has non degenerate total k-jet curvature but, in this case, some technical difficulties appear in the construction of the required singular hermitian metric h_k on $\mathcal{O}_{P_kT_X}(1)$ (see the proof below).

Proof. The hyperbolicity of (X,V) follows from 7.7 i) and Theorem 7.8. Now, the identity map defines a natural monomorphism $(X,V)\to(X,T_X)$ of directed manifolds and therefore induces an embedding $P_kV\hookrightarrow P_kT_X$ for each k. With respect to this embedding, we have

$$\begin{aligned}&\mathcal{O}_{P_kT_X}(1)_{\restriction P_kV}=\mathcal{O}_{P_kV}(1),\\ &\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)_{\restriction P_kV}=\mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)\end{aligned}$$

By our assumptions, $\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)$ is ample over P_kV and over the fibers of the projection $P_kT_X\to X$. Hence, we can find a smooth hermitian metric $h_{k,m,\boldsymbol{b}}$ on $\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)$ such that the curvature form is positive definite on a neighborhood U of P_kV and satisfies

$$\Theta\big(\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)\big)\geqslant -C\pi_{k,0}^\star\omega$$

for some Kähler metric ω over X. This metric $h_{k,m,\boldsymbol{b}}$ gives rise to a hermitian metric h_k on $\mathcal{O}_{P_kT_X}(1)$ with singularity set $\Sigma_{h_k}\subset P_k^{\rm sing}T_X$ and similar curvature properties, that is

$$\tag{8.5}\begin{cases}\Theta_{h_k}(\mathcal{O}_{P_kT_X}(1))\geqslant -C\pi_{k,0}^\star\omega & \text{on } P_kT_X,\\ \Theta_{h_k}(\mathcal{O}_{P_kT_X}(1))\geqslant \delta\omega_k\geqslant\delta'\pi_{k,0}^\star\omega & \text{on } U\supset P_kV,\end{cases}$$

where ω_k is a hermitian metric on P_kT_X and δ, $\delta'>0$. Now, assume that the conclusion of Prop. 8.4 is wrong. Then there would exist a sequence of curves (C_ℓ) and a sequence of positive numbers ε_ℓ converging to 0, such that

$$\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}\leqslant\varepsilon_\ell\deg_\omega(C_\ell),\qquad \mathrm{dev}_\omega^\infty(C_\ell/V)\big)\leqslant\varepsilon_\ell\deg_\omega(C_\ell)$$

where $C_{\ell,[k]}$ is the lifting of C_ℓ to P_kT_X [indeed, we have $\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}=-\chi(\overline{C}_\ell)-\sum(m_{k-1}(t)-1)$]. Let $\nu_\ell:\overline{C}_\ell\to X$ be the normalization map. As $\mathrm{dev}_\omega^\infty(C_\ell/V)\big)=\sup\nu_\ell^\star(\omega_{V^\perp})/d\widetilde{\sigma}$ where $d\sigma$ is the Poincaré metric and $d\widetilde{\sigma}$ the associated normalized metric, the second condition means

$$\sup\|\,\mathrm{pr}_{V^\perp}\nu_\ell'\|_{\sigma,\omega}^2=\sup\frac{\nu_\ell^\star(\omega_{V^\perp})}{d\sigma}\leqslant\frac{\varepsilon_\ell\deg_\omega(C_\ell)}{\int_{\overline{C}_\ell}d\sigma}=\varepsilon_\ell\frac{\int_{\overline{C}_\ell}\nu_\ell^\star\omega}{\int_{\overline{C}_\ell}d\sigma}.$$

In addition to this, we have

$$\frac{\int_{\overline{C}_\ell}\nu_\ell^\star\omega}{\int_{\overline{C}_\ell}d\sigma}\leqslant R_\ell^2:=\sup\|\nu_\ell'\|_{\sigma,\omega}^2$$

and $R=\sup R_\ell<+\infty$, otherwise the proof of Prop. 2.9 would produce a non constant entire curve $g:\mathbb{C}\to X$ tangent to V, contradicting the hyperbolicity of (X,V). An application of the Cauchy inequalities to the components of $\mathrm{pr}_{V^\perp}$ on sufficiently small disks in the universal covering of $\overline{C}_\ell$ and in suitable trivializations of T_X/V shows that there is a constant $M_k\geqslant 0$ such that

$$\sup_{1\leqslant j\leqslant k}\|\,\mathrm{pr}_{V^\perp}\nu_\ell^{(j)}\|_{\sigma,\omega}^2\leqslant M_k\sup\|\,\mathrm{pr}_{V^\perp}\nu_\ell'\|_{\sigma,\omega}^2\leqslant M_k\varepsilon_\ell\frac{\int_{\overline{C}_\ell}\nu_\ell^\star\omega}{\int_{\overline{C}_\ell}d\sigma}.$$

As $\int_{\overline{C}_\ell}\|\nu'_\ell\|^{-2}_{\sigma,\omega}\nu_\ell^\star\omega=\int_{\overline{C}_\ell}d\sigma$, we infer

$$\int_{\overline{C}_\ell}\frac{\sup_{1\leqslant j\leqslant k}\|\operatorname{pr}_{V^\perp}\nu_\ell^{(j)}\|^2_{\sigma,\omega}}{\|\nu'_\ell\|^2_{\sigma,\omega}}\nu_\ell^\star\omega\leqslant M_k\varepsilon_\ell\int_{\overline{C}\ell}\nu_\ell^\star\omega. \tag{8.6}$$

Since U is a neighborhood of P_kV, there exists a constant $\eta>0$ such that

$$\frac{\sup_{1\leqslant j\leqslant k}\|\operatorname{pr}_{V^\perp}\nu_\ell^{(j)}(t)\|^2_{\sigma,\omega}}{\|\nu'_\ell(t)\|^2_{\sigma,\omega}}<\eta\quad\Longrightarrow\nu_{\ell,[k]}(t)\in U$$

for any $t\in\overline{C}_\ell$. By the integral estimate (8.6), the set S_η of "bad points" $t\in\overline{C}_\ell$ at which the left hand inequality does not hold has area $<M_k\varepsilon_\ell\deg_\omega(C_\ell)/\eta$ with respect to $\nu_\ell^\star\omega$. By (8.5), we then get

$$\begin{aligned}\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}&=\int_{\overline{C}_\ell\smallsetminus S_\eta}\nu_{\ell,[k]}^\star\Theta(\mathcal{O}_{P_kT_X}(1))+\int_{S_\eta}\nu_{\ell,[k]}^\star\Theta(\mathcal{O}_{P_kT_X}(1))\\&\geqslant\delta'\int_{\overline{C}_\ell\smallsetminus S_\eta}\nu_\ell^\star\omega-C\int_{S_\eta}\nu_\ell^\star\omega\\&=\big(\delta'(1-M_k\varepsilon_\ell/\eta)-CM_k\varepsilon_\ell/\eta\big)\deg_\omega(C_\ell).\end{aligned}$$

This contradicts our initial hypothesis that $\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}\leqslant\varepsilon_\ell\deg_\omega(C_\ell)$ when ε_ℓ is small enough. □

The above results lead in a natural way to the following questions, dealing with the "directed manifold case" of Kleiman's criterion (Kleiman's criterion states that a line bundle L on X is ample if and only if there exists $\varepsilon>0$ such that $L\cdot C\geqslant\varepsilon\deg_\omega C$ for every curve $C\subset X$).

8.7. Questions. *Let (X,V) be a compact directed manifold and let L be a line bundle over X. Fix $p\in[2,+\infty]$.*

i) *Assume that*
$$\max\big(L\cdot C,\operatorname{dev}^p_\omega(C/V)\big)\geqslant\varepsilon\deg_\omega(C)$$
for every algebraic curve $C\subset X$ (and some $\varepsilon>0$). Does L admit a smooth hermitian metric h with $\Theta_h(L)_{\restriction V}$ positive definite ?

ii) *Assume more generally that there is an analytic subset $Y\supsetneq X$ such that* i) *holds for all curves $C\not\subset Y$. Does L admit a singular hermitian metric h with $\Theta_h(L)_{\restriction V}$ positive definite, and with degeneration set $\Sigma_h\subset Y$?*

iii) *Assume that there exists $\varepsilon>0$ such that every closed curve $C\subset X$ satisfies*

$$\max\Big(-\chi(\overline{C})-\sum_{t\in\overline{C}}(m_{k-1}(t)-1),\operatorname{dev}^p_\omega(C/V)\Big)\geqslant\varepsilon\deg_\omega(C).$$

Does it follow that (X,V) admits non degenerate negative k-jet (total) curvature ?

The answer to 8.7 i) is positive if V is the vertical tangent sheaf of a smooth map $X \to S$, and in that case one can even restrict oneself to curves that are tangent to V (i.e. vertical curves): this is just the relative version of Kleiman's criterion. However, in general, it is not sufficient to deal only with curves tangent to V (if X is an abelian variety and V is a constant line subbundle of T_X with non closed leaves, the condition required for algebraic curves C is void, hence L can be taken negative on X ; then, of course, the curvature cannot be made positive along V.)

§9. Proof of the Bloch theorem

The core of the result can be expressed as a characterization of the Zariski closure of an entire curve drawn on a complex torus. The proof will be obtained as a simple consequence of the Ahlfors-Schwarz lemma (more specifically Theorem 7.8), combined with a jet bundle argument. Our argument works in fact without any algebraicity assumption on the complex tori under consideration (only the case of abelian or semi-abelian varieties seems to have been treated earlier).

9.1. Theorem. *Let Z be a complex torus and let $f : \mathbb{C} \to Z$ be a holomorphic map. Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ is a translate of a subtorus, i.e. of the form $a + Z'$, $a \in Z$, where $Z' \subset Z$ is a subtorus.*

The converse is of course also true: for any subtorus $Z' \subset Z$, we can choose a dense line $L \subset Z'$, and the corresponding map $f : \mathbb{C} \simeq a + L \hookrightarrow Z$ has Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}} = a + Z'$.

Proof (based on the ideas of [GrGr80]). Let $f : \mathbb{C} \to Z$ be an entire curve and let X be the Zariski closure of $f(\mathbb{C})$. We denote by $Z_k = P_k(T_Z)$ the k-jet bundle of Z and by X_k the closure of $X_k^{\mathrm{reg}} = P_k(T_{X^{\mathrm{reg}}})$ in Z_k. As T_Z is trivial, we have $Z_k = Z \times \mathbb{R}_{n,k}$ where $\mathbb{R}_{n,k}$ is the rational variety introduced in § 5. By Proposition 6.16 iii), there is a weight $\boldsymbol{a} \in \mathbb{N}^k$ such that $\mathcal{O}_{Z_k}(\boldsymbol{a})$ is relatively very ample. This means that there is a very ample line bundle $\mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$ over $\mathbb{R}_{n,k}$ such that $\mathcal{O}_{Z_k}(\boldsymbol{a}) = \mathrm{pr}_2^\star\, \mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$. Consider the map $\Phi_k : X_k \to \mathbb{R}_{n,k}$ which is the restriction to X_k of the second projection $Z_k \to \mathbb{R}_{n,k}$. By fonctoriality, we have $\mathcal{O}_{X_k}(\boldsymbol{a}) = \Phi_k^\star \mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$.

Define $B_k \subset X_k$ to be the set of points $x \in X_k$ such that the fiber of Φ_k through x is positive dimensional. Assume that $B_k \neq X_k$. By Proposition 7.2 ii), $\mathcal{O}_{X_k}(\boldsymbol{a})$ carries a hermitian metric with degeneration set B_k and with strictly positive definite curvature on X_k (if necessary, blow-up X_k along the singularities and push the metric forward). Theorem 7.8 shows that $f_{[k]}(\mathbb{C}) \subset B_k$, and this is of course also true if $B_k = X_k$. The inclusion $f_{[k]}(\mathbb{C}) \subset B_k$ means that through every point $f_{[k]}(t_0)$ there is a germ of positive dimensional variety in the fiber $\Phi_k^{-1}(\Phi_k(f_{[k]}(t_0)))$, say a germ of curve $t' \mapsto u(t') = (z(t'), j_k) \in X_k \subset Z \times \mathbb{R}_{n,k}$ with $u(0) = f_{[k]}(t_0) = (z_0, j_k)$ and $z_0 = f(t_0)$. Then $(z(t'), j_k)$ is the image of $f_{[k]}(t_0)$ by the k-th lifting of the translation $\tau_s : z \mapsto z + s$ defined by $s = z(t') - z_0$. Now, we have $f(\mathbb{C}) \not\subset X^{\mathrm{sing}}$ since X is the Zariski closure of $f(\mathbb{C})$, and we may therefore choose t_0 so that $f(t_0) \in X^{\mathrm{reg}}$ and $f(t_0)$ is a regular point. Let us define

$$A_k(f) = \big\{ s \in Z \;:\; f_{[k]}(t_0) \in P_k(X) \cap P_k(\tau_{-s}(X)) \big\}.$$

Clearly $A_k(f)$ is an analytic subset of Z containing the curve $t' \mapsto s(t') = z(t') - z_0$ through 0. Since

$$A_1(f) \supset A_2(f) \supset \cdots \supset A_k(f) \supset \cdots,$$

the Noetherian property shows that the sequence stabilizes at some $A_k(f)$. Therefore, there is a curve $D(0,r) \to Z$, $t' \mapsto s(t')$ such that the infinite jet j_∞ defined by f at t_0 is $s(t')$-translation invariant for all t'. By uniqueness of analytic continuation, we conclude that $s(t') + f(t) \in X$ for all $t \in \mathbb{C}$ and $t' \in D(0,r)$. As X is the Zariski closure of $f(\mathbb{C})$, we must have $s(t') + X \subset X$ for all $t' \in D(0,r)$; also, X is irreducible, thus we have in fact $s(t') + X = X$. Define

$$W = \big\{s \in Z\,;\ s + X = X\big\}.$$

Then W is a closed positive dimensional subgroup of Z. Let $p : Z \to Z/W$ be the quotient map. As Z/W is a complex torus with $\dim Z/W < \dim Z$, we conclude by induction on dimension that the curve $\widehat{f} = p \circ f : \mathbb{C} \to Z/W$ has its Zariski closure $\widehat{X} := \overline{\widehat{f}(\mathbb{C})}^{\mathrm{Zar}} = p(X)$ equal to a translate $\widehat{s} + \widehat{T}$ of some subtorus $\widehat{T} \subset Z/W$. Since X is W-invariant, we get $X = s + p^{-1}(\widehat{T})$, where $p^{-1}(\widehat{T})$ is a closed subgroup of Z. This implies that X is a translate of a subtorus, as expected. □

We now state two simple corollaries, and then the "Bloch theorem" itself (see also [Och77], [Nog77, 81, 84], [Kaw80] for other approaches in the algebraic case).

