L^2 estimates for the $\overline{\partial}$-operator on complex manifolds

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Abstract. The main goal of these notes is to describe a powerful differential geometric method which yields precise existence theorems for solutions of equations $\overline{\partial}u = v$ on (pseudoconvex) complex manifolds. The main idea is to combine Hilbert space techniques with a geometric identity known as the Bochner-Kodaira-Nakano identity. The BKN identity relates the complex Laplace operators Δ' and Δ'' associated to ∂ and $\overline{\partial}$ with a suitable curvature tensor. The curvature tensor reflects the convexity properties of the manifold, from the viewpoint of complex geometry. In this way, under suitable convexity assumptions, one is able to derive existence theorems for holomorphic functions subject to certain constraints (in the form of L^2 estimates). The central ideas go back to Kodaira and Nakano (1954) in the case of compact manifolds, and to Androtti-Vesentini and Hörmander (1965) in the case of open manifolds with plurisubharmonic weights. Hörmander's estimates can be used for instance to give a quick solution of the Levi problem. They have many other important applications to complex analysis, complex geometry, local algebra and algebraic geometry... Important variants of these estimates have been developped in the last two decades. The first ones are the L^2 estimates of Skoda (1972, 1978), which deal with the problem of solving "Bezout identities" $\sum f_j g_j = h$ when g_j and h are given holomorphic functions and the f_j's are the unknowns. The last ones are the L^2 estimates of Ohsawa-Takegoshi (1987), which concern the problem of extending a holomorphic function given on a submanifold $Y \subset X$ to the whole manifold X. Our task will be to explain the main techniques leading to all three types of L^2-estimates (Hörmander, Skoda, Ohsawa-Takegoshi), and to present a few applications.

Contents

1. Non bounded operators on Hilbert spaces

A few preliminary results of functional analysis are needed. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be complex Hilbert spaces. We consider a linear operator T defined on a subspace $\mathrm{Dom}\,T \subset \mathcal{H}_1$ (called the domain of T) into $\mathcal{H}_2$. The operator T is said to be *densely defined* if $\mathrm{Dom}\,T$ is dense in $\mathcal{H}_1$, and *closed* if its graph

$$\mathrm{Gr}\,T = \big\{(x, Tx)\ ;\ x \in \mathrm{Dom}\,T\big\}$$

is closed in $\mathcal{H}_1 \times \mathcal{H}_2$.

Assume now that T is closed and densely defined. The adjoint $T^\star$ of T (in Von Neumann's sense) is constructed as follows: $\mathrm{Dom}\,T^\star$ is the set of $y \in \mathcal{H}_2$ such that the linear form

$$\mathrm{Dom}\,T \ni x \longmapsto \langle Tx, y\rangle_2$$

is bounded in $\mathcal{H}_1$-norm. Since $\mathrm{Dom}\,T$ is dense, there exists for every y in $\mathrm{Dom}\,T^\star$ a unique element $T^\star y \in \mathcal{H}_1$ such that $\langle Tx, y\rangle_2 = \langle x, T^\star y\rangle_1$ for all $x \in \mathrm{Dom}\,T^\star$. It is immediate to verify that $\mathrm{Gr}\,T^\star = \big(\mathrm{Gr}(-T)\big)^\perp$ in $\mathcal{H}_1 \times \mathcal{H}_2$. It follows that $T^\star$ is closed and that every pair $(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$ can be written

$$(u, v) = (x, -Tx) + (T^\star y, y), \qquad x \in \mathrm{Dom}\,T, \quad y \in \mathrm{Dom}\,T^\star.$$

Take in particular $u = 0$. Then

$$x + T^\star y = 0, \qquad v = y - Tx = y + TT^\star y, \qquad \langle v, y\rangle_2 = \|y\|_2^2 + \|T^\star y\|_1^2.$$

If $v \in (\mathrm{Dom}\,T^\star)^\perp$ we get $\langle v, y\rangle_2 = 0$, thus $y = 0$ and $v = 0$. This implies $(\mathrm{Dom}\,T^\star)^\perp = 0$, hence $T^\star$ is densely defined and our discussion yields

(1.1) Theorem (Von Neumann 1933)**.** *If $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a closed and densely defined operator, its adjoint $T^\star$ is also closed and densely defined and $(T^\star)^\star = T$. Furthermore, we have the relation* $\mathrm{Ker}\,T^\star = (\mathrm{Im}\,T)^\perp$ *and its dual* $(\mathrm{Ker}\,T)^\perp = \overline{\mathrm{Im}\,T^\star}$. □

Consider now two closed and densely defined operators T, S :

$$\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3$$

such that $S \circ T = 0$. By this, we mean that the range $T(\mathrm{Dom}\,T)$ is contained in $\mathrm{Ker}\,S \subset \mathrm{Dom}\,S$, in such a way that there is no problem for defining the composition $S \circ T$. The starting point of all L^2 estimates is the following abstract existence theorem.

(1.2) Theorem. *There are orthogonal decompositions*

$$\begin{aligned} \mathcal{H}_2 &= (\mathrm{Ker}\,S \cap \mathrm{Ker}\,T^\star) \oplus \overline{\mathrm{Im}\,T} \oplus \overline{\mathrm{Im}\,S^\star}, \\ \mathrm{Ker}\,S &= (\mathrm{Ker}\,S \cap \mathrm{Ker}\,T^\star) \oplus \overline{\mathrm{Im}\,T}. \end{aligned}$$

In order that $\mathrm{Im}\,T = \mathrm{Ker}\,S$*, it suffices that*

$$\|T^\star x\|_1^2 + \|Sx\|_3^2 \geqslant C\|x\|_2^2, \qquad \forall x \in \mathrm{Dom}\, S \cap \mathrm{Dom}\, T^\star \tag{1.3}$$

for some constant $C > 0$. In that case, for every $v \in \mathcal{H}_2$ such that $Sv = 0$, there exists $u \in \mathcal{H}_1$ such that $Tu = v$ and

$$\|u\|_1^2 \leqslant \frac{1}{C}\|v\|_2^2.$$

In particular

$$\overline{\mathrm{Im}\, T} = \mathrm{Im}\, T = \mathrm{Ker}\, S, \qquad \overline{\mathrm{Im}\, S^\star} = \mathrm{Im}\, S^\star = \mathrm{Ker}\, T^\star.$$

Proof. Since S is closed, the kernel $\mathrm{Ker}\, S$ is closed in $\mathcal{H}_2$. The relation $(\mathrm{Ker}\, S)^\perp = \overline{\mathrm{Im}\, S^\star}$ implies

$$\mathcal{H}_2 = \mathrm{Ker}\, S \oplus \overline{\mathrm{Im}\, S^\star} \tag{1.4}$$

and similarly $\mathcal{H}_2 = \mathrm{Ker}\, T^\star \oplus \overline{\mathrm{Im}\, T}$. However, the assumption $S \circ T = 0$ shows that $\overline{\mathrm{Im}\, T} \subset \mathrm{Ker}\, S$, therefore

$$\mathrm{Ker}\, S = (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^\star) \oplus \overline{\mathrm{Im}\, T}. \tag{1.5}$$

The first two equalities in Th. 1.2 are then equivalent to the conjunction of (1.4) and (1.5).

Now, under assumption (1.3), we are going to show that the equation $Tu = v$ is always solvable if $Sv = 0$. Let $x \in \mathrm{Dom}\, T^\star$. One can write

$$x = x' + x'' \quad \text{where} \quad x' \in \mathrm{Ker}\, S \quad \text{and} \quad x'' \in (\mathrm{Ker}\, S)^\perp \subset (\mathrm{Im}\, T)^\perp = \mathrm{Ker}\, T^\star.$$

Since $x, x'' \in \mathrm{Dom}\, T^\star$, we have also $x' \in \mathrm{Dom}\, T^\star$. We get

$$\langle v, x\rangle_2 = \langle v, x'\rangle_2 + \langle v, x''\rangle_2 = \langle v, x'\rangle_2$$

because $v \in \mathrm{Ker}\, S$ and $x'' \in (\mathrm{Ker}\, S)^\perp$. As $Sx' = 0$ and $T^\star x'' = 0$, the Cauchy-Schwarz inequality combined with (1.3) implies

$$|\langle v, x\rangle_2|^2 \leqslant \|v\|_2^2\ \|x'\|_2^2 \leqslant \frac{1}{C}\|v\|_2^2\ \|T^\star x'\|_1^2 = \frac{1}{C}\|v\|_2^2\ \|T^\star x\|_1^2.$$

This shows that the linear form $T_X^\star \ni x \longmapsto \langle x, v\rangle_2$ is continuous on $\mathrm{Im}\, T^\star \subset \mathcal{H}_1$ with norm $\leqslant C^{-1/2}\|v\|_2$. By the Hahn-Banach theorem, this form can be extended to a continuous linear form on $\mathcal{H}_1$ of norm $\leqslant C^{-1/2}\|v\|_2$, i.e. we can find $u \in \mathcal{H}_1$ such that $\|u\|_1 \leqslant C^{-1/2}\|v\|_2$ and

$$\langle x, v\rangle_2 = \langle T^\star x, u\rangle_1, \qquad \forall x \in \mathrm{Dom}\, T^\star.$$

This means that $u \in \mathrm{Dom}(T^\star)^\star = \mathrm{Dom}\, T$ and $v = Tu$. We have thus shown that $\mathrm{Im}\, T = \mathrm{Ker}\, S$, in particular $\mathrm{Im}\, T$ is closed. The dual equality $\mathrm{Im}\, S^\star = \mathrm{Ker}\, T^\star$ follows by considering the dual pair $(S^\star, T^\star)$. □

2. Basic concepts of complex analysis in several variables

For more details on the concepts introduced here, we refer to Thierry Bouche's lecture notes. Let X be a n-dimensional complex manifold and let $(z_1, \ldots, z_n)$ be holomorphic local coordinates on some open set $\Omega \subset X$ (we usually think of Ω as being just an open set in $\mathbb{C}^n$). We write $z_j = x_j + \mathrm{i}y_j$ and

$$dz_j = dx_j + \mathrm{i}dy_j, \qquad d\overline{z}_j = dx_j - \mathrm{i}dy_j. \tag{2.1}$$

(Complex) differential forms over X can be defined to be linear combinations

$$\sum c_{\alpha_1\ldots\alpha_\ell,\beta_1\ldots\beta_m} dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_\ell} \wedge dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_m}$$

with complex coefficients. Since $dx_j = \frac{1}{2}(dz_j + d\overline{z}_j)$ and $dy_j = \frac{1}{2\mathrm{i}}(dz_j - d\overline{z}_j)$, we can rearrange the wedge products as products in the complex linear forms dz_j (such that $dz_j(\xi) = \xi_j$) and the conjugate linear forms $d\overline{z}_j$ (such that $d\overline{z}_j(\xi) = \overline{\xi}_j$). A (p, q)-form is a differential form of total degree $p + q$ with complex coefficients, which can be written as

$$u(z) = \sum_{|I|=p,\, |J|=q} u_{IJ}(z)\, dz_I \wedge d\overline{z}_J, \tag{2.2}$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multiindices (arranged in increasing order) and

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \qquad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

The vector bundle of complex valued (p, q)-forms over X will be denoted by $\Lambda^{p,q}T^\star_X$. In this setting, the differential of a C^1 function f can be expressed as

$$df = \sum_{1\leqslant j\leqslant n} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = \sum_{1\leqslant j\leqslant n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

where

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} - \mathrm{i}\frac{\partial f}{\partial y_j}\Big), \qquad \frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + \mathrm{i}\frac{\partial f}{\partial y_j}\Big).$$

We thus get $df = d'f + d''f$ (or $df = \partial f + \overline{\partial} f$ in British-American style), where

$$d'f = \sum_{1\leqslant j\leqslant n} \frac{\partial f}{\partial z_j} dz_j, \qquad \text{resp.}\ \ d''f = \sum_{1\leqslant j\leqslant n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

is $\mathbb{C}$-linear (resp. conjugate $\mathbb{C}$-linear). We say that f is holomorphic if df is $\mathbb{C}$-linear, or, in an equivalent way, if $d''f = 0$ (*Cauchy-Riemann equation*). More generally, the exterior derivative du of the (p, q)-form u is

$$du = \sum_{|I|=p,\, |J|=q,\, 1\leqslant k\leqslant n} \Big(\frac{\partial u_{IJ}}{\partial z_k} dz_k + \frac{\partial u_{IJ}}{\partial \overline{z}_k} d\overline{z}_k\Big) dz_I \wedge d\overline{z}_J.$$

We may therefore write $du = d'u + d''u$ with uniquely defined forms $d'u$ of type $(p + 1, q)$ and $d''u$ of type $(p, q + 1)$, such that

$$(2.3') \qquad d'u = \sum_{|I|=p,\,|J|=q,\,1\leqslant k\leqslant n} \frac{\partial u_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$(2.3'') \qquad d''u = \sum_{|I|=p,\,|J|=q,\,1\leqslant k\leqslant n} \frac{\partial u_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

The operators $d'' = \overline{\partial}$ can be viewed as linear differential operators acting on the bundles of complex (p,q)-forms (see § 4). As

$$0 = d^2 = (d' + d'')^2 = d'^2 + (d'd'' + d''d') + d''^2$$

where each of the three components are of different types, we get the identities

$$(2.4) \qquad d'^2 = 0, \qquad d''^2 = 0, \qquad d'd'' + d''d' = 0.$$

Moreover, d' and d'' are *conjugate*, i.e., $\overline{d'u} = d''\overline{u}$ for any (p,q)-form u on X. A basic result is the so-called Dolbeault-Grothendieck lemma, which is the complex analogue of the Poincaré lemma.

(2.5) Dolbeault-Grothendieck lemma. *Let $v = \sum_{|J|=q} v_J d\overline{z}_J$, $q \geqslant 1$, be a smooth form of bidegree $(0,q)$ on a polydisk $\Omega = D(0,R) = D(0,R_1)\times\cdots\times D(0,R_n)$ in $\mathbb{C}^n$. Then there is a smooth $(0,q-1)$-form u on Ω such that $d''u = v$ on Ω.*

Proof. We first show that a solution u exists on any smaller polydisk $D(0,r) \Subset \Omega$, $r_j < R_j$. Let k be the smallest integer such that the monomials $d\overline{z}_J$ appearing in v only involve $d\overline{z}_1$, …, $d\overline{z}_k$. We prove by induction on k that the equation $d''u = v$ can be solved on the polydisk $D(0,r)$. If $k = 0$, then $v = 0$ and there is nothing to prove, whilst $k = n$ is the desired result. Now, assume that the result has been settled for $k-1$, that v only involves $d\overline{z}_1, \ldots, d\overline{z}_k$, and set

$$v = d\overline{z}_k \wedge f + g, \qquad f = \sum_{|J|=q-1} f_J d\overline{z}_J, \quad g = \sum_{|J|=q} g_J d\overline{z}_J$$

where f, g only involve $d\overline{z}_1$, …, $d\overline{z}_{k-1}$. The assumption $d''v = 0$ implies

$$d''v = -d\overline{z}_k \wedge d''f + d''g = 0$$

where $d\overline{z}_k \wedge d''f$ involves terms $\partial f_J/\partial\overline{z}_\ell \; d\overline{z}_k \wedge d\overline{z}_\ell \wedge d\overline{z}_J$, $\ell > k$, and $d''g$ can only involve one factor $d\overline{z}_\ell$ with an index $\ell \geqslant k$. From this we conclude that $\partial f_J/\partial z_\ell = 0$ for $\ell > k$. Hence the coefficients f_J are holomorphic in $z_{k+1}, \ldots, z_n$. Now, let us consider the $(0,q-1)$-form

$$F = \sum_{|J|=q-1} F_J d\overline{z}_J, \qquad F_J(z) = \big(\psi(z_k) f_J(z)\big) \star_k \Big(\frac{1}{\pi z_k}\Big),$$

where $\psi(z_k)$ is a cut-off function with support in $D(0,R_k)$, equal to 1 on some disk $D(0,r'_k)$, $r'_k \in \,]r_k, R_k[$, and $\star_k$ denotes a partial convolution with respect to z_k. In other words,

$$\begin{aligned} F_J(z) &= \int_{w\in D(0,R_j)} \psi(w) f_J(z_1,\ldots,z_{k-1},w,z_{k+1},\ldots,z_n)\frac{1}{\pi(z_k-w)}d\lambda(w). \\ &= \int_{w\in\mathbb{C}} \psi(z_k-w) f_J(z_1,\ldots,z_{k-1},z_k-w,z_{k+1},\ldots,z_n)\frac{1}{\pi w}d\lambda(w). \end{aligned}$$

It follows from differentiation under integral sign that F_J is a smooth function on Ω which is holomorphic in all variables $z_{k+1},\ldots,z_n$. Moreover, as $\frac{1}{\pi z}$ is a fundamental solution of $\frac{\partial}{\partial\overline{z}}$ in $\mathbb{C}$ (that is, $\frac{\partial}{\partial\overline{z}}(\frac{1}{\pi z}) = \delta_0$), we see that

$$\frac{\partial}{\partial\overline{z}_k}F_J(z) = \psi(z_k) f_J(z_1,\ldots,z_{k-1},z_k,z_{k+1},\ldots,z_n),$$

in particular $\frac{\partial}{\partial\overline{z}_k}F_J = f_J$ on some polydisk $D(0,r')$, $r'_j\in\,]r_j,R_j[$. Therefore

$$d''F = \sum_{|J|=q-1,\,1\leqslant\ell\leqslant k}\frac{\partial f_J}{\partial\overline{z}_\ell}d\overline{z}_\ell\wedge d\overline{z}_J = d\overline{z}_k\wedge f + g_1$$

where g_1 is a $(0,q)$ form which only involves $d\overline{z}_1,\ldots,d\overline{z}_{k-1}$. Hence

$$v_1 := v - d''F = (d\overline{z}_k\wedge f + g) - (d\overline{z}_k\wedge f + g_1) = g - g_1$$

only involves $d\overline{z}_1,\ldots,d\overline{z}_{k-1}$. As v_1 is again a d''-closed form, the induction hypothesis applied on $D(0,r')$ shows that we can find a smooth $(0,q-1)$-form u_1 on $D(0,r)$ such that $d''u_1 = v_1$. Therefore $v = d''(F+u_1)$ on $D(0,r)$, and we have thus found a solution $u = F + u_1$ on $D(0,r)\Subset\Omega$.

To conclude the proof, we now show by induction on q that one can find a solution u defined on all of $\Omega = D(0,R)$. Set $R_\nu = (R_1-2^{-\nu},\ldots,R_n-2^{-\nu})$. By what we have proved above, there exists a smooth solution $u_\nu\in D(0,R_{(\nu)})$ of the equation $d''u_\nu = v$. Now, if $q=1$, we get $d''(u_{\nu+1}-u_\nu) = 0$ on $D(0,R_{(\nu)})$, i.e., $u_{\nu+1}-u_\nu$ is holomorphic on $D(0,R_{(\nu)})$. By looking at its Taylor expansion at 0, we get a polynomial P_ν (equal to the sum of all terms in the Taylor expansion up to a certain degree) such that $|u_{\nu+1}-u_\nu-P_\nu|\leqslant 2^{-\nu}$ on $D(0,R_{(\nu-1)})\Subset D(0,R_{(\nu)})$. If we set $\widetilde{u}_\nu = u_\nu + P_1+\cdots+P_{\nu-1}$, then $\widetilde{u}_\nu$ is a uniform Cauchy sequence on every compact subset of $D(0,R)$. Since $\widetilde{u}_{\nu+1}-\widetilde{u}_\nu$ is holomorphic on $D(0,R_{(\nu)})$, we conclude that the limit u is smooth and satisfies $d''u = d''u_\nu = v$ on $D(0,R_{(\nu)})$ for every ν, QED. Now, if $q\geqslant 2$, the difference $u_{\nu+1}-u_\nu$ is d''-closed of degree $q-1\geqslant 1$ on $D(0,R_{(\nu)})$. Hence, by the induction hypothesis, we can find a $(0,q-2)$ form w_ν on $D(0,R_{(\nu)})$ such that $u_{\nu+1}-u_\nu = d''w_\nu$. If we replace inductively $u_{\nu+1}$ by $u_{\nu+1}-d''(\psi_\nu w_\nu)$ where ψ_ν is a cut-off function with support in $D(0,R_{(\nu)})$ equal to 1 on $D(0,R_{(\nu-1)})$, we see that we take arrange the sequence so that $u_{\nu+1}$ coincides with u_ν on $D(0,R_{(\nu-1)})$. Hence we get a stationary sequence converging towards a limit u such that $d''u = v$. □

We now introduce the concept of cohomology group. A *differential complex* is a graded module $K^\bullet = \bigoplus_{q\in\mathbb{Z}} K^q$ over some (commutative) ring R, together with a differential $d : K^\bullet \to K^\bullet$ of degree 1, that is, a R-linear map such that $d = d^q : K^q\to K^{q+1}$ on K^q and $d^2 = 0$ (i.e., $d^{q+1}\circ d^q = 0$ for every q). One defines the *cocycle* and *coboundary* modules to be

$$(2.6_Z) \qquad Z^q(K^\bullet) = \mathrm{Ker}(d^q : K^q \to K^{q+1}),$$
$$(2.6_B) \qquad B^q(K^\bullet) = \mathrm{Im}(d^{q-1} : K^{q-1} \to K^q).$$

The assumption $d^2 = 0$ immediately shows that $B^q(K^\bullet) \subset Z^q(K^\bullet)$, and one defines the *q-th cohomology group* of $K^\bullet$ to be

$$(2.7) \qquad H^q(K^\bullet) = Z^q(K^\bullet)/B^q(K^\bullet).$$

A basic example is the *De Rham complex* $K^q = C^\infty(X, \Lambda^q T^\star_X)$ together with the exterior derivative d, defined whenever X is a smooth differentiable manifold. Its cohomology groups are denoted $H^q_{\mathrm{DR}}(X, \mathbb{R})$ (resp. $H^q_{\mathrm{DR}}(X, \mathbb{C})$ in the case of complex valued forms) and are called the De Rham cohomology groups of X. Here, we will be rather concerned with the complex case. If X is a complex n-dimensional manifold, we consider for each integer p fixed the *Dolbeault complex* $(K^{p,\bullet}, d'')$ defined by $K^{p,q} = C^\infty(X, \Lambda^{p,q} T^\star_X)$ together with the d''-exterior differential; its cohomology groups $H^{p,q}(X)$ are called the Dolbeault cohomology groups of X. More generally, let us consider a holomorphic vector bundle $E \to X$. This means that we have a collection of trivializations $E_{\restriction U_j} \simeq U_j \times \mathbb{C}^r$, $r = \operatorname{rank} E$, such that the transition matrices $g_{jk}(z)$ are holomorphic. We consider the complex $K^{p,q}_E = C^\infty(X, \Lambda^{p,q} T^\star_X \otimes E)$ of E-valued smooth (p,q)-forms with values in E. Again, $K^{p,q}_E$ possesses a canonical d''-operator. Indeed, if u is a smooth (p,q)-section of E represented by forms $u_j \in C^\infty(U_j, \Lambda^{p,q} T^\star_X \otimes \mathbb{C}^r)$ over the open sets U_j, we have the transition relation $u_j = g_{jk} u_k$; this relation implies $d'' u_j = g_{jk} d'' u_k$ (since $d'' g_{jk} = 0$), hence the collection $(d'' u_j)$ defines a unique global $(p, q+1)$-section $d'' u$. By definition, the Dolbeault cohomology groups of X with values in E are

$$(2.8) \qquad H^{p,q}(X, E) = H^q(K^{p,\bullet}_E, d'').$$

An important observation is that the Dolbeault complex $K^{p,\bullet}_E$ is identical to the Dolbeault complex $K^{0,\bullet}_{\Lambda^p T^\star_X \otimes E}$, thanks to the obvious equality

$$\Lambda^{p,q} T^\star_X \otimes E = \Lambda^{0,q} T^\star_X \otimes (\Lambda^p T^\star_X \otimes E)$$

and the fact that $\Lambda^p T^\star_X$ is itself a holomorphic vector bundle. In particular, we get an equality

$$(2.9) \qquad H^{p,q}(X, E) = H^{0,q}(X, \Lambda^p T^\star_X \otimes E).$$

If $X = \Omega$ is an open subset of $\mathbb{C}^n$, the bundle $\Lambda^p T^\star_\Omega \simeq \mathcal{O}_\Omega^{\oplus\binom{n}{n}}$ is isomorphic to a direct sum of $\binom{n}{p}$ copies of the trivial line bundle $\mathcal{O}_\Omega$, hence we simply get

$$H^{p,q}(\Omega, E) = H^{0,q}(\Omega, E) \otimes_{\mathbb{C}} \Lambda^p(\mathbb{C}^n)^\star = H^{0,q}(\Omega, E)^{\oplus\binom{n}{p}}.$$

In this setting, the Dolbeault-Grothendieck lemma can be restated:

(2.10) Corollary. *On every polydisk $D(0,R) = D(0,R_1) \times \cdots \times D(0,R_n) \subset \mathbb{C}^n$, we have $H^{p,q}(D(0,R), \mathcal{O}_{D(0,R)}) = 0$ for all $p \geqslant 0$ and $q \geqslant 1$.* □

We finally discuss some basic properties of plurisubharmonic functions. In complex geometry, plurisubharmonic functions play exactly the same role as convex functions do in real (affine) geometry. A function $\varphi : \Omega \to [-\infty, +\infty[$ on an open subset $\Omega \subset \mathbb{C}^n$ is said to be *plurisubharmonic* (usually abbreviated as psh) if φ is upper semicontinuous and satisfies the mean value inequality

$$\varphi(z_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \varphi(z_0 + a\, e^{i\theta})\, d\theta \tag{2.11}$$

for every $a \in \mathbb{C}^n$ such that the closed disk $z_0 + a\overline{\mathbb{D}}$ is contained in Ω (here $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$).

(2.12) Example. Every convex function φ on Ω is psh, since convexity implies continuity, and since the convexity inequality

$$\varphi(z_0) \leqslant \frac{1}{2}\big(\varphi(z_0 + a\, e^{i\theta}) + \varphi(z_0 - a\, e^{i\theta})\big)$$

implies (2.11) by computing the average over $\theta \in [0, \pi]$.

Given a closed (euclidean) ball $\overline{B}(z_0, r) \subset \Omega$, the spherical mean value $(\sigma_{2n-1} r^{2n-1})^{-1} \int_{z \in S(z_0,r)} \varphi(z)\, d\sigma(z)$ is equal to the average of the mean values computed on each circle $z_0 + a\partial\mathbb{D}$, when a describes the sphere $S(0, r)$. Hence, (2.11) implies the weaker mean value inequality

$$\varphi(z_0) \leqslant \frac{1}{\sigma_{2n-1} r^{2n-1}} \int_{S(z_0,r)} \varphi(z)\, d\sigma(z) \tag{2.13}$$

for every ball $\overline{B}(z_0, r) \subset \Omega$, in other words, every psh function is subharmonic (with respect to the Euclidean metric). Notice that (2.13) still implies the apparently weaker inequality

$$\varphi(z_0) \leqslant \frac{1}{v_{2n} r^{2n}} \int_{B(z_0,r)} \varphi(z)\, dV(z) \tag{2.13$'$}$$

by averaging again over all radii in the range $]0, r[$, with respect to the density $2n\, r^{2n-1} dr$ (in fact, one can show that the mean value properties (2.13) and (2.13′) are equivalent). As a consequence, we get inclusions

$$\mathrm{Conv}(\Omega) \subset \mathrm{Psh}(\Omega) \subset \mathrm{Sh}(\Omega) \tag{2.14}$$

where $\mathrm{Conv}(\Omega)$, $\mathrm{Psh}(\Omega)$, $\mathrm{Sh}(\Omega)$ are the spaces of convex, psh and subharmonic functions, respectively. Now, if X is a complex manifold, we say that a function $\varphi : X \to [-\infty, +\infty[$ is psh if φ is psh on every holomorphic coordinate patch Ω, when viewed as a function of the corresponding coordinates. In fact, Property 2.15 j) below shows that the plurisubharmonicity property does not depend on the choice of complex coordinates; this contrasts with convexity or subharmonicity, which do require an additional linear or riemannian structure, respectively.

