On the approximate cohomology of quasi holomorphic line bundles

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Quasi holomorphic line bundles

Let $X$ be a compact complex manifold, and let

$$H^{p,q}_{BC}(X, \mathbb{C}) = \frac{\text{Ker} \partial \cap \text{Ker} \overline{\partial}}{\text{Im} \partial \overline{\partial}}$$

be the corresponding Bott-Chern cohomology groups.

Basic observation (cf. Laurent Laeng, PhD thesis 2002)

Given a class $\gamma \in H^{1,1}_{BC}(X, \mathbb{R})$ and a $(1,1)$-form $u$ representing $\gamma$, there exists an infinite subset $S \subset \mathbb{N}$ and $C^\infty$ Hermitian line bundles $(L_k, h_k)_{k \in S}$ equipped with Hermitian connections $\nabla_k$, such that the curvature 2-forms $\theta_k = \frac{i}{2\pi} \nabla_k^2$ satisfy $\theta_k = ku + \beta_k$ and

$$\beta_k = O(k^{-1/b_2}), \quad b_2 = b_2(X).$$

Proof. This is a consequence of Kronecker’s approximation theorem applied to the lattice $H^2(X, \mathbb{Z}) \hookrightarrow H^2_{DR}(X, \mathbb{R})$.

In fact $\beta_k$ can be chosen in a finite dimensional space of $C^\infty$ closed 2-forms isomorphic to $H^2_{DR}(X, \mathbb{R})$. 

Approximate holomorphic structure

Consequence
Let \( \nabla_k = \nabla_k^{1,0} + \nabla_k^{0,1} \). Then \( \theta_k = ku + \beta_k \) implies
\[
(\nabla_k^{0,1})^2 = \theta_k^{0,2} = \beta_k^{0,2} = O(k^{-1/b_2}).
\]
Thus the \( L_k \) are “closer and closer” to be holomorphic as \( k \to +\infty \).

Spectrum of the Laplace-Beltrami operator

Let \( \Box_k = \overline{\partial}_k \partial_k^* + \partial_k^* \overline{\partial}_k \) be the complex Laplace-Beltrami operator of \((L_k, h_k, \nabla_k)\) with respect to some Hermitian metric \( \omega \) on \( X \).

Let \( \Box_{k,E}^{p,q} \) the operator acting on \( C^\infty(X, \Lambda_X^{p,q} T^*_X \otimes L_k \otimes E) \), where \((E, h_E)\) is a holomorphic Hermitian vector bundle of rank \( r \).

We are interested in analyzing the (discrete) spectrum of the elliptic operator \( \Box_{k,E}^{p,q} \). Since the curvature is \( \theta_k \simeq ku \), it is better to renormalize and to consider instead \( \frac{1}{2\pi k} \Box_{k,E}^{p,q} \). For \( \lambda \in \mathbb{R} \), we define
\[
N_{p,q}^{k} (\lambda) = \dim \bigoplus \text{eigenspaces of } \frac{1}{2\pi k} \Box_{k,E}^{p,q} \text{ of eigenvalues } \leq \lambda.
\]

Let \( u_j(x), 1 \leq j \leq n \), be the eigenvalues of \( u(x) \) with respect to \( \omega(x) \) at any point \( x \in X \), ordered so that if \( s = \text{rank}(u(x)) \), then
\[
|u_1(x)| \geq \cdots \geq |u_s(x)| > |u_{s+1}(x)| = \cdots = |u_n(x)| = 0.
\]

For a multi-index \( J = \{ j_1 < j_2 < \ldots < j_q \} \subset \{1, \ldots, n\} \), set
\[
u_J(x) = \sum_{j \in J} u_j(x), \quad x \in X.
\]
Consider the “spectral density functions” $\nu_u, \overline{\nu}_u$ defined by

$$
\nu_u(\lambda) = 2^{s-n} \left| u_1 \cdots u_s \right| \frac{\left( \lambda - \sum (2p_j + 1)|u_j| \right)^{n-s}}{\Gamma(n-s+1)} \sum_{(p_1, \ldots, p_s) \in \mathbb{N}^s} \sum_{|J|=q} \nu_u\left( 2\lambda + u_{CJ} - u_J \right) dV_\omega \leq \liminf_{k \to +\infty} k^{-n} N_k^{p,q}(\lambda) \leq \limsup_{k \to +\infty} k^{-n} N_k^{p,q}(\lambda) \leq r \left( \frac{n}{p} \right) \sum_{|J|=q} \int_X \nu_u\left( 2\lambda + u_{CJ} - u_J \right) dV_\omega
$$

where $r = \text{rank}(E)$. By monotonicity, as $\nu_u(\lambda) = \lim_{\lambda \to 0^+} \nu_u(\lambda)$, all four terms are equal for $\lambda \in \mathbb{R} \setminus D$ with $D$ countable.