9.2. Corollary. *Let X be a complex analytic subvariety in a complex torus Z. Then X is hyperbolic if and only if X does not contain any translate of a subtorus.*

9.3. Corollary. *Let X be a complex analytic subvariety of a complex torus Z. Assume that X is not a translate of a subtorus. Then every entire curve drawn in X is analytically degenerate.*

9.4. Bloch theorem. *Let X be a compact complex Kähler variety such that the irregularity $q = h^0(X, \Omega^1_X)$ is larger than the dimension $n = \dim X$. Then every entire curve drawn in X is analytically degenerate.*

Here X may be singular and Ω^1_X can be defined in any reasonable way (direct image of the $\Omega^1_{\widehat{X}}$ of a desingularization $\widehat{X}$ or direct image of Ω^1_U where U is the set of regular points in the normalization of X).

Proof. By blowing-up, we may assume that X is smooth. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ sends X onto a proper subvariety $Y \subset \mathrm{Alb}(X)$ (as $\dim Y \leqslant \dim X < \dim \mathrm{Alb}(X)$), and Y is not a translate of a subtorus by the universal property of the Albanese map. Hence, for every entire curve $f : \mathbb{C} \to X$ we infer that $\alpha \circ f : \mathbb{C} \to Y$ is analytically degenerate; it follows that f itself is analytically degenerate. □

§10. Logarithmic jet bundles and a conjecture of Lang

We want to report here briefly about an important question raised by S. Lang, namely *whether the complement of an ample divisor in an Abelian variety is*

Kobayashi hyperbolic? This statement has been first settled in the affirmative by Siu and Yeung [SiYe96b], using an extension of some of the methods used to prove Bloch's theorem. We will adopt here a slightly different approach of G. Dethloff and S. Lu [DLu96], who followed a suggestion made during our Santa Cruz lectures in July 1995. Namely, there should exist a theory of *logarithmic jet bundles* extending Semple's construction, which would allow to study the hyperbolicity properties of open varieties of the form $X \smallsetminus D$ (D being a divisor in a projective variety X). We give here a short account of Dethloff and Lu's technique, referring to [DLu96] for details, and to [SiYe96b], [Nog96a] for alternative approaches.

Let (X, V) be a compact directed manifold and D a reduced divisor in X. Recall that the sheaf $\Omega^1_X\langle D\rangle$ of *holomorphic* 1*-forms with logarithmic poles along* D is defined to be the coherent sheaf generated by Ω^1_X and ds_j/s_j, where $s_j = 0$ are local equations for the irreducible components of D. It is locally free as soon as D is a normal crossing divisor (we may always suppose that this is the case after blowing up X along smooth centers contained in D). Similarly, one introduces the sheaf $\mathcal{O}(V^\star\langle D\rangle)$ to be the sheaf of holomorphic 1-forms along V with logarithmic poles along D (this is just the quotient of $\Omega^1_X\langle D\rangle$ by the conormal sheaf $V^o \subset V^\star$ of V). It is locally free as soon as D has normal crossings and its components $D_{(j)}$ are everywhere tranversal to V (by this we mean that $T_{D_{(j)}} + V = T_X$ along $D_{(j)}$). Under this assumption, we consider the dual (locally free) sheaves

$$\mathcal{O}(T_X\langle D\rangle) := (\Omega^1_X\langle D\rangle)^\star, \qquad \mathcal{O}(V\langle D\rangle) := (V^\star\langle D\rangle)^\star. \tag{10.1}$$

One easily checks that $\mathcal{O}(T_X\langle D\rangle)$ (resp. $\mathcal{O}(V\langle D\rangle)$) is the sheaf of germs of vector fields in $\mathcal{O}(T_X)$ (resp. $\mathcal{O}(V)$) which are tangent to each component of D. Now, one defines a sequence

$$(X_k, D_k, V_k) \tag{10.2}$$

of logarithmic k-jet bundles exactly in the same way as we proceeded in section § 4 and § 5: if $X_0 = X$, $D_0 = D$ and $V_0 = V\langle D\rangle$, one sets inductively $X_k = P(V_{k-1})$, $D_k = (\pi_{k,0})^{-1}(D)$, and V_k is the set of tangent vectors in $T_{X_k}\langle D_k\rangle$ which project into the line defined by the tautological line bundle $\mathcal{O}_{X_k}(-1) \subset \pi_k^\star V_{k-1}$. In this case, the direct image formula given in Theorem 6.8 reads

$$(\pi_{k,0})_\star \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m} V^\star\langle D\rangle), \tag{10.3}$$

where $\mathcal{O}(E_{k,m} V^\star\langle D\rangle)$ is the sheaf generated by all polynomial differential operators in the derivatives of order $1, 2, \ldots, k$ of the components $f_1, \ldots, f_n$, together with the extra function $\log s_j(f)$ along the j-th component of D.

Just as before, a *logarithmic k-jet metric* is just a singular hermitian metric on $\mathcal{O}_{X_k}(-1)$. Dethloff and Lu [DLu96] state the following results 10.4–10.9, which extend our results of sections § 7 and § 9 (most of these results can already be derived from [SiYe96b] as well).

10.4. Theorem. *Let (X, D, V) be as above. Let $\Sigma_{k,m}$ be the union of the base locus of $\mathcal{O}_{X_k}(m)$ and of the positive dimensional fibers of the canonical map defined by the corresponding linear system. Then*

i) *If $\Sigma_{k,m} \neq X_k$, there exists a logarithmic k-jet metric h_k with strictly negative jet curvature and $\Sigma_{h_k} = \Sigma_{k,m}$.*

ii) *For every entire map $f : \mathbb{C} \to X \smallsetminus D$ tangent to V, one has $f_{[k]}(\mathbb{C}) \subset \Sigma_{k,m}$.*

iii) *For every holomorphic map $f : \Delta^\star \to X \smallsetminus D$ tangent to V (where $\Delta^\star$ is the punctured disk), one has: either f extends to a holomorphic map $\overline{f} : \Delta \to X$ or $f_{[k]}(\Delta^\star) \subset \Sigma_{k,m}$.*

Consider now a semi-abelian variety Z (that is, a commutative algebraic group $\mathbb{C}^n/\Gamma$), and let $D \subset Z$ be a reduced algebraic divisor.

10.5. Theorem. *Let (Z, D) be as above.*

i) *For every entire curve $f : \mathbb{C} \to Z$, the Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ is a translate of an algebraic subgroup of Z.*

ii) *For every entire curve $f : \mathbb{C} \to Z \smallsetminus D$, we have $\overline{f(\mathbb{C})}^{\mathrm{Zar}} \cap D = \emptyset$.*

10.6. Corollary. *If D has non empty intersection with any translate of an algebraic subgroup of Z of positive dimension, then $Z \smallsetminus D$ is Brody hyperbolic. This is true e.g. if Z is abelian and D is ample.*

10.7. Remark. Theorem 10.5 and its corollary have been obtained independently by Noguchi [Nog96a], and also by Siu-Yeung [SiYe96b] in the case of abelian varieties. Both of their proofs use value distribution theory, whilst the present approach uses only negative curvature arguments. It is likely that Theorem 10.5 can be extended to arbitrary commutative (non necessarily algebraic) Lie groups $\mathbb{C}^n/\Gamma$.

10.8. Theorem. *The following properties hold true.*

A) *Let $f : \Delta^\star \to Z$ be a holomorphic map. Then either it extends to a holomorphic map $\overline{f} : \Delta \to \overline{Z}$ or there exists a maximal algebraic subgroup Z' of Z of positive dimension such that $\overline{f(\Delta^\star)}^{\mathrm{Zar}}$ is foliated by translates of Z'.*

B) *Let $f : \Delta^\star \to Z \smallsetminus D$ be a holomorphic map. Then one of the following holds:*

  i) *f extends to a holomorphic map $\overline{f} : \Delta \to Z$.*

  ii) *$\overline{f(\Delta^\star)}^{\mathrm{Zar}} \cap D = \emptyset$.*

  iii) *There exists an algebraic subgroup Z'' of Z' of positive dimension such that $\overline{f(\Delta^\star)}^{\mathrm{Zar}} \cap D$ is foliated by translates of Z''.*

C) *Assume here that Z is an abelian variety and let $f : \Delta^\star \to Z \smallsetminus D$ be a holomorphic map. Then one of the following holds:*

  i) *f extends to a holomorphic map $\overline{f} : \Delta \to Z$.*

  ii) *There exists an algebraic subgroup Z'' of Z' of positive dimension such that D is foliated by translates of Z''.*

Part A) of Theorem 10.8 is due to Noguchi [Nog96a] (again with a proof based on Nevanlinna theory).

10.9. Corollary. *If Z is abelian and D is ample, then every holomorphic map $f : \Delta^\star \to Z \smallsetminus D$ extends to a holomorphic map $\overline{f} : \Delta \to Z$.*

§11. Projective meromorphic connections and Wronskians

We describe here an important method introduced by Siu [Siu87] and later developped by Nadel [Nad89], which is powerful enough to provide explicit examples of algebraic hyperbolic surfaces. It yields likewise interesting results about the algebraic degeneration of entire curves in higher dimensions. The main idea is to use meromorphic connections with low pole orders, and the associated Wronskian operators. In this way, Nadel produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for any degree of the form $p = 6k + 3 \geqslant 21$. We present here a variation of Nadel's method, based on the more general concept of *partial projective connection*, which allows us to extend his result to all degrees $p \geqslant 11$. This approach is inspired from a recent work of J. El Goul [EG96], and is in some sense a formalization of his strategy.

Let X be a complex n-dimensional manifold. A *meromorphic connection* ∇ on T_X is a $\mathbb{C}$-linear sheaf morphism

$$\mathcal{M}(U, T_X) \longrightarrow \mathcal{M}(U, \Omega^1_X \otimes T_X)$$

(where $\mathcal{M}(U, \bullet)$ stands for meromorphic sections over U), satisfying the Leibnitz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

whenever $f \in \mathcal{M}(U)$ (resp. $s \in \mathcal{M}(U, T_X)$) is a meromorphic function (resp. section of T_X). Let $(z_1, \ldots, z_n)$ be holomorphic local coordinates on an open set $U \subset X$. The *Christoffel symbols* of ∇ with respect to these coordinates are the coefficients $\Gamma^\lambda_{j\mu}$ such that

$$\Gamma^\lambda_\mu = \sum_{1 \leqslant j \leqslant n} \Gamma^\lambda_{j\mu} dz_j = \lambda\text{-th component of } \nabla\Big(\frac{\partial}{\partial z_\mu}\Big).$$

The associated *connection form* on U is the tensor

$$\Gamma = \sum_{1 \leqslant j, \lambda, \mu \leqslant n} \Gamma^\lambda_{j\mu}\, dz_j \otimes dz_\mu \otimes \frac{\partial}{\partial z_\lambda} \in \mathcal{M}(U, T^\star_X \otimes T^\star_X \otimes T_X).$$

Then, for all local sections $v = \sum_{1 \leqslant \lambda \leqslant n} v_\lambda \frac{\partial}{\partial z_\lambda}$, $w = \sum_{1 \leqslant \lambda \leqslant n} w_\lambda \frac{\partial}{\partial z_\lambda}$ of $\mathcal{M}(U, T_X)$, we get

$$\nabla v = \sum_{1 \leqslant \lambda \leqslant n} \Big(dv_\lambda + \sum_{1 \leqslant \mu \leqslant n} \Gamma^\lambda_\mu v_\mu\Big) \frac{\partial}{\partial z_\lambda} = dv + \Gamma \cdot v,$$

$$\nabla_w v = \sum_{1 \leqslant j, \lambda \leqslant n} \Big(w_j \frac{\partial v_\lambda}{\partial z_j} + \sum_{1 \leqslant \mu \leqslant n} \Gamma^\lambda_{j\mu} w_j v_\mu\Big) \frac{\partial}{\partial z_\lambda} = d_w v + \Gamma \cdot (w, v).$$

The connection ∇ is said to be *symmetric* if it satisfies $\nabla_v w - \nabla_w v = [v, w]$, or equivalently, if the Christoffel symbols $\Gamma^\lambda_{j\mu} = \Gamma^\lambda_{\mu j}$ are symmetric in j, μ.

We now turn ourselves to the important concept of *Wronskian operator*. Let B be the divisor of poles of ∇, that is, the divisor of the least common multiple of all denominators occuring in the meromorphic functions $\Gamma^\lambda_{j\mu}$. If $\beta \in H^0(X, \mathcal{O}(B))$ is the canonical section of divisor B, then the operator $\beta\nabla$ has holomorphic coefficients. Given a holomorphic curve $f : D(0, r) \to X$ whose image does not lie in the support $|B|$ of B, one can define inductively a sequence of covariant derivatives

$$f', \quad f''_\nabla = \nabla_{f'}(f'),\ \ldots\ ,\ f^{(k+1)}_\nabla := \nabla_{f'}(f^{(k)}_\nabla).$$

These derivatives are given in local coordinates by the explicit inductive formula

$$f^{(k+1)}_\nabla(t)_\lambda = \frac{d}{dt}\big(f^{(k)}_\nabla(t)_\lambda\big) + \sum_{1\leqslant\mu\leqslant n} (\Gamma^\lambda_{j\mu}\circ f)\, f'_j\, f^{(k)}_\nabla(t)_\mu. \tag{11.1}$$

Therefore, if $\operatorname{Im} f \not\subset |B|$, one can define the *Wronskian* of f relative to ∇ as

$$W_\nabla(f) = f' \wedge f''_\nabla \wedge \cdots \wedge f^{(n)}_\nabla. \tag{11.2}$$

Clearly, $W_\nabla(f)$ is a meromorphic section of $f^\star(\Lambda^n T_X)$. By induction $\beta(f)^{k-1} f^{(k)}_\nabla$ is holomorphic for all $k \geqslant 1$. We infer that $\beta(f)^{n(n-1)/2} W_\nabla(f)$ is holomorphic and can be seen as a holomorphic section of the line bundle $f^\star(\Lambda^n T_X \otimes \mathcal{O}_X(\frac{1}{2}n(n-1)B)$. From (11.1) and (11.2) we see that $P = \beta^{n(n-1)/2} W_\nabla$ is a global holomorphic polynomial operator $f \mapsto P(f', f'', \ldots, f^{(n)})$ of order n and total degree $n(n+1)/2$, with values in $\Lambda^n T_X \otimes \mathcal{O}_X(\frac{1}{2}n(n-1)B)$. Moreover, if we take a biholomorphic reparametrization φ, we get inductively

$$(f\circ\varphi)^{(k)}_\nabla = (\varphi')^k f^{(k)}_\nabla \circ \varphi + \text{ linear combination of } f^{(j)}_\nabla \circ \varphi,\ \ j < k.$$

Therefore

$$W_\nabla(f\circ\varphi) = (\varphi')^{n(n+1)} W_\nabla(f)$$

and $\beta^{n(n-1)/2} W_\nabla$ can be viewed as a section

$$\beta^{n(n-1)/2} W_\nabla \in H^0(X, E_{n,n(n+1)/2} T^\star_X \otimes L^{-1}), \tag{11.3}$$

where L is the line bundle

$$L = K_X \otimes \mathcal{O}_X\Big(-\frac{1}{2}n(n-1)B\Big).$$

From this, we get the following theorem, which is essentially due to [Siu87] (with a more involved proof based on suitable generalizations of Nevanlinna's second main theorem).