(2.15) Basic properties of psh functions.

a) *For any decreasing sequence of psh functions $\varphi_k \in \mathrm{Psh}(X)$, the limit $\varphi = \lim \varphi_k$ is psh on X.*

b) *Let $(\varphi_j)_{j\in J}$ be a family of psh functions on X. Assume that $\varphi := \sup_{j\in J}\varphi_j$ is upper semicontinuous and locally bounded from above. Then φ is psh on X.*

c) *Let $\varphi_1,\ldots,\varphi_p \in \mathrm{Psh}(X)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1,\ldots,t_p)$ is increasing in each t_j. Then $\chi(\varphi_1,\ldots,\varphi_p)$ is psh on Ω. In particular $\varphi_1+\cdots+\varphi_p$, $\max\{\varphi_1,\ldots,\varphi_p\}$ and $\log(e^{\varphi_1}+\cdots+e^{\varphi_p})$ are psh on X.*

d) *Let $f\in\mathcal{O}(X)$ be a holomorphic function. Then $\log|f|$ is psh on X.*

e) *Let $f_1,\ldots,f_p\in\mathcal{O}(X)$ be holomorphic functions, and let $\gamma_1,\ldots,\gamma_p$ be positive numbers. Then*

$$\varphi = \log\big(|f_1|^{\gamma_1}+\cdots+|f_p|^{\gamma_p}\big)$$

is psh on X.

f) *If μ is a Radon measure on some compact space K and $(z,w)\mapsto \Phi(z,w)$ is an upper semicontinuous function on $X\times K$ such that $z\mapsto\Phi(z,w)$ is psh on X for μ-almost every $w\in K$, then $\varphi(z)=\int_{w\in K}\Phi(z,w)\,d\mu(w)$ is psh on X.*

g) *A function φ of class C^2 is psh if and only if the hermitian matrix of mixed derivatives $(\partial^2\varphi/\partial z_j\partial\overline{z}_k)_{1\leqslant j,k\leqslant n}$ is semipositive at every point. In particular, in dimension $n=1$, a function φ of class C^2 is (pluri)subharmonic if and only if $\Delta\varphi\geqslant 0$.*

h) *A function $\varphi\in L^1_{\mathrm{loc}}(X)$ is equal (almost everywhere) to a psh function φ_0 if and only if for every $a\in\mathbb{C}^n$ we have*

$$\sum_{1\leqslant j,k\leqslant n} a_j\overline{a}_k\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}\geqslant 0 \qquad (\textit{as a positive measure})$$

when $\partial^2\varphi/\partial z_j\partial\overline{z}_k$ is computed as a distribution.

i) *Let $\varphi\in\mathrm{Psh}(\Omega)$ where Ω is an open subset of $\mathbb{C}^n$ and assume that $\varphi\in L^1_{\mathrm{loc}}(\Omega)$. If (ρ_ε) is a family of smoothing kernels, then $\varphi\star\rho_\varepsilon$ is C^∞ and psh on*

$$\Omega_\varepsilon = \big\{z\in\Omega\,;\ d(z,\complement\Omega)>\varepsilon\big\}.$$

Moreover, the family $(\varphi\star\rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0}\varphi\star\rho_\varepsilon=\varphi$.

j) *If $F:Y\to X$ is a holomorphic map from a complex manifold Y to a complex manifold X and if $\varphi\in\mathrm{Psh}(X)$ then $\varphi\circ F\in\mathrm{Psh}(Y)$.*

k) *Assume that $\Omega=\omega\oplus \mathrm{i}\mathbb{R}^n$ is a "tube domain" of base an open subset $\omega\subset\mathbb{R}^n$. Let $\varphi(x+\mathrm{i}y)=\varphi(x)$ be a function depending only on $x\in\omega$. Then $z\mapsto\varphi(z)$ is psh on Ω if and only if $x\mapsto\varphi(x)$ is convex on ω.*

l) *Let φ be a psh function on an open subset $\Omega\subset\mathbb{C}^n$. Given a point $z_0\in\Omega$, let $R=d(z_0,\complement\Omega)$. Then the functions*

$$\log r\mapsto \sup_{B(z_0,r)}\varphi, \qquad \log r\mapsto\frac{1}{\pi^n r^{2n}/n!}\int_{z\in B(z_0,r)}\varphi(z)\,d\lambda(z)$$

are convex increasing functions on $]-\infty,\log R[$.

Proof. a) is just a consequence of the monotone convergence theorem, while b) follows from the obvious inequality $\sup\int\varphi_j\leqslant\int\sup\varphi_j$. Now, let us prove c). The conclusion is clearly true if $\chi(t) = \alpha(t) = a_1 t_1 + \cdots + a_p t_p + b$ is an affine function with all $a_j > 0$, for the function

$$\alpha(\varphi_1, \ldots, \varphi_p) = a_1\varphi_1 + \cdots + a_p\varphi_p + b$$

also satisfies the mean value inequality by taking positive linear combinations. However, it is well known that every convex function χ is equal to the upper envelope $\chi = \sup_{\alpha\in\mathcal{A}} \alpha$ where $\mathcal{A}$ is the family of all affine functions α such that $\alpha \leqslant \chi$; such functions α are necessarily increasing in each variable if χ is. Hence c) follows from b), and the case of $\log(e^{\varphi_1} + \cdots + e^{\varphi_p})$ is obtained by checking that $\chi(t_1, \ldots, t_p) = \log(e^{t_1} + \cdots + e^{t_p})$ is a convex increasing function (exercise: check that the matrix $(\partial^2\chi/\partial t_j \partial t_k)$ is semipositive of rank $p-1$ at any point $t \in \mathbb{R}^p$).

Property d) can be reduced easily to the Jensen formula in one variable: indeed the Jensen formula tells us that the average of $\log|f|$ on a circle of radius r is the sum of the value $\log|f(z_0)|$ at the center plus a term $\sum m_j \log(r/|w_j - z_0|) \geqslant 0$ where (w_j) are the zeros in the disk and m_j the multiplicities. Property e) is a special case of d) when we take $\varphi_j = \gamma_j \log|f_j|$, and f) is an immediate consequence of the Fubini theorem. Now, the convolution

$$\varphi \star \rho_\varepsilon(z) = \int_{B(0,\varepsilon)} \varphi(z-w)\rho_\varepsilon(w)\, d\lambda(w)$$

is a smooth function on Ω_ε, and f) shows that it is psh; hence the first part of i) follows. In dimension $n = 1$, the proof of g) is based on the elementary formula

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + r\, e^{\mathrm{i}\theta}) d\theta = \varphi(z_0) + \frac{1}{2\pi}\int_0^r \frac{d\rho}{\rho}\int_{D(z_0,\rho)} \Delta\varphi(z)\, dx\, dy. \tag{2.16}$$

(In fact, assuming $z_0 = 0$ for simplicity, the Green-Riemann formula yields

$$\begin{aligned}\frac{1}{\rho}\int_{D(0,\rho)} \Delta\varphi(z)\, dx\, dy &= \frac{1}{\rho}\int_{|z|=\rho} \frac{\partial\varphi}{\partial x}dy - \frac{\partial\varphi}{\partial y}dx \\ &= \int_{|z|=\rho} \frac{\partial\varphi}{\partial x}\cos\theta\, d\theta + \frac{\partial\varphi}{\partial y}\sin\theta\, d\theta = \frac{d}{d\rho}\int_0^{2\pi} \varphi(\rho e^{\mathrm{i}\theta}) d\theta,\end{aligned}$$

an we get (2.16) after an integration.) Now, if $\Delta\varphi \geqslant 0$, we infer from (2.16) that the mean value inequality holds; on the other hand, if $\Delta\varphi(z_0) < 0$, the mean value inequality fails for r small, QED. In higher dimensions, the conclusion is easily obtained by putting $\psi(w) = \varphi(z_0 + aw)$; we then get

$$\frac{1}{4}\Delta\psi(w) = \frac{\partial^2\psi}{\partial w \partial\overline{w}} = \sum_{1\leqslant j,k\leqslant n} a_j \overline{a}_k \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k}(z_0 + aw)$$

and everything follows.

i) (end of proof) Notice that (2.16) implies that the circular mean value $\int_0^{2\pi} \varphi(z_0 + r\, e^{\mathrm{i}\theta}) d\theta$ of a C^2 subharmonic function in $\Omega \subset \mathbb{C}$ is an increasing function of r. The same is true for spherical mean values of psh functions on open sets $\Omega \subset \mathbb{C}^n$, since we can compute them by averaging the circular mean values with respect to all complex directions. In particular, if φ is of class C^2, we conclude (through a use of polar coordinates) that

$$\varphi \star \rho_\varepsilon(z) = \int_{B(0,1)} \varphi(z - \varepsilon w)\rho_1(w)\, d\lambda(w)$$

is an increasing function of ε, for any $z \in \Omega_\varepsilon$ fixed). Since $\varphi \star \rho_\eta$ is a smooth psh function on Ω_η, we get that $(\varphi \star \rho_\eta) \star \rho_\varepsilon$ is increasing in ε whenever $z \in \Omega_{\varepsilon+\eta}$. By passing to the limit when $\eta \to 0$, we see that $\varphi \star \rho_\varepsilon(z)$ is always increasing in ε (even though φ is maybe not smooth). Since $\varphi \star \rho_\varepsilon(z) \geqslant \varphi(z)$ by the mean value inequality and $\limsup_{\varepsilon\to 0} \varphi\star\rho_\varepsilon(z) \leqslant \varphi(z)$ by the upper semicontinuity, we conclude that $\lim_{\varepsilon\to 0} \varphi \star \rho_\varepsilon(z) = \varphi(z)$ everywhere.

Let us now prove h). If φ_0 is psh and $\varphi = \varphi_0$ almost everywhere, then $\varphi \star \rho_\varepsilon = \varphi_0 \star \rho_\varepsilon$ is smooth and psh, hence

$$\sum_{1\leqslant j,k\leqslant n} a_j \overline{a}_k \frac{\partial^2 \varphi \star \rho_\varepsilon}{\partial z_j \partial \overline{z}_k} = \Big(\sum_{1\leqslant j,k\leqslant n} a_j \overline{a}_k \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \Big) \star \rho_\varepsilon \geqslant 0$$

for every $\lambda \in \mathbb{C}^n$ and every $\varepsilon > 0$. By passing to a weak limit, we conclude that the distribution $\sum a_j \overline{a}_k \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}$ is a positive measure. Conversely, if this is the case, the convolution $\varphi \star \rho_\eta$ is psh by g). Hence $(\varphi \star \rho_\eta) \star \rho_\varepsilon$ is an increasing function of ε, and by taking the limit as η tends to 0, we see again that $\varphi \star \rho_\varepsilon$ is increasing in ε. Therefore the decreasing limit

$$\varphi_0 = \lim_{k\to+\infty} \varphi \star \rho_{1/k}$$

is psh by a), and Lebesgue's theorem shows that $\varphi_0 = \varphi$ almost everywhere.

When φ is smooth, j) follows from the formula

$$\sum_{\ell,m} a_\ell \overline{a}_m \frac{\partial^2(\varphi \circ F)}{\partial w_\ell \partial \overline{w}_m} = \sum_{j,k} b_j \overline{b}_k \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \circ F, \qquad b_j = \sum_\ell a_\ell \frac{\partial F_j}{w_\ell}$$

in suitable coordinate systems (z_j) on X and (w_ℓ) on Y. In general, we conclude by considering regularizations $\varphi \star \rho_\varepsilon$ and passing to the limit.

k) is obvious when φ is smooth, since the convexity of $\varphi(x)$ and the plurisubharmonicity of $\varphi(z)$ are both characterized by the condition that $(\partial^2\varphi/\partial x_j \partial x_k)$ is semipositive everywhere. In general, we obtain the conclusion by using regularizations $\varphi \star \rho_\varepsilon$.

Finally, property l) follows from the following observation: the functions

$$\sigma(w) = \sup_{B(z_0, e^{\mathrm{Re}\, w})} \varphi = \sup_{a\in B(0,1)} \varphi(z_0 + a\, e^w),$$

$$\mu(w) = \frac{1}{\pi^n e^{2n\,\mathrm{Re}\, w}/n!} \int_{B(z_0, e^{\mathrm{Re}\, w})} \varphi(z)\, d\lambda(z) = \frac{1}{\pi^n/n!} \int_{a\in B(0,1)} \varphi(z_0 + a\, e^w)\, d\lambda(a)$$

are psh on the half-plane $\{\mathrm{Re}\, w < \log R\} \subset \mathbb{C}$, thanks to j), b) and f). As they only depend on $\mathrm{Re}\, w$, they must be convex in $\mathrm{Re}\, w$. Moreover, $\sigma(w)$ is clearly increasing with respect to $\mathrm{Re}\, w$, and the same is true for μ by (2.16). $\square$

(2.17) Definition. *A complex* $(1,1)$*-form*

$$u = \mathrm{i} \sum_{1 \leqslant j,k \leqslant n} u_{jk} dz_j \wedge d\overline{z}_k$$

is said to be (semi)positive if the hermitian matrix (u_{jk}) is (semi)positive.

Notice that u is real, i.e. $\overline{u} = u$, if and only if $\overline{u}_{jk} = u_{kj}$, i.e. iff the matrix is hermitian). In this setting, a real L^1_{loc} function φ is psh if and only if $\mathrm{i}d'd''\varphi \geqslant 0$ as a $(1,1)$-form.

3. Kähler metrics and Kähler manifolds

Let us recall that a *Riemannian metric* on a (real) differentiable manifold M is a positive definite symmetric bilinear form

$$g = \sum_{1 \leqslant j,k \leqslant n} g_{jk}(x)\, dx_j \otimes dx_k$$

on the tangent bundle T_M, where $(x_1, \ldots, x_n)$ are local coordinates on M. We usually assume that the coefficients $g_{jk}(x)$ are smooth. Then, for any tangent vector $\xi = \sum \xi_j \partial/\partial x_j \in T_{M,x}$, one defines its norm with respect to g by

$$|\xi|_g^2 = \sum_{1 \leqslant j,k \leqslant n} g_{jk}(x) \xi_j \xi_k. \tag{3.1}$$

If M is moreover assumed to be oriented, one defines a corresponding volume element

$$dV_g = \sqrt{\det(g_{jk}(x))}\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \tag{3.2}$$

whenever $(x_1, \ldots, x_n)$ fit with the given orientation. It is easy to check by the Jacobian formula that this definition of dV_g is independent of the choice of coordinates.

Now, we consider the complex case. Let X be a complex n-dimensional manifold. A *hermitian metric* on X is a positive definite hermitian form of class C^∞ on T_X; in a coordinate system $(z_1, \ldots, z_n)$, such a form can be written $h(z) = \sum_{1 \leqslant j,k \leqslant n} h_{jk}(z)\, dz_j \otimes d\overline{z}_k$, where (h_{jk}) is a positive hermitian matrix with C^∞ coefficients. Thanks to the hermitian condition $\overline{h_{jk}} = h_{kj}$, our form h can be written as $h = g - \mathrm{i}\omega$, where

$$\begin{aligned}
h(\xi, \eta) &= \sum_{1 \leqslant j,k \leqslant n} h_{jk}(z) \xi_j \overline{\eta}_k, \\
g(\xi, \eta) &= \operatorname{Re} h(\xi, \eta) = \frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} \big(h_{jk}(z) \xi_j \overline{\eta}_k + \overline{h_{jk}(z)}\, \overline{\xi}_j \eta_k\big) \\
&= \frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} h_{jk}(z) \big(\xi_j \overline{\eta}_k + \eta_j \overline{\xi}_k\big), \\
\omega(\xi, \eta) &= -\operatorname{Im} h(\xi, \eta) = \frac{\mathrm{i}}{2} \sum_{1 \leqslant j,k \leqslant n} h_{jk}(z) \big(\xi_j \overline{\eta}_k - \eta_j \overline{\xi}_k\big), \qquad \text{i.e.}
\end{aligned} \tag{3.3}$$

$$\omega = -\operatorname{Im} h = \frac{\mathrm{i}}{2} \sum_{1 \leqslant j,k \leqslant n} h_{jk}(z)\, dz_j \wedge d\overline{z}_k. \tag{3.4}$$

By definition, ω is the *fundamental* $(1,1)$*-form* associated with h. It is a positive definite $(1,1)$-form (according to the general definition). Since ω and h are "isomorphic" objects, we usually do not make any difference and will think of hermitian metrics as being positive $(1,1)$-forms.

(3.5) Definition.

a) *A hermitian manifold is a pair (X,ω) where ω is a C^∞ positive definite $(1,1)$-form on X.*

b) *The metric ω is said to be kähler if $d\omega = 0$.*

c) *X is said to be a Kähler manifold if X possesses at least one Kähler metric.*

Since ω is real, the conditions $d\omega = 0$, $d'\omega = 0$, $d''\omega = 0$ are all equivalent. In local coordinates we see that $d'\omega = 0$ if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \quad , \quad 1 \leqslant j,k,l \leqslant n.$$

A simple computation gives $\frac{\mathrm{i}}{2} dz_j \wedge d\overline{z}_j = dx_j \wedge dy_j$ and

$$\frac{\omega^n}{n!} = \det(h_{jk}) \bigwedge_{1\leqslant j\leqslant n} \Big(\frac{\mathrm{i}}{2} dz_j \wedge d\overline{z}_j\Big) = \det(h_{jk})\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_n = x_n + \mathrm{i}y_n$. Therefore the (n,n)-form

$$dV_\omega = \frac{1}{n!}\omega^n \tag{3.6}$$

is positive with respect to the canonical orientation of X. Since $\det(g) = \det(h)^2$ by (3.3), we see that $dV_\omega = dV_g$ coincides with the Riemannian volume element of X. If X is compact, then $\int_X \omega^n = n!\ \mathrm{Vol}_\omega(X) > 0$. This simple remark already implies that compact Kähler manifolds must satisfy some restrictive topological conditions:

(3.7) Consequence.

a) *If (X,ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X,\mathbb{R})$, then $\{\omega\}^n \neq 0$.*

b) *If X is compact Kähler, then $H^{2k}(X,\mathbb{R}) \neq 0$ for $0 \leqslant k \leqslant n$. In fact, $\{\omega\}^k$ is a non zero class in $H^{2k}(X,\mathbb{R})$.*

(3.8) Examples.

a) The most obvious example is $\mathbb{C}^n$ together with the standard Kähler metric

$$\omega = \frac{\mathrm{i}}{2} \sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j = \mathrm{i} \sum_{1\leqslant j\leqslant n} dx_j \wedge dy_j.$$

The volume element dV_ω coincides with the Lebesgue measure of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

b) The complex projective space $\mathbb{P}^n$ is Kähler. A natural Kähler metric ω on $\mathbb{P}^n$, called the *Fubini-Study metric*, is defined by

$$p^\star\omega = \frac{\mathrm{i}}{2}d'd''\log\big(|\zeta_0|^2+|\zeta_1|^2+\cdots+|\zeta_n|^2\big)$$

where $\zeta_0,\zeta_1,\ldots,\zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n$ is the projection. Let $z=(\zeta_1/\zeta_0,\ldots,\zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n=\{\zeta_0\neq 0\}\subset\mathbb{P}^n$. Then, since $d'd''\log|\zeta_0|^2=0$ on $\{\zeta_0\neq 0\}$, we see that

$$\begin{aligned}\omega &= \frac{\mathrm{i}}{2}d'd''\log(1+|z|^2)=\frac{\mathrm{i}}{2}\Big(\frac{d'd''|z|^2}{1+|z|^2}-\frac{\langle dz,z\rangle\wedge\overline{\langle dz,z\rangle}}{(1+|z|^2)^2}\Big),\\ |\xi|^2_\omega &= \frac{|\xi|^2}{1+|z|^2}-\frac{|\langle\xi,z\rangle|^2}{(1+|z|^2)^2}=\frac{|\xi|^2+|\xi\wedge z|^2}{(1+|z|^2)^2},\end{aligned}$$

thanks to Lagrange's identity $|\xi|^2|z|^2=|\langle\xi,z\rangle|^2+|\xi\wedge z|^2$. The eigenvalues of the Fubini-Study metric with respect to the standard Euclidean metric are $1/(1+|z|^2)^2$ in the radial direction $\mathbb{C}z$, and $1/(1+|z|^2)$ in the hyperplane $(\mathbb{C}z)^\perp$. From this we infer

$$dV_\omega=\frac{d\lambda(z)}{(1+|z|^2)^{n+1}}.$$

A computation shows that the global volume is $\mathrm{Vol}_\omega(\mathbb{P}^n)=\int_{\mathbb{P}^n}dV_\omega=\pi^n/n!$.

c) A *complex torus* is a quotient $X=\mathbb{C}^n/\Gamma$ by a lattice Γ of rank $2n$. Then X is a compact complex manifold. Any positive definite hermitian form with constant coefficients $\omega=\mathrm{i}\sum h_{jk}dz_j\wedge d\overline{z}_k$ defines a Kähler metric on X.

d) Every (complex analytic) submanifold Y of a Kähler manifold (X,ω) is Kähler with metric $\omega_{\restriction Y}$. Especially, all complex submanifolds of $\mathbb{P}^n$ are Kähler.

e) Consider the complex surface

$$X=(\mathbb{C}^2\setminus\{0\})/\Gamma$$

where $\Gamma=\{\lambda^n\ ;\ n\in\mathbb{Z}\}$, $\lambda<1$, acts as a group of homotheties. Since $\mathbb{C}^2\setminus\{0\}$ is diffeomorphic to $\mathbb{R}^\star_+\times S^3$, we have $X\simeq S^1\times S^3$. Therefore $H^2(X,\mathbb{R})=0$ by Künneth's formula, and property 3.7 b) shows that X is not Kähler. Hence there are compact complex surfaces which are not Kähler. □

The following Theorem shows that a hermitian metric ω on X is Kähler if and only if the metric ω is tangent at order 2 to a hermitian metric with constant coefficients at every point of X.

(3.9) Theorem. *Let ω be a C^∞ positive definite $(1,1)$-form on X. In order that ω be Kähler, it is necessary and sufficient that to every point $x_0\in X$ corresponds a holomorphic coordinate system $(z_1,\ldots,z_n)$ centered at x_0 such that*

$$\omega=\mathrm{i}\sum_{1\leqslant l,m\leqslant n}\omega_{lm}\,dz_l\wedge d\overline{z}_m,\qquad \omega_{lm}=\delta_{lm}+O(|z|^2).$$

Proof. It is clear that the existence of a Taylor expansion as above implies $d_{x_0}\omega = 0$, so the condition is sufficient. Conversely, assume that ω is Kähler. Then one can choose local holomorphic coordinates $(x_1, \ldots, x_n)$ such that $(dx_1, \ldots, dx_n)$ is an ω-orthonormal basis of $T^\star_{X,x_0}$. Therefore

$$\begin{aligned}\omega &= \mathrm{i} \sum_{1\leqslant l,m\leqslant n} \widetilde{\omega}_{lm}\, dx_l \wedge d\overline{x}_m, \qquad \text{where}\\ \widetilde{\omega}_{lm} &= \delta_{lm} + O(|x|) = \delta_{lm} + \sum_{1\leqslant j\leqslant n} (a_{jlm}x_j + a'_{jlm}\overline{x}_j) + O(|x|^2).\end{aligned}$$

Since ω is real, we have $a'_{jlm} = \overline{a}_{jml}$; on the other hand the Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at x_0 implies $a_{jlm} = a_{ljm}$. Set now

$$z_m = x_m + \frac{1}{2}\sum_{j,l} a_{jlm}x_jx_l, \qquad 1 \leqslant m \leqslant n.$$

Then (z_m) is a coordinate system at x_0, and

$$\begin{aligned}dz_m &= dx_m + \sum_{j,l} a_{jlm}x_jdx_l,\\ \mathrm{i}\sum_m dz_m \wedge d\overline{z}_m &= \mathrm{i}\sum_m dx_m \wedge d\overline{x}_m + \mathrm{i}\sum_{j,l,m} a_{jlm}x_j\, dx_l \wedge d\overline{x}_m\\ &\qquad + \mathrm{i}\sum_{j,l,m} \overline{a}_{jlm}\overline{x}_j\, dx_m \wedge d\overline{x}_l + O(|x|^2)\\ &= \mathrm{i}\sum_{l,m} \widetilde{\omega}_{lm}\, dx_l \wedge \overline{d}x_m + O(|x|^2) = \omega + O(|z|^2).\end{aligned}$$

Theorem 3.9 is proved. □

(3.10) Remark. When ω is Kähler, one can refine the above proof to shows that there are local coordinates $(z_1, \ldots, z_n)$ centered at x_0 such that $\omega = \frac{\mathrm{i}}{2}\sum_{lm}\omega_{lm}dz_l \wedge d\overline{z}_m$ with

$$\omega_{lm} = \delta_{lm} - \sum_{1\leqslant j,k\leqslant n} c_{jklm}\, z_j\overline{z}_k + O(|z|^3).$$

The coefficients c_{jklm} satisfy the symmetry relations

$$\overline{c}_{jklm} = c_{kjml}, \qquad c_{jklm} = c_{lkjm} = c_{jmlk} = c_{lmjk}.$$

(The c_{jklm} can be interpreted as the coefficients of the Levi-Civita curvature tensor of (T_X, ω), but we will not use this fact).

4. Differential operators on vector bundles

We first describe some basic concepts concerning differential operators (symbol, composition, adjunction, ellipticity), in the general setting of vector bundles. Let M

be a C^∞ differentiable manifold, $\dim_{\mathbb{R}} M = m$, and let E, F be $\mathbb{K}$-vector bundles over M, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} E = r$, $\operatorname{rank} F = r'$.

(4.1) Definition. *A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : C^\infty(M, E) \to C^\infty(M, F)$, $u \mapsto Pu$ of the form*

$$Pu(x) = \sum_{|\alpha| \leqslant \delta} a_\alpha(x) D^\alpha u(x),$$

where $E_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^r$, $F_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^{r'}$ are trivialized locally on some open chart $\Omega \subset M$ equipped with local coordinates $(x_1, \ldots, x_m)$, and

$$a_\alpha(x) = \big(a_{\alpha\lambda\mu}(x)\big)_{1 \leqslant \lambda \leqslant r',\, 1 \leqslant \mu \leqslant r}$$

are $r' \times r$-matrices with C^∞ coefficients on Ω. Here $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_m)^{\alpha_m}$ as usual, and $u = (u_\mu)_{1 \leqslant \mu \leqslant r}$, $D^\alpha u = (D^\alpha u_\mu)_{1 \leqslant \mu \leqslant r}$ are viewed as column matrices.

If $t \in \mathbb{K}$ is a parameter and $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, a simple calculation shows that $e^{-tf(x)} P(e^{tf(x)} u(x))$ is a polynomial of degree δ in t, of the form

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^\delta \sigma_P(x, df(x)) \cdot u(x) + \text{lower order terms } c_j(x) t^j,\ j < \delta,$$

where σ_P is the homogeneous polynomial map $T^\star_M \to \operatorname{Hom}(E, F)$ defined by

$$\text{(4.2)} \qquad T^\star_{M,x} \ni \xi \mapsto \sigma_P(x, \xi) \in \operatorname{Hom}(E_x, F_x), \qquad \sigma_P(x, \xi) = \sum_{|\alpha| = \delta} a_\alpha(x) \xi^\alpha.$$

Then $\sigma_P(x, \xi)$ is smooth on $T^\star_M$ as a function of (x, ξ), and is independent of the choice of coordinates or of the trivializations used for E, F. We say that σ_P is *the principal symbol* of P. The symbol of a composition $Q \circ P$ of differential operators is simply the product

$$\text{(4.3)} \qquad \sigma_{Q \circ P}(x, \xi) = \sigma_Q(x, \xi) \sigma_P(x, \xi),$$

computed as a product of matrices.

Now, assume that E is a euclidean or hermitian vector bundle. Recall that a *hermitian form* h on a complex vector bundle E if a collection of positive definite hermitian forms $h(x)$ on each fiber E_x, such that the map

$$E \to \mathbb{R}_+, \qquad E_x \ni \xi \mapsto |\xi|^2_h := h(x)(\xi)$$

is smooth. A *hermitian vector bundle* is a pair (E, h) where E is a complex vector bundle and h a hermitian metric on E. The notion of a euclidean (real) vector bundle is similar, so we leave the reader adapt our notations to that case. We assume in addition that M is oriented and is equipped with a smooth volume form $dV(x) = \gamma(x) dx_1 \wedge \cdots dx_m$, where $\gamma(x) > 0$ is a smooth density (usually, dV will be the volume element dV_g of some Riemannian metric). Then we get a Hilbert space $L^2(M, E)$ of global sections u of E with L^2 coefficients, by looking at all sections $x \mapsto u(x) \in E_x$ satisfying the L^2 estimate

$$\|u\|^2 = \int_M |u(x)|^2 \, dV(x) < +\infty. \tag{4.4}$$

We denote the corresponding global L^2 inner product by

$$\langle\!\langle u, v \rangle\!\rangle = \int_M \langle u(x), v(x) \rangle \, dV(x), \qquad u, v \in L^2(M, E) \tag{4.4$'$}$$

(4.5) Definition. *If $P : C^\infty(M, E) \to C^\infty(M, F)$ is a differential operator and both E, F are euclidean or hermitian, there exists a unique differential operator*

$$P^\star : C^\infty(M, F) \to C^\infty(M, E),$$

called the formal adjoint of P, such that for all sections $u \in C^\infty(M, E)$ and $v \in C^\infty(M, F)$ there is an identity

$$\langle\!\langle Pu, v \rangle\!\rangle = \langle\!\langle u, P^\star v \rangle\!\rangle, \qquad \text{whenever } \operatorname{Supp} u \cap \operatorname{Supp} v \Subset M.$$

Proof. The uniqueness is easy, using the density of smooth compactly supported forms in $L^2(M, E)$. By a partition of unity argument, it is enough to show the existence of $P^\star$ locally. Now, let $Pu(x) = \sum_{|\alpha| \leqslant \delta} a_\alpha(x) D^\alpha u(x)$ be the expansion of P with respect to trivializations of E, F given by orthonormal frames over some coordinate open set $\Omega \subset M$. When $\operatorname{Supp} u \cap \operatorname{Supp} v \Subset \Omega$ an integration by parts yields

$$\begin{aligned}
\langle\!\langle Pu, v \rangle\!\rangle &= \int_\Omega \sum_{|\alpha| \leqslant \delta, \lambda, \mu} a_{\alpha\lambda\mu} D^\alpha u_\mu(x) \overline{v}_\lambda(x) \, \gamma(x) \, dx_1, \ldots, dx_m \\
&= \int_\Omega \sum_{|\alpha| \leqslant \delta, \lambda, \mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x) \, \overline{a}_{\alpha\lambda\mu} v_\lambda(x)} \, dx_1, \ldots, dx_m \\
&= \int_\Omega \langle u, \sum_{|\alpha| \leqslant \delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha \big(\gamma(x) \, {}^t\overline{a}_\alpha v(x)\big) \rangle \, dV(x).
\end{aligned}$$

Hence we see that $P^\star$ exists and is uniquely defined by

$$P^\star v(x) = \sum_{|\alpha| \leqslant \delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha \big(\gamma(x) \, {}^t\overline{a}_\alpha v(x)\big). \square \tag{4.6}$$

It follows immediately from (4.6) that the principal symbol of $P^\star$ is

$$\sigma_{P^\star}(x, \xi) = (-1)^\delta \sum_{|\alpha| = \delta} {}^t\overline{a}_\alpha \xi^\alpha = (-1)^\delta \sigma_P(x, \xi)^\star. \tag{4.7}$$

(4.8) Hilbertian extensions of differential operators. Given a differential operator $P : C^\infty(M, E) \to C^\infty(M, F)$, we can extend it as an operator $\widetilde{P} : \mathcal{D}'(M, E) \to \mathcal{D}'(M, F)$ by computing Pu in the sense of distributions. Especially, we get an operator

$$P_{\mathcal{H}} : L^2(M, E) \to L^2(M, F),$$

which we call the maximal (Hilbertian) extension of P, such that $u \in \mathrm{Dom}(P_{\mathcal{H}})$ if and only if $u \in L^2(M,E)$ and $\widetilde{P}u \in L^2(M,F)$; we then set of course $P_{\mathcal{H}}u = \widetilde{P}u$.