**Theorem ([D] 1985)**

The spectrum of $\frac{1}{2\pi k} \Box_k^{p,q}$ on $C^\infty(X, \Lambda^{p,q} T^*_X \otimes L_k \otimes E)$ has an asymptotic distribution of eigenvalues such that $\forall \lambda \in \mathbb{R}$

$$
\sum_{|J|=q} \int_X \nu_u\left( 2\lambda + u_{CJ} - u_J \right) dV_\omega \leq \liminf_{k \to +\infty} k^{-n} N_k^{p,q}(\lambda) \leq \limsup_{k \to +\infty} k^{-n} N_k^{p,q}(\lambda) \leq r \left( \frac{n}{p} \right) \sum_{|J|=q} \int_X \overline{\nu}_u\left( 2\lambda + u_{CJ} - u_J \right) dV_\omega
$$

where $r = \text{rank}(E)$. By monotonicity, as $\overline{\nu}_u(\lambda) = \lim_{\lambda \to 0^+} \nu_u(\lambda)$, all four terms are equal for $\lambda \in \mathbb{R} \setminus D$ with $D$ countable.

**Approximate cohomology lower bounds**

**Proof.** One first estimates the spectrum of the total Laplacian $\Delta_{k,E} = \nabla_{k,E} \nabla_{k,E}^* + \nabla_{k,E}^* \nabla_{k,E}$ (harmonic oscillator with magnetic and electric fields), and then one uses a Bochner formula to relate $\Box_k^{p,q}$ and $\Delta_{k,E}$ ($\Box_k^{p,q} \simeq \frac{1}{2} \Delta_{k,E} + \text{curvature terms}$) for each $(p, q)$.

**Important special case $\lambda = 0$ (harmonic forms)**

$$
\sum_{|J|=q} \overline{\nu}_u\left( u_{CJ} - u_J \right) dV_\omega = (-1)^q \frac{u^n}{n!}.
$$

**Corollary (Laurent laeng, 2002)**

For $\lambda_k \to 0$ slowly enough, i.e. with $k^{2+2/b_2} \lambda_k \to +\infty$, one has

$$
\liminf_{k \to +\infty} k^{-n} N_0^{0,0}(\lambda_k) \geq \frac{r}{n!} \left( \int_X (u_n^0 + \int_X u_n) \right) \text{where}
$$

$$
\mathcal{X}(u, q) = q\text{-index set} = \{ x \in X / u(x) \text{ has signature } (n-q, q) \}.
$$
Proof of the lower bound

**Proof.** One uses the fact that for \( \delta' > \delta > 0 \) and \( k \gg 1 \), the composition \( \Pi \circ \overline{\partial}_k \) with an eigenspace projection yields an injection

\[
\bigoplus_{\lambda \in [\lambda_k, \delta]} \text{eigenspace}^{0,0}_\lambda \leftrightarrow \bigoplus_{\lambda \in [0, \delta'] \lambda} \text{eigenspace}^{0,1}_\lambda.
\]

In fact, in the holomorphic case \( \overline{\partial}^2_k = 0 \) implies \( \square_k \overline{\partial}_k^{0,0} = \overline{\partial}_k^{0,1} \overline{\partial}_k \), hence \( \overline{\partial}_k \) maps the \((0,0)\)-eigenspaces to the \((0,1)\)-eigenspaces for the same eigenvalues, and one can even take \( \lambda_k = 0, \delta' = \delta \).

In the quasi holomorphic case \( \overline{\partial}^2_k = O(k^{-1/b^2}) \), one can show that \( \square_k^{0,1} \overline{\partial}_k - \overline{\partial}_k \square_k^{0,0} = \overline{\partial}_k^{*} \overline{\partial}_k \) yields a small “deviation” of the eigenvalues to \([\lambda_k - \varepsilon, \delta + \varepsilon]\) with \( \varepsilon < \min(\lambda_k, \delta' - \delta) \), whence the injectivity.