11.4. Theorem (Y.T. Siu)**.** *Let X be a compact complex manifold equipped with a meromorphic connection ∇ of pole divisor B. If $K_X \otimes \mathcal{O}_X(-\frac{1}{2}n(n-1)B)$ is ample,* *then for every non constant entire curve $f : \mathbb{C} \to X$, one has either $f(\mathbb{C}) \subset |B|$ or $W_\nabla(f) \equiv 0$.*

Proof. By Corollary 7.9 applied with $P = \beta^{n(n-1)/2} W_\nabla$, we conclude that $\beta^{n(n-1)/2}(f) W_\nabla(f) \equiv 0$, whence the result. □

11.5. Basic observation. It is not necessary to know all Christoffel coefficients of the meromorphic connection ∇ in order to be able to compute its Wronskian W_∇. In fact, assume that $\widetilde{\nabla}$ is another connection such that there are meromorphic 1-forms α, β with

$$\begin{aligned} \widetilde{\nabla} &= \nabla + \alpha \otimes \mathrm{Id}_{T_X} + (\beta \otimes \mathrm{Id}_{T_X})_{\tau_{12}}, \qquad \text{i.e.,} \\ \widetilde{\nabla}_w v &= \nabla_w v + \alpha(w) v + \beta(v) w, \end{aligned}$$

where τ_{12} means transposition of first and second arguments in the tensors of $T_X^\star \otimes T_X^\star \otimes T_X$. Then $W_\nabla = W_{\widetilde{\nabla}}$. Indeed, the defining formula $f_{\widetilde{\nabla}}^{(k+1)} = \widetilde{\nabla}_{f'}(f_{\widetilde{\nabla}}^{(k)})$ implies that $f_{\widetilde{\nabla}}^{(k+1)} = \nabla_{f'}(f_{\widetilde{\nabla}}^{(k)}) + \alpha(f') f_{\widetilde{\nabla}}^{(k)} + \beta(f_{\widetilde{\nabla}}^{(k)}) f'$, and an easy induction then shows that the $\widetilde{\nabla}$ derivatives can be expressed as linear combinations with meromorphic coefficients

$$f_{\widetilde{\nabla}}^{(k)}(t) = f_{\nabla}^{(k)}(t) + \sum_{1 \leqslant j < k} \gamma_j(t)\, f_{\nabla}^{(j)}(t). \qquad \square$$

The essential consequence of Remark 11.5 is that we need only have a "partial projective connection" ∇ on X, in the following sense.

11.6. Definition. *A (meromorphic) partial projective connection ∇ on X is a section of the quotient sheaf of meromorphic connections modulo addition of meromorphic tensors in $(\Omega_X^1 \otimes \mathrm{Id}_{T_X}) \oplus (\Omega_X^1 \otimes \mathrm{Id}_{T_X})_{\tau_{12}}$. In other words, it can be defined as a collection of meromorphic connections ∇_j relative to an open covering (U_j) of X, satisfying the compatibility conditions*

$$\nabla_k - \nabla_j = \alpha_{jk} \otimes \mathrm{Id}_{T_X} + (\beta_{jk} \otimes \mathrm{Id}_{T_X})_{\tau_{12}}$$

for suitable meromorphic 1-forms α_{jk}, β_{jk} on $U_j \cap U_k$.

If we have similar more restrictive compatibility relations with $\beta_{jk} = 0$, the connection form Γ is just defined modulo $\Omega_X^1 \otimes \mathrm{Id}_{T_X}$ and can thus be seen as a 1-form with values in the Lie algebra $\mathfrak{pgl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C})$ rather than in $\mathfrak{gl}(n, \mathbb{C})$. Such objects are sometimes referred to as "projective connections", although this terminology has been also employed in a completely different meaning. In any event, Proposition 11.4 extends (with a completely identical proof) to the more general case where ∇ is just a partial projective connection. Accordingly, the pole divisor B can be taken to be the pole divisor of the trace free part

$$\Gamma^0 = \Gamma \mod (\Omega_X^1 \otimes \mathrm{Id}_{T_X}) \oplus (\Omega_X^1 \otimes \mathrm{Id}_{T_X})_{\tau_{12}}.$$

Such partial projective connections occur in a natural way when one considers quotient varieties under the action of a Lie group. Indeed, let W be a complex manifold in which a connected complex Lie group G acts freely and properly (on the left, say), and let $X = W/G$ be the quotient complex manifold. We denote by $\pi : W \to X$ the projection. Given a connection $\widetilde{\nabla}$ on W and a local section $\sigma : U \to W$ of π, one gets an induced connection on $T_{X|U}$ by putting

$$\nabla = \pi_\star \circ (\sigma^\star \widetilde{\nabla}), \tag{11.7}$$

where $\sigma^\star\widetilde{\nabla}$ is the induced connection on $\sigma^\star T_W$ and $\pi_\star : T_W \to \pi^\star T_X$ is the projection. Of course, the connection ∇ may depend on the choice of σ, but we nevertheless have the following simple criterion ensuring that it yields an intrinsic partial projective connection.

11.8. Lemma. *Let $\widetilde{\nabla} = d + \widetilde{\Gamma}$ be a meromorphic connection on W. Assume that $\widetilde{\nabla}$ satisfies the following conditions:*

i) *$\widetilde{\nabla}$ is G-invariant;*

ii) *there are meromorphic 1-forms α, $\beta \in \mathcal{M}(W, T_{W/X})$ along the relative tangent bundle of $X \to W$, such that for all G-invariant holomorphic vector fields v, τ on W (possibly only defined locally over X) such that τ is tangent to the G-orbits, the vector fields*

$$\widetilde{\nabla}_\tau v - \alpha(\tau)v, \qquad \widetilde{\nabla}_v \tau - \beta(\tau)v$$

are again tangent to the G-orbits (α and β are thus necessarily G-invariant, and $\alpha = \beta$ if $\widetilde{\nabla}$ is symmetric).

Then Formula (11.7) yields a partial projective connection ∇ which is globally defined on X and independent of the choice of the local sections σ.

Proof. Since the expected conclusions are local with respect to X, it is enough to treat the case when $W = X \times G$ and G acts on the left on the second factor. Then $W/G \simeq X$ and $\pi : W \to X$ is the first projection. If d_G is the canonical left-invariant connection on G, we can write $\widetilde{\nabla}$ as

$$\widetilde{\nabla} = d_X + d_G + \widetilde{\Gamma}, \qquad \widetilde{\Gamma} = \widetilde{\Gamma}(x, g), \quad x \in X,\ g \in G,$$

where d_X is some connection on X, e.g. the "coordinate derivative" taken with respect to given local coordinates $(z_1, \ldots, z_n)$ on X. Then $\widetilde{\nabla}$ is left invariant on $W = X \times G$ if and only if $\widetilde{\Gamma}(x, g) = \Gamma(x)$ is independent of $g \in G$ (this is meaningful since the tangent bundle to G is trivial), and condition ii) means that

$$\Gamma(x) \cdot (\tau, v) - \alpha(\tau)v \quad \text{and} \quad \Gamma(x) \cdot (v, \tau) - \beta(\tau)v$$

are tangent to the G-orbits. A local section $\sigma : U \to W$ of π can be written $\sigma(x) = (x, h(x))$ for some holomorphic function $h : U \to G$. Formula (11.7) says more explicitly that

$$\nabla_w v = \pi_\star\big((\sigma^\star\widetilde{\nabla})_w v\big) = \pi_\star\big(d_{\sigma_\star w}\sigma_\star v + (\widetilde{\Gamma} \circ \sigma) \cdot (\sigma_\star w, \sigma_\star v)\big).$$

Let $v = \sum v_j(z)\partial/\partial z_j$, $w = \sum w_j(z)\partial/\partial z_j$ be local vector fields on $U \subset X$. Since $\sigma_\star v = v + dh(v)$, we get

$$\begin{aligned}(\sigma^\star\widetilde{\nabla})_w v &= d_{w+dh(w)}(v + dh(v)) + \widetilde{\Gamma}(x, h(x)) \cdot \big(w + dh(w), v + dh(v)\big) \\ &= d_w v + d^2 h(w, v) + \Gamma(x) \cdot (w + dh(w), v + dh(v)).\end{aligned}$$

As v, w, $dh(v)$, $dh(w)$ depend only on X, they can be seen as G-invariant vector fields over W, and $dh(v)$, $dh(w)$ are tangent to the G-orbits. Hence

$$\Gamma(x)\cdot(dh(w), v) - \alpha(dh(w))v, \quad \Gamma(x)\cdot(w, dh(v)) - \beta(dh(v))w, \quad \Gamma(x)\cdot(dh(w), dh(v))$$

are tangent to the G-orbits, i.e., in the kernel of $\pi_\star$. We thus obtain

$$\nabla_w v = \pi_\star\big((\sigma^\star\widetilde{\nabla})_w v\big) = d_w v + \Gamma(x) \cdot (w, v) + \alpha(dh(w))v + \beta(dh(v))w.$$

From this it follows by definition that the local connections $\nabla_{\restriction U_j}$ defined by various sections $\sigma_j : U_j \to W$ can be glued together to define a global partial projective connection ∇ on X. □

11.9. Remark. Lemma 11.8 is also valid when $\widetilde{\nabla}$ is a partial projective connection. Hypothesis 11.8 ii) must then hold with local meromorphic 1-forms α_j, $\beta_j \in \mathcal{M}(\widetilde{U}_j, T_{W/X})$ relatively to some open covering $\widetilde{U}_j$ of W. □

In the special case $\mathbb{P}^n = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^\star$, we get

11.10. Corollary. *Let $\widetilde{\nabla} = d + \widetilde{\Gamma}$ be a meromorphic connection on $\mathbb{C}^{n+1}$. Let $\varepsilon = \sum z_j\partial/\partial z_j$ be the Euler vector field on $\mathbb{C}^{n+1}$ and $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ be the canonical projection. Then $\widetilde{\nabla}$ induces a meromorphic partial projective connection on $\mathbb{P}^n$ provided that*

i) *the Christoffel symbols $\Gamma^\lambda_{j\mu}$ are homogeneous rational functions of degree -1 (homothety invariance of the connection $\widetilde{\nabla}$) ;*

ii) *there are meromorphic functions α, β and meromorphic 1-forms γ, η such that*

$$\widetilde{\Gamma} \cdot (\varepsilon, v) = \alpha v + \gamma(v)\varepsilon, \qquad \widetilde{\Gamma} \cdot (w, \varepsilon) = \beta w + \eta(w)\varepsilon$$

for all vector fields v, w. □

Now, our goal is to study certain hypersurfaces Y of sufficiently high degree in $\mathbb{P}^n$. Assume for the moment that Y is an hypersurface in some n-dimensional manifold X, and that Y is defined locally by a holomorphic equation $s = 0$. We say that Y is *totally geodesic* with respect to a meromorphic connection ∇ on X if Y is not contained in the pole divisor $|B|$ of ∇, and for all pairs (v, w) of (local) vector fields tangent to Y the covariant derivative $\nabla_w v$ is again tangent to Y. (Notice that this concept also makes sense when ∇ is a partial projective connection.) If Y is totally geodesic, the ambient connection ∇ on T_X induces by restriction a connection $\nabla_{\restriction Y}$ on T_Y.

We now want to derive explicitly a condition for the hypersurface $Y = \{s = 0\}$ to be totally geodesic in (X, ∇). A vector field v is tangent to Y if and only if $ds \cdot v = 0$ along $s = 0$. By taking the differential of this identity along another vector field w tangent to Y, we find

$$d^2 s \cdot (w, v) + ds \cdot (d_w v) = 0 \tag{11.11}$$

along $s = 0$ (this is meaningful only with respect to some local coordinates). On the other hand, the condition that $\nabla_w v = d_w v + \Gamma \cdot (w, v)$ is tangent to Y is

$$ds \cdot \nabla_w v = ds \cdot (d_w v) + ds \circ \Gamma \cdot (w, v) = 0.$$

By subtracting the above from (11.11), we get the following equivalent condition: $(d^2 s - ds \circ \Gamma) \cdot (w, v) = 0$ for all vector fields v, w in the kernel of ds along $s = 0$. Therefore we obtain the

11.12. Characterization of totally geodesic hypersurfaces. *The hypersurface $Y = \{s = 0\}$ is totally geodesic with respect to ∇ if and only if there are holomorphic 1-forms $a = \sum a_j dz_j$, $b = \sum b_j dz_j$ and a 2-form $c = \sum c_{j\mu} dz_j \otimes dz_\mu$ such that*

$$\nabla^\star(ds) = d^2 s - ds \circ \Gamma = a \otimes ds + ds \otimes b + s\, c$$

in a neighborhood of every point of Y (here $\nabla^\star$ is the induced connection on $T^\star_Y$).

From this, we derive the following useful lemma.