(4.9) Proposition. *For any differential operator $P : C^\infty(M,E) \to C^\infty(M,F)$, its Hilbertian extension $P_{\mathcal{H}}$ is a closed and densely defined operator.*

Proof. First observe that $\mathrm{Dom}\, P_{\mathcal{H}}$ contains the space $\mathcal{D}(M,E)$ of smooth sections with compact support in M. As $\mathcal{D}(M,E)$ is dense in $L^2(M,E)$, we conclude that $P_{\mathcal{H}}$ has a dense domain. Now, let (u_ν, v_ν) be a sequence in the graph of $P_{\mathcal{H}}$, converging towards a limit (u,v) in $L^2(M,E) \times L^2(M,F)$. Then, $u_\nu \to u$ in $L^2(M,E)$ and in particular u_ν converges weakly to u in $\mathcal{D}'(M,E)$. As differentiations and multiplications by smooth functions are continuous for the weak topology, we infer that $v_\nu = Pu_\nu$ converges weakly to $\widetilde{P}u$ in $\mathcal{D}'(M,F)$. Since the weak topology is Hausdorff, we conclude that $v = \widetilde{P}u$. Hence $u \in \mathrm{Dom}\, P_{\mathcal{H}}$ and $v = P_{\mathcal{H}}u$, as desired. □

By the general results of § 1, we know that $P_{\mathcal{H}}$ admits a closed and densely defined adjoint $(P_{\mathcal{H}})^\star : L^2(M,F) \to L^2(M,E)$, called the *Hilbert adjoint* of $P_{\mathcal{H}}$, such that

$$\langle\!\langle P_{\mathcal{H}}u, v\rangle\!\rangle = \langle\!\langle u, (P_{\mathcal{H}})^\star v\rangle\!\rangle, \qquad \forall u \in \mathrm{Dom}\, P_{\mathcal{H}},\ \forall v \in \mathrm{Dom}(P_{\mathcal{H}})^\star.$$

In particular, this identity must hold true for all $u \in \mathcal{D}(M,E)$, and we conclude from this that $(P_{\mathcal{H}})^\star v$ coincides with the formal adjoint $\widetilde{P^\star} v$ computed in the sense of of distributions. Hence, if $(P^\star)_{\mathcal{H}}$ is the maximal Hilbertian extension of the formal adjoint (usually simply called the formal adjoint), we see that

$$\mathrm{Dom}(P_{\mathcal{H}})^\star \subset \mathrm{Dom}(P^\star)_{\mathcal{H}} \tag{4.10}$$

and that both operators $(P_{\mathcal{H}})^\star$ (Hilbert adjoint) and $(P^\star)_{\mathcal{H}}$ (formal adjoint) coincide on $\mathrm{Dom}(P_{\mathcal{H}})^\star$. However, the domains are in general distinct, as shown by the following simple example.

(4.11) Example. Consider $M =]0,1[$, $dV = dx$, together with the trivial hermitian vector bundles $E = F = M \times \mathbb{C}$, and the differential operator

$$P = \frac{d}{dx} : C^\infty(]0,1[,\mathbb{C}) \to C^\infty(]0,1[,\mathbb{C}).$$

Our general formula for the formal adjoint shows that

$$P^\star = -\frac{d}{dx} = -P.$$

Now, the domain of $P_{\mathcal{H}}$ consists of all $u \in L^2(]0,1[,\mathbb{C})$ such that $u' \in L^2(]0,1[,\mathbb{C})$ and is therefore nothing else by definition than the Sobolev space $W^1(]0,1[,\mathbb{C})$. However $W^1(]0,1[,\mathbb{C})$ injects continuously in $C^0(]0,1[,\mathbb{C})$, since $u' \in L^2(]0,1[,\mathbb{C}) \subset L^1(]0,1[,\mathbb{C})$ implies that u extends as a continuous function in $C^0([0,1],\mathbb{C})$. In particular, any $u \in \mathrm{Dom}\, P_{\mathcal{H}} = W^1(]0,1[,\mathbb{C})$ can be assigned well defined values

$u(0)$ and $u(1)$. Now, given $u \in \operatorname{Dom} P_{\mathcal{H}}$ and $v \in \operatorname{Dom}(P_{\mathcal{H}})^\star$, (4.10) shows that $u, v \in W^1(]0,1[,\mathbb{C}) \subset C^0([0,1],\mathbb{C})$. We then get the integration by part formula

$$\langle\!\langle P_{\mathcal{H}}u, v\rangle\!\rangle = \int_0^1 u'(x)\overline{v}(x)\,dx = u(1)v(1) - u(0)v(0) - \int_0^1 u(x)v'(x)\,dx,$$

which can be easily reduced to the case of smooth functions by using convolution with reglarizing kernels. As $v' \in L^2$, we conclude that the linear form $u \mapsto \langle\!\langle Pu, v\rangle\!\rangle$ is continuous in the L^2 topology if and only if $v(0) = v(1) = 0$ (and in that case we do have $(P_{\mathcal{H}})^\star v = -v' = (P^\star)_{\mathcal{H}} v$). Hence

$$\begin{aligned}\operatorname{Dom}(P_{\mathcal{H}})^\star = W^1_0(]0,1[,\mathbb{C}) &:= \big\{v \in W^1(]0,1[,\mathbb{C})\,;\ v(0) = v(1) = 0\big\}\\ &\subsetneq W^1(]0,1[,\mathbb{C}) = \operatorname{Dom}(P^\star)_{\mathcal{H}}.\end{aligned}$$ □

(4.12) Elliptic operators. Especially important in PDE theory are the so-called elliptic differential operators:

(4.13) Definition. *A differential operator P is said to be elliptic if $\sigma_P(x,\xi) \in \operatorname{Hom}(E_x, F_x)$ is injective for every $x \in M$ and $\xi \in T^\star_{M,x} \smallsetminus \{0\}$.*

The main result of elliptic PDE theory, which we only quote here (see e.g. (Hörmander 1963)), is

(4.14) Theorem. *Every solution $u \in \mathcal{D}'(M,E)$ of an elliptic equation $\widetilde{P}u = v$ with $v \in C^\infty(M,F)$ is in fact smooth, i.e., $u \in C^\infty(M,E)$. In fact, if P is of degree δ and v is in some Sobolev space $W^s_{\mathrm{loc}}(M,F)$, then $u \in W^s_{\mathrm{loc}}(M,E)$.*

5. Operators of Kähler geometry and commutation identities

In Kähler geometry, many linear differential operators are to be considered, together with their commutation relations. All these operators are $\mathbb{C}$-linear endomorphisms acting on the (bi)graded module $\bigoplus_{p,q\geqslant 0} C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E)$, where E is some hermitian vector bundle over X. These operators form a bigraded algebra $\mathcal{A} = \bigoplus \mathcal{A}_{r,s}$: an operator is called of *type* (or *bidegree*) (r,s) if it maps

$$C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E) \to C^\infty(X, \Lambda^{p+r,q+s}T^\star_X \otimes E) \tag{5.1}$$

for all p, q; the (*total*) *degree* of such an operator is by definition $k = r+s$, and we set $\mathcal{A}_k = \bigoplus_{r+s=k} \mathcal{A}_{r,s}$. Given homogeneous operators A, B of degrees a, b in a graded algebra $\mathcal{A} = \bigoplus \mathcal{A}_k$, their graded commutator is defined to be

$$[A,B] = AB - (-1)^{ab}BA. \tag{5.2}$$

If C is another endomorphism of degree c, the following *Jacobi identity* holds (as a purely formal computation shows):

$$(5.3) \qquad (-1)^{ca}\big[A,[B,C]\big]+(-1)^{ab}\big[B,[C,A]\big]+(-1)^{bc}\big[C,[A,B]\big]=0.$$

To every form $\alpha \in C^\infty(X, \Lambda^{p,q}T^\star_X)$ corresponds an endomorphism of type (p,q) on $C^\infty(X, \Lambda^{\bullet,\bullet}T^\star_X \otimes E)$, defined by $u \mapsto \alpha\wedge u$. We will often use again the notation α for this endomorphism, i.e., we will write $\alpha(u)=\alpha\wedge u$.

(5.4) Pointwise and global hermitian metrics on spaces of (p,q)-forms. From now on, we suppose that (X,ω) is a Kähler manifold and set $n=\dim_{\mathbb{C}} X$. The underlying Riemannian metric g defines a euclidean metric on the real tangent space $T^{\mathbb{R}}_X$, hence a hermitian metric on the complexified tangent space $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{R}}_X$. We get as well a hermitian metric on

$$\mathrm{Hom}_{\mathbb{R}}(T^{\mathbb{R}}_X,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{R}}_X,\mathbb{C}) = \Lambda^{1,0}T^\star_X \oplus \Lambda^{0,1}T^\star_X.$$

For instance, in $\mathbb{C}^n$ with the usual euclidean metric, we have

$$\Big|\frac{\partial}{\partial x_j}\Big| = \Big|\frac{\partial}{\partial y_j}\Big| = 1, \qquad \Big|\frac{\partial}{\partial z_j}\Big| = \Big|\frac{1}{2}\frac{\partial}{\partial x_j} - \frac{\mathrm{i}}{2}\frac{\partial}{\partial y_j}\Big| = \frac{1}{\sqrt{2}} = \Big|\frac{\partial}{\partial \overline{z}_j}\Big|,$$

$(\partial/\partial z_j)$, $(\partial/\partial \overline{z}_k)$ are orthogonal bases of $T^{1,0}_{\mathbb{C}^n}$, $T^{0,1}_{\mathbb{C}^n}$ (which are themselves mutually orthogonal). The dual metric on $\Lambda^{1,0}T^\star_{\mathbb{C}^n}\oplus\Lambda^{0,1}T^\star_{\mathbb{C}^n}$ is such that $|dz_j|=|dx_j+\mathrm{i}dy_j|=\sqrt{2}=|d\overline{z}_j|$. Now, for an arbitrary Kähler manifold (X,ω), we can use the Gram-Schmidt orthogonalization process in order to construct on any coordinate open set $\Omega\subset X$ an orthonormal frame $(\xi_1,\ldots,\xi_n)$ of $T^{0,1}_{X|\Omega}$ for the metric ω. The dual basis $(\xi^\star_j)$ defines an orthonormal frame of $\Lambda^{1,0}T^\star_X$ for the dual metric, furthermore, any (p,q)-form can be written in a unique way

$$u = \sum_{|I|=p,\,|J|=q} u_{IJ}\,\xi^\star_I\wedge\overline{\xi}^\star_J.$$

We define the (pointwise) hermitian norm of u to be $|u|^2_\omega = \sum_{I,\,J}|u_{IJ}|^2$. In this way, we get a hermitian metric on $\Lambda^{p,q}T^\star_X$, which is actually independent of the initial choice of the orthonormal frame (ξ_j). One can check this by observing that the corresponding hermitian inner product satisfies the intrinsic property

$$\langle u_1\wedge\cdots\wedge u_p\wedge\overline{v}_1\cdots\wedge v_q,\, u'_1\wedge\cdots\wedge u'_p\wedge\overline{v}'_1\cdots\wedge\overline{v}'_q\rangle_\omega = \det(\langle u_j,u'_k\rangle_\omega)\,\overline{\det(\langle v_j,v'_k\rangle_\omega)},$$

which characterizes the inner product in a unique way. Now, we consider the Hilbert space $L^2(X,\Lambda^{p,q}T^\star_X)$ of global (p,q)-forms u with measurable complex coefficients such that

$$(5.5) \qquad \|u\|^2_\omega = \int_X |u|^2_\omega\, dV_\omega < +\infty.$$

The corresponding global L^2 inner product is

$$\langle\!\langle u,v\rangle\!\rangle_\omega = \int_X \langle u,v\rangle_\omega\, dV_\omega.$$

Unless there are several Kähler metrics under consideration, we will usually omit the subscripts in the notation of the norms or inner products.

(5.6) Contraction of a differential form by a tangent vector. Let u be a (p,q)-form on Ω, viewed as an antisymmetric $\mathbb{R}$-linear form of degree $k=p+q$. Given a complex tangent vector $\xi=\sum\xi'_j\,\partial/\partial z_j+\sum\xi''_j\,\partial/\partial\overline{z}_j$, we define the *contraction* $\xi\,\lrcorner\,u$ to be the differential form of degree $k-1=p+q-1$ such that

$$(\xi\,\lrcorner\,u)(\eta_1,\ldots,\eta_{k-1})=u(\xi,\eta_1,\ldots,\eta_{p-1})$$

for all tangent vectors η_j. Then $(\xi,u)\longmapsto\xi\,\lrcorner\,u$ is bilinear, and from the fact that $u\wedge v$ is the antisymmetrization of $u\otimes v$ one easily sees that contraction by a tangent vector is a *derivation*, i.e.

$$\xi\,\lrcorner\,(u\wedge v)=(\xi\,\lrcorner\,u)\wedge v+(-1)^{\deg u}u\wedge(\xi\,\lrcorner\,v).$$

From this and the obvious rule $\frac{\partial}{\partial z_\ell}\,\lrcorner\,dz_j=\delta_{j\ell}$ we derive the explicit formulas

$$\begin{aligned}\frac{\partial}{\partial z_\ell}\,\lrcorner\,(dz_I\wedge d\overline{z}_J)&=\begin{cases}0&\text{if }\ \ell\notin I,\\(-1)^{s-1}dx_{I\smallsetminus\{\ell\}}&\text{if }\ \ell=i_s\in I,\end{cases}\\\frac{\partial}{\partial\overline{z}_\ell}\,\lrcorner\,(dz_I\wedge d\overline{z}_J)&=\begin{cases}0&\text{if }\ \ell\notin J,\\(-1)^{p+s-1}dx_{I\smallsetminus\{\ell\}}&\text{if }\ \ell=j_s\in J,\end{cases}\end{aligned}$$

whenever $|I|=p$ and $|J|=q$. An easy check shows that the interior product $\xi\,\lrcorner\,\bullet$ is the adjoint of the wedge multiplication $\xi^\star\wedge\bullet$, where $\xi^\star=\langle\bullet,\xi\rangle_\omega$ is the $(1,0)$-form associated with ξ, i.e.

$$\langle\xi\,\lrcorner\,u,v\rangle=\langle u,\xi^\star\wedge v\rangle$$

for any pair of forms (u,v) of respective degrees k, $k-1$. Of course, a similar formula also holds for global inner products $\langle\!\langle\ ,\ \rangle\!\rangle$, since we need only integrate the above pointwise formula.

(5.7) Operators of Kähler geometry. Here is a short list of the operators we will have to deal with:

a) The operators $d=d'+d''$ acting on $C^\infty(X,\Lambda^{\bullet,\bullet}T^\star_X)$, which are all three of degree 1 (d' being of bidegree $(1,0)$ and d'' of bidegree $(0,1)$).

b) Their adjoints $d^\star=d'^\star+d''^\star$, computed with respect to the global L^2 inner product. We have for instance

$$\langle\!\langle d''u,v\rangle\!\rangle=\langle\!\langle u,d''^\star v\rangle\!\rangle$$

for all smooth forms u of type $(p,q-1)$ and v of type (p,q) with $\operatorname{Supp}u\cap\operatorname{Supp}v$ compact. Hence d' is of type $(-1,0)$ and d'' is of type $(0,-1)$. (More generally, the adjoint of an operator of type (r,s) is of type $(-r,-s)$.)

c) The *Laplace-Beltrami* operators

$$\begin{aligned}\Delta&=dd^\star+d^\star d=[d,d^\star],\\\Delta'&=d'd'^\star+d'^\star d'=[d',d'^\star],\\\Delta''&=d''d''^\star+d''^\star d''=[d'',d''^\star].\end{aligned}$$

d) Two other important operators are the operators L of type $(1,1)$ defined by 2

$$Lu=\omega\wedge u,$$

and its adjoint $\Lambda = L^\star$, which is obtained by taking the pointwise adjoint

$$\langle u, \Lambda v\rangle = \langle Lu, v\rangle. \qquad \square$$

(5.8) Special case of the flat hermitian metric on open subsets of $\mathbb{C}^n$. Assume that $X = \Omega \subset \mathbb{C}^n$ is an open subset and that ω is the standard Kähler metric of $\mathbb{C}^n$, *multiplied by* 2, i.e.

$$\omega = \mathrm{i} \sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j$$

(the reason for multiplying the standard metric by 2 is that we get in this way $|\partial/\partial z_j| = 1$, and this allows us to avoid annoying constants 2 or 1/2 in the computations). For any form $u \in C^\infty(\Omega, \Lambda^{p,q}T^\star_\Omega)$ we have

$$d'u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J = \sum_{1\leqslant k\leqslant n} dz_k \wedge \Big(\frac{\partial u}{\partial z_k}\Big),$$
$$d''u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J = \sum_{1\leqslant k\leqslant n} d\overline{z}_k \wedge \Big(\frac{\partial u}{\partial \overline{z}_k}\Big),$$

where $\partial u/\partial z_k$ and $\partial u/\partial \overline{z}_k$ are the differentiations of u in z_k, $\overline{z}_k$, taken componentwise on each coefficient u_{IJ}. From this we easily get

(5.9) Lemma. *On any open subset $\omega \subset \mathbb{C}^n$ equipped with the flat hermitian metric ω, we have*

$$d'^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} \frac{\partial}{\partial z_k} \lrcorner\ (dz_I \wedge d\overline{z}_J),$$
$$d''^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner\ (dz_I \wedge d\overline{z}_J).$$

These formulas can be written more briefly as

$$d'^\star u = -\sum_{1\leqslant k\leqslant n} \frac{\partial}{\partial z_k} \lrcorner\ \Big(\frac{\partial u}{\partial \overline{z}_k}\Big), \qquad d''^\star u = -\sum_{1\leqslant k\leqslant n} \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial u}{\partial z_k}\Big).$$

Proof. The adjoint of $dz_j \wedge \bullet$ is $\frac{\partial}{\partial z_j} \lrcorner \bullet$. In the case of $d'^\star$, for instance, we get

$$\begin{aligned}\langle\!\langle d'u, v\rangle\!\rangle &= \int_\Omega \langle \sum_{1\leqslant k\leqslant n} dz_k \wedge \Big(\frac{\partial u}{\partial z_k}\Big), v\rangle\, dV = \int_\Omega \sum_{1\leqslant k\leqslant n} \langle \frac{\partial u}{\partial z_k}, \frac{\partial}{\partial z_k} \lrcorner\ v\rangle\, dV \\ &= \int_\Omega \langle u, -\sum_{1\leqslant k\leqslant n} \frac{\partial}{\partial \overline{z}_k}\Big(\frac{\partial}{\partial z_k} \lrcorner\ v\Big)\rangle\, dV = \int_\Omega \langle u, -\sum_{1\leqslant k\leqslant n} \frac{\partial}{\partial z_k} \lrcorner\ \Big(\frac{\partial v}{\partial \overline{z}_k}\Big)\rangle\, dV\end{aligned}$$

whenever u (resp. v) is a $(p-1,q)$-form (resp. (p,q)-form), with $\operatorname{Supp} u \cap \operatorname{Supp} v \Subset \Omega$. The third equality is simply obtained through an integration par parts, and amounts to observe that the formal adjoint of $\partial/\partial z_k$ is $-\partial/\partial \overline{z}_k$. $\square$

We now prove a basic lemma due to (Akizuki and Nakano 1954).

(5.10) Lemma. *In $\mathbb{C}^n$, we have $[d''^\star, L] = \mathrm{i}d'$.*

Proof. Using Lemma 5.9, we find

$$\begin{aligned}[d''^\star, L]u &= d''^\star(\omega\wedge u) - \omega\wedge d''^\star u\\ &= -\sum_k \frac{\partial}{\partial\overline{z}_k} \lrcorner \Big(\frac{\partial}{\partial z_k}(\omega\wedge u)\Big) + \omega\wedge\sum_k \frac{\partial}{\partial\overline{z}_k} \lrcorner \Big(\frac{\partial u}{\partial z_k}\Big).\end{aligned}$$

Since ω has constant coefficients, we have $\frac{\partial}{\partial z_k}(\omega\wedge u) = \omega\wedge\frac{\partial u}{\partial z_k}$ and therefore

$$\begin{aligned}[d''^\star, L]\,u &= -\sum_k \Big(\frac{\partial}{\partial\overline{z}_k} \lrcorner \Big(\omega\wedge\frac{\partial u}{\partial z_k}\Big) - \omega\wedge\Big(\frac{\partial}{\partial\overline{z}_k} \lrcorner \frac{\partial u}{\partial z_k}\Big)\Big)\\ &= -\sum_k \Big(\frac{\partial}{\partial\overline{z}_k} \lrcorner\, \omega\Big)\wedge\frac{\partial u}{\partial z_k}\end{aligned}$$

by the derivation property of $\lrcorner$. Clearly

$$\frac{\partial}{\partial\overline{z}_k} \lrcorner\, \omega = \frac{\partial}{\partial\overline{z}_k} \lrcorner\, \mathrm{i}\sum_{1\leqslant j\leqslant n} dz_j\wedge d\overline{z}_j = -\mathrm{i}\,dz_k,$$

hence

$$[d''^\star, L]\,u = \mathrm{i}\sum_k dz_k\wedge\frac{\partial u}{\partial z_k} = \mathrm{i}\,d'u. \qquad\square$$

We are now ready to derive the basic commutation relations in the case of an arbitrary Kähler manifold (X,ω).

(5.11) Theorem. *If (X,ω) is Kähler, then*

$$\begin{aligned}&[d''^\star, L] = \mathrm{i}\,d', &\qquad &[d'^\star, L] = -\mathrm{i}\,d'',\\ &[\Lambda, d''] = -\mathrm{i}\,d'^\star, &\qquad &[\Lambda, d'] = \mathrm{i}\,d''^\star.\end{aligned}$$

Proof. It is sufficient to verify the first relation, because the second one is the conjugate of the first, and the relations of the second line are the adjoint of those of the first line. According to Theorem 3.9, let (z_j) be a coordinate system at a point $x_0\in X$, chosen such that $\omega_{\ell m} = \delta_{\ell m} + O(|z|^2)$. For any (p,q)-forms u, v with compact support in a neighborhood of x_0, we get

$$\langle\!\langle u, v\rangle\!\rangle = \int_M \Big(\sum_{I,J} u_{IJ}\overline{v}_{IJ} + \sum_{I,J,K,L} a_{IJKL}\,u_{IJ}\overline{v}_{KL}\Big)\,dV,$$

with $a_{IJKL}(z) = O(|z|^2)$ at x_0. An integration by parts as in the proof of Lemma 5.9 yields

$$d''^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner\ (dz_I \wedge d\overline{z}_J) + \sum_{I,J,K,L} b_{IJKL}\, u_{IJ}\, dz_K \wedge d\overline{z}_L,$$

where the coefficients b_{IJKL} are obtained by derivation of the a_{IJKL}'s. Therefore $b_{IJKL} = O(|z|)$. Since $\partial\omega/\partial z_k = O(|z|)$, the proof of Lemma 5.10 implies here $[d''^\star, L]u = \mathrm{i}d'u + O(|z|)$, in particular both terms coincide at every given point $x_0 \in X$. □

(5.12) Corollary. *If (X,ω) is Kähler, the complex Laplace-Beltrami operators satisfy*

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

Proof. It will be first shown that $\Delta'' = \Delta'$. We have

$$\Delta'' = [d'', d''^\star] = -\mathrm{i}\big[d'', [\Lambda, d']\big].$$

Since $[d', d''] = 0$, the Jacobi identity implies

$$-\big[d'', [\Lambda, d']\big] + \big[d', [d'', \Lambda]\big] = 0,$$

hence $\Delta'' = \big[d', -\mathrm{i}[d'', \Lambda]\big] = [d', d'^\star] = \Delta'$. On the other hand

$$\Delta = [d' + d'', d'^\star + d''^\star] = \Delta' + \Delta'' + [d', d''^\star] + [d'', d'^\star].$$

Thus, it is enough to prove:

(5.13) Lemma. $[d', d''^\star] = 0,\ [d'', d'^\star] = 0.$

Proof. We have $[d', d''^\star] = -\mathrm{i}\big[d', [\Lambda, d']\big]$ and the Jacobi identity implies

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^\star] = 0$. The second relation $[d'', d'^\star] = 0$ is the adjoint of the first. □

From the above, we also get the following result, which is of fundamental importance in Hodge theory.

(5.14) Theorem. *Δ commutes with all operators d', d'', $d'^\star$, $d''^\star$, L, Λ.*

Proof. The identities $[d', \Delta'] = [d'^\star, \Delta'] = 0$, $[d'', \Delta''] = [d''^\star, \Delta''] = 0$ are immediate. Furthermore, the equality $[d', L] = d'\omega = 0$ together with the Jacobi identity implies

$$[L, \Delta'] = \big[L, [d', d'^\star]\big] = -\big[d', [d'^\star, L]\big] = \mathrm{i}[d', d''] = 0.$$

By adjunction, we also get $[\Delta', \Lambda] = 0$. □

6. Connections and curvature

The goal of this section is to recall the most basic definitions of hermitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let E be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on E is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T^\star_M \otimes E) \to C^\infty(M, \Lambda^{q+1} T^\star_M \otimes E)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{6.1}$$

for all forms $f \in C^\infty(M, \Lambda^p T^\star_M)$, $u \in C^\infty(X, \Lambda^q T^\star_M \otimes E)$. On an open set $\Omega \subset M$ where E admits a trivialization $\theta : E_{\restriction \Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T^\star_M \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T^\star_M \otimes \mathrm{Hom}(E, E)) \tag{6.2}$$

such that $D^2 u = \Theta(D) \wedge u$ for every form u with values in E.

Assume now that E is endowed with a C^∞ hermitian metric along the fibers and that the isomorphism $E_{\restriction \Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing

$$\begin{aligned} C^\infty(M, \Lambda^p T^\star_M \otimes E) \times C^\infty(M, \Lambda^q T^\star_M \otimes E) &\longrightarrow C^\infty(M, \Lambda^{p+q} T^\star_M \otimes \mathbb{C}) \\ (u, v) &\longmapsto \{u, v\} \end{aligned} \tag{6.3}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \qquad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u} \{u, Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^\star = -\Gamma$. In this case $\Theta(D)^\star = -\Theta(D)$, thus

$$\mathrm{i}\Theta(D) \in C^\infty(M, \Lambda^2 T^\star_M \otimes \mathrm{Herm}(E, E)).$$

(6.4) Special case. For a bundle E of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients $\Gamma = \mathrm{i}A$ (A

real). Then we have $\Theta(D) = d\Gamma = \mathrm{i}dA$. In particular $\mathrm{i}\Theta(E)$ is a closed 2-form. The *First Chern class* of E is defined to be the cohomology class

$$c_1(E)_{\mathbb{R}} = \Big\{\frac{\mathrm{i}}{2\pi}\Theta(D)\Big\} \in H^2_{\mathrm{DR}}(M,\mathbb{R}).$$

The cohomology class is actually independent of the connection, since any other connection D_1 differs by a global 1-form, $D_1 u = Du + B\wedge u$, so that $\Theta(D_1) = \Theta(D) + dB$. (*Note*: the normalizing factor 2π is introduced in such a way that $c_1(E)_{\mathbb{R}}$ becomes the image in $H^2(M,\mathbb{R})$ of an integral class $c_1(E)\in H^2(M,\mathbb{Z})$; see e.g. (Griffiths-Harris 1978) for details). □

We now concentrate ourselves on the complex analytic case. If $M = X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle E can be split in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D' + D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$D'u \simeq_\theta d'u + \Gamma'\wedge u, \tag{6.5$'$}$$

$$D''u \simeq_\theta d''u + \Gamma''\wedge u, \tag{6.5$''$}$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^\star$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the bundle E itself has a *holomorphic* structure. The unique hermitian connection for which D'' is the d'' operator defined in § 1 is called the *Chern connection* of E. In a local holomorphic frame (e_λ) of $E_{\restriction\Omega}$, the metric is given by the hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. We have

$$\{u,v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu = u^\dagger\wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u, and easy computations yield

$$\begin{aligned} d\{u,v\} &= (du)^\dagger\wedge H\overline{v} + (-1)^{\deg u} u^\dagger\wedge(dH\wedge\overline{v} + H\overline{dv}) \\ &= \big(du + \overline{H}^{-1}d'\overline{H}\wedge u\big)^\dagger\wedge H\overline{v} + (-1)^{\deg u} u^\dagger\wedge\overline{\big(dv + \overline{H}^{-1}d'\overline{H}\wedge v\big)} \end{aligned}$$

using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$\begin{cases} Du \simeq_\theta du + \overline{H}^{-1}d'\overline{H}\wedge u, \\ D' \simeq_\theta d' + \overline{H}^{-1}d'\overline{H}\wedge\bullet = \overline{H}^{-1}d'(\overline{H}\bullet), \qquad D'' = d''. \end{cases} \tag{6.6}$$

It is clear from this relations that $D'^2 = D''^2 = 0$. Consequently D^2 is given by to $D^2 = D'D'' + D''D'$, and the curvature tensor $\Theta(D)$ is of type $(1,1)$. Since $d'd'' + d''d' = 0$, we get

$$\begin{aligned} (D'D'' + D''D')u &\simeq_\theta \overline{H}^{-1}d'\overline{H}\wedge d''u + d''(\overline{H}^{-1}d'\overline{H}\wedge u) \\ &= d''(\overline{H}^{-1}d'\overline{H})\wedge u. \end{aligned}$$

(6.7) Proposition. *The Chern curvature tensor $\Theta(E) := \Theta(D)$ is such that*

$$\mathrm{i}\,\Theta(E) \in C^\infty(X, \Lambda^{1,1}T_X^\star \otimes \mathrm{Herm}(E,E)).$$

If $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $E_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta(E) \simeq_\theta \mathrm{i}\,d''(\overline{H}^{-1}d'\overline{H}) \quad \text{on } \Omega. \qquad \square$$

Let $(z_1,\ldots,z_n)$ be holomorphic coordinates on X and let $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ be an orthonormal frame of E. Writing

$$\mathrm{i}\Theta(E) = \sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu,$$

we can identify the curvature tensor to a hermitian form

$$\widetilde{\Theta}(E)(\xi\otimes v, \xi\otimes v) = \sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu \tag{6.8}$$

on $T_X \otimes E$. This leads in a natural way to positivity concepts, following definitions introduced by Kodaira (Kodaira 1953), (Nakano 1955) and (Griffiths 1966).