This implies

\[
N_{k,E}^{0,1}(\delta') \geq N_{k,E}^{0,0}(\delta) - N_{k,E}^{0,0}(\lambda_k)
\]

thus

\[
N_{k,E}^{0,0}(\lambda_k) \geq N_{k,E}^{0,0}(\delta) - N_{k,E}^{0,1}(\delta'), \quad \text{QED}
\]

Transcendental holomorphic Morse inequalities

**Conjecture on Morse inequalities**

Let \( \gamma \in H^{1,1}_{BC}(X, \mathbb{R}) \). Then

\[
\operatorname{Vol}(\gamma) \geq \sup_{u \in \gamma} \int_X (u, \leq 1) u^n.
\]

(One could even suspect equality, an even stronger conjecture!)

If one sets by definition

\[
\operatorname{Vol}(\gamma) = \sup_{u \in \gamma} \lim_{\lambda \to 0^+} \liminf_{k \to +\infty} N_k^{0,0}(\lambda)
\]

for the eigenspaces of the sequence \((L_k, h_k, \nabla_k)\) approximating \(ku\), then the above expected lower bound is a theorem!

There is however a stronger & more usual definition of the volume.

**Definition**

For \( \gamma \in H^{1,1}_{BC}(X, \mathbb{R}) \), set \( \operatorname{Vol}(\gamma) = 0 \) if \( \gamma \not\ni \) any current \( T \geq 0 \), and otherwise set

\[
\operatorname{Vol}(\gamma) = \sup_{T \in \gamma} \int_X \frac{T_{ac}^n}{T_{ac}}, \quad u_0 \in C^\infty.
\]
The conjecture on Morse inequalities is known to be true when $\gamma = c_1(L)$ is an integral class ([D-1985]). In fact, one then gets a Hermitian holomorphic line bundle $(L, h)$ and its multiples $L^\otimes k$. The spectral estimates provide many holomorphic sections $\sigma_{k,\ell}$, and one gets positive currents right away by putting

$$T_k = \frac{i}{2k\pi} \partial \bar{\partial} \log \sum \ell |\sigma_{k,\ell}|^2 h + \frac{i}{2\pi} \Theta_{L,h} \geq 0$$

(the volume estimate can be derived from there by Fujita).

In the “quasi-holomorphic” case, one only gets eigenfunctions $\sigma_{k,\ell}$ with small eigenvalues, and the positivity of $T_k$ is a priori lost.

**Conjectural corollary (fundamental volume estimate)**

Let $X$ be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef cohomology classes. Then

$$\text{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$  

**Known results on holomorphic Morse inequalities**

The conjectural corollary is derived from the main conjecture by an elementary symmetric function argument. In fact, one has a pointwise inequality of forms

$$1_{X(\alpha - \beta, \leq 1)}(\alpha - \beta)^n \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$ 

Again, the corollary is known for $\gamma = \alpha - \beta$ when $\alpha, \beta$ are integral classes (by [D-1993] and independently [Trapani, 1993]).

Recently (2016), the volume estimate for $\gamma = \alpha - \beta$ transcendental has been established by D. Witt-Nyström when $X$ is projective, using deep facts on Monge-Ampère operators and upper envelopes.

Xiao and Popovici also proved in the Kähler case that

$$\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \implies \text{Vol}(\alpha - \beta) > 0$$

and $\alpha - \beta$ contains a Kähler current.

(The proof is short, once the Calabi-Yau theorem is taken for granted).
Projective vs Kähler vs non Kähler varieties

**Problem.** Investigate positivity for general compact manifolds. Obviously, non projective varieties do not carry any ample line bundle. In the Kähler case, a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$, $\omega > 0$, may sometimes be used as a substitute for a polarization. What for non Kähler compact complex manifolds?

**Surprising facts (?)**

- Every compact complex manifold $X$ carries a “very ample” complex Hilbert bundle, produced by means of a natural Bergman space construction.
- The curvature of this bundle is strongly positive in the sense of Nakano, and is given by a universal formula.