11.13. Lemma. *Let $Y \subset X$ be an analytic hypersurface which is totally geodesic with respect to a meromorphic connection ∇, and let $n = \dim X = \dim Y + 1$. Let $f : D(0, R) \to X$ be a holomorphic curve such that $W_\nabla(f) \equiv 0$. Assume that there is a point $t_0 \in D(0, R)$ such that*

i) *$f(t_0)$ is not contained in the poles of ∇;*

ii) *the system of vectors $(f'(t), f''_\nabla(t), \ldots, f^{(n-1)}_\nabla(t))$ achieves its generic rank (i.e. its maximal rank) at $t = t_0$;*

iii) *$f(t_0) \in Y$ and $f'(t_0)$, $f''_\nabla(t_0), \ldots, f^{(n-1)}_\nabla(t_0) \in T_{Y, f(t_0)}$.*

Then $f(D(0, R)) \subset Y$.

Proof. Since $W_\nabla(f) \equiv 0$, the vector fields f', $f''_\nabla, \ldots, f^{(n)}_\nabla$ are linearly dependent and satisfy a non trivial relation

$$u_1(t) f'(t) + u_2(t) f''_\nabla(t) + \cdots + u_n(t) f^{(n)}_\nabla(t) = 0$$

with suitable meromorphic coefficients $u_j(t)$ on $D(0, R)$. If u_n happens to be $\equiv 0$, we take ∇-derivatives in the above relation so as to reach another relation with $u_n \not\equiv 0$. Hence we can always write

$$f^{(n)}_\nabla = v_1 f' + v_2 f''_\nabla + \cdots + v_{n-1} f^{(n-1)}_\nabla$$

for some meromorphic functions $v_1, \ldots, v_{n-1}$. We can even prescribe the v_j to be 0 eXcept for indices $j = j_k \in \{1, \ldots, n-1\}$ such that $(f_\nabla^{(j_k)}(t))$ is a minimal set of generators at $t = t_0$. Then the coefficients v_j are uniquely defined and are holomorphic near t_0. By taking further derivatives, we conclude that $f_\nabla^{(k)}(t_0) \in T_{X,f(t_0)}$ for all k. We now use the assumption that X is totally geodesic to prove the following claim: if $s = 0$ is a local equation of Y, the k-th derivative $\frac{d^k}{dt^k}(s \circ f(t))$ can be expressed as a holomorphic linear combination

$$\frac{d^k}{dt^k}\big(s \circ f(t)\big) = \gamma_{0k}(t)\, s \circ f(t) + \sum_{1 \leqslant j \leqslant k} \gamma_{jk}(t)\, ds_{f(t)} \cdot f_\nabla^{(j)}(t)$$

on a neighborhood of t_0. This will imply $\frac{d^k}{dt^k}(s \circ f)(t_0) = 0$ for all $k \geqslant 0$, hence $s \circ f \equiv 0$. Now, the above claim is clearly true for $k = 0, 1$. By taking the derivative and arguing inductively, we need only show that

$$\frac{d}{dt}\big(ds_{f(t)} \cdot f_\nabla^{(j)}(t)\big)$$

is again a linear combination of the same type. However, Leibnitz's rule for covariant differentiations together with 11.12 yield

$$\begin{aligned}
\frac{d}{dt}\big(ds_{f(t)} \cdot f_\nabla^{(j)}(t)\big) &= ds_{f(t)} \cdot \Big(\frac{\nabla}{dt} f_\nabla^{(j)}(t)\Big) + \nabla^\star(ds)_{f(t)} \cdot \big(f'(t), f_\nabla^{(j)}(t)\big) \\
&= ds \cdot f_\nabla^{(j+1)}(t) + (a \cdot f'(t))\big(ds \cdot f_\nabla^{(j)}(t)\big) \\
&\qquad + (ds \cdot f'(t))\big(b \cdot f_\nabla^{(j)}(t)\big) + (s \circ f(t))\big(c \cdot (f'(t), f_\nabla^{(j)}(t))\big),
\end{aligned}$$

as desired. □

If $Y = \{s = 0\} \subset X$ is given and a connection ∇ on X is to be found so that Y is totally geodesic, condition 11.12 amounts to solving a highly underdetermined linear system of equations

$$\frac{\partial^2 s}{\partial z_j \partial z_\mu} - \sum_{1 \leqslant \lambda \leqslant n} \Gamma_{j\mu}^\lambda \frac{\partial s}{\partial z_\lambda} = a_j \frac{\partial s}{\partial z_\mu} + b_\mu \frac{\partial s}{\partial z_j} + s\, c_{j\mu}, \qquad 1 \leqslant j, \mu \leqslant n,$$

in terms of the unknowns $\Gamma_{j\mu}^\lambda$, a_j, b_μ and $c_{j\mu}$. Nadel's idea is to take advantage of this indeterminacy to achieve that all members in a large linear system (Y_α) of hypersurfaces are totally geodesic with respect to ∇. The following definition is convenient.

11.14. Definition. *For any $(n+2)$-tuple of integers $(p, k_0, k_1 \ldots, k_n)$ with $0 < k_j < p/2$, let $\mathcal{S}_{p;\, k_0,\ldots,k_n}$ be the space of homogeneous polynomials $s \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ of degree p such that every monomial of s is a product of a power $z_j^{p-k_j}$ of one of the variables with a lower degree monomial of degree k_j. Any polynomial $s \in \mathcal{S}_{p;\, k_0,\ldots,k_n}$ admits a unique decomposition*

$$s = s_0 + s_1 + \cdots + s_n, \qquad s_j \in \mathcal{S}_{p;\, k_0,\ldots,k_n}$$

where s_j is divisible by $z_j^{p-k_j}$.

Given a homogeneous polynomial $s = s_0 + s_1 + \cdots + s_n \in \mathcal{S}_{p;\,k_0,\ldots,k_n}$, we consider the linear system

$$Y_\alpha = \big\{\alpha_0 s_0 + \alpha_1 s_1 + \cdots + \alpha_n s_n = 0\big\}, \quad \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^n. \tag{11.15}$$

Our goal is to study smooth varieties Z which arise as complete intersections $Z = Y_{\alpha^1} \cap \cdots \cap Y_{\alpha^q}$ of members in the linear system (the α^j being linearly independent elements in $\mathbb{C}^{n+1}$). For this, we want to construct a (partial projective) meromorphic connection ∇ on $\mathbb{P}^n$ such that all Y_α are totally geodesic. Corollary 11.10 shows that it is enough to construct a meromorphic connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ on $\mathbb{C}^{n+1}$ satisfying 11.10 i) and ii), such that the conic affine varieties $\widetilde{Y}_\alpha \subset \mathbb{C}^{n+1}$ lying over the Y_α are totally geodesic with respect to $\widetilde{\nabla}$. Now, Characterization 11.12 yields a sufficient condition in terms of the linear system of equations

$$\sum_{0 \leqslant \lambda \leqslant n} \widetilde{\Gamma}^\lambda_{j\mu} \frac{\partial s_\kappa}{\partial z_\lambda} = \frac{\partial^2 s_\kappa}{\partial z_j \partial z_\mu}, \qquad 0 \leqslant j, \kappa, \mu \leqslant n. \tag{11.16}$$

(We just fix the choice of a_j, b_μ and $c_{j\mu}$ to be 0). This linear system can be considered as a collection of decoupled linear systems in the unknowns $(\widetilde{\Gamma}^\lambda_{j\mu})_\lambda$, when j and μ are fixed. Each of these has format $(n+1) \times (n+1)$ and can be solved by Cramer's rule if the principal determinant

$$\delta := \det\Big(\frac{\partial s_\kappa}{\partial z_\lambda}\Big)_{0 \leqslant \kappa, \lambda \leqslant n} \not\equiv 0 \tag{11.17}$$

is not identically zero. We always assume in the sequel that this non degeneracy assumption is satisfied. As $\partial s_\kappa / \partial z_\lambda$ is homogeneous of degree $p-1$ and $\partial^2 s_\kappa / \partial z_j \partial z_\mu$ is homogeneous of degree $p-2$, the solutions $\widetilde{\Gamma}^\lambda_{j\mu}(z)$ are homogeneous rational functions of degree -1 (condition 11.10 i)). Moreover, $\widetilde{\nabla}$ is symmetric, for $\partial^2 s / \partial z_j \partial z_\mu$ is symmetric in j, μ. Finally, if we multiply (11.16) by z_j and take the sum, Euler's identity yields

$$\sum_{0 \leqslant j, \lambda \leqslant n} z_j \widetilde{\Gamma}^\lambda_{j\mu} \frac{\partial s_\kappa}{\partial z_\lambda} = \sum_{0 \leqslant j \leqslant n} z_j \frac{\partial^2 s_\kappa}{\partial z_j \partial z_\mu} = (p-1) \frac{\partial s_\kappa}{\partial z_\mu}, \qquad 0 \leqslant \kappa, \mu \leqslant n.$$

The non degeneracy assumption implies $(\sum_j z_j \widetilde{\Gamma}^\lambda_{j\mu})_{\lambda\mu} = (p-1)\,\mathrm{Id}$, hence

$$\widetilde{\Gamma}(\varepsilon, v) = \widetilde{\Gamma}(v, \varepsilon) = (p-1)v$$

and condition 11.10 ii) is satisfied. From this we infer

11.18. Proposition. *Let $s = s_0 + \cdots + s_n \in \mathcal{S}_{p;\,k_0,\ldots,k_n}$ be satisfying the non degeneracy condition $\delta := \det(\partial s_\kappa / \partial z_\lambda)_{0 \leqslant \kappa, \lambda \leqslant n} \not\equiv 0$. Then the solution $\widetilde{\Gamma}$ of the linear system* (11.16) *provides a partial projective meromorphic connection on $\mathbb{P}^n$ such that all hypersurfaces*

$$Y_\alpha = \{\alpha_0 s_0 + \cdots + \alpha_n s_n = 0\}$$

are totally geodesic. Moreover, the divisor of poles B of ∇ has degree at most equal to $n+1+\sum k_j$.

Proof. Only the final degree estimate on poles has to be checked. By Cramer's rule, the solutions are expressed in terms of ratios

$$\widetilde{\Gamma}^\lambda_{j\mu} = \frac{\delta^\lambda_{j\mu}}{\delta},$$

where $\delta^\lambda_{j\mu}$ is the determinant obtained by replacing the column of index λ in $\det(\partial s_\kappa/\partial z_\lambda)_{0\leqslant\kappa,\lambda\leqslant n}$ by the column $(\partial^2 s_\kappa/\partial z_j\partial z_\mu)_{0\leqslant\kappa\leqslant n}$. Now, $\partial s_\kappa/\partial z_\lambda$ is a homogeneous polynomial of degree $p-1$ which is divisible by $z_\kappa^{p-k_\kappa-1}$, hence δ is a homogeneous polynomial of degree $(n+2)(p-1)$ which is divisible by $\prod z_j^{p-k_j-1}$. Similarly, $\partial^2 s_\kappa/\partial z_j\partial z_\mu$ has degree $p-2$ and is divisible by $z_\kappa^{p-k_\kappa-2}$. This implies that $\delta^\lambda_{j\mu}$ is divisible by $\prod z_j^{p-k_j-2}$. After removing this common factor in the numerator and denominator, we are left with a denominator of degree

$$\sum_{0\leqslant j\leqslant n}\big((p-1)-(p-k_j-2)\big) = \sum(k_j+1) = n+1+\sum k_j,$$

as stated. □

An application of Theorem 11.4 then yields the following theorem on certain complete intersections in projective spaces.

11.19. Theorem. *Let $s\in\mathcal{S}_{p;\,k_0,\ldots,k_{n+q}}\subset\mathbb{C}[z_0,z_1,\ldots,z_{n+q}]$ be a homogeneous polynomial satisfying the non degeneracy assumption $\det(\partial s_\kappa/\partial z_\lambda)\not\equiv 0$ in $\mathbb{C}^{n+q+1}$. Let*

$$Y_\alpha = \big\{\alpha_0 s_0+\alpha_1 s_1+\cdots+\alpha_{n+q}s_{n+q}=0\big\}\subset\mathbb{P}^{n+q}$$

be the corresponding linear system, and let

$$Z = Y_{\alpha^1}\cap\cdots\cap Y_{\alpha^q}\subset\mathbb{P}^{n+q}$$

be a smooth n-dimensional complete intersection, for some linearly independent elements $\alpha^j\in\mathbb{C}^{n+q+1}$ such that $ds_{\alpha^1}\wedge\cdots\wedge ds_{\alpha^q}$ does not vanish along Z. Assume that Z is not contained in the set of poles $|B|$ of the meromorphic connection ∇ defined by (11.16)*, nor in any of the coordinate hyperplanes $z_j=0$, and that*

$$pq > n+q+1+\frac12 n(n-1)\Big(n+q+1+\sum k_j\Big).$$

Then every nonconstant entire curve $f:\mathbb{C}\to Z$ is algebraically degenerate and satisfies either

i) $f(\mathbb{C})\subset Z\cap|B|$ *or*

ii) $f(\mathbb{C})\subset Z\cap Y_\alpha$ *for some member Y_α which does not contain Z.*

Proof. By Proposition 11.18, the pole divisor of ∇ has degree at most equal to $n+q+1+\sum k_j$, hence, if we let $B = \mathcal{O}(n+q+1+\sum k_j)$, we can find a section $\beta \in H^0(\mathbb{P}^{n+q}, B)$ such that the operator $f \mapsto \beta^{n(n+1)/2}(f)\, W_{Z,\nabla}(f)$ is holomorphic. Moreover, as Z is smooth, the adjunction formula yields

$$K_Z = \big(K_{\mathbb{P}^{n+q}} \otimes \mathcal{O}(pq)\big)_{\restriction Z} = \mathcal{O}_Z(pq-n-q-1).$$

By (11.3), the differential operator $\beta^{n(n-1)/2}(f)\, W_{Z,\nabla}(f)$ defines a section in $H^0(Z, E_{n,n(n+1)/2} T_Z^\star \otimes L^{-1})$ with

$$\begin{aligned} L &= K_Z \otimes \mathcal{O}_Z\Big(-\frac{1}{2}n(n-1)B\Big) \\ &= \mathcal{O}_Z\Big(pq-n-q-1-\frac{1}{2}n(n-1)\Big(n+q+1+\sum k_j\Big)\Big). \end{aligned}$$