(6.9) Definition. *The hermitian vector bundle E is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}(E)(\tau,\tau) = \sum c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau_{k\mu}} > 0$ for all non zero tensors $\tau = \sum \tau_{j\lambda}\partial/\partial z_j \otimes e_\lambda \in T_X \otimes E$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}(E)(\xi\otimes v, \xi\otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in T_X \otimes E$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities. We will write $E >_{\mathrm{Nak}} 0$, $E \geqslant_{\mathrm{Nak}} 0$, $E >_{\mathrm{Grif}} 0$, $E \geqslant_{\mathrm{Grif}} 0$ to express that E possesses a smooth hermitian metric with the corresponding (semi)positivity properties.

(6.10) Special case of rank 1 bundles. Assume that E is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : E_{\restriction\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. In this case Prop. 6.10 shows that the curvature form $\Theta(E)$ can be identified with the $(1,1)$-form $d'd''\varphi$, and

$$\mathrm{i}\Theta(E) = \mathrm{i}\,d'd''\varphi$$

is a real $(1,1)$-form. Hence E is semipositive (in either Nakano or Griffiths sense) if and only if φ is psh, resp. positive definite if and only if φ is *strictly psh* (in the sense that $\mathrm{i}\,d'd''\varphi \gg 0$).

7. Bochner-Kodaira-Nakano identity and inequality

We now proceed to explain the basic ideas of the Bochner technique used to prove existence theorems for solutions of d''. Let (X,ω) be a Kähler manifold (the assumption that ω is Kähler is not absolutely necessary, but considerably simplifies the computations). Let (E,h) be a holomorphic vector bundle on X, and let D be the associated Chern connection. We first prove a lemma which will reduce the situation to the case of a trivial vector bundle.

(7.1) Lemma. *For every point $x_0\in X$ and every coordinate system $(z_j)_{1\leqslant j\leqslant n}$ at x_0, there exists a holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ in a neighborhood of x_0 such that*

$$\langle e_\lambda(z),e_\mu(z)\rangle=\delta_{\lambda\mu}-\sum_{1\leqslant j,k\leqslant n}c_{jk\lambda\mu}\,z_j\overline{z}_k+O(|z|^3)$$

where $(c_{jk\lambda\mu})$ are the coefficients of the Chern curvature tensor $\Theta(E)_{x_0}$. Such a frame (e_λ) is called a normal coordinate frame at x_0.

Proof. Let (ε_λ) be a holomorphic frame of E. After replacing (ε_λ) by suitable linear combinations with constant coefficients, we may assume that $\big(\varepsilon_\lambda(x_0)\big)$ is an orthonormal basis of E_{x_0}. Then the inner products $\langle\varepsilon_\lambda,\varepsilon_\mu\rangle$ have an expansion

$$\langle\varepsilon_\lambda(z),\varepsilon_\mu(z)\rangle=\delta_{\lambda\mu}+\sum_j(a_{j\lambda\mu}\,z_j+a'_{j\lambda\mu}\,\overline{z}_j)+O(|z|^2)$$

for some complex coefficients $a_{j\lambda\mu}$, $a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu}=\overline{a}_{j\mu\lambda}$. Set first

$$\eta_\lambda(z)=\varepsilon_\lambda(z)-\sum_{j,\mu}a_{j\lambda\mu}\,z_j\,\varepsilon_\mu(z).$$

Then there are coefficients $a_{jk\lambda\mu}$, $a'_{jk\lambda\mu}$, $a''_{jk\lambda\mu}$ such that

$$\begin{aligned}\langle\eta_\lambda(z),\eta_\mu(z)\rangle&=\delta_{\lambda\mu}+O(|z|^2)\\&=\delta_{\lambda\mu}+\sum_{j,k}\big(a_{jk\lambda\mu}\,z_j\overline{z}_k+a'_{jk\lambda\mu}\,z_jz_k+a''_{jk\lambda\mu}\overline{z}_j\overline{z}_k\big)+O(|z|^3).\end{aligned}$$

The holomorphic frame (e_λ) we are looking for is

$$e_\lambda(z)=\eta_\lambda(z)-\sum_{j,k,\mu}a'_{jk\lambda\mu}\,z_jz_k\,\eta_\mu(z).$$

Since $a''_{jk\lambda\mu}=\overline{a}'_{jk\mu\lambda}$, we easily find

$$\begin{aligned}\langle e_\lambda(z),e_\mu(z)\rangle&=\delta_{\lambda\mu}+\sum_{j,k}a_{jk\lambda\mu}\,z_j\overline{z}_k+O(|z|^3),\\ d'\langle e_\lambda,e_\mu\rangle&=\{D'e_\lambda,e_\mu\}=\sum_{j,k}a_{jk\lambda\mu}\,\overline{z}_k\,dz_j+O(|z|^2),\\ \Theta(E)\cdot e_\lambda&=D''(D'e_\lambda)=\sum_{j,k,\mu}a_{jk\lambda\mu}\,d\overline{z}_k\wedge dz_j\otimes e_\mu+O(|z|),\end{aligned}$$

therefore $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$. □

(7.2) Extended commutation relations. *Let (X, ω) be a Kähler manifold and let L be the operators defined by $Lu = \omega \wedge u$ and $\Lambda = L^\star$. Then*

$$[D''^\star, L] = \mathrm{i}D', \qquad [D'^\star, L] = -\mathrm{i}D'',$$
$$[\Lambda, D''] = -\mathrm{i}D'^\star, \qquad [\Lambda, D'] = \mathrm{i}D''^\star.$$

Proof. Fix a point x_0 in X and a coordinate system $z = (z_1, \ldots, z_n)$ centered at x_0. Then Lemma 7.1 shows the existence of a normal coordinate frame (e_λ) at x_0. Given any section $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda \in C^\infty_{p,q}(X, E)$, it is easy to check that the operators D, $D''^\star$, ... have Taylor expansions of the type

$$Ds = \sum_\lambda d\sigma_\lambda \otimes e_\lambda + O(|z|), \qquad D''^\star s = \sum_\lambda d''^\star \sigma_\lambda \otimes e_\lambda + O(|z|), \ \ldots$$

in terms of the scalar valued operators d, $d''^\star$, ... Here the terms $O(|z|)$ depend on the curvature coefficients of E. The proof of Th. 7.2 is then reduced to the case of operators with values in the trivial bundle $X \times \mathbb{C}$, which is granted by Theorem 5.11. □

(7.3) Bochner-Kodaira-Nakano identity. *If (X, ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on E-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(E), \Lambda].$$

Proof. The last equality in (7.2) yields $D''^\star = -\mathrm{i}[\Lambda, D']$, hence

$$\Delta'' = [D'', D''^\star] = -\mathrm{i}\big[D'', [\Lambda, D']\big].$$

By the Jacobi identity we get

$$\big[D'', [\Lambda, D']\big] = \big[\Lambda, [D', D'']\big] + \big[D', [D'', \Lambda]\big] = [\Lambda, \Theta(E)] + \mathrm{i}[D', D'^\star],$$

taking into account that $[D', D''] = D^2 = \Theta(E)$. The formula follows. □

Assume that X is compact and that $u \in C^\infty(X, \Lambda^{p,q}T^\star X \otimes E)$ is an arbitrary (p, q)-form. An integration by parts yields

$$\langle \Delta' u, u \rangle = \|D'u\|^2 + \|D'^\star u\|^2 \geqslant 0$$

and similarly for Δ'', hence we get the basic inequalities

$$\|D''u\|^2 + \|D''^\star u\|^2 = \|D'u\|^2 + \|D'^\star u\|^2 + \int_X \langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle dV_\omega,$$

$$(7.4) \qquad \|D''u\|^2 + \|D''^\star u\|^2 \geqslant \int_X \langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle dV_\omega.$$

This a priori inequality is known as the *Bochner-Kodaira-Nakano* inequality (see (Bochner 1948), (Kodaira 1953), (Nakano 1955)). Thanks to the general functional analysis results of § 1 (see (1.3)), this inequality can be used to obtain existence theorem for solutions of d''-equations.

Now, one of the main points is to compute the curvature term $\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle$. Unfortunately, as we will soon see, this term turns out to be rather intricate. Fix $x_0\in X$ and local coordinates $(z_1,\ldots,z_n)$ such that $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ is an orthonormal basis of $(T_X^{1,0},\omega)$ at x_0. One can write

$$\begin{aligned}\omega_{x_0}&=\mathrm{i}\sum_{1\leqslant j\leqslant n}dz_j\wedge d\overline{z}_j,\\ \mathrm{i}\Theta(E)_{x_0}&=\mathrm{i}\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}\,dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu\end{aligned}$$

where $(e_1,\ldots,e_r)$ is an orthonormal basis of E_{x_0}. Let

$$u=\sum_{|J|=p,\,|K|=q,\,\lambda}u_{J,K,\lambda}\,dz_J\wedge d\overline{z}_K\otimes e_\lambda\in\big(\Lambda^{p,q}T_X^\star\otimes E\big)_{x_0}.$$

As $\Lambda=L^\star=\Big(-\mathrm{i}\sum d\overline{z}_k\wedge(dz_k\wedge\bullet)\Big)^\star=\mathrm{i}\sum\frac{\partial}{\partial z_k}\lrcorner\Big(\frac{\partial}{\partial\overline{z}_k}\lrcorner\bullet\Big)$, a simple computation gives

$$\begin{aligned}\Lambda u&=\mathrm{i}(-1)^p\sum_{J,K,\lambda,s}u_{J,K,\lambda}\Big(\frac{\partial}{\partial z_s}\lrcorner dz_J\Big)\wedge\Big(\frac{\partial}{\partial\overline{z}_s}\lrcorner d\overline{z}_K\Big)\otimes e_\lambda,\\ \mathrm{i}\Theta(E)\wedge u&=\mathrm{i}(-1)^p\sum_{j,k,\lambda,\mu,J,K}c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_j\wedge dz_J\wedge d\overline{z}_k\wedge d\overline{z}_K\otimes e_\mu,\\ [\mathrm{i}\Theta(E),\Lambda]u&=\mathrm{i}\Theta(E)\wedge(\Lambda u)-\Lambda\big(\mathrm{i}\Theta(E)\wedge u\big)\\ &=\sum_{j,k,\lambda,\mu,J,K}c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_j\wedge\Big(\frac{\partial}{\partial z_k}\lrcorner dz_J\Big)\wedge d\overline{z}_K\otimes e_\mu\\ &\quad+\sum_{j,k,\lambda,\mu,J,K}c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_J\wedge d\overline{z}_k\wedge\Big(\frac{\partial}{\partial\overline{z}_j}\lrcorner d\overline{z}_K\Big)\otimes e_\mu\\ &\quad-\sum_{j,\lambda,\mu,J,K}c_{jj\lambda\mu}\,u_{J,K,\lambda}\,dz_J\wedge d\overline{z}_K\otimes e_\mu.\end{aligned}$$

We extend the definition of $u_{J,K,\lambda}$ to non increasing multi-indices $J=(j_s)$, $K=(k_s)$ by deciding that $u_{J,K,\lambda}=0$ if J or K contains identical components repeated and that $u_{J,K,\lambda}$ is alternate in the indices (j_s), (k_s). Then the above equality can be written

$$\begin{aligned}\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle=&\sum_{j,k,\lambda,\mu,J,S}c_{jk\lambda\mu}\,u_{J,jS,\lambda}\,\overline{u_{J,kS,\mu}}\\ &+\sum_{j,k,\lambda,\mu,R,K}c_{jk\lambda\mu}\,u_{kR,K,\lambda}\,\overline{u_{jR,K,\mu}}\\ &-\sum_{j,\lambda,\mu,J,K}c_{jj\lambda\mu}\,u_{J,K,\lambda}\,\overline{u_{J,K,\mu}},\end{aligned}\tag{7.5}$$

where the sum is extended to all indices $1\leqslant j,k\leqslant n$, $1\leqslant\lambda,\mu\leqslant r$ and multiindices $|R|=p-1$, $|S|=q-1$ (here the notation $u_{JK\lambda}$ is extended to non necessarily increasing multiindices by making it alternate with respect to permutations). It is usually hard to decide the sign of the curvature term (7.5), except in some special cases.

The easiest case is when $p = n$. Then all terms in the second summation of (7.5) must have $j = k$ and $R = \{1, \dots, n\} \smallsetminus \{j\}$, therefore the second and third summations are equal. Formula (7.5) takes the simpler form

$$\langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle = \sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu}\, u_{J,jS,\lambda}\, \overline{u_{J,kS,\mu}}, \tag{7.6}$$

and it follows that $[\mathrm{i}\Theta(E), \Lambda]$ is positive on (n, q)-forms under the assumption that E is *positive in the sense of Nakano* (we will see later in § 10 refined sufficient conditions).

Another tractable case is the case where E is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms $\omega(x)$ and $\mathrm{i}\Theta(E)(x)$, in such a way that

$$\omega(x) = \mathrm{i} \sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j, \qquad \mathrm{i}\Theta(E)(x) = \mathrm{i} \sum_{1\leqslant j\leqslant n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \leqslant \ldots \leqslant \gamma_n$. The curvature eigenvalues $\gamma_j = \gamma_j(x)$ are then uniquely defined and depend continuously on x. With our previous notation, we have $\gamma_j = c_{jj11}$ and all other coefficients $c_{jk\lambda\mu}$ are zero. For any (p, q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned} \langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle &= \sum_{|J|=p,\,|K|=q} \Big(\sum_{j\in J} \gamma_j + \sum_{j\in K} \gamma_j - \sum_{1\leqslant j\leqslant n} \gamma_j\Big)|u_{JK}|^2 \\ &\geqslant (\gamma_1 + \cdots + \gamma_q - \gamma_{n-p+1} - \cdots - \gamma_n)|u|^2. \end{aligned} \tag{7.7}$$

8. L^2 estimates for solutions of d''-equations

Our goal here is to prove a central L^2 existence theorem, which is essentially due to (Hörmander 1965, 1966), and (Andreotti-Vesentini 1965). We will only outline the main ideas, referring e.g. to [(Demailly 1982) for a more detailed exposition of the technical situation considered here. We start with a Kähler manifold (X, ω) and denote by δ_ω the geodesic distance associated with ω. One says that ω is *complete* if δ_ω is complete. The proof is based on the following two observations.

(8.1) Hopf-Rinow lemma. *If δ_ω is complete, then all δ_ω-balls $\overline{B}(x_0, r)$ are compact (and conversely). Moreover, under this hypothesis, there exist a sequence of compact sets K_ν with $X = \bigcup K_\nu$ and $K_\nu \subset K^\circ_{\nu+1}$, and a sequence of cut-off functions ψ_ν such that $|d\psi_\nu|_\omega \leqslant 1$, $\psi_\nu = 1$ on K_ν and* $\mathrm{Supp}\,\psi_\nu \subset K_{\nu+1}$.

Proof (abridged). Take the infimum r_0 of radii r such that the ball $\overline{B}(x_0, r)$ is not compact, if any; then $\overline{B}(x_0, r_0)$ is non compact (otherwise the local compactness of X would imply that some slightly larger ball is still compact). Hence, there is a sequence of points $y_\nu \in \overline{B}(x_0, r_0)$ without any accumulation point. For each k, we can select (along a suitable path from x_0 to y_ν) a point $z_{\nu,k}$ such that $\delta(x_0, z_{\nu,k}) \leqslant r_0 - 2^{-k}$ and $\delta(z_{\nu,k}, y_\nu) \leqslant 2^{-k+1}$. Since $\overline{B}(x_0, r_0 - 2^{-k})$ is compact, we may assume after taking subsequences and a diagonal subsequence, that all sequences $(z_{\nu,k})_\nu$ converge to a point $z_k \in \overline{B}(x_0, r_0 - 2^{-k})$. One then sees that (y_ν) is a Cauchy sequence without any accumulation point, contradiction. Now, the functions (ψ_ν) can be defined by

$$\psi_\nu(x) = \theta\big(3^{-\nu} d(x_0, x)\big)$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a smooth function with $-0.9 \leqslant \theta' \leqslant 0$ such that $\theta(t) = 1$ for $t \leqslant 1.1$, $\theta(t) = 0$ for $t \geqslant 2.9$, if we set $K_\nu = \overline{B}(x_0, 3^\nu)$. As the distance function $x \mapsto \delta_\omega(x_0, x)$ is a 1-Lipschitz function, it is almost everywhere differentiable with a differential of norm $\leqslant 1$, hence $|d\psi_\nu| \leqslant 0.9$. It still remains to get ψ_ν smooth, and this can be achieved by taking suitable convolutions with regularizing kernels. $\square$

(8.2) Lemma (Andreotti-Vesentini). *Assume that ω is complete. For every measurable form u on X with values in $\Lambda^{p,q}T^\star_X \otimes E$ such that*

a) $u \in L^2$ *and* $D''u \in L^2$, *resp.*

b) $u \in L^2$ *and* $D''^\star u \in L^2$, *resp.*

c) $u \in L^2$, $D''u \in L^2$ *and* $D''^\star u \in L^2$,

there exists a sequence of smooth forms u_ν with compact support, such that $u_\nu \to u$ in L^2 in all cases a,b,c) *and $D''u_\nu \to D''u$ in L^2 in case* a) (*resp. $D''^\star u_\nu \to D''^\star u$ in L^2, resp. $D''u_\nu \to D''u$ in L^2 and $D''^\star u_\nu \to D''^\star u$ in L^2 in cases* b) *and* c)).

Proof. Let ψ_ν be a sequence of cut-off functions as in Lemma 8.1. If $u \in L^2$ and $D''u \in L^2$, then $\psi_\nu u \in L^2$ and

$$D''(\psi_\nu u) = \psi_\nu D''u + d''\psi_\nu \wedge u \in L^2.$$

Moreover $\psi_\nu D''u \to D''u$ and $d''\psi_\nu \wedge u \to 0$ in L^2 by Lebesgue's bounded convergence theorem (as $\psi_\nu \to 1$ and $|d''\psi_\nu|_\omega \to 0$ pointwise, with 1 as a uniform bound). The above formula shows that $\psi_\nu D'' = D''(\psi_\nu \bullet) - d''\psi_\nu \wedge \bullet$, hence by adjunction

$$D''^\star(\psi_\nu \bullet) = \psi_\nu D''^\star - (\operatorname{grad}\psi_\nu)^{0,1} \lrcorner \bullet, \qquad D''^\star(\psi_\nu u) = \psi_\nu D''^\star u - (\operatorname{grad}\psi_\nu)^{0,1} \lrcorner u.$$

We infer as before that $D''^\star(\psi_\nu u) \to D''^\star u$ in L^2 as soon as $u \in L^2$ and $D''^\star u \in L^2$. We have thus been able to approximate u by the compactly supported elements $\psi_\nu u$. In order to get smooth approximants u_ν, we need only use convolution by regularizing kernels, i.e., $u_\nu = (\psi_\nu u) \star \rho_\varepsilon$ (possibly after using a partition of unity so as to divide the support of $\psi_\nu u$ in small pieces contained in coordinate open sets). $\square$

(8.3) Corollary. *If ω is complete, the Von Neumann adjoint $(D''_{\mathcal{H}})^\star$ and the hilbertian extension $(D''^\star)_{\mathcal{H}}$ of the formal adjoint coincide.*

Proof. The result is equivalent to proving that $\langle\!\langle D''u, v\rangle\!\rangle = \langle\!\langle u, D''^\star v\rangle\!\rangle$ whenever u, $v \in L^2$ and $D''u \in L^2$, $D''^\star v \in L^2$ (with these operators being computed in the sense of distributions). However we certainly have the equality

$$\langle\!\langle D''u_\nu, v_\nu\rangle\!\rangle = \langle\!\langle u_\nu, D''^\star v_\nu\rangle\!\rangle$$

for any smooth approximants u_ν, v_ν of u and v (as in Lemma 8.2 a) and b)). The desired equality is obtained by taking the limits in L^2. □

(8.4) Theorem. *Let (X,ω) be a Kähler manifold. Here X is not necessarily compact, but we assume that the metric ω is complete on X. Let E be a hermitian vector bundle of rank r over X, and assume that the curvature operator $A = A^{p,q}_{E,\omega} = [\mathrm{i}\Theta(E), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^\star_X \otimes E$, for some $q \geqslant 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^\star_X \otimes E)$ satisfying $D''g = 0$ and $\int_X \langle A^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes E)$ such that $D''f = g$ and*

$$\int_X |f|^2\, dV_\omega \leqslant \int_X \langle A^{-1}g, g\rangle\, dV_\omega.$$

Proof. Lemma 8.2 c) shows that the basic a priori inequality (7.4) extends to arbitrary forms u such that $u \in L^2$, $D''u \in L^2$ and $D''^\star u \in L^2$. Now, consider the Hilbert space orthogonal decomposition

$$L^2(X, \Lambda^{p,q}T^\star_X \otimes E) = \mathrm{Ker}\, D'' \oplus (\mathrm{Ker}\, D'')^\perp,$$

observing that $\mathrm{Ker}\, D''$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}^{p,q}(X,E)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\mathrm{Ker}\, D'')^\perp \subset \mathrm{Ker}\, D''^\star$ by duality and g, $v_1 \in \mathrm{Ker}\, D''$ by hypothesis, we get $D''^\star v_2 = 0$ and

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \leqslant \int_X \langle A^{-1}g, g\rangle\, dV_\omega \int_X \langle Av_1, v_1\rangle\, dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (7.4) applied to $u = v_1$ yields

$$\int_X \langle Av_1, v_1\rangle\, dV_\omega \leqslant \|D''v_1\|^2 + \|D''^\star v_1\|^2 = \|D''^\star v_1\|^2 = \|D''^\star v\|^2.$$

Combining both inequalities, we find

$$|\langle g, v\rangle|^2 \leqslant \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big) \|D''^\star v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w = D''^\star v \longmapsto \langle v, g\rangle, \qquad L^2(X, \Lambda^{p,q-1}T^\star_X \otimes E) \supset D''^\star(\mathcal{D}^{p,q}(E)) \longrightarrow \mathbb{C}$$

on the range of $D''^\star$. This linear form is continuous in L^2 norm and has norm $\leqslant C$ with

$$C = \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes E)$ with $\|f\| \leqslant C$, such that $\langle v, g\rangle = \langle D''^\star v, f\rangle$ for every v, hence $D''f = g$ in the sense of distributions. The inequality $\|f\| \leqslant C$ is equivalent to the last estimate in the theorem. □

(8.5) Remark. One can always select a solution f satisfying the additional property that $f \in (\operatorname{Ker} D'')^\perp$ (otherwise, just replace f by its orthogonal projection on $(\operatorname{Ker} D'')^\perp$). This solution is clearly unique and is precisely the solution of minimal L^2 norm of the equation $D''f = g$. Since $(\operatorname{Ker} D'')^\perp = \overline{\operatorname{Im} D''^\star} \subset \operatorname{Ker} D''^\star$, we see that the minimal L^2 solution satisfies the additional equation

$$D''^\star f = 0.$$

Consequently $\Delta'' f = D''^\star D'' f = D''^\star g$. If g is of class C^∞, the ellipticity of the Δ''-operator shows that f is automatically smooth. □

The above L^2 existence theorem can be applied in the fairly general context of *weakly* (*strongly*) *pseudoconvex* manifolds (these manifolds are frequently referred to as weakly (resp. strongly) 1-complete manifolds in the literature, but we feel that this terminology is a bit misleading).

(8.6) Definition. *A complex manifold X is said to be weakly* (*resp. strongly*) *pseudoconvex if there exists a smooth psh* (*resp. strongly psh*) *exhaustion function ψ on X* (*ψ is said to be an exhaustion if for every $c > 0$ the sublevel set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z tends to "infinity" in X, with respect to the filter of complements of compact sets*)*.*

For example, every closed analytic submanifold of $\mathbb{C}^N$ is strongly pseudoconvex (take $\psi(z) = |z|^2$). Convex open subsets of $\mathbb{C}^n$ are likewise strongly pseudoconvex (take $\psi(z) = |z|^2 + (1-\gamma_a(z))^{-1}$ where γ_a is the "gauge function" with center $a \in \Omega$, namely the unique nonnegative function which is equal to 1 on the boundary of Ω and linear on each half-ray through a ; if γ_a is not smooth, one can take a small "convolution" $z \mapsto \int \rho_\varepsilon(w)\gamma_{a-w}(z)d\lambda(w)$ to get a smooth function). Examples of weakly pseudoconvex manifolds are compact manifolds (just take $\psi \equiv 0$ in that case !), products of such with strongly pseudoconvex manifolds, closed submanifolds of weakly pseudoconvex manifolds, etc. Now, a basic observation is

(8.7) Proposition. *Every weakly pseudoconvex Kähler manifold (X, ω) carries a complete Kähler metric $\widehat{\omega}$.*

Proof. Let $\psi \geqslant 0$ be a psh exhaustion function and set

$$\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi \circ \psi) = \omega + \mathrm{i}\chi' \circ \psi \; d'd''\psi + \mathrm{i}\chi'' \circ \psi \; d'\psi \wedge d''\psi$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a convex increasing function. For any function $\rho : \mathbb{R} \to \mathbb{R}$, we get

$$|d'(\rho \circ \psi)|_{\widehat{\omega}} \leqslant \frac{|\rho' \circ \psi|}{\sqrt{\chi'' \circ \psi}}.$$

If we assume $\int_0^{+\infty}\sqrt{\chi''(t)}\,dt=+\infty$ (as is the case e.g. if $\chi(t)=t^2$ or $\chi(t)=t-\log t$ for $t\geqslant 1$), the function $\rho(t)=\int_0^t\sqrt{\chi''(\tau)}\,d\tau$ tends to $+\infty$ as $t\to+\infty$, and we have $|d(\rho\circ\psi)|_{\widehat{\omega}}\leqslant\sqrt{2}$. By integrating along paths, this bound yields

$$|\rho\circ\psi(x)-\rho\circ\psi(y)|\leqslant\sqrt{2}\,\delta_{\widehat{\omega}}(x,y).$$

It follows easily from the fact that ψ is an exhaustion that all closed geodesic balls are compact. Therefore $\delta_{\widehat{\omega}}$ is complete. □

If we apply the main L^2 existence theorem for the complete Kähler metrics ω_ε, we see by passing to the limit that the theorem even applies to the non necessarily complete metric ω. An important special consequence is the following

(8.8) Theorem. *Let (X,ω) be a Kähler manifold with* $\dim X=n$ *(ω is not assumed to be complete). Assume that X is weakly pseudoconvex. Let E be a hermitian holomorphic line bundle and let*

$$\gamma_1(x)\leqslant\ldots\leqslant\gamma_n(x)$$

be the curvature eigenvalues (i.e. the eigenvalues of $\mathrm{i}\Theta(E)$ *with respect to the metric ω) at every point. Assume that the curvature is positive, i.e. $\gamma_1>0$ everywhere. Then for any form $g\in L^2_{\mathrm{loc}}(X,\Lambda^{n,q}T^\star_X\otimes E)$, $q\geqslant 1$, satisfying $D''g=0$ and $\int_X\langle(\gamma_1+\cdots+\gamma_q)^{-1}|g|^2\,dV_\omega<+\infty$, there exists $f\in L^2(X,\Lambda^{n,q-1}T^\star_X\otimes E)$ such that $D''f=g$ and*

$$\int_X|f|^2\,dV_\omega\leqslant\int_X(\gamma_1+\cdots+\gamma_q)^{-1}|g|^2\,dV_\omega.$$

Proof. Indeed, for $p=n$, Eormula 7.7 shows that

$$\langle Au,u\rangle\geqslant(\gamma_1+\cdots+\gamma_q)|u|^2,$$

hence $\langle A^{-1}u,u\rangle\leqslant(\gamma_1+\cdots+\gamma_q)^{-1}|u|^2$. The assumption that $g\in L^2_{\mathrm{loc}}(X,\Lambda^{n,q}T^\star_X\otimes E)$ instead of $g\in L^2(X,\Lambda^{n,q}T^\star_X\otimes E)$ is not a real problem, since we may restrict ourselves to $X_c=\{x\in X\,;\,\psi(x)<c\}\Subset X$ where ψ is a psh exhaustion function on X. Then X_c itself is weakly pseudoconvex (with psh exhaustion function $\psi_c=1/(c-\psi)$), hence X_c can be equipped with a complete Kähler metric $\omega_{c,\varepsilon}=\omega+\varepsilon\,\mathrm{i}\,d'd''(\psi_c^2)$. For each (c,ε), Theorem 8.4 yields a solution $f_{c,\varepsilon}\in L^2_{\omega_{c,\varepsilon}}(X_c,\Lambda^{n,q-1}T^\star_X\otimes E)$ of the equation $D''f_{c,\varepsilon}=g$ on X_c, such that

$$\int_{X_c}|f_{c,\varepsilon}|^2_{\omega_{c,\varepsilon}}\,dV_{\omega_{c,\varepsilon}}\leqslant\int_{X_c}\langle(A^{n,q}_{E,\omega_{c,\varepsilon}})^{-1}g,g\rangle_{\omega_{c,\varepsilon}}\,dV_{\omega_{c,\varepsilon}}$$

A simple computation shows that the integral in the right hand side is monotonic decreasing with respect to ω, hence

$$\begin{aligned}\int_{X_c}\langle(A^{n,q}_{E,\omega_{c,\varepsilon}})^{-1}g,g\rangle_{\omega_{c,\varepsilon}}\,dV_{\omega_{c,\varepsilon}}&\leqslant\int_{X_c}\langle(A^{n,q}_{E,\omega})^{-1}g,g\rangle_\omega\,dV_\omega\\&\leqslant\int_X(\gamma_1+\cdots+\gamma_q)^{-1}|g|^2\,dV_\omega.\end{aligned}$$

Therefore the solutions $f_{c,\varepsilon}$ are uniformly bounded in L^2 norm on every compact subset of X. Since the closed unit ball of an Hilbert space is weakly compact (and metrizable if the Hilbert space is separable), we can extract a subsequence

$$f_{c_k,\varepsilon_k} \to f \in L^2_{\mathrm{loc}}$$

converging weakly in L^2 on any compact subset $K \subset X$, for some $c_k \to +\infty$ and $\varepsilon_k \to 0$. By the weak continuity of differentiations, we get again in the limit $D''f = g$. Also, for every compact set $K \subset X$ we get

$$\int_K |f|^2_\omega \, dV_\omega \leqslant \liminf_{k\to+\infty} \int_K |f_{c_k,\varepsilon_k}|^2_{\omega_{c_k,\varepsilon_k}} \, dV_{\omega_{c_k,\varepsilon_k}}$$

by weak L^2_{loc} convergence (closed balls, and more generally closed convex sets, are closed in the weak topology of any Banach space). Finally, we let K increase to X and conclude that the desired estimate holds on all of X. □

An important observation is that the above theorem still applies when the hermitian metric on E is a singular metric with positive curvature in the sense of currents.