In the sequel of this lecture, we aim to investigate this construction and look for potential applications, especially to transcendental holomorphic Morse inequalities ...

**Tubular neighborhoods (thanks to Grauert)**

Let $X$ be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_X = \mathcal{O}_{\overline{X}}$.

The diagonal of $X \times \overline{X}$ is totally real, and by Grauert, we know that it possesses a fundamental system of Stein tubular neighborhoods.

Assume that $X$ is equipped with a real analytic hermitian metric $\gamma$, and let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.
Lemma
Denote by $\text{exph}$ the “holomorphic” part of $\exp$, so that for $z \in X$ and $\xi \in T_X, z$

$$\exp_z(\xi) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha \beta}(z) \xi^\alpha \xi^\beta, \quad \text{exph}_z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha 0}(z) \xi^\alpha.$$ 

Then $d\xi \exp_z(\xi)\xi = 0 = d\xi \text{exph}_z(\xi)\xi = 0 = \text{Id}_{T_X}$, and so $\text{exph}$ is a diffeomorphism from a neighborhood $V$ of the 0 section of $T_X$ to a neighborhood $V'$ of the diagonal in $X \times X$.

Notation
With the identification $\overline{X} \simeq_{\text{diff}} X$, let $\text{logh} : X \times \overline{X} \supset V' \to T_{\overline{X}}$ be the inverse diffeomorphism of $\text{exph}$ and

$$U_\varepsilon = \{(z, w) \in V' \subset X \times \overline{X} ; |\text{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$ 

Then, for $\varepsilon \ll 1$, $U_\varepsilon$ is Stein and $\text{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

Such tubular neighborhoods are Stein

In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n ; |z - w| < \varepsilon\}$. It is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2 \text{Re} \sum z_j w_j$$

and $(z, w) \mapsto \text{Re} \sum z_j w_j$ is pluriharmonic.
Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (pr_1)|_{U_\varepsilon} : U_\varepsilon \to X, \quad \overline{p} = (pr_2)|_{U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections.

Bergman sheaves

**Definition of the Bergman sheaf $B_\varepsilon$**

The Bergman sheaf $B_\varepsilon = B_{\gamma,\varepsilon}$ is by definition the $L^2$ direct image

$$B_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})),$$

i.e. the space of sections over an open subset $V \subset X$ defined by $B_\varepsilon(V) = \text{holomorphic sections } f \text{ of } \overline{p}^*\mathcal{O}(K_{\overline{X}}) \text{ on } p^{-1}(V),

$$f(z, w) = f_1(z, w) \, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V:

$$\int_{p^{-1}(K)} i^n f(z, w) \wedge \overline{f(z, w)} \wedge \gamma(z)^n < +\infty, \quad \forall K \Subset V.$$

(This $L^2$ condition is the reason we speak of “$L^2$ direct image”).

Clearly, $B_\varepsilon$ is an $\mathcal{O}_X$-module over $X$, but since it is a space of functions in $w$, it is of infinite rank.
**Definition of the associated Bergman bundle $B_{\varepsilon}$**

We consider the vector bundle $B_{\varepsilon} \to X$ whose fiber $B_{\varepsilon, z_0}$ consists of all holomorphic functions $f$ on $p^{-1}(z_0) \subset U_{\varepsilon}$ such that

$$\|f(z_0)\|^2 = \int_{p^{-1}(z_0)} i^n f(z_0, w) \wedge \bar{f}(z_0, w) < +\infty.$$ 

Then $B_{\varepsilon}$ is a real analytic locally trivial Hilbert bundle whose fiber $B_{\varepsilon, z_0}$ is isomorphic to the Hardy-Bergman space $H^2(B(0, \varepsilon))$ of $L^2$ holomorphic $n$-forms on $p^{-1}(z_0) \simeq B(0, \varepsilon) \subset \mathbb{C}^n$.

The Ohsawa-Takegoshi extension theorem implies that every $f \in B_{\varepsilon, z_0}$ can be extended as a germ $\tilde{f}$ in the sheaf $B_{\varepsilon, z_0}$.

Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $B_{\varepsilon', z_0} \to B_{\varepsilon, z_0}$ such that $B_{\varepsilon, z_0}$ is the $L^2$ completion of $B_{\varepsilon', z_0} / m_{z_0} B_{\varepsilon', z_0}$.