Hence, if $f(\mathbb{C}) \not\subset |B|$, we know by Theorem 11.4 that $W_{Z,\nabla}(f) \equiv 0$. Fix a point $t_0 \in \mathbb{C}$ such that $f(t_0) \notin |B|$ and $(f'(t_0), f''_\nabla(t_0), \ldots, f^{(n)}_\nabla(t_0))$ is of maximal rank $r < n$. There must exist an hypersurface $Y_\alpha \not\supset Z$ such that

$$f(t_0) \in Y_\alpha, \quad f'(t_0),\ f''_\nabla(t_0), \ldots,\ f^{(n)}_\nabla(t_0) \in T_{Y_\alpha, f(t_0)}.$$

In fact, these conditions amount to solve a linear system of equations

$$\sum_{0 \leqslant j \leqslant n+q} \alpha_j s_j(f(t_0)) = 0, \qquad \sum_{0 \leqslant j \leqslant n+q} \alpha_j ds_j(f^{(j)}_\nabla(t_0)) = 0$$

in the unknowns $(\alpha_0, \alpha_1, \ldots, \alpha_{n+q}) = \alpha$, which has rank $\leqslant r+1 \leqslant n$. Hence the solutions form a vector space Sol of dimension at least $q+1$, and we can find a solution α which is linearly independent from $\alpha^1, \ldots, \alpha^q$. We complete $(\alpha, \alpha^1, \ldots, \alpha^q)$ into a basis of $\mathbb{C}^{n+q+1}$ and use the fact that the determinant $\delta = \det(\partial s_\kappa / \partial s_\lambda)$ does not vanish identically on Z, since

$$Z \cap \{\delta = 0\} \subset Z \cap \big(|B| \cup \{\textstyle\prod z_j = 0\}\big) \subsetneq Z.$$

From this we see that $\sum \alpha_j ds_j$ does not vanish identically on Z, in particular $Z \not\subset Y_\alpha$. By taking a generic element $\alpha \in$ Sol, we get a smooth n-dimensional hypersurface $Z_\alpha = Y_\alpha \cap Y_{\alpha^2} \cap \cdots \cap Y_{\alpha^q}$ in $W = Y_{\alpha^2} \cap \cdots \cap Y_{\alpha^q}$. Lemma 11.13 applied to the pair (Z_α, W) shows that $f(\mathbb{C}) \subset Z_\alpha$, hence $f(\mathbb{C}) \subset Z \cap Z_\alpha = Z \cap Y_\alpha$, as desired. □

If we want to decide whether Z is hyperbolic, we are thus reduced to decide whether the hypersurfaces $Z \cap |B|$ and $Z \cap Y_\alpha$ are hyperbolic. This may be a very hard problem, especially if $Z \cap |B|$ and $Z \cap Y_\alpha$ are singular. (In the case of a smooth intersection $Z \cap Y_\alpha$, we can of course apply the theorem again to $Z' = Z \cap Y_\alpha$ and try to argue by induction). However, when Z is a surface, $Z \cap |B|$ and $Z \cap Y_\alpha$ are curves and the problem can in principle be solved directly through explicit genus calculations.

11.20. Examples.

i) Consider the Fermat hypersurface of degree p

$$Z = \{z_0^p + z_1^p + \cdots + z_{n+1}^p = 0\}$$

in $\mathbb{P}^{n+1}$, which is defined by an element in $\mathcal{S}_{p;\,0,\ldots,0}$. A simple calculation shows that $\delta = p^{n+2}\prod z_j^{p-1} \not\equiv 0$ and that the Christoffel symbols are given by $\widetilde{\Gamma}_{jj}^j = (p-1)/z_j$ (with all other coefficients being equal to 0). Theorem 11.19 shows that all nonconstant entire curves $f : \mathbb{C} \to Y$ are algebraically degenerate when

$$p > n + 2 + \frac{1}{2}n(n-1)(n+2).$$

In fact the term $\frac{1}{2}n(n-1)(n+2)$ coming from the pole order estimate of the Wronskian is by far too pessimistic. A more precise calculation shows in that case that $(z_0 \cdots z_{n+1})^{n-1}$ can be taken as a denominator for the Wronskian. Hence the algebraic degeneracy occurs for $p > n + 2 + (n+2)(n-1)$, i.e., $p \geqslant (n+1)^2$. However, the Fermat hypersurfaces are not hyperbolic. For instance, when $n = 2$, they contain rational lines $z_1 = \omega z_0$, $z_3 = \omega' z_2$ associated with any pair (ω, ω') of p-th roots of -1.

ii) Following J. El Goul ([EG96, 97]), let us consider surfaces $Z \subset \mathbb{P}^3$ of the form

$$Z = \{z_0^p + z_1^p + z_2^p + z_3^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + z_3^2) = 0\},$$

defined by the element in $\mathcal{S}_{p;\,0,0,0,2}$ such that $s_3 = z_3^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + z_3^2)$ and $s_j = z_j^p$ for $0 \leqslant j \leqslant 2$. One can check that Z is smooth provided that

$$\sum_{j\in J} \varepsilon_j^{\frac{p}{p-2}} \neq \frac{2}{p-2}\Big(-\frac{p}{2}\Big)^{\frac{p}{p-2}}, \qquad \forall J \subset \{0,1,2\}, \tag{11.21}$$

for any choice of complex roots of order $p-2$. The connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ is computed by solving linear systems with principal determinant $\delta = \det(\partial s_\kappa/\partial z_\lambda)$ equal to

$$\begin{vmatrix} pz_0^{p-1} & 0 & 0 & 0 \\ 0 & pz_1^{p-1} & 0 & 0 \\ 0 & 0 & pz_2^{p-1} & 0 \\ 2\varepsilon_0 z_0 z_3^{p-2} & 2\varepsilon_1 z_1 z_3^{p-2} & 2\varepsilon_2 z_2 z_3^{p-2} & (p-2)z_3^{p-3}\big(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \frac{p}{p-2}z_3^2\big) \end{vmatrix}$$
$$= p^3(p-2)z_0^{p-1}z_1^{p-1}z_2^{p-1}z_3^{p-3}\Big(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \frac{p}{p-2}z_3^2\Big) \not\equiv 0.$$

The numerator of $\widetilde{\Gamma}_{j\mu}^\lambda$ is obtained by replacing the column of index λ of δ by $(\partial^2 s_\kappa/\partial z_j \partial z_\mu)_{0\leqslant\kappa\leqslant 3}$, and $z_0^{p-2}z_1^{p-2}z_2^{p-2}z_3^{p-4}$ cancels in all terms. Hence the pole order of $\widetilde{\nabla}$ and of $W_{\widetilde{\nabla}}$ is 6 (as given by Proposition 11.18), with

$$z_0 z_1 z_2 z_3\Big(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \frac{p}{p-2}z_3^2\Big)$$

as the denominator, and its zero divisor as the divisor B. The condition on p we get is $p > n+2+6 = 10$. An explicit calculation shows that all curves $Z \cap |B|$ and $Z \cap Y_\alpha$ have geometric genus $\geqslant 2$ under the additional hypothesis

$$\text{(11.22)}\qquad \begin{cases} \text{none of the pairs } (\varepsilon_i, \varepsilon_j) \text{ is equal to } (0,0), \\ \varepsilon_i/\varepsilon_j \neq -\theta^2 \text{ whenever } \theta \text{ is a root of } \theta^p = -1. \end{cases}$$

[(11.22) excludes the existence of lines in the intersections $Z \cap Y_\alpha$.]

11.23. Corollary. *Under conditions* (11.21) *and* (11.22), *the algebraic surface*

$$Z = \big\{z_0^p + z_1^p + z_2^p + z_3^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + z_3^2) = 0\big\} \subset \mathbb{P}^3$$

is smooth and hyperbolic for all $p \geqslant 11$. □

Another question which has raised considerable interest is to decide when the complement $\mathbb{P}^2 \smallsetminus C$ of a plane curve C is hyperbolic. If $C = \{\sigma = 0\}$ is defined by a polynomial $\sigma(z_0, z_1, z_2)$ of degree p, we can consider the surface X in $\mathbb{P}^3$ defined by $z_3^p = \sigma(z_0, z_1, z_2)$. The projection

$$\rho : X \to \mathbb{P}^2, \qquad (z_0, z_1, z_2, z_3) \mapsto (z_0, z_1, z_2)$$

is a finite $p:1$ morphism, ramified along C. It follows that $\mathbb{P}^2 \smallsetminus C$ is hyperbolic if and only if its unramified covering $X \smallsetminus \rho^{-1}(C)$ is hyperbolic; hence a sufficient condition is that X itself is hyperbolic. If we take $\varepsilon_2 = 0$ in Cor. 11.23 and exchange the roles of z_2, z_3, we get the following

11.24. Corollary. *Consider the plane curve*

$$C = \big\{z_0^p + z_1^p + z_2^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + z_2^2) = 0\big\} \subset \mathbb{P}^2, \qquad \varepsilon_0,\, \varepsilon_1 \in \mathbb{C}^\star.$$

Assume that neither of the numbers ε_0, ε_1, $\varepsilon_0 + \varepsilon_1$ is equal to $\frac{2}{p-2}\big(-\frac{p}{2}\big)^{\frac{p}{p-2}}$ and that $\varepsilon_1/\varepsilon_0 \neq -\theta^2$ whenever $\theta^p = -1$. Then $\mathbb{P}^2 \smallsetminus C$ is hyperbolic. □

§12. Decomposition of jets in irreducible representations

Let us first briefly recall the definition of the Schur fonctors $\Gamma^\bullet$ (they are frequently denoted $S_\bullet$ in the literature, but we want to avoid any confusion with ordinary symmetric powers). Let V be a complex vector space of dimension r. To the set of nonincreasing r-tuples $(a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$, one associates in a fonctorial way a collection of vector spaces $\Gamma^{(a_1, a_2, \ldots, a_r)}V$ which provide the list of all irreducible representations of the linear group $\mathrm{GL}(V)$, up to isomorphism (here, $(a_1, \ldots, a_r)$ is the highest weight of the action of a maximal torus $(\mathbb{C}^\star)^r \subset \mathrm{GL}(V)$). The Schur fonctors can be defined in an elementary way as follows. Let $\mathbb{U}_r = \left\{ \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} \right\}$ be the group of lower triangular unipotent $r \times r$ matrices. If all a_j are nonnegative, one defines

$$\Gamma^{(a_1, a_2, \ldots, a_r)}V \subset S^{a_1}V \otimes \cdots \otimes S^{a_r}V$$

to be the set of polynomials $P(\xi_1,\ldots,\xi_r)$ on $(V^\star)^r$ which are homogeneous of degree a_j with respect to ξ_j and which are invariant under the left action of $\mathbb{U}_r$ on $(V^\star)^r=\mathrm{Hom}(V,\mathbb{C}^r)$, namely such that

$$P(\xi_1,\ldots,\xi_{j-1},\xi_j+\xi_k,\xi_{j+1},\ldots,\xi_r)=P(\xi_1,\ldots,\xi_r)\qquad\forall k<j.$$

We agree that $\Gamma^{(a_1,a_2,\ldots,a_r)}V=0$ unless $(a_1,a_2,\ldots,a_r)$ is nonincreasing. As a special case, we recover symmetric and exterior powers

$$\begin{aligned}S^kV&=\Gamma^{(k,0,\ldots,0)}V,\\ \Lambda^kV&=\Gamma^{(1,\ldots,1,0,\ldots,0)}V,\qquad\text{(with k indices 1)}\\ \det V&=\Gamma^{(1,\ldots,1)}V.\end{aligned}\tag{12.1}$$

The Schur fonctors satisfy the well-known formula

$$\Gamma^{(a_1+\ell,\ldots,a_r+\ell)}V=\Gamma^{(a_1,\ldots,a_r)}V\otimes(\det V)^\ell.\tag{12.2}$$

This formula can of course be used to define $\Gamma^{(a_1,\ldots,a_r)}V$ if any of the a_j's happens to be negative.

Now, by what we saw in section §6, the group $\mathbb{G}'_k$ of germs of reparametrizations $\varphi(t)=t+b_2t^2+\cdots+b_kt^k+O(t^{k+1})$ tangent to identity acts on k-tuples $(f',f'',\ldots,f^{(k)})$ of derivatives of f at 0 by the formulas

$$(f\circ\varphi)'=f',\quad(f\circ\varphi)''=f''+2b_2f',\quad(f\circ\varphi)'''=f'''+3b_2f''+3b_3f',\ \ldots.$$

This is clearly a unipotent action, induced by the action of $\mathbb{U}_k$ through an embedding

$$\mathbb{G}'_k\hookrightarrow\mathbb{U}_k,\qquad\varphi\longmapsto\begin{pmatrix}1&0&0&\cdots&0&0\\ 2b_2&1&0&\cdots&0&0\\ 3b_3&3b_2&1&\cdots&0&0\\ \vdots&&&&\vdots&\vdots\\ &&&\cdots&1&0\\ kb_k&&&\cdots&kb_2&1\end{pmatrix}.$$

By formula (6.5), we find that the graded bundle of $E_{k,m}V^\star$ is

$$\mathrm{Gr}^\bullet E_{k,m}V^\star=\Bigg(\bigoplus_{\ell\in\mathbb{N}^k,\,\ell_1+2\ell_2+\cdots+k\ell_k=m}S^{\ell_1}V^\star\otimes S^{\ell_2}V^\star\otimes\cdots\otimes S^{\ell_k}V^\star\Bigg)^{\mathbb{G}'_k}.$$

Since the action of $\mathbb{G}'_k$ does not preserve each individual component in the summation, the computation of the invariants is quite difficult in general. We will see however that everything is easy if $k\leqslant 2$. In fact, if $k=1$, then

$$E_{1,m}V^\star=E^{\mathrm{GG}}_{1,m}V^\star=S^mV^\star.\tag{12.3}$$

If $k = 2$, the effect of a parameter change $(f', f'') \mapsto (f', f'' + \lambda f')$ on a weighted homogeneous polynomial $Q(f', f'') = \sum_{|\alpha_1|+2|\alpha_2|=m} a_{\alpha_1\alpha_2}(f')^{\alpha_1}(f'')^{\alpha_2}$ is to replace each monomial $(f')^{\alpha_1}(f'')^{\alpha_2}$ by a sum

$$\sum_\beta C_\beta \lambda^{|\beta|}(f')^{\alpha_1+\beta}(f'')^{\alpha_2-\beta}.$$

It follows that terms $(f')^{\alpha_1}(f'')^{\alpha_2}$ corresponding to different values of the pair $(|\alpha_1|, |\alpha_2|) =: (\ell_1, \ell_2)$ cannot produce monomials with the same multidegree and the same exponent $|\beta|$ of λ. Hence the various components $S^{\ell_1}V^\star \otimes S^{\ell_2}V^\star$ do not mix up and we get

$$(12.4)\quad \mathrm{Gr}^\bullet E_{2,m}V^\star = \bigoplus_{\ell_1+2\ell_2=m} \big(S^{\ell_1}V^\star \otimes S^{\ell_2}V^\star\big)^{\mathbb{G}'_k} = \bigoplus_{\ell_1+2\ell_2=m} \Gamma^{(\ell_1,\ell_2,0,\ldots,0)}V^\star.$$

In the special case when $r = \operatorname{rank} V = 2$, (12.1) and (12.2) yield $\Gamma^{(\ell_1,\ell_2)}V^\star = S^{\ell_1-\ell_2}V^\star \otimes (\det V^\star)^{\ell_2}$. Hence we get the simpler formula

$$(12.5)\qquad \mathrm{Gr}^\bullet E_{2,m}V^\star = \bigoplus_{0\leqslant j\leqslant m/3} S^{m-3j}V^\star \otimes (\det V^\star)^j \qquad (k = r = 2).$$

Similar calculations can be done for low values of k and m, but it is a major unsolved problem to compute the decomposition formula of $\mathrm{Gr}^\bullet E_{k,m}V^\star$ for arbitrary k and m.