(8.9) Corollary. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let E be a hermitian holomorphic line bundle and let $\varphi \in L^1_{\mathrm{loc}}$ be a weight function (no further regularity assumption is made on φ). Suppose that*

$$\mathrm{i}\Theta(E) + \mathrm{i}d'd''\varphi \geqslant \gamma\omega$$

for some positive continuous function $\gamma > 0$ on X. Then for any form $g \in L^2_{\mathrm{loc}}(X, \Lambda^{n,q}T^\star_X \otimes E)$, $q \geqslant 1$, satisfying $D''g = 0$ and $\int_X \gamma^{-1}|g|^2 e^{-\varphi}\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes E)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi}\, dV_\omega \leqslant \frac{1}{q}\int_X \gamma^{-1}|g|^2 e^{-\varphi}\, dV_\omega.$$

Proof (sketch) . The general proof is based on regularization techniques for psh functions (see e.g. (Demailly 1982)). It is technically involved essentially because the required regularization techniques are difficult in the case of arbitrary manifolds. We will therefore just explain the proof in the simple case when $X = \Omega$ is a weakly pseudoconvex open set in $\mathbb{C}^n$ with a psh exhaustion function ψ. Then the function $\varphi_\varepsilon = \varphi \star \rho_\varepsilon$ is well defined, smooth on $\Omega_c = \{x \in \Omega\,;\, \psi(x) < c\}$ for ε small enough. Moreover, it satisfies a lower bound of the form

$$\mathrm{i}d'd''\varphi_\varepsilon \geqslant (\gamma\omega - \mathrm{i}\Theta(E)) \star \rho_\varepsilon \geqslant \gamma_\varepsilon\omega - \mathrm{i}\Theta(E)$$

for some continuous function γ_ε converging uniformly to γ on compact subsets of Ω as $\varepsilon \to 0$. We define new hermitian metrics h_ε on the line bundle E by multiplying the original metric h with the weight $e^{-\varphi_\varepsilon}$, i.e., we set $h_\varepsilon = h\, e^{-\varphi_\varepsilon}$. Then

$$\mathrm{i}\Theta_{h_\varepsilon}(E) = \mathrm{i}\Theta_h(E) + \mathrm{i}\, d'd''\varphi_\varepsilon \geqslant \gamma_\varepsilon\omega.$$

We thus get solutions $f_{c,\varepsilon}$ on X_c such that

$$\int_{X_c} |f_{c,\varepsilon}|^2 e^{-\varphi_\varepsilon}\, dV_\omega \leqslant \frac{1}{q}\int_{X_c} \gamma_\varepsilon^{-1}|g|^2 e^{-\varphi_\varepsilon}\, dV_\omega$$

whenever $\gamma_\varepsilon > 0$ on $\overline{X}_c$. As $\varphi_\varepsilon \geqslant \varphi$ converges to φ monotonically, we conclude by extracting weak limits and applying Lebesgue's monotone convergence theorem as before. □

The next corollaries are simple special cases of Hörmander's estimates which are especially convenient in the case of bounded or unbounded pseudoconvex domains in $\mathbb{C}^n$.

(8.10) Corollary. *Let Ω be a bounded (weakly) pseudoconvex open set in $\mathbb{C}^n$, and let φ be a psh function on Ω. Then for any form $g \in L^2(\Omega, \Lambda^{p,q}T^\star_\Omega)$, $q \geqslant 1$, such that $d''g = 0$, there exists $f \in L^2(\Omega, \Lambda^{p,q-1}T^\star_\Omega)$ such that $d''f = g$ and*

$$\int_X |f|^2 e^{-\varphi}\, dV_\omega \leqslant \frac{e}{2q}(\operatorname{diam}\Omega)^2 \int_X |g|^2 e^{-\varphi}\, dV_\omega$$

(All norms being computed with the standard metric $\frac{\mathrm{i}}{2}d'd''|z|^2$.)

Proof. This is a special case of Corollary 8.9 when we take $E = \Omega \times \mathbb{C}$ to be the trivial bundle equipped with the weight function

$$\varphi_\varepsilon(z) = \varphi(z) + \varepsilon|z - z_0|^2, \qquad z_0 \in \Omega, \quad \varepsilon = 1/(\operatorname{diam}\Omega)^2$$

instead of φ. We then find $\varphi \leqslant \varphi_\varepsilon \leqslant \varphi+1$ and $\mathrm{i}d'd''\varphi_\varepsilon \geqslant 2\varepsilon\omega$ on Ω. The L^2 estimate follows immediately. □

(8.11) Corollary. *Let Ω be a (weakly) pseudoconvex open set in $\mathbb{C}^n$, and let φ be a psh function on Ω. Then for any form $g \in L^2_{\mathrm{loc}}(\Omega, \Lambda^{p,q}T^\star_\Omega)$, $q \geqslant 1$, satisfying $d''g = 0$ and $\int_\Omega |g|^2(1+|z|^2)^{2-\varepsilon}e^{-\varphi}\, dV < +\infty$ for some $\varepsilon \in \,]0,+\infty[$, there exists $f \in L^2_{\mathrm{loc}}(\Omega, \Lambda^{p,q-1}T^\star_\Omega)$ such that $d''f = g$ and*

$$\int_X |f|^2(1+|z|^2)^{-\varepsilon} e^{-\varphi}\, dV_\omega \leqslant \frac{1}{2\varepsilon q}\int_X |g|^2(1+|z|^2)^{2-\varepsilon}e^{-\varphi}\, dV_\omega$$

(All norms being computed with the standard metric $\frac{\mathrm{i}}{2}d'd''|z|^2$.)

Proof. The proof is essentially the same as before, except that we take

$$\varphi_\varepsilon(z) = \varphi(z) + \varepsilon\log(1+|z|^2)$$

Then the computations made in example 3.8 b) gives

$$\mathrm{i}d'd''\varphi_\varepsilon \geqslant \varepsilon\, \mathrm{i}d'd''\log(1+|z|^2) \geqslant \frac{\varepsilon\, \mathrm{i}d'd''|z|^2}{(1+|z|^2)^2} = \frac{2\varepsilon}{(1+|z|^2)^2}\,\omega.$$ □

9. Some applications of Hörmander's L^2 estimates

The main applications concern three principal items: vanishing theorems for Dolbeault cohomology groups, existence and approximation theorems for holomorphic functions (these aspects are in fact intimately related, as we will see).

We first list four important vanishing theorems.

(9.1) Nakano vanishing theorem (Nakano 1955 for the compact case, Nakano 1973 in general)**.** *Let E we a hermitian holomorphic vector bundle on a weakly pseudoconvex complex manifold X,* $\dim X = n$*, such that* $\mathrm{i}\Theta(E) > 0$ *in the sense of Nakano. Then $H^{n,q}(X, E) = 0$ for all $q \geqslant 1$.*

Proof. Indeed, (7.5) implies that $\langle[\mathrm{i}\Theta(E), \Lambda]u, u\rangle$ is positive definite. Theorem 8.4 shows that the equation $D''f = g$ can be solved whenever g is D''-closed and satisfies a suitable L^2-condition (moreover Remark 8.5 implies that f is smooth if g is smooth). In fact, we want to solve the equation for a given smooth D''-closed form, whatever is its growth at infinity. For this, we let ψ be a smooth psh exhaustion function on X and multiply the metric of E by the weight factor $e^{-\chi\circ\psi}$ where $\chi : \mathbb{R} \to \mathbb{R}$ is a convex increasing function. If the resulting curvature tensor of E is denoted $\Theta_\chi(E)$, we find

$$\mathrm{i}\Theta_\chi(E) = \mathrm{i}\Theta_0(E) + \mathrm{i}\, d'd''(\chi\circ\psi) = \mathrm{i}\Theta_0(E) + \mathrm{i}(\chi'\circ\psi\, d'd''\psi + \chi''\circ\psi\, d'\psi\wedge d''\psi),$$

and both terms $d'd''\psi$ and $d'\psi\wedge d''\psi$ yield nonnegative contributions (in the sense of Nakano) to the curvature tensor. In particular the resulting curvature operator A_χ on (n,q)-forms satisfies $A_\chi \geqslant A$, $A_\chi^{-1} \leqslant A^{-1}$ and we get

$$\int_X \langle A_\chi^{-1}g, g\rangle e^{-\chi\circ\psi} dV_\omega \leqslant \int_X \langle A^{-1}g, g\rangle e^{-\chi\circ\psi} dV_\omega < +\infty$$

when χ grows quickly enough [take e.g. χ so that

$$e^{-\chi(k)} \int_{\{k\leqslant\psi\leqslant k+1\}} \langle A^{-1}g, g\rangle dV_\omega \leqslant 2^{-k}$$

for every integer $k \geqslant 0$]. We then get a smooth (minimal) L^2 solution f. This implies $H^{n,q}(X, E) = 0$ for $q \geqslant 1$, as desired. □

(9.2) Cartan theorem B (1953)**.** *Let X be a strongly pseudoconvex manifold. Then $H^{p,q}(X, E) = 0$ for every holomorphic vector bundle E and every $q \geqslant 1$.*

Proof. Fix an arbitrary hermitian metric h on E. By the above formula

$$\mathrm{i}\Theta_\chi(E) = \mathrm{i}\Theta_0(E) + \mathrm{i}(\chi'\circ\psi\, d'd''\psi + \chi''\circ\psi\, d'\psi\wedge d''\psi),$$

we see that the curvature of E can be made positive definite if the first derivative of χ grows fast enough. We then conclude that $H^{n,q}(X, E) = 0$ for $q \geqslant 1$ as for Theorem 9.1. To obtain the conclusion for (p,q)-forms as well, we just observe that we have a canonical duality pairing

$$\Lambda^k T_X \otimes \Lambda^k T_X^\star \longrightarrow \mathbb{C},$$

hence a (p,q)-form with values in E can be viewed as a section of

$$\Lambda^{p,q} T_X^\star \otimes E = \Lambda^{0,q} T_X^\star \otimes \Lambda^p T_X^\star \otimes E = \Lambda^{n,q} T_X^\star \otimes \widetilde{E}$$

where $\widetilde{E}$ is the holomorphic vector bundle

$$\widetilde{E} = \Lambda^n T_X \otimes \Lambda^p T_X^\star \otimes E = \Lambda^{n-p} T_X \otimes E,$$

through the contraction pairing

$$\Lambda^n T_X \otimes \Lambda^p T_X^\star \xrightarrow{\simeq} \Lambda^{n-p} T_X.$$

Moreover the Dolbeault complex $\Lambda^{p,\bullet} T_X^\star \otimes E$ is isomorphic to the Dolbeault complex $\Lambda^{n,\bullet} T_X^\star \otimes \widetilde{E}$, hence

$$H^{p,q}(X,E) = H^{n,q}(X,\widetilde{E}) = 0. \qquad \square$$

(9.3) Akizuki-Kodaira-Nakano theorem (1954, also referred to as “precise vanishing theorem”)**.** *Let E be a hermitian holomorphic line bundle on a compact complex manifold X,* $\dim X = n$*, such that* $\mathrm{i}\Theta(E) > 0$*. Then the Dolbeault cohomology groups vanish in the range $p+q \geqslant n+1$, i.e.*

$$H^{p,q}(X,E) = 0 \qquad \textit{for } p+q \geqslant n+1.$$

Proof. Since $\mathrm{i}\Theta(E)$ is a closed positive $(1,1)$-form, one can select $\omega = \mathrm{i}\Theta(E)$ as the basic Kähler metric on X (in particular, X is automatically Kähler). Then the eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω are

$$\gamma_1 = \cdots = \gamma_n = 1,$$

and (7.7) implies

$$\langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle \geqslant (q-(n-p))|u|^2 = (p+q-n)|u|^2.$$

We then get the conclusion from Theorem 8.4. $\square$

(9.4) Kodaira-Serre vanishing theorem (1953, “unprecise vanishing theorem”)**.** *Let E we a hermitian holomorphic line bundle on a compact complex manifold X such that* $\mathrm{i}\Theta(E) > 0$*. Then for every holomorphic vector bundle F, there exists an integer $k_0 = k_0(F)$ such that*

$$H^{p,q}(X, E^{\otimes k} \otimes F) = 0 \qquad \textit{for all } p \geqslant 0,\ q \geqslant 1,\ k \geqslant k_0.$$

Proof. If $p = n$, we reduce (9.4) to the Nakano vanishing theorem. In fact, for any pair of hermitian holomorphic vector bundles F, G, the Chern connections of F, G and $F \otimes G$ are related by

$$D_{F\otimes G}(u \otimes v) = D_F u \otimes v + (-1)^{\deg u} u \otimes D_G v,$$

and this implies easily

$$\Theta(F \otimes G) = \Theta(F) \otimes \mathrm{Id}_G + \mathrm{Id}_F \otimes G. \tag{9.5}$$

In particular, as E is a line bundle, we have $\Theta(E^{\otimes k}) = k\,\Theta(E)$ (with the identification $\mathrm{End}(E) = \mathbb{C}$), hence

$$\Theta(E^{\otimes k} \otimes F) = k\,\Theta(E) \otimes \mathrm{Id}_F + \Theta(F),$$

again with the identifications $\mathrm{End}(E) = \mathrm{End}(E^{\otimes k}) = \mathbb{C}$. In other words, the associated hermitian form of $T_X \otimes F$ satisfies

$$\widetilde{\Theta}(E^{\otimes k} \otimes F)(\xi \otimes v, \xi \otimes v) = k\,\widetilde{\Theta}(F)(\xi, \xi)|v|^2 + \widetilde{\Theta}(F)(\xi \otimes v, \xi \otimes v)$$
$$\widetilde{\Theta}(E^{\otimes k} \otimes F)(\tau, \tau) \geqslant k|\tau|^2 + \widetilde{\Theta}(F)(\tau, \tau)$$

for all elements $\xi \otimes v$, $\tau \in T_X \otimes F$, when $\omega = \mathrm{i}\,\Theta(E)$ is taken as the Kähler metric on X. Hence $E^{\otimes k} \otimes F$ is Nakano positive for $k \geqslant k_0$ large enough, and we infer $H^{n,q}(X, E^{\otimes k} \otimes F) = 0$ for $q \geqslant 1$ and $k \geqslant k_0$. The case of (p,q)-cohomology groups is obtained by replacing F with

$$\widetilde{F} = \Lambda^{n-p} T_X \otimes F$$

as in the proof of 9.2. □

The next application of L^2 estimates is the solution of the so-called Levi problem. In vague terms, the Levi problem asserts that the existence of holomorphic functions on a complex manifold X is intimately related to its pseudoconvexity properties. Complex analysts became aware of the question with the foundational paper of (E.E. Levi, 1910). The final solution for domains of $\mathbb{C}^n$ has been finally settled in three independent papers by (Oka 1953), (Norguet 1954) and (Bremermann, 1954). The generalization to complex manifolds is due to (Grauert, 1958); it gives a characterization of the so called "Stein manifolds", which were introduced by K. Stein and H. Cartan in the early fifties.

(9.6) Concept of holomorphic convexity. *Let X be a complex manifold and let $A \subset X$ be a closed subset. The holomorphic hull $\widehat{A}_{\mathcal{O}(X)}$ is defined to be*

$$\widehat{A}_{\mathcal{O}(X)} = \big\{x \in X\,;\ |f(x)| \leqslant \sup_A |f|\big\}.$$

The subset A is said to be holomorphically convex (in X) if $\widehat{A}_{\mathcal{O}(X)} = A$. The manifold X is said to be holomorphically convex if $\widehat{K}$ is compact for every compact set K, or equivalently, if X can be exhausted by holomorphically convex compact sets K_ν (we say that X is "exhausted" by the K_ν's if $X = \bigcup K_\nu$ and $K_\nu \subset K^\circ_{\nu+1}$ for all ν).

Observe that $\widehat{A}_{\mathcal{O}(X)}$ is a closed subset of X and that $\widehat{\widehat{A}}_{\mathcal{O}(X)} = \widehat{A}_{\mathcal{O}(X)}$, i.e., $\widehat{A}_{\mathcal{O}(X)}$ is holomorphically convex in X. Hence, if X is holomorphically convex, we get inductively an exhausting and strictly increasing sequence of holomorphically convex compact sets K_ν by putting

$$K_\nu = \widehat{(S_\nu)}_{\mathcal{O}(X)}, \qquad S_\nu = \text{a compact neighborhood of } K_{\nu-1} \cup L_\nu,$$

where L_ν is any exhausting sequence of compact sets.

A similar concept of "pseudoconvex hull" $\widehat{A}_{P^\infty(X)}$ with respect to the class $P^\infty(X) = \mathrm{Psh}(X) \cap C^\infty(X)$ exists, namely one can set

$$\widehat{A}_{P^\infty(X)} = \big\{x \in X\,;\ \varphi(x) \leqslant \sup_A \varphi\big\}. \tag{9.7}$$

Since $\varphi = |f|^2 \in P^\infty(X)$ for every $f \in \mathcal{O}(X)$, the inclusion $\widehat{A}_{P^\infty(X)} \subset \widehat{A}_{\mathcal{O}(X)}$ always holds. It is not hard to see that a manifold X is *weakly pseudoconvex* if and only if $\widehat{K}_{P^\infty}(X)$ is compact for every compact set K in X ; in fact if ψ is a psh exhaustion function, then $K_\nu = \{x \in X\,;\ \psi(x) \leqslant \nu\}$ is an exhausting sequence of pseudoconvex compact sets; conversely if such a sequence K_ν with $\widehat{(K_\nu)}_{P^\infty(X)} = K_\nu$ exists, we define a psh exhaustion ψ by putting

$$\psi = \sum_{\nu=0}^{+\infty} 2^\nu \chi \circ \varphi_\nu$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function such that $\chi(t) = 0$ for $t \leqslant 0$, $\chi(t) \geqslant 1$ for $t \geqslant 1$, and $\varphi_\nu \in P^\infty(X)$ is chosen such that $\varphi_\nu \leqslant 0$ on K_ν and $\varphi_\nu \geqslant 1$ on $K_{\nu+2} \smallsetminus K^\circ_{\nu+1}$ (a finite "regularized" maximum $\varphi_\nu = (\max\star\rho_\varepsilon)(\varphi_{\nu,1}, \ldots, \varphi_{\nu,N})$ with $\varphi_{\nu,j}(x_j) > 1 > 0 > \sup_{K_\nu} \varphi_{\nu,j}$ at sufficiently many points $x_j \in K_{\nu+2} \smallsetminus K^\circ_{\nu+1}$ will do by the Borel-Lebesgue lemma).

(9.8) Stein manifolds. *Let X be a n-dimensional complex manifold. Then X is said to be a Stein manifold if it satisfies the following two properties.*

a) *X is holomorphically convex*;

b) *$\mathcal{O}(X)$ locally separates points, in the sense that every point $x \in X$ has a neighborhood V such that for any $y \in V \smallsetminus \{x\}$ there exists $f \in \mathcal{O}(X)$ with $f(y) \neq f(x)$.*

We first prove the "easy direction" in the Levi problem, namely the implication

$$\text{Stein} \Rightarrow \text{strongly pseudoconvex},$$

which depends only on elementary considerations about psh functions. The converse (deeper) implication $\Leftarrow$ can be proved using L^2 estimates.

(9.9) Theorem. *Let X be a complex manifold.*

a) *If X is holomorphically convex, then X is weakly pseudoconvex.*

b) *If $\mathcal{O}(X)$ satisfies the local separation property* 9.8 b)*, there exists a smooth non-negative strictly plurisubharmonic function $u \in \mathrm{Psh}(X)$.*

c) *If X is Stein, then X is strongly pseudoconvex.*

Proof. a) If X is holomorphically convex, we have seen that there is an exhausting sequence (K_ν) of holomorphically convex compact sets of X. These compact sets then satisfy $(\widehat{K_\nu})_{\mathcal{O}(X)} = (\widehat{K_\nu})_{P^\infty(X)} = K_\nu$, hence X is weakly pseudoconvex.

b) Fix $x_0 \in X$. We first show that there exists a smooth nonnegative function $u_0 \in \mathrm{Psh}(X)$ which is strictly plurisubharmonic on a neighborhood of x_0. Let $(z_1, \ldots, z_n)$ be local analytic coordinates centered at x_0, and if necessary, replace z_j by λz_j so that the closed unit ball $\overline{B} = \{\sum |z_j|^2 \leqslant 1\}$ is contained in the neighborhood $V \ni x_0$ on which (6.16 b) holds. Then, for every point $y \in \partial B$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(y) \neq f(x_0)$. Replacing f with $\lambda(f - f(x_0))$, we can achieve $f(x_0) = 0$ and $|f(y)| > 1$. By compactness of ∂B, we find finitely many functions $f_1, \ldots, f_N \in \mathcal{O}(X)$ such that $v_0 = \sum |f_j|^2$ satisfies $v_0(x_0) = 0$, while $v_0 \geqslant 1$ on ∂B. Now, we set

$$u_0(z) = \begin{cases} v_0(z) & \text{on } X \smallsetminus B, \\ \max_\varepsilon\{v_0(z), (|z|^2 + 1)/3\} & \text{on } B. \end{cases}$$

where $\max_\varepsilon = \max(\bullet, \bullet) \star \rho_\varepsilon$ is a regularized max function in $\mathbb{R}^2$. Then u_0 is smooth and plurisubharmonic, coincides with v_0 near ∂B and with $(|z|^2 + 1)/3$ on a neighborhood of x_0. We can cover X by countably many neighborhoods $(V_j)_{j \geqslant 1}$, for which we have a smooth plurisubharmonic functions $u_j \in \mathrm{Psh}(X)$ such that u_j is strictly plurisubharmonic on V_j. Then select a sequence $\varepsilon_j > 0$ converging to 0 so fast that $u = \sum \varepsilon_j u_j \in C^\infty(X)$. The function u is nonnegative and strictly plurisubharmonic everywhere on X.

c) Select ψ as in a) and u as in b). Then $\psi + u$ is a strictly psh exhaustion function of X. $\square$

Conversely, we have the following existence theorem derived from L^2 estimates.

(9.10) Theorem. *Let X be a strongly pseudonconvex manifold. For every locally finite sequence (x_ν) of distinct points of X and every sequence of polynomials $P_\nu(z^\nu)$ relative to local coordinates $z^\nu = (z_1^\nu, \ldots, z_n^\nu)$ around x_ν (with given bounds $\deg P_\nu \leqslant m_\nu$ for the degrees), there is a global holomorphic function $f \in \mathcal{O}(X)$ such that the Taylor expansion of order m_ν of f at x_ν coincides with P_ν.*

Proof. The main idea is to use a L^2 estimate with a weight assuming a logarithmic pole at each point x_ν. Let U_ν be the open coordinate patch where z^ν is defined and let $\theta_\nu \in \mathcal{D}(U_\nu)$ be a cut-off function such that $\theta_\nu = 1$ on a neighborhood of x_ν and $0 \leqslant \theta_\nu \leqslant 1$. Then

$$\varphi_0 = \sum_\nu 2(n + m_\nu)\theta_\nu \log |z_n u|$$

is psh in a neighborhood of x_ν and φ_0 is smooth on $X \smallsetminus \{x_\nu\}$. It follows that the negative part of $\mathrm{i}\, d'd''\varphi_0$ is locally bounded below everywhere. Hence, if ψ is a strictly psh exhaustion function on X, there exists a convex increasing function $\chi_1 : \mathbb{R} \to \mathbb{R}$ such that $\varphi_1 = \varphi_0 + \chi_1 \circ \psi$ is psh (with an arbitrarily large positive preassigned Hessian form $\mathrm{i}\, d'd''\varphi_1$). Now, the function $F = \sum \theta_\nu P_\nu$ is a smooth function which has the prescribed Taylor expansions at all points x_ν. The idea is to solve the equation

$$d''u = v := d''F$$

where v is a $(0,1)$-form with complex values, that is, a $(n,1)$-form with values in $\Lambda^n T_X$. We assume that $\mathrm{i}\Theta(\Lambda^n T_X) + \mathrm{i}\, d'd''\varphi_1 \geqslant \omega := \mathrm{i}\, d'd''\psi$. Then, for every convex increasing function χ and $\varphi = \varphi_1 + \chi\circ\psi = \varphi_0 + (\chi+\chi_1)\circ\psi$, we get a solution u such that

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leqslant \int_X |v|^2 e^{-\varphi} dV_\omega = \sum_\nu \int_{U_\nu} |P_\nu|^2 |d''\theta_\nu|^2 e^{-2(n+m_\nu)\theta_\nu \log|z^\nu| + \chi\circ\psi} dV_\omega,$$

provided that the right hand side is convergent. However, each term in the sum is convergent since the support of $d''\theta_\nu$ avoids a neighborhood of x_ν, and the global convergence is easily guaranteed if χ grows fast enough. As d'' is elliptic in bidegree $(0,0)$, we get a smooth solution u such that

$$\int_{U_\nu} \frac{|u|^2}{|z^\nu|^{2(n+m_\nu)}} dV_\omega < +\infty$$

for each ν. From this one concludes that $D^\alpha u(x_\nu) = 0$ for $|\alpha| \leqslant m_\nu$, hence $f = F - u$ is a holomorphic function on X with P_ν as its Taylor expansion of order m_ν at x_ν. □

(9.11) Corollary. *Every strongly pseudoconvex manifold X is Stein. Moreover, the functions in $\mathcal{O}(X)$ separate any pair of distinct points in X, and for $x_0 \in X$ given, there are functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $(f_1, \ldots, f_n)$ is a local coordinate system at x_0.*

Proof. Let X be a complex manifold. The holomorphic convexity property of X is formally equivalent to the following assertion: for every sequence (x_ν) in X, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $(f(x_\nu))$ is unbounded (the equivalence can be seen more or less by the same argument as in 9.9 a)). By Theorem 9.10 we need only take a function f which interpolates the values $f(x_\nu) = \nu$. The property of local separation of points 9.9 b) is also clear, as well as the stronger properties asserted in Corollary 9.11. □

We end this section by proving a general n-dimensional version of the Runge theorem.