**Question**

Is there a “complex structure” on $B_{\varepsilon}$ such that “$B_{\varepsilon} = \mathcal{O}(B_{\varepsilon})$”? 

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**Bergman Dolbeault complex**

For this, consider the “Bergman Dolbeault” complex $\bar{\partial} : \mathcal{F}_{\varepsilon}^q \to \mathcal{F}_{\varepsilon}^{q+1}$ over $X$, with $\mathcal{F}_{\varepsilon}^q(V) = \text{smooth } (n, q)$-forms

$$f(z, w) = \sum_{|J|=q} f_J(z, w) dw_1 \wedge ... \wedge dw_n \wedge d\bar{z}_J, \quad (z, w) \in U_{\varepsilon} \cap (V \times \overline{X}),$$

such that $f_J(z, w)$ is holomorphic in $w$, and for all $K \subseteq V$ one has

$$f(z, w) \in L^2(p^{-1}(K)) \quad \text{and} \quad \bar{\partial}_z f(z, w) \in L^2(p^{-1}(K)).$$

An immediate consequence of this definition is:

**Proposition**

$\bar{\partial} = \bar{\partial}_z$ yields a complex of sheaves $(\mathcal{F}^\bullet_{\varepsilon}, \bar{\partial})$, and the kernel $\text{Ker } \bar{\partial} : \mathcal{F}_{\varepsilon}^0 \to \mathcal{F}_{\varepsilon}^1$ coincides with $B_{\varepsilon}$.

If we define $\mathcal{O}_{L^2}(B_{\varepsilon})$ to be the sheaf of $L^2_{\text{loc}}$ sections $f$ of $B_{\varepsilon}$ such that $\bar{\partial} f = 0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_{\varepsilon}) = B_{\varepsilon}$ as a sheaf.
Bergman sheaves are “very ample”

Theorem

Assume that \( \varepsilon > 0 \) is taken so small that \( \psi(z, w) := |\log h_z(w)|^2 \) is strictly plurisubharmonic up to the boundary on the compact set \( \overline{U}_\varepsilon \subset X \times \overline{X} \). Then the complex of sheaves \( (\mathcal{F}_\varepsilon^\bullet, \overline{\partial}) \) is a resolution of \( \mathcal{B}_\varepsilon \) by soft sheaves over \( X \) (actually, by \( \mathcal{C}^\infty_X \)-modules), and for every holomorphic vector bundle \( E \to X \) we have

\[
H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q(\Gamma(X, \mathcal{F}_\varepsilon^\bullet \otimes \mathcal{O}(E)), \overline{\partial}) = 0, \quad \forall q \geq 1.
\]

Moreover the fibers \( \mathcal{B}_{\varepsilon, z} \otimes E_z \) are always generated by global sections of \( H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) \).

In that sense, \( \mathcal{B}_\varepsilon \) is a “very ample holomorphic vector bundle” (as a Hilbert bundle of infinite dimension).

The proof is a direct consequence of Hörmander’s \( L^2 \) estimates.

Caution !!

\( \mathcal{B}_\varepsilon \) is NOT a locally trivial holomorphic bundle.

Embedding into a Hilbert Grassmannian

Corollary of the very ampleness of Bergman sheaves

Let \( X \) be an arbitrary compact complex manifold, \( E \to X \) a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space \( \mathbb{H} = H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) \). Then one gets a “holomorphic embedding” into a Hilbert Grassmannian,

\[
\Psi : X \to \text{Gr}(\mathbb{H}), \quad z \mapsto S_z,
\]

mapping every point \( z \in X \) to the infinite codimensional closed subspace \( S_z \) consisting of sections \( f \in \mathbb{H} \) such that \( f(z) = 0 \) in \( \mathcal{B}_{\varepsilon, z} \), i.e. \( f|_{p^{-1}(z)} = 0 \).

The main problem with this “holomorphic embedding” is that the holomorphicity is to be understood in a weak sense, for instance the map \( \Psi \) is not even continuous with respect to the strong metric topology of \( \text{Gr}(\mathbb{H}) \), given by \( d(S, S') = \text{Hausdorff distance of the unit balls of } S, S' \).
Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \bar{\partial}$ connection on $B_\varepsilon$, and a natural hermitian metric as well, it follows from the usual formalism that $B_\varepsilon$ can be equipped with a unique Chern connection.