12.6. Special case. Assume that X is a surface and consider the absolute case $V = T_X$. We find

$$\mathrm{Gr}^\bullet E_{2,m}T_X^\star = \bigoplus_{0\leqslant j\leqslant m/3} S^{m-3j}T_X^\star \otimes K_X^j,$$

where $E_{1,m}T_X^\star = S^m T_X^\star$ is a subbundle of $E_{2,m}T_X^\star$. We thus get an exact sequence

$$0 \to S^m T_X^\star \to E_{2,m}T_X^\star \to Q_m \to 0,$$

and Q_m admits a filtration with

$$\mathrm{Gr}^\bullet Q_m = \bigoplus_{1\leqslant j\leqslant m/3} S^{m-3j}T_X^\star \otimes K_X^j.$$

The simplest case is $m = 3$, which yields the interesting exact sequence

$$0 \to S^3T_X^\star \to E_{2,3}T_X^\star \to K_X \to 0.$$

12.7. Complement. Assume that X is a surface of degree d in $\mathbb{P}^3$. Then $K_X = \mathcal{O}_X(d-4)$. As $T_X^\star$ is a quotient bundle of $T^\star_{\mathbb{P}^3|X}$ and as $T^\star_{\mathbb{P}^n|X} \otimes \mathcal{O}(2)$ is generated by sections, we conclude that $S^{m-3j}T^\star_X \otimes K^j_X$ is (very) ample whenever $j(d-4) > 2(m-3j)$. This condition is most restrictive when $j = 1$. In particular, Q_m is ample for $d > 2m-2$, and we see that there is at most a "very small part" of $E_{2,m}T^\star_X$, namely $S^mT^\star_X$, which need not be ample when the degree d is large. By contrast, the Green-Griffiths graded bundle

$$G^\bullet E^{\mathrm{GG}}_{2,m}T^\star_X = \sum_{\ell_1+2\ell_2=m} S^{\ell_1}T^\star_X \otimes S^{\ell_2}T^\star_X$$

does not such exhibit such strong positivity properties. This is one of the main reasons for which we believe that the invariant jet bundles $E_{k,m}T^\star_X$ are more appropriate to the study of hyperbolicity questions.

§13. Riemann-Roch calculations and study of the base locus

In view of the Green-Griffiths conjecture 3.7 concerning algebraic degeneration of entire curves, the main point is to compute the base loci

$$B_k = \bigcap_{m>0} \mathrm{Bs}\left(H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi^\star_{k,0}\mathcal{O}(-A))\right) \subset X_k \tag{13.1}$$

where $X_k = P_kT_X$ and A is an ample divisor over X. By corollary 7.9, every nonconstant entire curve $f : \mathbb{C} \to X$ must satisfy $f_{[k]}(\mathbb{C}) \subset B_k$. If the set $Y = \bigcap_{k>0} \pi_{k,0}(B_k)$ is distinct from X, then $f(\mathbb{C}) \subset Y \subsetneq X$ and every entire curve is thus algebraically degenerate. We will call Y the *Green-Griffiths locus of* X, although Green and Griffiths did use ordinary jet bundles in place of the Semple jet bundles. Unfortunately, it turns out that Y is extremely hard to compute, especially in the case when X is an hypersurface or complete intersection in projective space. (However, an important breakthrough has been achieved in [SiYe96a] for the case of complements of curves in $\mathbb{P}^2$; noticeably, the authors obtain an explicit construction of global jet differentials of order 1 and 2, which allows them to show that the base locus is small enough.) Here, we will derive a few sufficient conditions for the existence of sections, mostly based on Riemann-Roch computations and a use of (semi-)stability inequalities. From now on, we restrict ourselves to the case when X is an algebraic surface of general type.

The easiest case is the case of order 1 jets $E_{1,m}T^\star_X = S^mT^\star_X$, namely symmetric differentials. The Riemann-Roch formula then gives

$$\chi(X, S^mT^\star_X \otimes \mathcal{O}(-A)) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2), \tag{13.2}$$

where c_1 and c_2 are the Chern classes of X. This can be seen e.g. by computing h^3 for the hyperplane bundle first Chern class $h = c_1(\mathcal{O}_{PT_X}(1))^3$ and using the identity $h^2 + c_1h + c_2 = 0$. By the Bogomolov vanishing theorem 14.1 of the Appendix, we get $h^2(X, S^mT^\star_X \otimes \mathcal{O}(-A)) = 0$ for m large, thus

$$h^0(X, S^mT^\star_X \otimes \mathcal{O}(-A)) \geqslant \frac{m^3}{6}(c_1^2 - c_2) - O(m^2). \tag{13.3}$$

As a consequence, if $c_1^2 > c_2$, there are non trivial symmetric differentials σ with values in $\mathcal{O}(-A)$, and every entire curve must satisfy the corresponding order 1 differential equation $\sigma(f') = 0$. This is especially interesting in connection with the following result of Jouanolou [Jou78].

13.4. Theorem (Jouanolou). *Let Z be a compact complex manifold such that the Hodge spectral sequence degenerates in E_2, and let $\mathcal{L} \subset \Omega_Z^1$ be a rank 1 coherent subsheaf such that $\Omega_Z^1/\mathcal{L}$ has no torsion. Let $\mathcal{V} \subset \mathcal{O}(T_X)$ be the dual distribution of hyperplanes in T_Z. Then either $\mathcal{V}$ is the relative tangent sheaf of a meromorphic fibration from Z to a curve, or there are only finitely many compact hypersurfaces tangent to $\mathcal{V}$.*

(Jouanolou [Jou78] even obtains a precise upper bound for the number of hypersurfaces which may occur in terms of $h^0(X, \Omega_X^2 \otimes \mathcal{L}^{-1})$ and of the Picard number of X). As a consequence, one recovers the following result due to Bogomolov [Bog77].

13.5. Theorem (Bogomolov). *On a surface X of general type such that $c_1^2 > c_2$, there are only finitely many rational or elliptic curves.*

Proof. By the results of § 7, these curves must be integral curves of some multivalued distribution of lines in X, associated with the zero divisor $Z \subset P(T_X)$ of any nonzero section in

$$H^0(P(T_X), \mathcal{O}_{P(T_X)}(m) \otimes \pi_{1,0}^\star \mathcal{O}(-A)).$$

At a generic point of Z over a point $x \in X$, this distribution defines a unique line in $T_{X,x}$, and we thus get a rank 1 subsheaf of $T_{\tilde{Z}}$ (or $\Omega_{\tilde{Z}}^1$) on any desingularization $\widetilde{Z}$ of Z. By Jouanolou's result applied to $\widetilde{Z}$, either these integral curves form a family or there are only a finite number of them. If they form a family, not all of them can be rational or elliptic, otherwise X would be a ruled or elliptic surface; hence the general fiber has genus at least 2. In both cases, there are only finitely many rational or elliptic curves. □

The above result of Bogomolov does not give information on transcendental curves, essentially because very little is known on transcendental leaves of a randomly chosen meromorphic foliation (e.g., one does not know how to decide whether there are only finitely many integral curves of parabolic type). As observed by Lu and Yau [LuYa90], one can say more if the *topological index* $c_1^2 - 2c_2$ is positive, using the following result of Schneider-Tancredi [ScTa85] (the special case when $E = T_X^\star$ is due to Miyaoka [Miy82]).

13.6. Theorem (Schneider-Tancredi). *Let E be rank 2 vector bundle over a projective algebraic surface X. Assume that* $\det E$ *is nef and big* (*i.e.* $c_1(E)$ *is numerically nonnegative and* $c_1(E)^2 > 0$), *that E is* ($\det E$)*-semistable and that $c_1(E)^2 - 2\,c_2(E) > 0$. Then E is almost ample in the sense that $S^m E$ generates all 1-jets of sections outside a finite union of curves in X, when m is large enough.*

Proof (sketch). Let $P = P(E^\star)$ be the hyperplane bundle of E and $H = \mathcal{O}_P(1)$. Then P is a ruled 3-fold and the hypotheses imply $c_1(H)^3 = c_1(E)^2 - c_2(E) > 0$.

Hence by Riemann-Roch and Serre duality, either $h^0(X, S^m E)$ or $h^0(X, S^m E^\star)$ grow fast. The latter case is impossible by the assumption on semistability and the assumption $\det E$ nef. Therefore H is big. Fix an ample divisor A on P. We have to show that the base locus of $mH - A$ in P projects to a curve in X when m is large. Otherwise, let D be an irreducible component of a divisor in the linear system $|mH - A|$. In the Picard group $\mathrm{Pic}(P) = \mathrm{Pic}(X) \oplus \mathbb{Z}[H]$ we then have $D = kH - \pi^\star F$ for some integer $k > 0$ and some divisor F on X. Observing that the multiplication by the canonical section of $H^0(P, \mathcal{O}(D))$ yields an injection of sheaves

$$\mathcal{O}(F) \hookrightarrow \pi_\star \mathcal{O}(kH) = \mathcal{O}(S^k E),$$

we find by semistability

$$c_1(F) \cdot c_1(E) \leqslant \frac{1}{k+1} c_1(S^k E) \cdot c_1(E) = \frac{k}{2} c_1(E)^2.$$

From this, we infer

$$\begin{aligned} H^2 \cdot D = H^2 \cdot (kH - \pi^\star F) &= k(c_1(E)^2 - c_2(E)) - c_1(E) \cdot c_1(F) \\ &\geqslant \frac{k}{2}\big(c_1(E)^2 - 2\, c_2(E)\big) > 0, \end{aligned}$$

therefore $(mH - A)^2 \cdot D > 0$ for m large. By Riemann-Roch, either

$$h^0(D, p(mH - A)_{|D}) \quad \text{or} \quad h^2(D, p(mH - A)_{|D})$$

grows fast as p goes to infinity. By stability again, the latter case cannot occur, as we see by looking at the exact sequence

$$0 \to \mathcal{O}(-D) \otimes \mathcal{O}(p(mH - A)) \to \mathcal{O}_P \otimes \mathcal{O}(p(mH - A)) \to \mathcal{O}_D \otimes \mathcal{O}(p(mH - A)) \to 0,$$

and descending everything at the h^2 and h^3 level down to X by the Leray spectral sequence. Hence $H_{|D}$ is big and the claim follows by 7.2 iv). □

13.7. Theorem ([LuYa90])**.** *Let X be a smooth algebraic surface of general type such that $c_1^2 - 2c_2 > 0$. Then there are only finitely many rational or elliptic curves in X, and every non constant entire curve $f : \mathbb{C} \to X$ maps to one of these.*

Proof. One may assume that X is minimal, i.e. that K_X is nef (and big). By the work of Bogomolov [Bog79], $T_X^\star$ is semi-stable. The result of Schneider and Tancredi now implies that $T_X^\star$ is almost ample. Theorem 7.8 concludes the proof. □

We now turn ourselves to the case of jet differentials of degree 2. A simple Riemann-Roch computation based on Formula 12.6 shows that

$$\chi(X, E_{2,m} T_X^\star) = \chi(X, \mathrm{Gr}^\bullet E_{2,m} T_X^\star) = \frac{m^4}{648}(13\, c_1^2 - 9\, c_2) + O(m^3) \tag{13.8}$$

where c_1, c_2 are the Chern classes of X (only the terms of bidegree $(2,2)$ in $\mathrm{Ch}(\mathrm{Gr}^\bullet E_{2,m} T_X^\star)$ play a role). This formula should be put in perspective with the one obtained by Green and Griffiths [GrGr80] for the jet bundles $E^{\mathrm{GG}}_{k,m}T^\star_X$. In the case of surfaces, they obtain

$$\chi(X, E^{\mathrm{GG}}_{k,m}T^\star_X) = \frac{m^{2k+1}}{(k!)^2(2k+1)!}(\alpha_k\, c_1^2 - \beta_k\, c_2) + O(m^{2k}),$$

where $\alpha_k \sim \frac{1}{2}(\log k)^2$ and $\beta_k = O(\log k)$ (especially $\lim \beta_k/\alpha_k = 0$). In the special case $n = k = 2$, their formula yields

$$\chi(X, E^{\mathrm{GG}}_{2,m}T^\star_X) = \frac{m^5}{384}(7\, c_1^2 - 5\, c_2) + O(m^4).$$

This is weaker than formula (13.8) in the sense that the ratio $5/7$ is larger than $9/13$. In general, we expect analogous estimates of the form

$$\chi(X, E_{k,m}T^\star_X) \sim m^{k+2}(\gamma_k\, c_1^2 - \delta_k\, c_2) + O(m^{k+1})$$

with $\lim \delta_k/\gamma_k = 0$ (and even similar higher dimensional estimates with a leading term of the form $c_{n,k}m^{(n-1)k+n}(-c_1)^n$ when $m \gg k \gg 1$). Unfortunately, our lack of knowledge of the combinatorics of the Schur representations involved makes the computation hard to achieve.