(9.12) Runge approximation theorem. *Let E be a holomorphic vector bundle on a Stein manifold X. Let φ be a psh exhaustion of X and let*

$$K = K_c = \{x \in X\,;\ \varphi(x) \leqslant c\}$$

for some c (φ need not be strictly psh). Then every holomorphic section g defined on a neighborhood of K is a uniform limit on a neighborhood of K of a sequence of global holomorphic sections f_ν of E over X.

Proof. Fix c_i, $i = 1,2,3,4$ such that $c < c_1 < c_2 < c_3 < c_4$ and g is holomorphic on $\Omega_{c_4} = \{x \in X\,;\ \varphi(x) \leqslant c_4\}$. Fix a cut-off function $\theta \in \mathcal{D}(X)$ with $\mathrm{Supp}\,\theta \subset \Omega_{c_4}$

and $\theta = 1$ on Ω_{c_3}. We view $g\theta$ as a smooth function on X (by defining it to be 0 on $X \smallsetminus \Omega_{c_4}$), and solve the equation

$$d''u = v := d''(g\theta) = g\,d''\theta$$

with a weight of the form $\varphi_\nu = \nu\varphi + \chi\circ\psi$, where ψ is a strictly psh exhaustion function and χ a convex increasing function such that the resulting curvature eigenvalues are $\geqslant 1$. We then get a solution $u = u_\nu$ such that

$$\int_X |u_\nu|^2 e^{-\nu\varphi-\chi\circ\psi} dV_\omega \leqslant \int_X |g|^2 |d''\theta|^2 e^{-\nu\varphi-\chi\circ\psi} dV_\omega.$$

As $d''\theta$ has support in $\Omega_{c_4} \smallsetminus \Omega_{c_3} \subset \{\varphi \geqslant c_3\}$, the right hand side is bounded by $C\,e^{-\nu c_3}$. From the L^2 estimate, we infer

$$\int_{\Omega_{c_2}} |u_\nu|^2 dV_\omega \leqslant C' e^{\nu c_2} \int_{\Omega_{c_2}} |u_\nu|^2 e^{-\nu\varphi-\chi\circ\psi} dV_\omega \leqslant C'' e^{\nu c_2} e^{-\nu c_3} \underset{\nu\to+\infty}{\longrightarrow} 0.$$

Finally, as $d''u_\nu = g\,d''\theta = 0$ on Ω_{c_3}, the function $|u_\nu|^2$ is plurisubharmonic on Ω_{c_2}. By the mean value inequality, we see that u_ν converges uniformly to 0 on Ω_{c_1}. Hence $f_\nu = g\theta - u_\nu \in \mathcal{O}(X)$ converges uniformly to f on Ω_{c_1}. $\square$

(9.13) Remark. The assumption that $K = \{\psi \leqslant c\}$ for some psh exhaustion function ψ is satisfied if and only if $\widehat{K}_{P^\infty(X)} = K$ (in which case we say that K is *pseudoconvex*). In fact, we have the following lemma.

(9.14) Lemma. *Let X be a weakly (resp. strongly) pseudoconvex manifold with a weakly (strictly) psh exhaustion function ψ_0. If $K \subset X$ is a pseudoconvex compact set, i.e., if $\widehat{K}_{P^\infty(X)} = K$, there exists an exhaustion function $\varphi \in P^\infty(X)$ such that*

a) $\varphi = 0$ *on* X;

b) $\varphi > 0$ *and* φ *is weakly (strictly) psh on* $X \smallsetminus K$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing with $\chi(t) = 0$ for $t \leqslant 0$ and $\chi(t) > 0$ for $t > 0$. For any $x \notin K$, there is a function $\varphi_x \in P^\infty(X)$ such that

$$\varphi_x(x) > 0 > \sup_K \varphi_x.$$

We set $u_x = \chi(\varphi_x + \delta\psi_0)$ ($\delta > 0$ small), so as to get a nonnegative function $u_x \in P^\infty(X)$ such that $u_x = 0$ on K and $u_x > 0$ (strictly psh) on a neighborhood V_x of x. Then $X \smallsetminus K$ can be covered by countably many such neighborhoods V_{x_ν} and we get a function φ with the required properties by setting

$$\varphi = \chi(\psi_0 - C) + \sum \varepsilon_\nu u_{x_\nu}$$

for some large constant C. $\square$

(9.15) Corollary. *If X is strongly pseudoconvex, then*

$$\widehat{K}_{P^\infty(X)} = \widehat{K}_{\mathcal{O}(X)}$$

for every compact set $K \subset X$.

Proof. Set $\widetilde K = \widehat K_{P^\infty}(X)$. Then $\widetilde K$ is pseudoconvex and we thus get by Lemma 9.14 a function $\varphi \in P^\infty(X)$ with $\varphi = 0$ on K and $\varphi > 0$ strictly psh on $X \smallsetminus K$. Fix an arbitrary point $x_0 \in X \smallsetminus K$, a coordinate system $z = (z_1, \ldots, z_n)$ centered at x_0, and a cut-off function θ equal to 1 on a neighborhood of x_0, with support disjoint from K. For $0 < \varepsilon \ll 1$, the function

$$\varphi_\varepsilon(x) = \varphi(x) + \varepsilon\theta(x)\log(|z|^2 + e^{-1/\varepsilon^2})$$

is psh everywhere and $\varphi_\varepsilon(x_0) = \varphi(x_0) - 1/\varepsilon < 0$. Thus $K_\varepsilon = \{\varphi_\varepsilon \leqslant 0\}$ is equal to K union a small neighborhood $\overline V_\varepsilon$ of x_0 disjoint from K. We define a holomorphic function g on a neighborhood of K_ε by taking $g = 0$ on a neighborhood of K and $g = 1$ on a neighborhood of $\overline V_\varepsilon$. The Runge approximation theorem provides a global holomorphic function $f \in \mathcal{O}(X)$ such that $|f - g| \leqslant 1/3$ on K_ε. We thus get $|f| \leqslant 1/3$ on K and $|f(x_0)| \geqslant 2/3$. This implies that every $x_0 \notin \widetilde K$ is not either in the holomorphic hull of $\widetilde K$, hence $\widetilde K$ is holomorphically convex. From this we infer

$$\widehat K_{\mathcal{O}(X)} \subset \widehat{(\widetilde K)}_{\mathcal{O}(X)} = \widetilde K = \widehat K_{P^\infty(X)},$$

and the opposite inclusion is clear. □

10. Further preliminary results of hermitian differential geometry

In the course of the proof of Skoda's L^2 estimates, we will have to deal with dual bundles and exact sequences of hermitian vector bundles. The following fundamental differential geometric lemma will be needed.

(10.1) Lemma. *Let E be a hermitian holomorphic vector bundle of rank r on a complex n-dimensional manifold X. Then the Chern connections of E and $E^\star$ are related by $\Theta(E^\star) = -{}^t\Theta(E)$ where ^{t} denotes transposition. In other words, the associated hermitian forms $\widetilde\Theta(E)$ and $\widetilde\Theta(E^\star)$ are related by*

$$\widetilde\Theta(E)(\tau,\tau) = \sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\tau_{j\lambda}\overline\tau_{k\mu}, \qquad \tau = \sum_{j,\lambda}\tau_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e_\lambda,$$

$$\widetilde\Theta(E^\star)(\tau,\tau) = -\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r} c_{jk\mu\lambda}\tau^\star_{j\lambda}\overline\tau^\star_{k\mu}, \qquad \tau^\star = \sum_{j,\lambda}\tau^\star_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e^\star_\lambda.$$

In particular $E >_{\mathrm{Grif}} 0$ if and only if $E^\star <_{\mathrm{Grif}} 0$.

Notice that the corresponding duality statement for Nakano positivity is wrong (because of the twist of indices, which is fortunately irrelevant in the case of decomposable tensors).

Proof. The Chern connections of E and $E^\star$ are related by the Leibnitz rule

$$d(\sigma \wedge s) = (D_{E^\star}\sigma) \wedge s + (-1)^{\deg \sigma} \sigma \wedge D_E s$$

whenever s, σ are forms with values in E, $E^\star$ respectively, and $\sigma \wedge s$ is computed using the pairing $E^\star \otimes E \to \mathbb{C}$. If we differentiate a second time, this yields the identity

$$0 = (D_{E^\star}^2 \sigma) \wedge s + \sigma \wedge D_E^2 s,$$

which is equivalent to the formula $\Theta(E^\star) = -{}^t\Theta(E)$. All other assertions follow. $\square$

(10.2) Lemma. *Let*

$$0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0$$

be an exact sequence of holomorphic vector bundles. Assume that E is equipped with a smooth hermitian metric, and that S and Q are endowed with the metrics (restriction-metric and quotient-metric) induced by that of E. Then

$$j^\star \oplus g : E \to S \oplus Q, \qquad j \oplus g^\star : S \oplus Q \to E \tag{10.3}$$

are C^∞ isomorphisms of bundles, which are inverse of each other. In the C^∞-splitting $E \simeq S \oplus Q$, the Chern connection of E admits a matrix decomposition

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix} \tag{10.4}$$

in terms of the Chern connections of S and Q, where

$$\beta \in C^\infty\big(X, \Lambda^{1,0}T_X^\star \otimes \mathrm{Hom}(S,Q)\big), \qquad \beta^\star \in C^\infty\big(X, \Lambda^{0,1}T_X^\star \otimes \mathrm{Hom}(Q,S)\big).$$

The form β is called the second fundamental form associated with the exact sequence. It is uniquely defined by each of the two formulas

$$D'_{\mathrm{Hom}(S,E)} j = g^\star \circ \beta, \qquad j \circ \beta^\star = -D''_{\mathrm{Hom}(Q,E)} g^\star. \tag{10.5}$$

We have $D'_{\mathrm{Hom}(S,Q)}\beta = 0$, $D''_{\mathrm{Hom}(Q,S)}\beta^\star = 0$, and the curvature form of E splits as

$$\Theta(E) = \begin{pmatrix} \Theta(S) - \beta^\star \wedge \beta & -D'_{\mathrm{Hom}(Q,S)}\beta^\star \\ D''_{\mathrm{Hom}(S,Q)}\beta & \Theta(Q) - \beta \wedge \beta^\star \end{pmatrix}, \tag{10.6}$$

and the curvature forms of S and Q can be expressed as

$$\Theta(S) = \Theta(E)_{\restriction S} + \beta^\star \wedge \beta, \qquad \Theta(Q) = \Theta(E)_{\restriction Q} + \beta \wedge \beta^\star, \tag{10.7}$$

where $\Theta(E)_{\restriction S}$, $\Theta(E)_{\restriction Q}$ stand for $j^\star \circ \Theta(E) \circ j$ and $g \circ \Theta(E) \circ g^\star$.

Proof. Because of the uniqueness property of Chern connections, it is easy to see that we have a Leibnitz formula

$$D_F(f \wedge u) = (D_{\mathrm{Hom}(E,F)} f) \wedge u + (-1)^{\deg f} f \wedge D_E u$$

whenever u, f are forms with values in hermitian vector bundles E and $\mathrm{Hom}(E,F)$ (where $\mathrm{Hom}(E,F) = E^\star \otimes F$ is equipped with the tensor product metric and $f \wedge u$ incorporates the evaluation mapping $\mathrm{Hom}(E,F) \otimes E \to F$). In our case, given a form u with values in E, we write $u = ju_S + g^\star u_Q$ where $u_S = j^\star u$ and $u_Q = gu$ are the projections of u on S and Q. We then get

$$\begin{aligned} D_E u &= D_E(ju_S + g^\star u_Q) \\ &= (D_{\mathrm{Hom}(S,E)} j) \wedge u_S + j \cdot D_S u_S + (D_{\mathrm{Hom}(Q,E)} g^\star) \wedge u_Q + g^\star \cdot D_Q u_Q. \end{aligned}$$

Since j is holomorphic as well as $j^\star \circ j = \mathrm{Id}_S$, we find $D''_{Hom(S,E)} j = 0$ and

$$D''_{Hom(S,S)} \mathrm{Id}_S = 0 = D''_{\mathrm{Hom}(E,S)} j^\star \circ j.$$

By taking the adjoint, we see that $j^\star \circ D'_{\mathrm{Hom}(S,E)} j = 0$, hence $D'_{\mathrm{Hom}(S,E)} j$ takes values in $g^\star Q$ and we thus have a unique form β as in the Lemma such that $D'_{\mathrm{Hom}(S,E)} j = g^\star \circ \beta$. Similarly, g and $g \circ g^\star = \mathrm{Id}_Q$ are holomorphic, thus

$$D''_{Hom(Q,Q)} \mathrm{Id}_Q = 0 = g \circ D''_{\mathrm{Hom}(Q,E)} g^\star$$

and there is a form $\gamma \in C^\infty\big(X, \Lambda^{0,1} T^\star_X \otimes \mathrm{Hom}(Q,S)\big)$ such that $D''_{\mathrm{Hom}(Q,E)} g^\star = j \circ \gamma$. By adjunction, we get $D'_{\mathrm{Hom}(E,Q)} g = \gamma^\star \circ j^\star$ and $D''_{Hom(E,Q)} g = 0$ implies $D'_{\mathrm{Hom}(Q,E)} g^\star = 0$. If we differentiate $g \circ j = 0$ we then get

$$0 = D'_{\mathrm{Hom}(E,Q)} g \circ j + g \circ D'_{\mathrm{Hom}(S,E)} j = \gamma^\star \circ j^\star \circ j + g \circ g^\star \circ \beta = \gamma^\star + \beta,$$

thus $\gamma = -\beta^\star$ and $D''_{\mathrm{Hom}(Q,E)} g^\star = -j \circ \beta^\star$. Combining all this, we get

$$\begin{aligned} D_E u &= g^\star \beta \wedge u_S + j \cdot D_S u_S - j\beta^\star \wedge u_Q + g^\star \cdot D_Q u_Q \\ &= j\big(D_S u_S - \beta^\star \wedge u_Q\big) + g^\star\big(\beta \wedge u_S + D_Q u_Q\big), \end{aligned}$$

and the asserted matrix decomposition formula follows. By squaring the matrix, we get

$$D_E^2 = \begin{pmatrix} D_S^2 - \beta^\star \wedge \beta & -D_S \circ \beta^\star - \beta^\star \circ D_Q \\ D_Q \circ \beta + \beta \circ D_S & D_Q^2 - \beta \wedge \beta^\star \end{pmatrix}.$$

As $D_Q \circ \beta + \beta \circ D_S = D_{\mathrm{Hom}(S,Q)} \beta$ and $D_S \circ \beta^\star + \beta^\star \circ D_Q = D_{\mathrm{Hom}(Q,S)} \beta^\star$ by the Leibniz rule, the curvature formulas follow (observe, since the Chern curvature form is of type $(1,1)$, that we must have $D'_{\mathrm{Hom}(S,Q)} \beta = 0$, $D''_{\mathrm{Hom}(Q,S)} \beta^\star = 0$). $\square$

(10.8) Corollary. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then*

a) $E \geqslant_{\mathrm{Grif}} 0 \implies Q \geqslant_{\mathrm{Grif}} 0$,

b) $E \leqslant_{\mathrm{Grif}} 0 \implies S \leqslant_{\mathrm{Grif}} 0$,

c) $E \leqslant_{\mathrm{Nak}} 0 \implies S \leqslant_{\mathrm{Nak}} 0$,

and analogous implications hold true for strict positivity.

Proof. If β is written $\sum dz_j \otimes \beta_j$, $\beta_j \in \mathrm{Hom}(S,Q)$, then formulas (10.7) yield

$$\begin{aligned} \mathrm{i}\Theta(S) &= \mathrm{i}\Theta(E)_{\restriction S} - \sum dz_j \wedge d\overline{z}_k \otimes \beta_k^\star \beta_j, \\ \mathrm{i}\Theta(Q) &= \mathrm{i}\Theta(E)_{\restriction Q} + \sum dz_j \wedge d\overline{z}_k \otimes \beta_j \beta_k^\star. \end{aligned}$$

Since $\beta\cdot(\xi\otimes s)=\sum\xi_j\beta_j\cdot s$ and $\beta^\star\cdot(\xi\otimes s)=\sum\overline{\xi}_k\beta_k^\star\cdot s$ we get

$$\widetilde{\Theta}(S)(\xi\otimes s,\xi'\otimes s')=\widetilde{\Theta}(E)(\xi\otimes s,\xi'\otimes s')-\sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_j\cdot s,\beta_k\cdot s'\rangle,$$

$$\widetilde{\Theta}(S)(u,u)=\widetilde{\Theta}(E)(u,u)-|\beta\cdot u|^2,$$

$$\widetilde{\Theta}(Q)(\xi\otimes s,\xi'\otimes s')=\widetilde{\Theta}(E)(\xi\otimes s,\xi'\otimes s')+\sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_k^\star\cdot s,\beta_j^\star\cdot s'\rangle,$$

$$\widetilde{\Theta}(Q)(\xi\otimes s,\xi\otimes s)=\widetilde{\Theta}(E)(\xi\otimes s,\xi\otimes s)=|\beta^\star\cdot(\xi\otimes s)|^2. \qquad \square$$

Next, we need positivity properties which somehow interpolate between Griffiths and Nakano positivity. This leads to the concept of m-tensor positivity.

(10.9) Definition. *Let T and E be complex vector spaces of dimensions n, r respectively, and let Θ be a hermitian form on $T\otimes E$.*

a) *A tensor $u\in T\otimes E$ is said to be of rank m if m is the smallest $\geqslant 0$ integer such that u can be written*

$$u=\sum_{j=1}^{m}\xi_j\otimes s_j,\qquad \xi_j\in T,\ \ s_j\in E.$$

b) *Θ is said to be m-tensor positive (resp. m-tensor semi-positive) if $\Theta(u,u)>0$ (resp. $\Theta(u,u)\geqslant 0$) for every tensor $u\in T\otimes E$ of rank $\leqslant m$, $u\neq 0$. In this case, we write*

$$\Theta>_m 0\qquad (\textit{resp. }\ \Theta\geqslant_m 0).$$

We say that a hermitian vector bundle E is m-tensor positive if $\widetilde{\Theta}(E)>_m 0$. Griffiths positivity corresponds to $m=1$ and Nakano positivity to $m\geqslant\min(n,r)$. Recall from (7.6) that we have

$$\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle=\sum_{|S|=q-1}\ \sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}\,u_{jS,\lambda}\overline{u}_{kS,\mu}$$

for every (n,q)-form $u=\sum u_{K,\lambda}\,dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z}_K\otimes e_\lambda$ with values in E. Since $u_{jS,\lambda}=0$ for $j\in S$, the rank of the tensor $(u_{jS,\lambda})_{j,\lambda}\in\mathbb{C}^n\otimes\mathbb{C}^r$ is in fact $\leqslant\min\{n-q+1,r\}$. We obtain therefore:

(10.10) Lemma. *Assume that $E\geqslant_m 0$ (resp. $E>_m 0$). Then the hermitian operator $[\mathrm{i}\Theta(E),\Lambda]$ is semipositive (resp. positive definite) on $\Lambda^{n,q}T^\star X\otimes E$ for $q\geqslant 1$ and $m\geqslant\min\{n-q+1,r\}$.*

The Nakano vanishing theorem can then be improved as follows.

(10.11) Theorem. *Let X be a weakly pseudoconvex Kähler manifold of dimension n and let E a hermitian vector bundle of rank r such that $\widetilde{\Theta}(E)>_m 0$ over X. Then*

$$H^{n,q}(X,E)=0 \qquad \text{for} \quad q\geqslant 1 \quad \text{and} \quad m\geqslant \min\{n-q+1,r\}.$$

We next study some important relations which exist between the various positivity concepts. Our starting point is the following result of (Demailly-Skoda 1979).

(10.12) Theorem. *For any hermitian vector bundle E,*

$$E>_{\rm Grif} 0 \quad\Longrightarrow\quad E\otimes\det E>_{\rm Nak} 0.$$

To prove this result, we use the fact that

$$\Theta(\det E)=\mathrm{Tr}_E\,\Theta(E) \tag{10.13}$$

where $\mathrm{Tr}_E : \mathrm{Hom}(E,E)\to\mathbb{C}$ is the trace map, together with the identity

$$\Theta(E\otimes\det E)=\Theta(E)+\mathrm{Tr}_E(\Theta(E))\otimes \mathrm{Id}_E,$$

which is a special case of formula (9.5). Formula (10.13) is easily obtained by differentiating twice a wedge product, according to the formula

$$D_{\Lambda^pE}(s_1\wedge\cdots s_p)=\sum_{j=1}^p(-1)^{\deg s_1+\cdots+\deg s_{j-1}}s_1\wedge\cdots\wedge s_{j-1}\wedge D_Es_j\wedge\cdots\wedge s_p.$$

We then get $\Theta(E\otimes\det E)=\Theta(E)+\mathrm{Tr}_E\,\Theta(E)\otimes\mathrm{Id}_E$. The corresponding hermitian forms on $T_X\otimes E$ are thus related by

$$\widetilde\Theta(E\otimes\det E)=\widetilde\Theta(E)+\mathrm{Tr}_E\,\widetilde\Theta(E)\otimes h,$$

where h denotes the hermitian metric on E and $\mathrm{Tr}_E\,\widetilde\Theta(E)$ is the hermitian form on T_X defined by

$$\mathrm{Tr}_E\,\widetilde\Theta(E)(\xi,\xi)=\sum_{1\leqslant\lambda\leqslant r}\widetilde\Theta(E)(\xi\otimes e_\lambda,\xi\otimes e_\lambda),\qquad \xi\in T_X,$$

for any orthonormal frame $(e_1,\ldots,e_r)$ of E. Theorem 10.12 is now a consequence of the following simple property of hermitian forms on a tensor product of complex vector spaces.

(10.14) Proposition. *Let T,E be complex vector spaces of respective dimensions n,r, and h a hermitian metric on E. Then for every hermitian form Θ on $T\otimes E$*

$$\Theta>_{\rm Grif}0\quad\Longrightarrow\quad \Theta+\mathrm{Tr}_E\,\Theta\otimes h>_{\rm Nak}0.$$

We first need a lemma analogous to Fourier inversion formula for discrete Fourier transforms.

(10.15) Lemma. *Let q be an integer $\geqslant 3$, and x_λ, y_μ, $1\leqslant\lambda,\mu\leqslant r$, be complex numbers. Let σ describe the set U_q^r of r-tuples of q-th roots of unity and put*

$$x'_\sigma = \sum_{1\leqslant\lambda\leqslant r} x_\lambda\overline{\sigma}_\lambda, \qquad y'_\sigma = \sum_{1\leqslant\mu\leqslant r} y_\mu\overline{\sigma}_\mu, \qquad \sigma\in U_q^r.$$

Then for every pair (α,β), $1\leqslant\alpha,\beta\leqslant r$, *the following identity holds:*

$$q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma\overline{y}'_\sigma\sigma_\alpha\overline{\sigma}_\beta = \begin{cases} x_\alpha\overline{y}_\beta & \textit{if}\ \ \alpha\neq\beta,\\ \displaystyle\sum_{1\leqslant\mu\leqslant r} x_\mu\overline{y}_\mu & \textit{if}\ \ \alpha=\beta.\end{cases}$$

Proof. The coefficient of $x_\lambda\overline{y}_\mu$ in the summation $q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma\overline{y}'_\sigma\sigma_\alpha\overline{\sigma}_\beta$ is given by

$$q^{-r}\sum_{\sigma\in U_q^r}\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu.$$

This coefficient equals 1 when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are equal (in which case $\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu=1$ for any one of the q^r elements of U_q^r). Hence, it is sufficient to prove that

$$\sum_{\sigma\in U_q^r}\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = 0$$

when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are distinct.

If $\{\alpha,\mu\}\neq\{\beta,\lambda\}$, then one of the elements of one of the pairs does not belong to the other pair. As the four indices α,β,λ,μ play the same role, we may suppose for example that $\alpha\notin\{\beta,\lambda\}$. Let us apply to σ the substitution $\sigma\mapsto\tau$, where τ is defined by

$$\tau_\alpha = e^{2\pi\mathrm{i}/q}\sigma_\alpha,\ \ \tau_\nu=\sigma_\nu \quad\text{for}\quad \nu\neq\alpha.$$

We get

$$\sum_\sigma \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = \sum_\tau = \begin{cases} e^{2\pi\mathrm{i}/q}\displaystyle\sum_\sigma & \text{if}\ \ \alpha\neq\mu,\\ e^{4\pi\mathrm{i}/q}\displaystyle\sum_\sigma & \text{if}\ \ \alpha=\mu,\end{cases}$$

Since $q\geqslant 3$ by hypothesis, it follows that

$$\sum_\sigma \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = 0.$$

Proof of Proposition 10.14. Let $(t_j)_{1\leqslant j\leqslant n}$ be a basis of T, $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ an orthonormal basis of E and $\xi=\sum_j\xi_j t_j\in T$, $u=\sum_{j,\lambda}u_{j\lambda}\,t_j\otimes e_\lambda\in T\otimes E$. The coefficients $c_{jk\lambda\mu}$ of Θ with respect to the basis $t_j\otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu}=c_{kj\mu\lambda}$, and we have the formulas

$$\Theta(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}u_{j\lambda}\overline{u}_{k\mu},$$

$$\mathrm{Tr}_E\,\Theta(\xi,\xi) = \sum_{j,k,\lambda} c_{jk\lambda\lambda}\xi_j\overline{\xi}_k,$$

$$(\Theta+\mathrm{Tr}_E\,\Theta\otimes h)(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}u_{j\lambda}\overline{u}_{k\mu} + c_{jk\lambda\lambda}u_{j\mu}\overline{u}_{k\mu}.$$

For every $\sigma \in U_q^r$ (cf. Lemma 10.15), put

$$u'_{j\sigma} = \sum_{1\leqslant\lambda\leqslant r} u_{j\lambda}\overline{\sigma}_\lambda \in \mathbb{C},$$
$$\widehat{u}_\sigma = \sum_j u'_{j\sigma} t_j \in T \quad , \quad \widehat{e}_\sigma = \sum_\lambda \sigma_\lambda e_\lambda \in E.$$

Lemma 10.15 implies

$$q^{-r} \sum_{\sigma\in U_q^r} \Theta(\widehat{u}_\sigma \otimes \widehat{e}_\sigma, \widehat{u}_\sigma \otimes \widehat{e}_\sigma) = q^{-r} \sum_{\sigma\in U_q^r} c_{jk\lambda\mu} u'_{j\sigma}\overline{u}'_{k\sigma}\sigma_\lambda\overline{\sigma}_\mu$$
$$= \sum_{j,k,\lambda\neq\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} u_{j\mu}\overline{u}_{k\mu}.$$

The Griffiths positivity assumption shows that the left hand side is $\geqslant 0$, hence

$$(\Theta + \mathrm{Tr}_E\,\Theta \otimes h)(u,u) \geqslant \sum_{j,k,\lambda} c_{jk\lambda\lambda} u_{j\lambda}\overline{u}_{k\lambda} \geqslant 0$$

with strict positivity if $\Theta >_{\mathrm{Grif}} 0$ and $u \neq 0$. $\square$

We now relate Griffiths positivity to m-tensor positivity. The most useful result is the following

(10.16) Proposition. *Let T be a complex vector space and (E,h) a hermitian vector space of respective dimensions n,r with $r\geqslant 2$. Then for any hermitian form Θ on $T\otimes E$ and any integer $m\geqslant 1$*

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad m\,\mathrm{Tr}_E\,\Theta\otimes h - \Theta >_m 0.$$

Proof. Let us distinguish two cases.

a) $m=1$. Let $u\in T\otimes E$ be a tensor of rank 1. Then u can be written $u=\xi_1\otimes e_1$ with $\xi_1\in T$, $\xi_1\neq 0$, and $e_1\in E$, $|e_1|=1$. Complete e_1 into an orthonormal basis $(e_1,\ldots,e_r)$ of E. One gets immediately

$$(\mathrm{Tr}_E\,\Theta\otimes h)(u,u) = \mathrm{Tr}_E\,\Theta(\xi_1,\xi_1) = \sum_{1\leqslant\lambda\leqslant r} \Theta(\xi_1\otimes e_\lambda, \xi_1\otimes e_\lambda)$$
$$> \Theta(\xi_1\otimes e_1, \xi_1\otimes e_1) = \Theta(u,u).$$

b) $m\geqslant 2$. Every tensor $u\in T\otimes E$ of rank $\leqslant m$ can be written

$$u = \sum_{1\leqslant\lambda\leqslant q} \xi_\lambda\otimes e_\lambda \quad , \quad \xi_\lambda\in T,$$

with $q=\min(m,r)$ and $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ an orthonormal basis of E. Let F be the vector subspace of E generated by $(e_1,\ldots,e_q)$ and Θ_F the restriction of Θ to $T\otimes F$. The first part shows that

$$\Theta' := \mathrm{Tr}_F\, \Theta_F \otimes h - \Theta_F >_{\mathrm{Grif}} 0.$$

Proposition 10.14 applied to Θ' on $T \otimes F$ yields

$$\Theta' + \mathrm{Tr}_F\, \Theta' \otimes h = q\, \mathrm{Tr}_F\, \Theta_F \otimes h - \Theta_F >_q 0.$$

Since $u \in T \otimes F$ is of rank $\leqslant q \leqslant m$, we get (for $u \neq 0$)

$$\begin{aligned}\Theta(u,u) = \Theta_F(u,u) &< q(\mathrm{Tr}_F\, \Theta_F \otimes h)(u,u)\\ &= q \sum_{1 \leqslant j,\lambda \leqslant q} \Theta(\xi_j \otimes e_\lambda, \xi_j \otimes e_\lambda) \leqslant m\, \mathrm{Tr}_E\, \Theta \otimes h(u,u). \qquad \square\end{aligned}$$