**Model case:** $X = \mathbb{C}^n$, $\gamma =$ standard hermitian metric.

Then one sees that an orthonormal frame of $B_\varepsilon$ is given by

$$e_\alpha(z, w) = \pi^{-n/2} e^{-|\alpha|-n} \sqrt{(|\alpha| + n)!} \alpha_1! \cdots \alpha_n! (w - z)^\alpha, \quad \alpha \in \mathbb{N}^n.$$

This frame is non holomorphic! The $(0, 1)$-connection $\nabla^{0,1} = \bar{\partial}$ is given by

$$\nabla^{0,1} e_\alpha = \bar{\partial}_z e_\alpha(z, w) = \varepsilon^{-1} \sum_{1 \leq j \leq n} \sqrt{\alpha_j(|\alpha| + n)} d\bar{z}_j \otimes e_{\alpha-c_j}$$

where $c_j = (0, ..., 1, ..., 0) \in \mathbb{N}^n$.

Curvature of Bergman bundles

Let $\Theta_{B_\varepsilon, h} = \nabla^2$ be the curvature tensor of $B_\varepsilon$ with its natural Hilbertian metric $h$. Remember that

$$\Theta_{B_\varepsilon, h} = \nabla^{1,0} \nabla^{0,1} + \nabla^{0,1} \nabla^{1,0} \in C^\infty(X, \Lambda^{1,1} T^*_X \otimes \text{Hom}(B_\varepsilon, B_\varepsilon)),$$

and that one gets an associated quadratic Hermitian form on $T_X \otimes B_\varepsilon$ such that

$$\tilde{\Theta}_\varepsilon(v \otimes \xi) = \langle \Theta_{B_\varepsilon, h} \sigma(v, Jv) \xi, \xi \rangle_h$$

for $v \in T_X$ and $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

**Definition**

One says that the curvature tensor is **Griffiths positive** if

$$\tilde{\Theta}_\varepsilon(v \otimes \xi) > 0, \quad \forall 0 \neq v \in T_X, \forall 0 \neq \xi \in B_\varepsilon,$$

and **Nakano positive** if

$$\tilde{\Theta}_\varepsilon(\tau) > 0, \quad \forall 0 \neq \tau \in T_X \otimes B_\varepsilon.$$
Calculation of the curvature tensor for \( X = \mathbb{C}^n \)

A simple calculation of \( \nabla^2 \) in the orthonormal frame \((e_\alpha)\) leads to:

**Formula**

In the model case \( X = \mathbb{C}^n \), the curvature tensor of the Bergman bundle \((B_\varepsilon, h)\) is given by

\[
\tilde{\Theta}_\varepsilon(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \left( \left| \sum_j \sqrt{\alpha_j} \xi_{\alpha - c_j} v_j \right|^2 + \sum_j (|\alpha| + n) |\xi_{\alpha}|^2 |v_j|^2 \right).
\]

**Consequence**

In \( \mathbb{C}^n \), the curvature tensor \( \Theta_\varepsilon(v \otimes \xi) \) is Nakano positive.

On should observe that \( \tilde{\Theta}_\varepsilon(v \otimes \xi) \) is an **unbounded** quadratic form on \( B_\varepsilon \) with respect to the standard metric \( \|\xi\|^2 = \sum_{\alpha} |\xi_{\alpha}|^2 \).

However there is convergence for all \( \xi = \sum_{\alpha} \xi_{\alpha} e_\alpha \in B_{\varepsilon'}, \varepsilon' > \varepsilon \), since then

\[
\sum_{\alpha} (\varepsilon'/\varepsilon)^2 |\alpha| |\xi_{\alpha}|^2 < +\infty.
\]

Curvature of Bergman bundles (general case)

**Bergman curvature formula on a general hermitian manifold**

Let \( X \) be a compact complex manifold equipped with a \( C^\omega \) hermitian metric \( \gamma \), and \( B_\varepsilon = B_{\gamma, \varepsilon} \) the associated Bergman bundle.