In the special case when X is a surface of degree d in $\mathbb{P}^3$, we have $c_1 = (4-d)h$ and $c_2 = (d^2 - 4d + 6)h^2$ where $h = c_1(\mathcal{O}(1)_{|X})$, $h^2 = d$, thus

$$\chi(X, E_{2,m}T^\star_X) = \frac{m^4}{648}\, d(4\, d^2 - 68\, d + 154) + O(m^3).$$

This estimate is especially useful in combination with vanishing theorems for holomorphic tensor fields (see Theorem 14.1 in the Appendix).

13.9. Corollary. *If X is an algebraic surface of general type and A an ample line bundle over X, then*

$$h^0(X, E_{2,m}T^\star_X \otimes \mathcal{O}(-A)) \geqslant \frac{m^4}{648}(13\, c_1^2 - 9\, c_2) - O(m^3).$$

In particular, every smooth surface $X \subset \mathbb{P}^3$ of degree $d \geqslant 15$ admits non trivial sections of $E_{2,m}T^\star_X \otimes \mathcal{O}(-A)$ for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

Proof. First note that the leading term in the Riemann-Roch estimate does not depend on taking a tensor product by a line bundle $\mathcal{O}(-A)$. The claim will follow from the computation of the Euler characteristic made in (13.8) if we check that $h^2(X, E_{2,m}T^\star_X \otimes \mathcal{O}(-A)) = 0$ for m large. However

$$H^2(X, E_{2,m}T^\star_X \otimes \mathcal{O}(-A)) = H^0(X, K_X \otimes (E_{2,m}T^\star_X)^\star \otimes \mathcal{O}(A))$$

by Serre duality. Since $K_X \otimes (E_{2,m}T^\star_X)^\star \otimes \mathcal{O}(A)$ admits a filtration with graded pieces

$$S^{m-3j}T_X \otimes K_X^{\otimes 1-j} \otimes \mathcal{O}(A),$$

we easily deduce the vanishing of global sections from Bogomolov's result 14.1, using the fact that $K_X^{\otimes\nu} \otimes \mathcal{O}(-A)$ is big for $\nu \geqslant \nu_0$ large enough. □

Other approach using weighted line bundles $\mathcal{O}_{X_k}(\boldsymbol{a})$. We show here how a use of the weighted line bundles $\mathcal{O}_{X_k}(\boldsymbol{a})$ may yield further information on the base locus. Consider a directed manifold (X,V) with $\dim X = n$ and $\operatorname{rank} V = r$. We set $u_k = c_1(\mathcal{O}_{X_k}(1))$ and let

$$c_\bullet^{[k]} = 1 + c_1^{[k]} + \cdots + c_r^{[k]} := c_\bullet(V_k)$$

be the total chern class of V_k. Then the cohomology ring of $X_k = P(V_{k-1})$ is defined in terms of generators and relations as the polynomial algebra $H^\bullet(X)[u_1,\ldots,u_k]$ with relations

$$u_j^r + c_1^{[j-1]}u_j^{r-1} + \cdots + c_{r-1}^{[j-1]}u_j + c_r^{[j-1]} = 0, \qquad 1 \leqslant j \leqslant k \tag{13.10}$$

(we omit all pull-backs $\pi_j^\star$ for simplicity of notation). Moreover, the exact sequences (5.4) and (5.4′) yield the inductive formula

$$\begin{aligned}
&c_\bullet^{[k]} = c_\bullet(\mathcal{O}_{X_k}(-1))\,c_\bullet(T_{X_k/X_{k-1}}) = (1-u_k)\,c_\bullet(T_{X_k/X_{k-1}}),\\
&c_\bullet(T_{X_k/X_{k-1}}) = c_\bullet(\pi_k^\star V_{k-1}\otimes\mathcal{O}_{X_k}(1)) = \sum_{0\leqslant j\leqslant r} c_j^{[k-1]}(1+u_k)^{r-j},
\end{aligned}$$

in other words

$$c_\bullet^{[k]} = (1-u_k)\sum_{0\leqslant j\leqslant r} c_j^{[k-1]}(1+u_k)^{r-j}. \tag{13.11}$$

In particular, if $r = \operatorname{rank} V = 2$, we find

$$\begin{aligned}
&u_k^2 + c_1^{[k-1]}u_1 + c_2^{[k-1]} = 0,\\
&c_1^{[k]} = c_1^{[k-1]} + u_k, \qquad c_2^{[k]} = c_2^{[k-1]} - u_k^2,
\end{aligned}\tag{13.12}$$

hence

$$c_1^{[k]} = c_1^{[0]} + u_1 + \cdots + u_k, \qquad c_2^{[k]} = c_2^{[0]} - u_1^2 - \cdots - u_k^2. \tag{13.13}$$

From now on, we concentrate again on the surface case. The 2-jet bundle

$$X_2 \to X_1 \to X$$

is a 2-step tower of $\mathbb{P}^1$-bundles over X and therefore has dimension 4. The exact sequence (5.4) shows that V_1 has splitting type $V_{1\restriction F_1} = \mathcal{O}(2)\oplus\mathcal{O}(-1)$ along the fibers F_1 of $X_1 \to X$, since $T_{X_1/X\restriction F_1} = \mathcal{O}(2)$. Hence the fibers F_2 of $X_2 \to X$ are Hirzebruch surfaces $P(\mathcal{O}(2)\oplus\mathcal{O}(-1)) \simeq P(\mathcal{O}\oplus\mathcal{O}(-3))$ and

$$\mathcal{O}_{X_2}(1)_{\restriction F_2} = \mathcal{O}_{P(\mathcal{O}(2)\oplus\mathcal{O}(-1))}(1).$$

The weighted line bundle $\mathcal{O}_{X_2}(2,1)$ is relatively nef over X, as follows from our general result (6.16 ii) or from the equality

$$\mathcal{O}_{X_2}(2,1)_{\restriction F_2} = \mathcal{O}_{P(\mathcal{O}(2)\oplus\mathcal{O}(-1))}(1) \otimes \pi^\star\mathcal{O}_{\mathbb{P}^1}(2) = \mathcal{O}_{P(\mathcal{O}\oplus\mathcal{O}(-3))}(1).$$

Its multiples have zero higher order direct images $R^q(\pi_{2,0})_\star\mathcal{O}_{X_2}(2m,m)$, $q \geqslant 1$, and lower order direct images

$$(\pi_{2,0})_\star\mathcal{O}_{X_2}(2m,m) = (\pi_{2,0})_\star\mathcal{O}_{X_2}(3m) = E_{2,3m}T^\star_X$$

[either apply (6.16 i) or observe that

$$\begin{aligned}&(\pi_{1,0})_\star\big(\mathcal{O}_{P(\mathcal{O}\oplus\mathcal{O}(-3))}(m)\big) = S^m\big(\mathcal{O}\oplus\mathcal{O}(3)\big),\\ &(\pi_{1,0})_\star\big(\mathcal{O}_{P(\mathcal{O}(2)\oplus\mathcal{O}(-1))}(3m)\big) = S^{3m}\big(\mathcal{O}(-2)\oplus\mathcal{O}(1)\big)\end{aligned}$$

have the same sections over $\mathbb{P}^1$]. By the Leray spectral sequence, we conclude that

$$h^q(X_2,\mathcal{O}_{X_2}(2m,m)) = h^q(X,E_{2,3m}T^\star_X), \qquad 0 \leqslant q \leqslant 2,$$

in particular the Euler characteristics are equal and grow as $\frac{1}{8}m^4(13\,c_1^2-9c_2)$ when $m \to +\infty$. This can also be checked directly by computing $\frac{1}{4!}(2u_1+u_2)^4$. In fact, (13.12) and (13.13) easily provide

$$u_1^4 = 0, \quad u_1^3u_2 = c_1^2 - c_2, \quad u_1^2u_2^2 = c_2, \quad u_1u_2^3 = c_1^2 - 3c_2, \quad u_1^4 = 5c_2 - c_1^2.$$

The main difficulty when trying to check the hyperbolicity of X is to show that the base locus of $\mathcal{O}_{X_2}(2,1)$ is small enough. Proving that the base locus is one dimensional would imply that X only admits a finite number of rational and elliptic curves, and that every entire curve $f:\mathbb{C}\to X$ maps into one of these. A possibility for this would be to check that $(2u_1+u_2)^3\cdot Y > 0$ for every 3-fold $Y \subset X_2$ and $(2u_1+u_2)^2\cdot S > 0$ for every surface $S \subset X_2$. Unfortunately, such estimates are rather hard to check, since we would need to evaluate the numerical cones of effective codimension 1 and codimension 2 cycles in the 4-fold X_2. The codimension 1 case, however, can be treated by using semi-stability arguments (although possibly the conditions obtained in this way are far from being optimal). The following computation is due to J. El Goul [EG97].

13.14. Proposition ([EG97]). *Let X be a minimal algebraic surface of general type. If $c_1^2 - \frac{9}{7}c_2 > 0$, then $\mathcal{O}_{X_2}(2,1)$ is almost ample on X_2 with a base locus of dimension 2 at most.*

Proof (sketch). We proceed as in the proof of the result by Miyaoka and Schneider-Tancredi. Let Y be a 3-dimensional irreducible component of the base locus, if any. In $\mathrm{Pic}(X_2) = \mathrm{Pic}(X)\oplus\mathbb{Z}u_1\oplus\mathbb{Z}u_2$, we then find an equality $Y = a_1u_1+a_2u_2-\pi^\star F$ for some integers a_1, $a_2 \in \mathbb{Z}$ and some divisor F on X. As Y is effective, we must have $a_1 \geqslant 0$, $a_2 \geqslant 0$. Moreover, $\mathcal{O}(F)$ can be viewed as a subsheaf of $\pi_\star(\mathcal{O}_{X_2}(a_1,a_2)) \subset E_{2,m}T^\star_X$ where $m = a_1 + a_2$. Thus there is a non trivial morphism $\mathcal{O}(F) \hookrightarrow S^{m-3j}T^\star_X \otimes K^j_X$ for some j, and the semistability inequality yields

$$F \cdot K_X \leqslant \frac{m-j}{2} K^2_X \leqslant \frac{m}{2} c_1^2.$$

A short computation now yields

$$(2u_1 + u_2)^2 \cdot Y = (a_1 + a_2)(13\, c_1^2 - 9\, c_2) - 12\, c_1 \cdot F \geqslant m(7\, c_1^2 - 9\, c_2). \qquad \square$$

Logarithmic case. Similar computations can be made in this situation. In fact, if X is a surface and D is a smooth effective divisor in X, the bundle $E_{2,m}T^\star_X\langle D\rangle$ of logarithmic jet differentials of order 2 and degree m admits a filtration with

$$\mathrm{Gr}^\bullet\, E_{2,m}T^\star_X\langle D\rangle = \bigoplus_{0\leqslant j\leqslant m/3} S^{m-3j}(T^\star_X\langle D\rangle) \otimes \det(T^\star_X\langle D\rangle)^j. \tag{13.15}$$

We thus get

$$\begin{aligned} h^0(X, E_{2,m}T^\star_X\langle D\rangle) &\geqslant \chi(X, E_{2,m}T^\star_X\langle D\rangle) \\ &\geqslant \frac{m^4}{648}\big(13\, c_1^2(T_X\langle D\rangle) - 9\, c_2(T_X\langle D\rangle)\big) - O(m^3). \end{aligned} \tag{13.16}$$

The exact sequence $0 \to T_X\langle D\rangle \to T_X \to (i_D)_\star N_{X/D} \to 0$ yields

$$c_\bullet(T_X\langle D\rangle) = c_\bullet(T_X) c_\bullet((i_D)_\star N_{X/D})^{-1} = (1 + c_1 + c_2)(1+\delta)^{-1},$$

where $\delta = c_1(\mathcal{O}_X(D))$ and $c_\bullet((i_D)_\star N_{X/D}) = c_\bullet(\mathcal{O}_X(D)) = 1 + \delta$, thus

$$c_1(T_X\langle D\rangle) = c_1 - \delta, \qquad c_2(T_X\langle D\rangle) = c_2 - c_1 \cdot \delta + \delta^2.$$

Moreover, the expected vanishing theorem for $h^2(X, E_{2,m}T^\star_X\langle D\rangle)$ still holds since $T_X\langle D\rangle$ is a subbundle of T_X. In particular, if $X = \mathbb{P}^2$ and D is a smooth curve of degree d, we find $c_1(T_X\langle D\rangle) = 3 - d$, $c_2(T_X\langle D\rangle) = 3 - 3d + d^2$ and

$$h^0(X, E_{2,m}T^\star_{\mathbb{P}^2}\langle D\rangle) \geqslant \frac{m^4}{648}(4\, d^2 - 51\, d + 90) - O(m^3).$$

From this, one infers that every entire curve $f : \mathbb{C} \to \mathbb{P}^2 \smallsetminus D$ must satisfy a non trivial algebraic differential equation of order 2 if $d \geqslant 11$.

§14. Appendix: a vanishing theorem for holomorphic tensor fields

In this appendix, we prove a basic vanishing theorem for holomorphic tensor fields on minimal varieties of general type. It has been observed since a long time that the existence of holomorphic tensor fields on a compact Kähler manifold is governed in a rather precise way by the sign of the Ricci curvature (in case the Ricci curvature does admit some definite sign, semipositive or seminegative). See for instance the papers [Li67, 71] by Lichnerowicz for the case of sections of $\Lambda^k T_X$ or $\Lambda^k T_X^\star$, and S. Kobayashi's articles [Kob80, 81] for the more general case of tensors in $\Gamma^a T_X$. However, we want here to consider the situation of varieties of general type (i.e. with K_X big), and it is unknown whether K_X should be semipositive even if K_X is assumed to be big and nef. On the other hand, it is a consequence of Bogomolov's work [Bog79] (dealing with the so-called "Bogomolov stability" concept), that such vanishing theorems hold when T_X is semistable; this is the case for instance if X is a minimal surface of general type. Tsuji [Tsu88] has proved more generally that the tangent bundle T_X is semistable for any minimal nonsingular projective variety of general type (here, X "minimal" means that K_X is nef) . Thus, the following theorem 14.1 below can be obtained as a combination of the above mentioned results of Bogomolov and Tsuji. For the convenience of the reader, we give instead a direct proof based on a use of approximate Kähler-Einstein metrics in combination with the Bochner formula. Our hope is that similar a priori estimates could produce as well vanishing theorems for higher degree cohomology groups H^q.