Proposition 10.16 is of course also true in the semi-positive case. From these facts, we deduce

(10.17) Theorem. *Let E be a Griffiths (semi-)positive bundle of rank $r \geqslant 2$. Then for any integer $m \geqslant 1$*

$$E^\star \otimes (\det E)^m >_m 0 \qquad (\textit{resp.} \quad \geqslant_m 0).$$

Proof. We apply Prop. 10.16 to $\Theta = -\Theta(E^\star) = {}^t\Theta(E) \geqslant_{\mathrm{Grif}} 0$ on $T_X \otimes E^\star$ and observe that

$$\Theta(\det E) = \mathrm{Tr}_E\, \Theta(E) = \mathrm{Tr}_{E^\star}\, \Theta.$$

(10.18) Theorem. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then for any $m \geqslant 1$*

$$E >_m 0 \quad \Longrightarrow \quad S \otimes (\det Q)^m >_m 0.$$

Proof. Formulas (10.7) imply

$$\mathrm{i}\Theta(S) >_m \mathrm{i}\beta^\star \wedge \beta \quad , \quad \mathrm{i}\Theta(Q) >_m \mathrm{i}\beta \wedge \beta^\star,$$

$$\mathrm{i}\Theta(\det Q) = \mathrm{Tr}_Q(\mathrm{i}\Theta(Q)) > \mathrm{Tr}_Q(\mathrm{i}\beta \wedge \beta^\star).$$

If we write $\beta = \sum dz_j \otimes \beta_j$ as in the proof of Corollary 10.8, then

$$\begin{aligned}\mathrm{Tr}_Q(\mathrm{i}\beta \wedge \beta^\star) &= \sum i dz_j \wedge d\overline{z}_k\, \mathrm{Tr}_Q(\beta_j \beta_k^\star)\\ &= \sum i dz_j \wedge d\overline{z}_k\, \mathrm{Tr}_S(\beta_k^\star \beta_j) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta).\end{aligned}$$

Furthermore, it has been already proved that $-\mathrm{i}\beta^\star \wedge \beta \geqslant_{\mathrm{Nak}} 0$. By Prop. 8.1 applied to the corresponding hermitian form Θ on $T_X \otimes S$, we get

$$m\, \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\beta^\star \wedge \beta \geqslant_m 0,$$

and Theorem 10.18 follows. $\square$

(10.19) Corollary. *Let X be a weakly pseudoconvex Kähler n-dimensional manifold, E a holomorphic vector bundle of rank $r \geqslant 2$ and $m \geqslant 1$ an integer. Then*

a) $E >_{\mathrm{Grif}} 0 \Rightarrow H^{n,q}(X, E \otimes \det E) = 0$ *for* $q \geqslant 1$;

b) $E >_{\mathrm{Grif}} 0 \Rightarrow H^{n,q}\big(X, E^\star \otimes (\det E)^m\big) = 0$ *for* $q \geqslant 1$ *and* $m \geqslant \min\{n-q+1, r\}$;

c) *Let* $0 \to S \to E \to Q \to 0$ *be an exact sequence of vector bundles and* $m = \min\{n-q+1, \mathrm{rk}\, S\}$, $q \geqslant 1$. *If* $E >_m 0$ *and if* L *is a line bundle such that* $L \otimes (\det Q)^{-m} \geqslant 0$, *then*

$$H^{n,q}(X, S \otimes L) = 0.$$

Proof. Immediate consequence of Theorem 10.11, in combination with 10.12 for a), 10.17 for b) and 10.18 for c). □

11. Skoda's L^2 estimates for surjective bundle morphisms

Let (X, ω) be a Kähler manifold, $\dim X = n$, and let $g : E \to Q$ a holomorphic morphism of hermitian vector bundles over X. Assume in the first instance that g is *surjective*. We are interested in conditions insuring that the induced morphisms $g : H^{n,k}(X, E) \longrightarrow H^{n,k}(X, Q)$ are also surjective (dealing with $(n, \bullet)$ bidegrees is always easier, since we have to understand positivity conditions for the curvature term). For that purpose, it is natural to consider the subbundle $S = \mathrm{Ker}\, g \subset E$ and the exact sequence

$$0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0 \tag{11.1}$$

where $j : S \to E$ is the inclusion. In fact, we need a little more flexibility to handle the curvature terms, so we take the tensor product of the exact sequence by a holomorphic line bundle L (whose properties will be specified later):

$$0 \longrightarrow S \otimes L \longrightarrow E \otimes L \xrightarrow{\ g\ } Q \otimes L \longrightarrow 0. \tag{11.2}$$

(11.3) Theorem. *Let k be an integer such that $0 \leqslant k \leqslant n$. Set $r = \mathrm{rk}\, E$, $q = \mathrm{rkQ}$, $s = \mathrm{rk}\, S = r - q$ and*

$$m = \min\{n-k, s\} = \min\{n-k, r-q\}.$$

Assume that (X, ω) possesses also a complete Kähler metric $\widehat{\omega}$, that $E \geqslant_m 0$, and that $L \longrightarrow X$ is a hermitian holomorphic line bundle such that

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \geqslant 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that $\|f\| < +\infty$, there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g \cdot h$ and

$$\|h\|^2 \leqslant (1 + m/\varepsilon)\, \|f\|^2.$$

The idea of the proof is essentially due to (Skoda 1978), who actually proved the special case $k = 0$. The general case appeared in (Demailly 1982).

Proof. Let $j : S \to E$ be the inclusion morphism, $g^\star : Q \to E$ and $j^\star : E \to S$ the adjoints of g, j, and the matrix of D_E with respect to the orthogonal splitting $E \simeq S \oplus Q$ (cf. Lemma 10.2). Then $g^\star f$ is a lifting of f in $E \otimes L$. We will try to find h under the form

$$h = g^\star f + ju, \qquad u \in L^2(X, \Lambda^{n,k} T_X^\star \otimes S \otimes L).$$

As the images of S and Q in E are orthogonal, we have $|h|^2 = |f|^2 + |u|^2$ at every point of X. On the other hand $D''_{Q\otimes L} f = 0$ by hypothesis and $D'' g^\star = -j \circ \beta^\star$ by (10.5), hence

$$D''_{E\otimes L} h = -j(\beta^\star \wedge f) + j\, D''_{S\otimes L} = j(D''_{S\otimes L} - \beta^\star \wedge f).$$

We are thus led to solve the equation

$$D''_{S\otimes L} u = \beta^\star \wedge f, \tag{11.4}$$

and for that, we apply Th. 8.4 to the $(n, k+1)$-form $\beta^\star \wedge f$. One now observes that the curvature of $S \otimes L$ can be expressed in terms of β. This remark will be used to prove:

(11.5) Lemma. *Let $A_k = [\mathrm{i}\Theta(S \otimes L), \Lambda]$ be the curvature operator acting on $(n, k+1)$-forms. Then $\langle A_k^{-1}(\beta^\star \wedge f), (\beta^\star \wedge f)\rangle \leqslant (m/\varepsilon)\,|f|^2$.*

If the Lemma is taken for granted, Th. 8.4 yields a solution u of (11.4) in $L^2(X, \Lambda^{n,q} T_X^\star \otimes S \otimes L)$ such that $\|u\|^2 \leqslant (m/\varepsilon)\,\|f\|^2$. As $\|h\|^2 = \|f\|^2 + \|u\|^2$, the proof of Th. 11.3 is complete. $\square$

Proof of Lemma 11.5. Exactly as in the proof of Th. 10.18 formulas (10.7) yield

$$\mathrm{i}\Theta(S) \geqslant_m \mathrm{i}\beta^\star \wedge \beta, \qquad \mathrm{i}\Theta(\det Q) \geqslant \mathrm{Tr}_Q(\mathrm{i}\beta \wedge \beta^\star) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta).$$

Since $C^\infty(X, \Lambda^{1,1} T_X^\star \otimes \mathrm{Herm}\, S) \ni \Theta := -\mathrm{i}\beta^\star \wedge \beta \geqslant_{\mathrm{Grif}} 0$, Prop. 10.16 implies

$$m\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\beta^\star \wedge \beta \geqslant_m 0.$$

From the hypothesis on the curvature of L we get

$$\begin{aligned} \mathrm{i}\Theta(S\otimes L) &\geqslant_m \mathrm{i}\Theta(S) \otimes \mathrm{Id}_L + (m+\varepsilon)\,\mathrm{i}\Theta(\det Q) \otimes \mathrm{Id}_{S\otimes L} \\ &\geqslant_m \big(\mathrm{i}\beta^\star \wedge \beta + (m+\varepsilon)\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S\big) \otimes \mathrm{Id}_L \\ &\geqslant_m (\varepsilon/m)\,(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S \otimes \mathrm{Id}_L\,. \end{aligned}$$

For any $v \in \Lambda^{n,k+1} T_X^\star \otimes S \otimes L$, Lemma 10.10 implies

$$\langle A_k v, v\rangle \geqslant (\varepsilon/m)\,\langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle, \tag{11.6}$$

because $\mathrm{rk}(S\otimes L) = s$ and $m = \min\{n-k, s\}$. Let $(dz_1, \ldots, dz_n)$ be an orthonormal basis of $T_X^\star$ at a given point $x_0 \in X$ and set

$$\beta = \sum_{1\leqslant j\leqslant n} dz_j \otimes \beta_j, \qquad \beta_j \in \mathrm{Hom}(S,Q).$$

The adjoint of the operator $\beta^\star \wedge \bullet = \sum d\overline{z}_j \wedge \beta_j^\star \bullet$ is the contraction operator $\beta \mathbin{\lrcorner} \bullet$ defined by

$$\beta \mathbin{\lrcorner} v = \sum \frac{\partial}{\partial \overline{z}_j} \mathbin{\lrcorner} (\beta_j v) = \sum -\mathrm{i}dz_j \wedge \Lambda(\beta_j v) = -\mathrm{i}\beta \wedge \Lambda v.$$

We get consequently $\langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle = |\beta \mathbin{\lrcorner} v|^2$ and (11.6) implies

$$|\langle \beta^\star \wedge f, v\rangle|^2 = |\langle f, \beta \mathbin{\lrcorner} v\rangle|^2 \leqslant |f|^2\,|\beta \mathbin{\lrcorner} v|^2 \leqslant (m/\varepsilon)\langle A_k v, v\rangle\,|f|^2.$$

This is equivalent to the estimate asserted in the lemma. □

If X has a plurisubharmonic exhaustion function ψ, we can select a convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ and multiply the metric of L by the weight $\exp(-\chi\circ\psi)$ in order to make the L^2 norm of f converge. Theorem 11.3 implies therefore:

(11.7) Corollary. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, $g : E \to Q$ a surjective bundle morphism with $r = \mathrm{rk}\,E$, $q = \mathrm{rk}\,Q$, and $L \to X$ a hermitian holomorphic line bundle. We set $m = \min\{n-k, r-q\}$ and assume that $E \geqslant_m 0$ and*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\,\mathrm{i}\Theta(\det Q) \geqslant 0$$

for some $\varepsilon > 0$. Then g induces a surjective map

$$H^{n,k}(X, E\otimes L) \longrightarrow H^{n,k}(X, Q\otimes L).$$

The most remarkable feature of this result is that it does not require any strict positivity assumption on the curvature (for instance E can be a flat bundle). A careful examination of the proof shows that it amounts to verify that the image of the coboundary morphism

$$-\beta^\star \wedge \bullet \;:\; H^{n,k}(X, Q\otimes L) \longrightarrow H^{n,k+1}(X, S\otimes L)$$

vanishes; however the cohomology group $H^{n,k+1}(X, S\otimes L)$ itself does not necessarily vanish, as it would do under a strict positivity assumption (cf. Cor. 10.19 c)).

We want now to get also estimates when Q is endowed with a metric given a priori, that can be distinct from the quotient metric of E by g. Then the map $g^\star(gg^\star)^{-1} \;:\; Q \longrightarrow E$ is the lifting of Q orthogonal to $S = \mathrm{Ker}\,g$. The quotient metric $|\bullet|'$ on Q is therefore defined in terms of the original metric $|\bullet|$ by

$$|v|'^2 = |g^\star(gg^\star)^{-1}v|^2 = \langle (gg^\star)^{-1}v, v\rangle = \det(gg^\star)^{-1}\,\langle \widetilde{gg^\star}v, v\rangle$$

where $\widetilde{gg^\star} \in \mathrm{End}(Q)$ denotes the endomorphism of Q whose matrix is the transposed of the comatrix of $gg^\star$. For every $w \in \det Q$, we find

$$|w|'^2 = \det(gg^\star)^{-1}\,|w|^2.$$

If Q' denotes the bundle Q with the quotient metric, we get

$$\mathrm{i}\Theta(\det Q') = \mathrm{i}\Theta(\det Q) + \mathrm{i}d'd'' \log\det(gg^\star).$$

In order that the hypotheses of Th. 11.3 be satisfied, we are led to define a new metric $|\bullet|'$ on L by $|u|'^2 = |u|^2 \left(\det(gg^\star)\right)^{-m-\varepsilon}$. Then

$$\mathrm{i}\Theta(L') = \mathrm{i}\Theta(L) + (m+\varepsilon)\,\mathrm{i}d'd'' \log\det(gg^\star) \geqslant (m+\varepsilon)\,\mathrm{i}\Theta(\det Q').$$

Theorem 11.3 applied to (E, Q', L') can now be reformulated:

(11.8) Theorem. *Let X be a complete Kähler manifold equipped with a Kähler metric ω on X, let $E \to Q$ be a surjective morphism of hermitian vector bundles and let $L \to X$ be a hermitian holomorphic line bundle. Set $r = \mathrm{rk}\ E$, $q = \mathrm{rk}\ Q$ and $m = \min\{n-k, r-q\}$, and assume that $E \geqslant_m 0$ and*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \geqslant 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that

$$I = \int_X \langle \widetilde{gg^\star} f, f\rangle\, (\det gg^\star)^{-m-1-\varepsilon}\, dV < +\infty,$$

there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g \cdot h$ and

$$\int_X |h|^2\, (\det gg^\star)^{-m-\varepsilon}\, dV \leqslant (1 + m/\varepsilon)\, I. \qquad \square$$

Our next goal is to extend Th. 11.8 in the case when $g : E \longrightarrow Q$ is only *generically* surjective; this means that the analytic set

$$Y = \{x \in X\ ;\ g_x : E_x \longrightarrow Q_x \text{ is not surjective }\}$$

defined by the equation $\Lambda^q g = 0$ is nowhere dense in X. Here $\Lambda^q g$ is a section of the bundle $\mathrm{Hom}(\Lambda^q E, \det Q)$. The idea is to apply the above Theorem 11.8 to $X \smallsetminus Y$. For this, we have to know whether $X \smallsetminus Y$ has a complete Kähler metric.

(11.9) Lemma. *Let (X,ω) be a Kähler manifold, and $Y = \sigma^{-1}(0)$ an analytic subset defined by a section of a hermitian vector bundle $E \to X$. If X is weakly pseudoconvex and exhausted by $X_c = \{x \in X\ ;\ \psi(x) < c\}$, then $X_c \smallsetminus Y$ has a complete Kähler metric for all $c \in \mathbb{R}$. The same conclusion holds for $X \smallsetminus Y$ if (X,ω) is complete and if for some constant $C \geqslant 0$ we have $\Theta_E \leqslant_{\mathrm{Grif}} C\,\omega \otimes \langle\ ,\ \rangle_E$ on X.*

Proof. Set $\tau = \log|\sigma|^2$. Then $d'\tau = \{D'\sigma, \sigma\}/|\sigma|^2$ and $D''D'\sigma = D^2\sigma = \Theta(E)\sigma$, thus

$$\mathrm{i}d'd''\tau = \mathrm{i}\frac{\{D'\sigma, D'\sigma\}}{|\sigma|^2} - \mathrm{i}\frac{\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\}}{|\sigma|^4} - \frac{\{\mathrm{i}\Theta(E)\sigma, \sigma\}}{|\sigma|^2}.$$

For every $\xi \in T_X$, we find therefore

$$\begin{aligned} H\tau(\xi) &= \frac{|\sigma|^2\,|D'\sigma\cdot\xi|^2 - |\langle D'\sigma\cdot\xi,\sigma\rangle|^2}{|\sigma|^4} - \frac{\widetilde{\Theta}(E)(\xi\otimes\sigma,\xi\otimes\sigma)}{|\sigma|^2} \\ &\geqslant -\frac{\widetilde{\Theta}(E)(\xi\otimes\sigma,\xi\otimes\sigma)}{|\sigma|^2} \end{aligned}$$

by the Cauchy-Schwarz inequality. If C is a bound for the coefficients of $\widetilde{\Theta}(E)$ on the compact subset $\overline{X}_c$, we get $\mathrm{i}d'd''\tau \geqslant -C\omega$ on X_c. Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be a convex increasing function. We set

$$\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi\circ\tau) = \omega + \mathrm{i}\big(\chi'\circ\tau\ d'd''\tau + \chi''\circ\tau\ d'\tau\wedge d''\tau\big).$$

We thus see that $\widehat{\omega}$ is positive definite if $\chi' \leqslant 1/2C$, and by a computation similar to that in proposition 8.7, we check that $\widehat{\omega}$ is complete near $Y = \tau^{-1}(-\infty)$ as soon as

$$\int_{-\infty}^{0} \sqrt{\chi''(t)}\,dt = +\infty.$$

One can choose for example χ such that $\chi(t) = \frac{1}{5C}(t - \log|t|)$ for $t \leqslant -1$. In order to obtain a complete Kähler metric on $X_c \smallsetminus Y$, we also need the metric to be complete near ∂X_c. If $\widehat{\omega}$ is not, such a metric can be defined by

$$\begin{aligned} \widetilde{\omega} &= \widehat{\omega} + \mathrm{i}d'd''\log(c-\psi)^{-1} = \widehat{\omega} + \frac{\mathrm{i}d'd''\psi}{c-\psi} + \frac{\mathrm{i}d'\psi\wedge d''\psi}{(c-\psi)^2} \\ &\geqslant \mathrm{i}d'\log(c-\psi)^{-1}\wedge d''\log(c-\psi)^{-1}\ ; \end{aligned}$$

$\widetilde{\omega}$ is complete on $X_c \smallsetminus \Omega$ because $\log(c-\psi)^{-1}$ tends to $+\infty$ on ∂X_c. □

We also need another elementary lemma dealing with the extension of partial differential equalities across analytic sets.

(11.10) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega\smallsetminus Y$ (in the sense of distribution theory). Then $d''v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it is sufficient to prove the result in a neighborhood of a regular point $a \in Y$. By using a local analytic isomorphism, the proof is reduced to the case where Y is contained in the hyperplane $z_1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that $\lambda(t) = 0$ for $t \leqslant \frac{1}{2}$ and $\lambda(t) = 1$ for $t \geqslant 1$. We must show that

$$\int_\Omega w\wedge\alpha = (-1)^{p+q}\int_\Omega v\wedge d''\alpha \tag{11.11}$$

for all $\alpha \in \mathcal{D}(\Omega, \Lambda^{n-p,n-q}T^\star_\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(|z_1|/\varepsilon)$ and replace α in the integral by $\lambda_\varepsilon\alpha$. Then $\lambda_\varepsilon\alpha \in \mathcal{D}(\Omega\smallsetminus Y, \Lambda^{n-p,n-q}T^\star_\Omega)$ and the hypotheses imply

$$\int_\Omega w\wedge\lambda_\varepsilon\alpha = (-1)^{p+q}\int_\Omega v\wedge d''(\lambda_\varepsilon\alpha) = (-1)^{p+q}\int_\Omega v\wedge(d''\lambda_\varepsilon\wedge\alpha + \lambda_\varepsilon d''\alpha).$$

As w and v have $L^1_{\rm loc}$ coefficients on Ω, the integrals of $w \wedge \lambda_\varepsilon \alpha$ and $v \wedge \lambda_\varepsilon d''\alpha$ converge respectively to the integrals of $w \wedge \alpha$ and $v \wedge d''\alpha$ as ε tends to 0. The remaining term can be estimated by means of the Cauchy-Schwarz inequality:

$$\Big|\int_\Omega v \wedge d''\lambda_\varepsilon \wedge \alpha\Big|^2 \leqslant \int_{|z_1|\leqslant\varepsilon} |v\wedge\alpha|^2\,dV.\ \int_{\mathrm{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV\,;$$

as $v \in L^2_{\rm loc}(\Omega)$, the integral $\int_{|z_1|\leqslant\varepsilon} |v\wedge\alpha|^2\,dV$ converges to 0 with ε, whereas

$$\int_{\mathrm{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV \leqslant \frac{C}{\varepsilon^2}\,\mathrm{Vol}\big(\mathrm{Supp}\ \alpha \cap \{|z_1| \leqslant \varepsilon\}\big) \leqslant C'.$$

Equality (11.11) follows when ε tends to 0. □

(11.12) Theorem. *The existence statement and the estimates of Th.* 11.8 *remain true for a generically surjective morphism $g : E \to Q$, provided that X is weakly pseudoconvex.*

Proof. Apply Th. 11.8 to each relatively compact domain $X_c \smallsetminus Y$ (these domains are complete Kähler by Lemma 11.9). From a sequence of solutions on $X_c \smallsetminus Y$ we can extract a subsequence converging weakly on $X \smallsetminus Y$ as c tends to $+\infty$. One gets a form h satisfying the estimates, such that $D''h = 0$ on $X \smallsetminus Y$ and $f = g \cdot h$. In order to see that $D''h = 0$ on X, it suffices to apply Lemma 11.10 and to observe that h has $L^2_{\rm loc}$ coefficients on X by our estimates. □

A very special but interesting case is obtained for the trivial bundles $E = \Omega \times \mathbb{C}^r$, $Q = \Omega \times \mathbb{C}$ over a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Then the morphism g is given by a r-tuple $(g_1, \ldots, g_r)$ of holomorphic functions on Ω. Let us take $k = 0$ and $L = \Omega \times \mathbb{C}$ with the metric given by a weight $e^{-\varphi}$. If we observe that $\widetilde{gg^\star} = \mathrm{Id}$ when rk $Q = 1$, Th. 11.8 applied on $X = \Omega \smallsetminus g^{-1}(0)$ and Lemmas 11.9, 11.10 give:

(11.13) Theorem (Skoda 1972b). *Let Ω be a complete Kähler open subset of $\mathbb{C}^n$ and φ a plurisubharmonic function on Ω. Set $m = \min\{n, r-1\}$. Then for every holomorphic function f on Ω such that*

$$I = \int_{\Omega\smallsetminus Z} |f|^2\,|g|^{-2(m+1+\varepsilon)} e^{-\varphi}\,dV < +\infty,$$

where $Z = g^{-1}(0)$, there exist holomorphic functions $(h_1, \ldots, h_r)$ on Ω such that $f = \sum g_j h_j$ and

$$\int_{\Omega\smallsetminus Y} |h|^2\,|g|^{-2(m+\varepsilon)} e^{-\varphi}\,dV \leqslant (1 + m/\varepsilon) I.$$ □

This last theorem can be used in order to obtain interesting results about domains of holomorphy in $\mathbb{C}^n$ and the relation with the existence of complete Kähler metrics. Recall that an open set $\Omega \subset \mathbb{C}^n$ is said to be a domain of holomorphy if for every connected open subset U such that $U \cap \partial\Omega \neq \emptyset$ and every connected component W of $U \cap \Omega$ there exists a holomorphic function h on Ω such that $h_{\restriction W}$ cannot be continued to U. It is well known that Ω is a domain of holomorphy if and only if Ω is Stein [the latter condition being of course trivially sufficient; see e.g. (Hörmander 1966), chap. 2].

(11.14) Theorem (Diederich-Pflug 1981). *Let $\Omega \subset \mathbb{C}^n$ be an open subset. If $(\overline{\Omega})^\circ = \Omega$ and Ω admits a complete Kähler metric $\widehat{\omega}$, then Ω is a domain of holomorphy.*

Note that the statement becomes false if the assumption $(\overline{\Omega})^\circ = \Omega$ is omitted: in fact $\mathbb{C}^n \smallsetminus \{0\}$ is complete Kähler by Lemma 10.9, but it is not a domain of holomorphy if $n \geqslant 2$.

Proof. Since $(\overline{\Omega})^\circ = \Omega$, the set $U \smallsetminus \overline{\Omega}$ is not empty. We select $a \in U \smallsetminus \overline{\Omega}$. Then the integral

$$\int_\Omega |z-a|^{-2(n+\varepsilon)}\, dV(z)$$

converges. By Th. 11.13 applied to $f(z) = 1$, $g_j(z) = z_j - a_j$ and $\varphi = 0$, there exist holomorphic functions h_j on Ω such that $\sum (z_j - a_j)\, h_j(z) = 1$. This shows that at least one of the functions h_j cannot be analytically continued at $a \in U$. □

(11.15) Remark. Skoda's theorem 11.13 can also be used to prove the implication

$$\Omega \text{ pseudoconvex} \implies \Omega \text{ domain of holomorphy},$$

which is equivalent to the "interesting implication" in the Levi problem (modulo the equivalence of domains of holomorphy with Stein open sets, an easy property). In fact, if Ω is pseudoconvex, it can be shown that the function $z \mapsto -\log d(z, \complement\Omega)$ is psh (see again Hörmander 1966, chap. 2). Given any open connected set U such that $U \cap \partial\Omega \neq \emptyset$, select $a \in U \cap \partial\Omega$. The weight function

$$\varphi(z) = (n+\varepsilon)\log(1+|z|^2) - 2(n+\varepsilon)\log d(z, \complement\Omega)$$

is psh on Ω. As $|z-a| \geqslant d(z, \complement\Omega)$, we see that the integral

$$\int_\Omega |z-a|^{-2(n+\varepsilon)}\, e^{-\varphi(z)}\, dV(z) \leqslant \int_\Omega (1+|z|^2)^{-n-\varepsilon}\, dV(z)$$

converges, and we conclude as in the proof of 11.14. □

12. Application of Skoda's L^2 estimates to local algebra

We show here how Theorem 11.13 can be applied to get deep results concerning ideals of the local ring $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ of germs of holomorphic functions on $(\mathbb{C}^n, 0)$. Let $\mathcal{I} = (g_1, \ldots, g_r) \neq (0)$ be an ideal of $\mathcal{O}_n$.

(12.1) Definition. *Let $k \in \mathbb{R}_+$. We associate to $\mathcal{I}$ the following ideals:*

a) *the ideal* $\overline{\mathcal{I}}^{(k)}$ *of germs* $u \in \mathcal{O}_n$ *such that* $|u| \leqslant C|g|^k$ *for some constant* $C \geqslant 0$, *where* $|g|^2 = |g_1|^2 + \cdots + |g_r|^2$.

b) *the ideal* $\widehat{\mathcal{I}}^{(k)}$ *of germs* $u \in \mathcal{O}_n$ *such that*

$$\int_\Omega |u|^2 \, |g|^{-2(k+\varepsilon)} \, dV < +\infty$$

on a small ball Ω *centered at* 0, *if* $\varepsilon > 0$ *is small enough.*

(12.2) Proposition. *For all* $k, l \in \mathbb{R}_+$ *we have*

a) $\overline{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k)}$;

b) $\mathcal{I}^k \subset \overline{\mathcal{I}}^{(k)}$ *if* $k \in \mathbb{N}$;

c) $\overline{\mathcal{I}}^{(k)}.\overline{\mathcal{I}}^{(l)} \subset \overline{\mathcal{I}}^{(k+l)}$;

d) $\overline{\mathcal{I}}^{(k)}.\widehat{\mathcal{I}}^{(l)} \subset \widehat{\mathcal{I}}^{(k+l)}$.

All properties are immediate from the definitions except a) which is a consequence of the integrability of $|g|^{-\varepsilon}$ for $\varepsilon > 0$ small (exercise to the reader!). Before stating the main result, we need a simple lemma.

(12.3) Lemma. *If* $\mathcal{I} = (g_1, \ldots, g_r)$ *and* $r > n$, *we can find elements* $\widetilde{g}_1, \ldots, \widetilde{g}_n \in \mathcal{I}$ *such that* $C^{-1}|g| \leqslant |\widetilde{g}| \leqslant C|g|$ *on a neighborhood of* 0. *Each* $\widetilde{g}_j$ *can be taken to be a linear combination*

$$\widetilde{g}_j = a_j . g = \sum_{1 \leqslant k \leqslant r} a_{jk} g_k, \qquad a_j \in \mathbb{C}^r \smallsetminus \{0\}$$

where the coefficients $([a_1], \ldots, [a_n])$ *are chosen in the complement of a proper analytic subset of* $(\mathbb{P}^{r-1})^n$.

It follows from the Lemma that the ideal $\mathcal{J} = (\widetilde{g}_1, \ldots, \widetilde{g}_n) \subset \mathcal{I}$ satisfies $\overline{\mathcal{J}}^{(k)} = \overline{\mathcal{I}}^{(k)}$ and $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$ for all k.