Then its curvature is given by an asymptotic expansion

\[
\tilde{\Theta}_\varepsilon(z, v \otimes \xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v \otimes \xi), \quad v \in T_X, \xi \in B_\varepsilon
\]

where \( Q_0(z, v \otimes \xi) = Q_0(v \otimes \xi) \) is given by the model case \( \mathbb{C}^n \):

\[
Q_0(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \left( \left| \sum_j \sqrt{\alpha_j} \xi_{\alpha - c_j} v_j \right|^2 + \sum_j (|\alpha| + n) |\xi_{\alpha}|^2 |v_j|^2 \right).
\]

The other terms \( Q_p(z, v \otimes \xi) \) are real analytic; \( Q_1 \) and \( Q_2 \) depend respectively on the torsion and curvature tensor of \( \gamma \).

In particular \( Q_1 = 0 \) is \( \gamma \) is Kähler.

A consequence of the above formula is that \( B_\varepsilon \) is strongly Nakano positive for \( \varepsilon > 0 \) small enough.
Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by Wang Xu, expressing the curvature of weighted Bergman bundles $\mathcal{H}_t$ attached to a smooth family $\{D_t\}$ of strongly pseudoconvex domains. Wang’s formula is however in integral form and not completely explicit.

Here, one simply uses the real analytic Taylor expansion of $\log h : X \times \overline{X} \to T_X$ (inverse diffeomorphism of $\exp h$)

$$\log h_z(w) = w - z + \sum z_j a_j(w - z) + \sum \overline{z}_j a'_j(w - z)$$

$$+ \sum z_j z_k b_{jk}(w - z) + \sum \overline{z}_j \overline{z}_k b'_{jk}(w - z)$$

$$+ \sum z_j \overline{z}_k c_{jk}(w - z) + O(|z|^3),$$

which is used to compute the difference with the model case $\mathbb{C}^n$, for which $\log h_z(w) = w - z$.

Back to holomorphic Morse inequalities

**Idea for the general case.** Let $\gamma \in H^{1,1}_{BC}(X, \mathbb{R})$ and $u \in \gamma$ a smooth form. As we have seen, one can find a sequence of Hermitian line bundles $(L_k, h_k, \nabla_k)$ such that

$$\theta_k = \frac{i}{2\pi} \nabla_k^2 = ku + \beta_k, \quad \beta_k = O(k^{-1/b_2}).$$

Then $d\theta_k = 0 \Rightarrow \overline{\partial} \beta_k = 0$, and as $U_\varepsilon$ is Stein, $pr_1^* \beta_k = \overline{\partial} \eta_k$ with a $C^\infty (0,1)$-form $\eta_k = O(k^{-1/b_2})$. This shows that $\tilde{L}_k := pr_1^* L_k$ becomes a holomorphic line bundle when equipped with the connection

$$\tilde{\nabla}_k = pr_1^* \nabla_k - \eta_k,$$

which has a curvature form

$$\Theta_{\tilde{L}_k, \tilde{\nabla}_k} = k pr_1^* u + O(k^{-1/b_2}).$$

Two possibilities emerge:

- correct the small eigenvalue eigenfunctions $pr_1^* \sigma_{k,\ell}$ given by Laeng’s method to actually get holomorphic sections of $\tilde{L}_k$ on $U_\varepsilon$.

- directly deal with the Hilbert Dolbeault complex of $(pr_1)^*_L (\mathcal{O}_{U_\varepsilon} (\tilde{L}_k))$, and use Bergman estimates instead of dimension counts in Morse inequalities.
Other potential target: invariance of plurigenera for polarized families of compact Kähler manifolds?

**Conjecture**

Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base $S$. Assume that the family admits a polarization, i.e. a closed smooth $(1,1)$-form $\omega$ such that $\omega|_{X_t}$ is positive definite on each fiber $X_t := \pi^{-1}(t)$. Then the plurigenera

$$p_m(X_t) = h^0(X_t, mK_{X_t})$$

are independent of $t$ for all $m \geq 0$.

The conjecture is known to be true for a projective family $\mathcal{X} \to S$:

- Siu and Kawamata (1998) in the case of varieties of general type

The proof is based on an iterated application of the Ohsawa-Takegoshi $L^2$ extension theorem w.r.t. an ample line bundle $A$ on $\mathcal{X}$: replace $A$ by a Bergman bundle in the Kähler case?

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End of 27/28

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Thank you for your attention