14.1. Theorem. *Let X be a projective algebraic manifold, $n = \dim X$, and let L be a holomorphic line bundle over X. Assume that X is of general type and minimal (i.e. K_X is big and nef), and let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $a_1 \geqslant \cdots \geqslant a_n$, be a weight. If either L is pseudoeffective and $|a| = \sum a_j > 0$, or L is big and $|a| \geqslant 0$, then*

$$H^0(X, \Gamma^a T_X \otimes L^\star) = 0.$$

Recall that a line bundle L is said to be pseudoeffective if $c_1(L)$ belongs to the closure of the cone of effective divisors, or equivalently, if L carries a singular hermitian metric h with curvature current $\Theta_h(L) \geqslant 0$. Also notice that the result is invariant by modifications, hence it extends to the case when X is of general type and possesses a smooth minimal model $\widetilde{X}$; this is always the case when X is a surface. On the other hand, it is likely that the result holds for all varieties X of general type, in view of Mori's minimal model conjecture (however, the differential geometric proof given below might be difficult to extend to the case when the minimal model is singular).

Proof of Theorem 14.1. We will use the following notation: if all a_j are nonnegative integers, $\Gamma^a T_X$ is viewed as a subbundle of $(T_X)^{\otimes p}$ with $p = |a|$. In particular, given coordinates $(z_1, \ldots, z_n)$ on X, any tensor of $\Gamma^a T_X$ can be expressed as a linear combination of the elements

$$(\partial/\partial z)^I := \frac{\partial}{\partial z_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial z_{i_p}}, \qquad I = (i_1, \ldots, i_p), \quad 1 \leqslant i_k \leqslant n,$$

which form a basis of $(T_X)^{\otimes p}$. If some a_j is negative, we use instead the identity

$$\Gamma^{(a_1,\ldots,a_r)} T_X = \Gamma^{(a_1+\ell,\ldots,a_r+\ell)} T_X \otimes (\det T_X)^{-\ell}$$

with $\ell = \max(-a_j)$, and consider the basis elements $(\partial/\partial z)^J \otimes (dz_1 \wedge \cdots \wedge dz_n)^\ell$ with $|J| = p + n\ell$. Same notation with the elements of $\Gamma^a T_X^\star$ in terms of the basis $(dz)^I = dz_{i_1} \otimes \cdots \otimes dz_{i_p}$, resp. $(dz)^J \otimes (dz_1 \wedge \cdots \wedge dz_n)^{-\ell}$.

14.2. Lemma. *Let L be a holomorphic line bundle over X equipped with a smooth hermitian metric h, and let ω be a Kähler metric over X. We denote by $\#$ the conjugate linear C^∞-isomorphism $T_X \to T_X^\star$, $v \mapsto i\overline{v} \lrcorner\, \omega$, defined by contracting $(0,1)$-vectors with the Kähler metric ω. Denote also by $\# : \Gamma^a T_X \otimes L \to \Gamma^a T_X^\star \otimes L^\star$ the induced C^∞ isomorphism on the Schur tensor powers of T_X and $T_X^\star$, combined with the conjugate linear (metric) isomorphism $L \to L^\star$. Then for an arbitrary smooth section v of $\Gamma^a T_X \otimes L$ we have*

$$\int_X \|\overline{\partial}(\# v)\|^2 dV_\omega = \int_X \|\overline{\partial} v\|^2 dV_\omega + \int_X \langle \mathcal{R}_a(v), v\rangle + \gamma |v|^2\, dV_\omega$$

where dV_ω is the Kähler element of volume, γ the trace (= sum of eigenvalues) of $\Theta_h(L)$ with respect to ω, and $\mathcal{R}_a$ is the hermitian operator

$$v = \sum_{|I|=p} v_I (\partial/\partial z)^I \otimes s \longmapsto \mathcal{R}_a(v) = \sum_{|I|=p} \Big(\sum_{1\leqslant k\leqslant p} \rho_{i_k}\Big) v_I (\partial/\partial z)^I \otimes s,$$

$$(\textit{resp. } \mathcal{R}_a v = \mathcal{R}_{a+\ell(1,\ldots,1)} v - \ell(\textstyle\sum_j \rho_j) v \textit{ with } \ell = \max(-a_j), \textit{ if } a \notin \mathbb{N}^n)$$

associated with the Ricci curvature form: ρ_k denotes the eigenvalues of Ricci(ω) *and $(\partial/\partial z_k)$, s are supposed to be orthonormal frames of (T_X, ω) and (L, h).*

Proof. We first make a pointwise calculation of $\overline{\partial}^\star \overline{\partial} v$ and $\overline{\partial}^\star \overline{\partial}(\# v)$ in a normal coordinate system for the Kähler metric ω and in a normalized holomorphic frame (s) for (L, h). In suitable such coordinates we can write

$$\omega = i \sum_{1\leqslant m\leqslant n} dz_m \wedge d\overline{z}_m - i \sum_{1\leqslant j,k,\ell,m\leqslant n} c_{jk\ell m} z_j \overline{z}_k dz_\ell \wedge d\overline{z}_m + O(|z|^3),$$

$$|s|^2 = 1 - \sum_{1\leqslant j\leqslant n} \gamma_{jk} z_j \overline{z}_k + O(|z|^3)$$

where $(c_{jk\ell m})$ is the curvature tensor of T_X with respect to ω, and the γ_{jk}'s are the coefficients of $\Theta_h(L)$. The Kähler property shows that we have the symmetry relations $c_{jk\ell m} = c_{\ell kjm} = c_{jm\ell k}$, and the Ricci tensor $R = \sum R_{\ell m} dz_\ell \wedge d\overline{z}_m$ is obtained as the trace: $R_{\ell m} = \sum_j c_{jj\ell m}$. Since ω is tangent of order 2 to a flat metric at the center x_0 of the chart, we easily see that the first order operator $\overline{\partial}^\star$ has the same formal expression at x_0 as in the case of the flat metric on $\mathbb{C}^n$: if w if a smooth $(0,q)$-form with values in a holomorphic vector bundle E trivialized locally by a holomorphic frame (e_λ) such that $(e_\lambda(x_0))$ is orthonormal and $De_\lambda(x_0) = 0$, we have at x_0 the formula

$$w = \sum_{|J|=q,\, 1\leqslant\lambda\leqslant r} w_{J,\lambda}\, d\overline{z}_J \otimes e_\lambda, \qquad \overline{\partial}^\star w = - \sum_{|J|=q,\, \lambda,\, k} \frac{\partial w_{J,\lambda}}{\partial z_k} \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\, d\overline{z}_J \Big) \otimes e_\lambda.$$

We apply this to smooth sections $v = \sum v_I\, (\partial/\partial z)^I \otimes s$ of $\Gamma^a T_X \otimes L$ and $w = \sum w_I\, (dz)^I \otimes s^\star$ of $\Gamma^a T_X^\star \otimes L^\star$ where $s^\star$ denotes the holomorphic section of $L^\star$ such that $s^\star(s) = 1$. We get

$$\overline{\partial}^\star \overline{\partial} v = - \sum_{I,k} \frac{\partial^2 v_I}{\partial z_k \partial \overline{z}_k}\, (\partial/\partial z)^I \otimes s, \qquad \overline{\partial}^\star \overline{\partial} w = - \sum_{I,k} \frac{\partial^2 w_I}{\partial z_k \partial \overline{z}_k}\, (dz)^I \otimes s^\star$$

at x_0. Now, we find

$$\begin{aligned}
\#\,\frac{\partial}{\partial z_m} &= i\frac{\partial}{\partial \overline{z}_m} \lrcorner\, \omega = dz_m - \sum_{j,k,\ell} c_{jk\ell m} z_j \overline{z}_k dz_\ell + O(|z|^3),\\
\#\, s &= \Big(1 - \sum_{1\leqslant j,k\leqslant n} \gamma_{jk} z_j \overline{z}_k + O(|z|^3)\Big) s^\star\\
\#\, v &= \sum_I \overline{v}_I (dz)^I \otimes s^\star - \sum_{I,j,k,\ell,m} \overline{v}_I c_{jk\ell m} z_j \overline{z}_k \Big(dz_\ell \otimes \frac{\partial}{\partial z_m}\Big) \lrcorner\, (dz)^I \otimes s^\star,\\
&\qquad - \sum_{I,j,k} \overline{v}_I\, \gamma_{jk} z_j \overline{z}_k (dz)^I \otimes s^\star + O(|z|^3)
\end{aligned}$$

where (by definition)

$$\Big(dz_\ell \otimes \frac{\partial}{\partial z_m}\Big) \lrcorner\, (dz)^I := \sum_{1\leqslant k\leqslant p,\; i_k=m} dz_{i_1} \otimes \cdots \otimes dz_{i_{k-1}} \otimes dz_\ell \otimes dz_{i_{k+1}} \otimes \cdots \otimes dz_{i_p}.$$

Computing $\overline{\partial}^\star\overline{\partial}(\#\, v)$ at x_0 we obtain

$$\begin{aligned}
\overline{\partial}^\star\overline{\partial}(\#\, v) &= -\sum_{I,k} \frac{\partial^2 \overline{v}_I}{\partial z_k \partial \overline{z}_k} (dz)^I \otimes s^\star + \sum_{I,k,\ell,m} \overline{v}_I c_{kk\ell m} \Big(dz_\ell \otimes \frac{\partial}{\partial z_m}\Big) \lrcorner\, (dz)^I \otimes s^\star\\
&\qquad + \sum_j \gamma_{jj} \sum_I \overline{v}_I (dz)^I \otimes s^\star\\
&= \#\,(\overline{\partial}^\star\overline{\partial} v) + \sum_{I,\ell,m} \overline{v}_I R_{\ell m} \Big(dz_\ell \otimes \frac{\partial}{\partial z_m}\Big) \lrcorner\, (dz)^I \otimes s^\star + \gamma(\#\, v)\\
&= \#\,(\overline{\partial}^\star\overline{\partial} v) + \#\, \mathcal{R}_a(v) + \gamma(\#\, v)
\end{aligned}$$

where $\gamma = \sum_j \gamma_{jj}$. Lemma 14.2 then follows from this identity by writing

$$\int_X \|\overline{\partial}(\#\, v)\|^2 dV_\omega = \int_X \langle \overline{\partial}^\star\overline{\partial}(\#\, v), \#\, v\rangle\, dV_\omega = \int_X \langle \overline{\partial}^\star\overline{\partial} v + \mathcal{R}_a(v) + \gamma v, v\rangle\, dV_\omega. \quad \square$$

Proof of Theorem 14.1 (end). Our goal is to apply the Bochner formula of Lemma 14.2 to show that every section v of $H^0(X, \Gamma^a T_X \otimes L^\star)$ must vanish. We first make a reduction to the case when L is ample. In fact, by raising v to some tensor power, we get a section $v^m \in H^0(X, \Gamma^{ma} T_X \otimes L^{\otimes -m})$. If L is big, some power $L^{\otimes m}$ can be written as $\mathcal{O}(A + D)$ where A is an ample divisor and D an effective divisor. It is then enough to prove the vanishing of $H^0(\Gamma^{ma} T_X \otimes \mathcal{O}(-A))$. If L is just pseudoeffective, then $|a| > 0$ by hypothesis and we write

$$\Gamma^{ma} T_X \otimes L^{\otimes -m} = \Gamma^{ma-(1,\ldots,1)} T_X \otimes (L^{\otimes m} \otimes K_X)^{-1}$$

where $|ma - (1,\ldots,1)| = m|a| - n > 0$ for $m > n$ and $L^{\otimes m} \otimes K_X$ is big. We can now proceed as before to reduce the situation to the case of an ample L.

If, in addition to this, K_X is also ample, the statement is a straightforward consequence of the Aubin-Calabi-Yau theorem ([Aub77], [Yau77]). In fact, we can choose ω to be Kähler-Einstein, i.e., $\mathrm{Ricci}(\omega) = -\omega$. Then, for any holomorphic section $v \in H^0(X, \Gamma^a T_X \otimes L^\star$, Lemma 14.2 yields

$$\begin{aligned}0 &= \int_X \|\overline{\partial} v\|^2 dV_\omega = \int_X \|\overline{\partial}(\# v)\|^2 dV_\omega \int_X -\langle \mathcal{R}_a(v), v\rangle + \gamma |v|^2\, dV_\omega \\ &\geqslant \int_X (|a| + \gamma)|v|^2\, dV_\omega\end{aligned}$$

(γ becomes $-\gamma$ since we changed L into $L^\star$, and all Ricci eigenvalues are equal to -1 in that case). As $|a| \geqslant 0$ and $\gamma > 0$ by the ampleness of L, we get the desired conclusion.

If K_X is only big and nef, we take ω to be a Kähler form in the positive class $c_1(K_X) + \varepsilon c_1(L) = -c_1(X) + \varepsilon c_1(L)$, such that

$$\mathrm{Ricci}(\omega) = -\omega + \varepsilon\theta$$

where $\theta = \Theta_h(L) > 0$ (the existence of such ω is is a well-known consequence of the theory of Monge-Ampère equations). Then the Ricci curvature eigenvalues satisfy $\rho_j = -1 + \varepsilon\gamma_j \leqslant -1 + \varepsilon\gamma$ and we get

$$\langle -\mathcal{R}_a(v), v\rangle + \gamma|v|^2 \geqslant (|a| + \gamma - N\varepsilon\gamma)|v|^2$$

where N is an integer depending only on the weight $a = (a_1, \ldots, a_n)$; for instance, $N = |a|$ works if all a_j are nonnegative, otherwise we can take $N = |a| + n\max(-a_j)$. □

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(version of April 7, 1997, printed on May 23, 2017, 10:04)

Université Joseph Fourier Grenoble I
Institut Fourier (Mathématiques)
UMR 5582 du C.N.R.S., BP 74
38402 Saint-Martin d'Hères, France
e-mail: demailly@fourier.ujf-grenoble.fr