Proof. Assume that $g \in \mathcal{O}(\Omega)^r$. Consider the analytic subsets in $\Omega \times (\mathbb{P}^{r-1})^n$ defined by

$$A = \big\{(z, [w_1], \ldots, [w_n])\,;\, w_j . g(z) = 0\big\},$$
$$A^\star = \bigcup \text{irreducible components of } A \text{ not contained in } g^{-1}(0) \times (\mathbb{P}^{r-1})^n.$$

For $z \notin g^{-1}(0)$ the fiber $A_z = \{([w_1], \ldots, [w_n])\,;\, w_j . g(z) = 0\} = A^\star_z$ is a product of n hyperplanes in $\mathbb{P}^{r-1}$, hence $A \cap (\Omega \smallsetminus g^{-1}(0)) \times (\mathbb{P}^{r-1})^n$ is a fiber bundle with base $\Omega \smallsetminus g^{-1}(0)$ and fiber $(\mathbb{P}^{r-2})^n$. As $A^\star$ is the closure of this set in $\Omega \times (\mathbb{P}^{r-1})^n$, we have

$$\dim A^\star = n + n(r-2) = n(r-1) = \dim(\mathbb{P}^{r-1})^n.$$

It follows that the zero fiber

$$A_0^\star = A^\star \cap \big(\{0\} \times (\mathbb{P}^{r-1})^n\big)$$

is a proper subset of $\{0\} \times (\mathbb{P}^{r-1})^n$. Choose $(a_1, \ldots, a_n) \in (\mathbb{C}^r \smallsetminus \{0\})^n$ such that $(0, [a_1], \ldots, [a_n])$ is not in $A_0^\star$. By an easy compactness argument the set $A^\star \cap \big(\overline{B}(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big)$ is disjoint from the neighborhood $B(0,\varepsilon) \times \prod[B(a_j,\varepsilon)]$ of $(0, [a_1], \ldots, [a_n])$ for ε small enough. For $z \in B(0,\varepsilon)$ we have $|a_j.\,g(z)| \geqslant \varepsilon|g(z)|$ for some j, otherwise the inequality $|a_j.\,g(z)| < \varepsilon|g(z)|$ would imply the existence of $h_j \in \mathbb{C}^r$ with $|h_j| < \varepsilon$ and $a_j.\,g(z) = h_j.\,g(z)$. Since $g(z) \neq 0$, we would have

$$(z, [a_1 - h_1], \ldots, [a_n - h_n]) \in A^\star \cap \big(B(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big),$$

a contradiction. We obtain therefore

$$\varepsilon|g(z)| \leqslant \max|a_j.\,g(z)| \leqslant (\max|a_j|)\,|g(z)| \quad \text{on} \;\; B(0,\varepsilon). \qquad \square$$

(12.4) Theorem (Briançon-Skoda 1974). *Set* $p = \min\{n-1, r-1\}$. *Then*

a) $\widehat{\mathcal{I}}^{(k+1)} = \mathcal{I}\widehat{\mathcal{I}}^{(k)} = \overline{\mathcal{I}}\widehat{\mathcal{I}}^{(k)}$ *for* $k \geqslant p$.

b) $\overline{\mathcal{I}}^{(k+p)} \subset \widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$ *for all* $k \in \mathbb{N}$.

Proof. a) The inclusions $\mathcal{I}\widehat{\mathcal{I}}^{(k)} \subset \overline{\mathcal{I}}\widehat{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k+1)}$ are obvious thanks to Prop. 12.2, so we only have to prove that $\widehat{\mathcal{I}}^{(k+1)} \subset \mathcal{I}\widehat{\mathcal{I}}^{(k)}$. Assume first that $r \leqslant n$. Let $f \in \widehat{\mathcal{I}}^{(k+1)}$ be such that

$$\int_\Omega |f|^2\,|g|^{-2(k+1+\varepsilon)}\,dV < +\infty.$$

For $k \geqslant p-1$, we can apply Th. 11.13 with $m = r-1$ and with the weight $\varphi = (k-m)\log|g|^2$. Hence f can be written $f = \sum g_j h_j$ with

$$\int_\Omega |h|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty,$$

thus $h_j \in \widehat{\mathcal{I}}^{(k)}$ and $f \in \mathcal{I}\widehat{\mathcal{I}}^{(k)}$. When $r > n$, Lemma 12.3 shows that there is an ideal $\mathcal{J} \subset \mathcal{I}$ with n generators such that $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$. We find

$$\widehat{\mathcal{I}}^{(k+1)} = \widehat{\mathcal{J}}^{(k+1)} \subset \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} \subset \mathcal{I}\widehat{\mathcal{I}}^{(k)} \quad \text{for} \;\; k \geqslant n-1.$$

b) Property a) implies inductively $\widehat{\mathcal{I}}^{(k+p)} = \mathcal{I}^k\,\widehat{\mathcal{I}}^{(p)}$ for all $k \in \mathbb{N}$. This gives in particular $\widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$. $\square$

(12.5) Corollary.

a) *The ideal* $\overline{\mathcal{I}}$ *is the integral closure of* $\mathcal{I}$, *i.e. by definition the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{I}^s, \qquad 1 \leqslant s \leqslant d.$$

b) *Similarly,* $\overline{\mathcal{I}}^{(k)}$ *is the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{I}^{\lceil ks \rceil}, \qquad 1 \leqslant s \leqslant d,$$

where $\lceil t\rceil$ denotes the smallest integer $\geqslant t$.

As the ideal $\overline{\mathcal{I}}^{(k)}$ is finitely generated, property b) shows that there always exists a rational number $l \geqslant k$ such that $\overline{\mathcal{I}}^{(l)} = \overline{\mathcal{I}}^{(k)}$.

Proof. a) If $u \in \mathcal{O}_n$ satisfies a polynomial equation with coefficients $a_s \in \mathcal{I}^s$, then clearly $|a_s| \leqslant C_s\,|g|^s$ and the usual elementary bound

$$|\mathrm{roots}| \leqslant 2 \max_{1\leqslant s\leqslant d} |a_s|^{1/s}$$

for the roots of a monic polynomial implies $|u| \leqslant C\,|g|$.

Conversely, assume that $u \in \overline{\mathcal{I}}$. The ring $\mathcal{O}_n$ is Noetherian, so the ideal $\widehat{\mathcal{I}}^{(p)}$ has a finite number of generators $v_1, \ldots, v_N$. For every j we have $uv_j \in \overline{\mathcal{I}}\,\widehat{\mathcal{I}}^{(p)} = \mathcal{I}\,\widehat{\mathcal{I}}^{(p)}$, hence there exist elements $b_{jk} \in \mathcal{I}$ such that

$$uv_j = \sum_{1\leqslant k\leqslant N} b_{jk} v_k.$$

The matrix $(u\delta_{jk} - b_{jk})$ has the non zero vector (v_j) in its kernel, thus u satisfies the equation $\det(u\delta_{jk} - b_{jk}) = 0$, which is of the required type.

b) Observe that $v_1, \ldots, v_N$ satisfy simultaneously some integrability condition $\int_\Omega |v_j|^{-2(p+\varepsilon)} < +\infty$, thus $\widehat{\mathcal{I}}^{(p)} = \widehat{\mathcal{I}}^{(p+\eta)}$ for $\eta \in [0, \varepsilon[$. Let $u \in \overline{\mathcal{I}}^{(k)}$. For every integer $m \in \mathbb{N}$ we have

$$u^m v_j \in \overline{\mathcal{I}}^{(km)}\,\widehat{\mathcal{I}}^{(p+\eta)} \subset \widehat{\mathcal{I}}^{(km+\eta+p)}.$$

If $k \notin \mathbb{Q}$, we can find m such that $d(km + \varepsilon/2, \mathbb{Z}) < \varepsilon/2$, thus $km + \eta \in \mathbb{N}$ for some $\eta \in]0, \varepsilon[$. If $k \in \mathbb{Q}$, we take m such that $km \in \mathbb{N}$ and $\eta = 0$. Then

$$u^m v_j \in \widehat{\mathcal{I}}^{(N+p)} = \mathcal{I}^N\,\widehat{\mathcal{I}}^{(p)} \quad \text{with} \quad N = km + \eta \in \mathbb{N},$$

and the reasoning made in a) gives $\det(u^m\delta_{jk} - b_{jk}) = 0$ for some $b_{jk} \in \mathcal{I}^N$. This is an equation of the type described in b), where the coefficients a_s vanish when s is not a multiple of m and $a_{ms} \in \mathcal{I}^{Ns} \subset \mathcal{I}^{\lceil kms\rceil}$. □

Let us mention that Briançon and Skoda's result 12.4 b) is optimal for $k = 1$. Take for example $\mathcal{I} = (g_1, \ldots, g_r)$ with $g_j(z) = z_j^r$, $1 \leqslant j \leqslant r$, and $f(z) = z_1 \ldots z_r$. Then $|f| \leqslant C|g|$ and 12.4 b) yields $f^r \in \mathcal{I}$; however, it is easy to verify that $f^{r-1} \notin \mathcal{I}$. The theorem also gives an answer to the following conjecture made by J. Mather.

(12.6) Corollary. *Let $f \in \mathcal{O}_n$ and $\mathcal{I}_f = (z_1\partial f/\partial z_1, \ldots, z_n\partial f/\partial z_n)$. Then $f \in \overline{\mathcal{I}}_f$, and for every integer $k \geqslant 0$, $f^{k+n-1} \in \mathcal{I}_f^k$.*

The Corollary is also optimal for $k = 1$: for example, one can verify that the function $f(z) = (z_1 \ldots z_n)^3 + z_1^{3n-1} + \cdots + z_n^{3n-1}$ is such that $f^{n-1} \notin \mathcal{I}_f$.

Proof. Set $g_j(z) = z_j\,\partial f/\partial z_j$, $1 \leqslant j \leqslant n$. By 12.4 b), it suffices to show that $|f| \leqslant C|g|$. For every germ of analytic curve $\mathbb{C} \ni t \longmapsto \gamma(t)$, $\gamma \not\equiv 0$, the vanishing order of $f \circ \gamma(t)$ at $t = 0$ is the same as that of

$$t\,\frac{d(f\circ\gamma)}{dt} = \sum_{1\leqslant j\leqslant n} t\,\gamma_j'(t)\,\frac{\partial f}{\partial z_j}\big(\gamma(t)\big).$$

We thus obtain

$$|f\circ\gamma(t)| \leqslant C_1\,|t|\,\left|\frac{d(f\circ\gamma)}{dt}\right| \leqslant C_2 \sum_{1\leqslant j\leqslant n} |t\,\gamma_j'(t)|\,\left|\frac{\partial f}{\partial z_j}\big(\gamma(t)\big)\right| \leqslant C_3\,|g\circ\gamma(t)|$$

and conclude by the following elementary lemma. □

(12.7) Curve selection lemma. *Let $f, g_1, \ldots, g_r \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at* 0*. Then we have $|f| \leqslant C|g|$ for some constant C if and only if for every germ of analytic curve γ through* 0 *there exists a constant C_γ such that $|f\circ\gamma| \leqslant C_\gamma|g\circ\gamma|$.*

Proof. If the inequality $|f| \leqslant C|g|$ does not hold on any neighborhood of 0, the germ of analytic set $(A, 0) \subset (\mathbb{C}^{n+r}, 0)$ defined by

$$g_j(z) = f(z)z_{n+j}, \qquad 1 \leqslant j \leqslant r,$$

contains a sequence of points $\big(z_\nu, g_j(z_\nu)/f(z_\nu)\big)$ converging to 0 as ν tends to $+\infty$, with $f(z_\nu) \neq 0$. Hence $(A, 0)$ contains an irreducible component on which $f \not\equiv 0$ and there is a germ of curve $\widetilde{\gamma} = (\gamma, \gamma_{n+j}) : (\mathbb{C}, 0) \to (\mathbb{C}^{n+r}, 0)$ contained in $(A, 0)$ such that $f \circ \gamma \not\equiv 0$. We get $g_j \circ \gamma = (f \circ \gamma)\gamma_{n+j}$, hence $|g \circ \gamma(t)| \leqslant C|t|\,|f \circ \gamma(t)|$ and the inequality $|f \circ \gamma| \leqslant C_\gamma|g \circ \gamma|$ does not hold. □

13. The Ohsawa-Takegoshi L^2 extension theorem

We address here the following extension problem: let Y be a complex analytic submanifold of a complex manifold X ; given a holomorphic function f on Y satisfying suitable L^2 conditions on Y, find a holomorphic extension F of f to X, together with a good L^2 estimate for F on X. The first satisfactory solution has been obtained only rather recently by (Ohsawa-Takegoshi 1987). We follow here a more geometric approach due to (Manivel 1993), which provides a generalized extension theorem in the general framework of vector bundles. As in Ohsawa-Takegoshi's fundamental paper, the main idea is to use a modified Bochner-Kodaira-Nakano inequality. Such inequalities were originally introduced in the work of (Donnelly-Fefferman 1983) and (Donnelly-Xavier 1984). The main a priori inequality we are going to use is a simplified (and slightly extended) version of the original Ohsawa-Takegoshi a priori inequality, as proposed recently by (Ohsawa 1995); see also (Berndtsson 1995) for related calculations in the special case of domains in $\mathbb{C}^n$.

(13.1) Lemma (Ohsawa 1995)**.** *Let E be a hermitian vector bundle on a complex manifold X equipped with a Kähler metric ω. Let η, $\lambda > 0$ be smooth functions on X. Then for every form $u \in \mathcal{D}(X, \Lambda^{p,q}T_X^\star \otimes E)$ with compact support we have*

$$\begin{aligned}\|(\eta^{\frac12}+\lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12}D''u\|^2 + \|\lambda^{\frac12}D'u\|^2 + 2\|\lambda^{-\frac12}d'\eta\wedge u\|^2 \\ \geqslant \langle\langle[\eta\,\mathrm{i}\Theta(E) - \mathrm{i}\,d'd''\eta - \mathrm{i}\lambda^{-1}d'\eta\wedge d''\eta, \Lambda]u, u\rangle\rangle.\end{aligned}$$

Proof. Let us consider the "twisted" Laplace-Beltrami operators

$$\begin{aligned}D'\eta D'^\star + D'^\star\eta D' &= \eta[D', D'^\star] + [D',\eta]D'^\star + [D'^\star,\eta]D' \\ &= \eta\Delta' + (d'\eta)D'^\star - (d'\eta)^* D', \\ D''\eta D''^\star + D''^\star\eta D'' &= \eta[D'', D''^\star] + [D'',\eta]D''^\star + [D''^\star,\eta]D'' \\ &= \eta\Delta'' + (d''\eta)D''^\star - (d''\eta)^* D'',\end{aligned}$$

where η, $(d'\eta)$, $(d''\eta)$ are abbreviated notations for the multiplication operators $\eta\bullet$, $(d'\eta)\wedge\bullet$, $(d''\eta)\wedge\bullet$. By subtracting the above equalities and taking into account the Bochner-Kodaira-Nakano identity $\Delta'' - \Delta' = [\mathrm{i}\Theta(E), \Lambda]$, we get

$$\begin{aligned}&D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D' \\ &\qquad = \eta[\mathrm{i}\Theta(E),\Lambda] + (d''\eta)D''^\star - (d''\eta)^\star D'' + (d'\eta)^\star D' - (d'\eta)D'^\star. \qquad (13.2)\end{aligned}$$

Moreover, the Jacobi identity yields

$$[D'', [d'\eta, \Lambda]] - [d'\eta, [\Lambda, D'']] + [\Lambda, [D'', d'\eta]] = 0,$$

whilst $[\Lambda, D''] = -\mathrm{i}D'^\star$ by the basic commutation relations 7.2. A straightforward computation shows that $[D'', d'\eta] = -(d'd''\eta)$ and $[d'\eta, \Lambda] = \mathrm{i}(d''\eta)^\star$. Therefore we get

$$\mathrm{i}[D'', (d''\eta)^\star] + \mathrm{i}[d'\eta, D'^\star] - [\Lambda, (d'd''\eta)] = 0,$$

that is,

$$[\mathrm{i}\,d'd''\eta, \Lambda] = [D'', (d''\eta)^\star] + [D'^\star, d'\eta] = D''(d''\eta)^\star + (d''\eta)^\star D'' + D'^\star(d'\eta) + (d'\eta)D'^\star.$$

After adding this to (13.2), we find

$$\begin{aligned}&D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D' + [\mathrm{i}\,d'd''\eta, \Lambda] \\ &\qquad = \eta[\mathrm{i}\Theta(E),\Lambda] + (d''\eta)D''^\star + D''(d''\eta)^\star + (d'\eta)^\star D' + D'^\star(d'\eta).\end{aligned}$$

We apply this identity to a form $u \in \mathcal{D}(X, \Lambda^{p,q}T_X^\star \otimes E)$ and take the inner bracket with u. Then

$$\langle\langle(D''\eta D''^\star)u, u\rangle\rangle = \langle\langle \eta D''^\star u, D''^\star u\rangle\rangle = \|\eta^{\frac12}D''^\star u\|^2,$$

and likewise for the other similar terms. The above equalities imply

$$\begin{aligned}&\|\eta^{\frac12}D''^\star u\|^2 + \|\eta^{\frac12}D''u\|^2 - \|\eta^{\frac12}D'u\|^2 - \|\eta^{\frac12}D'^\star u\|^2 = \\ &\quad \langle\langle[\eta\,\mathrm{i}\Theta(E) - \mathrm{i}\,d'd''\eta, \Lambda]u, u\rangle\rangle + 2\,\mathrm{Re}\,\langle\langle D''^\star u, (d''\eta)^\star u\rangle\rangle + 2\,\mathrm{Re}\,\langle\langle D'u, d'\eta\wedge u\rangle\rangle.\end{aligned}$$

By neglecting the negative terms $-\|\eta^{\frac12}D'u\|^2 - \|\eta^{\frac12}D'^\star u\|^2$ and adding the squares

$$\|\lambda^{\frac{1}{2}}D''^{\star}u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D''^{\star}u, (d''\eta)^{\star}u\rangle\!\rangle + \|\lambda^{-\frac{1}{2}}(d''\eta)^{\star}u\|^2 \geqslant 0,$$
$$\|\lambda^{\frac{1}{2}}D'u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle + \|\lambda^{-\frac{1}{2}}d'\eta\wedge u\|^2 \geqslant 0$$

we get

$$\|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^{\star}u\|^2 + \|\eta^{\frac{1}{2}}D''u\|^2 + \|\lambda^{\frac{1}{2}}D'u\|^2 + \|\lambda^{-\frac{1}{2}}d'\eta\wedge u\|^2 + \|\lambda^{-\frac{1}{2}}(d''\eta)^{\star}u\|^2$$
$$\geqslant \langle\!\langle[\eta\,\mathrm{i}\Theta(E) - \mathrm{i}\,d'd''\eta, \Lambda]u, u\rangle\!\rangle.$$

Finally, we use the identities

$$(d'\eta)^{\star}(d'\eta) - (d''\eta)(d''\eta)^{\star} = \mathrm{i}[d''\eta, \Lambda](d'\eta) + \mathrm{i}(d''\eta)[d'\eta, \Lambda] = [\mathrm{i}d''\eta\wedge d'\eta, \Lambda],$$
$$\|\lambda^{-\frac{1}{2}}d'\eta\wedge u\|^2 - \|\lambda^{-\frac{1}{2}}(d''\eta)^{\star}u\|^2 = -\langle\!\langle[\mathrm{i}\lambda^{-1}d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle,$$

The inequality asserted in Lemma 13.1 follows by adding the second identity to our last inequality. □

In the special case of (n,q)-forms, the forms $D'u$ and $d'\eta\wedge u$ are of bidegree $(n+1,q)$, hence the estimate takes the simpler form

$$\text{(13.3)}\quad \|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^{\star}u\|^2+\|\eta^{\frac{1}{2}}D''u\|^2 \geqslant \langle\!\langle[\eta\,\mathrm{i}\Theta(E)-\mathrm{i}\,d'd''\eta-\mathrm{i}\lambda^{-1}\,d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle.$$

(13.4) Proposition. *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let E be a hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $\lambda > 0$ on X such that the (hermitian) curvature operator $B = B^{n,q}_{E,\omega,\eta} = [\eta\,\mathrm{i}\Theta(E) - \mathrm{i}\,d'd''\eta - \mathrm{i}\lambda^{-1}d'\eta\wedge d''\eta, \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{n,q}T^{\star}_X\otimes E$, for some $q\geqslant 1$. Then for every form $g\in L^2(X,\Lambda^{n,q}T^{\star}_X\otimes E)$ such that $D''g=0$ and $\int_X\langle B^{-1}g,g\rangle\,dV_\omega < +\infty$, there exists $f\in L^2(X,\Lambda^{n,q-1}T^{\star}_X\otimes E)$ such that $D''f=g$ and*

$$\int_X(\eta+\lambda)^{-1}|f|^2\,dV_\omega \leqslant 2\int_X\langle B^{-1}g,g\rangle\,dV_\omega.$$

Proof. The proof is almost identical to the proof of Theorem 8.4, except that we use (13.4) instead of (7.4). Assume first that ω is complete. With the same notation as in 7.4, we get for every $v = v_1 + v_2 \in (\mathrm{Ker}\,D'')\oplus(\mathrm{Ker}\,D'')^{\perp}$ the inequalities

$$|\langle g,v\rangle|^2 = |\langle g,v_1\rangle|^2 \leqslant \int_X\langle B^{-1}g,g\rangle\,dV_\omega\int_X\langle Bv_1,v_1\rangle\,dV_\omega,$$

and

$$\int_X\langle Bv_1,v_1\rangle\,dV_\omega \leqslant \|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^{\star}v_1\|^2 + \|\eta^{\frac{1}{2}}D''v_1\|^2 = \|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^{\star}v\|^2$$

provided that $v\in\mathrm{Dom}\,D''^{\star}$. Combining both, we find

$$|\langle g,v\rangle|^2 \leqslant \Big(\int_X\langle B^{-1}g,g\rangle\,dV_\omega\Big)\|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^{\star}v\|^2.$$

This shows the existence of an element $w\in L^2(X,\Lambda^{n,q}T^{\star}_X\otimes E)$ such that

$$\|w\|^2 \leqslant \int_X \langle B^{-1}g, g\rangle\, dV_\omega \qquad \text{and}$$
$$\langle\!\langle v, g\rangle\!\rangle = \langle\!\langle (\eta^{\frac12} + \lambda^{\frac12}) D''^\star v, w\rangle\!\rangle \qquad \forall g \in \operatorname{Dom} D'' \cap \operatorname{Dom} D''^\star.$$

As $(\eta^{1/2} + \lambda^{\frac12})^2 \leqslant 2(\eta+\lambda)$, it follows that $f = (\eta^{1/2} + \lambda^{\frac12})w$ satisfies $D''f = g$ as well as the desired L^2 estimate. If ω is not complete, we set $\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$ with some complete Kähler metric $\widehat{\omega}$. The final conclusion is then obtained by passing to the limit and using a monotonicity argument (the integrals are monotonic with respect to ε). □

(13.5) Remark. We will also need a variant of the L^2-estimate, so as to obtain approximate solutions with weaker requirements on the data: given $\delta > 0$ and $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists an approximate solution $f \in L^2(X, \Lambda^{n,q-1}T^\star_X \otimes E)$ and a correcting term $h \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''f + \delta^{1/2}h = g$ and

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_\omega + \int_X |h|^2\, dV_\omega \leqslant 2\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega.$$

The proof is almost unchanged, we rely instead on the estimates

$$|\langle g, v_1\rangle|^2 \leqslant \int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega \int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega \leqslant \|(\eta^{\frac12} + \lambda^{\frac12})D''^\star v\|^2 + \delta\|v\|^2.$$ □

(13.6) Theorem. *Let X be a weakly pseudoconvex n-dimensional complex manifold equipped with a Kähler metric ω, let L (resp. E) be a hermitian holomorphic line bundle (resp. a hermitian holomorphic vector bundle of rank r over X), and s a global holomorphic section of E. Assume that s is generically transverse to the zero section, and let*

$$Y = \{x \in X\,;\ s(x) = 0, \Lambda^r ds(x) \neq 0\}, \qquad p = \dim Y = n - r.$$

Moreover, assume that the $(1,1)$-form $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2$ is semipositive and that there is a continuous function $\alpha \geqslant 1$ such that the following two inequalities hold everywhere on X :

a) $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \geqslant \alpha^{-1}\dfrac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}$,

b) $|s| \leqslant e^{-\alpha}$.

Then for every smooth D''-closed $(0,q)$-form f over Y with values in the line bundle $\Lambda^nT^\star_X \otimes L$ (restricted to Y), such that $\int_Y |f|^2|\Lambda^r(ds)|^{-2}dV_\omega < +\infty$, there exists a D''-closed $(0,q)$-form F over X with values in $\Lambda^nT^\star_X \otimes L$, such that F is smooth over $X \smallsetminus \{s = \Lambda^r(ds) = 0\}$, satisfies $F_{\restriction Y} = f$ and

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \leqslant C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}\,,$$

where C_r is a numerical constant depending only on r.

Observe that the differential ds (which is intrinsically defined only at points where s vanishes) induces a vector bundle isomorphism $ds : T_X/T_Y \to E$ along Y, hence a non vanishing section $\Lambda^r(ds)$, taking values in

$$\Lambda^r(T_X/T_Y)^\star \otimes \det E \subset \Lambda^r T_X^\star \otimes \det E.$$

The norm $|\Lambda^r(ds)|$ is computed here with respect to the metrics on $\Lambda^r T_X^\star$ and $\det E$ induced by the Kähler metric ω and by the given metric on E. Also notice that if hypothesis a) is satisfied for some α, one can always achieve b) by multiplying the metric of E with a sufficiently small weight $e^{-\chi\circ\psi}$ (with ψ a psh exhaustion on X and χ a convex increasing function; property a) remains valid after we multiply the metric of L by $e^{-(r+\alpha_0^{-1})\chi\circ\psi}$, where $\alpha_0 = \inf_{x\in X}\alpha(x)$.

Proof. Let us first assume that the singularity set $\Sigma = \{s = 0\} \cap \{\Lambda^r(ds) = 0\}$ is empty, so that Y is closed and nonsingular. We claim that there exists a smooth section

$$F_\infty \in C^\infty(X, \Lambda^{n,q} T_X^\star \otimes L) = C^\infty(X, \Lambda^{0,q} T_X^\star \otimes \Lambda^n T_X^\star \otimes L)$$

such that

(a) F_∞ coincides with f in restriction to Y,

(b) $|F_\infty| = |f|$ at every point of Y,

(c) $D'' F_\infty = 0$ at every point of Y.

For this, consider coordinates patches $U_j \subset X$ biholomorphic to polydiscs such that

$$U_j \cap Y = \{z \in U_j\,;\ z_1 = \ldots = z_r = 0\}$$

in the corresponding coordinates. We can find a section $\widetilde f$ in $C^\infty(X, \Lambda^{n,q} T_X^\star \otimes L)$ which achieves a) and b), since the restriction map $(\Lambda^{0,q} T_X^*)_{\restriction Y} \to \Lambda^{0,q} T_Y^*$ can be viewed as an orthogonal projection onto a C^∞-subbundle of $(\Lambda^{0,q} T_X^*)_{\restriction Y}$. It is enough to extend this subbundle from $U_j \cap Y$ to U_j (e.g. by extending each component of a frame), and then to extend f globally via local smooth extensions and a partition of unity. For any such extension $\widetilde f$ we have

$$(D''\widetilde f)_{\restriction Y} = (D''\widetilde f_{\restriction Y}) = D'' f = 0.$$

It follows that we can divide $D''\widetilde f = \sum_{1\leqslant\lambda\leqslant r} g_{j,\lambda}(z) \wedge d\overline z_\lambda$ on $U_j \cap Y$, with suitable smooth $(0,q)$-forms $g_{j,\lambda}$ which we also extend arbitrarily from $U_j \cap Y$ to U_j. Then

$$F_\infty := \widetilde f - \sum_j \theta_j(z) \sum_{1\leqslant\lambda\leqslant r} \overline z_\lambda g_{j,\lambda}(z)$$

coincides with $\widetilde f$ on Y and satisfies (c). Since we do not know about F_∞ except in an infinitesimal neighborhood of Y, we will consider a truncation F_ε of F_∞ with support in a small tubular neighborhood $|s| < \varepsilon$ of Y, and solve the equation $D'' u_\varepsilon = D'' F_\varepsilon$ with the constraint that u_ε should be 0 on Y. As $\operatorname{codim} Y = r$, this will be the case if we can guarantee that $|u_\varepsilon|^2 |s|^{-2r}$ is locally integrable near Y. For this, we will apply Proposition 13.4 with a suitable choice of the functions η and λ, and an additional weight $|s|^{-2r}$ in the metric of L.

Let us consider the smooth strictly convex function $\chi_0 :]-\infty,0] \to]-\infty,0]$ defined by $\chi_0(t) = t - \log(1-t)$ for $t \leqslant 0$, which is such that $\chi_0(t) \leqslant t$, $1 \leqslant \chi_0' \leqslant 2$ and $\chi_0''(t) = 1/(1-t)^2$. We set

$$\sigma_\varepsilon = \log(|s|^2+\varepsilon^2), \qquad \eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon).$$

As $|s| \leqslant e^{-\alpha} \leqslant e^{-1}$, we have $\sigma_\varepsilon \leqslant 0$ for ε small, and

$$\eta_\varepsilon \geqslant \varepsilon - \sigma_\varepsilon \geqslant \varepsilon - \log(e^{-2\alpha}+\varepsilon^2).$$

Given a relatively compact subset $X_c = \{\psi < c\} \Subset X$, we thus have $\eta_\varepsilon \geqslant 2\alpha$ for $\varepsilon < \varepsilon(c)$ small enough. Simple calculations yield

$$\begin{aligned}
\mathrm{i}\,d'\sigma_\varepsilon &= \frac{\mathrm{i}\{D's,s\}}{|s|^2+\varepsilon^2},\\
\mathrm{i}\,d'd''\sigma_\varepsilon &= \frac{\mathrm{i}\{D's,D's\}}{|s|^2+\varepsilon^2} - \frac{\mathrm{i}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\geqslant \frac{\varepsilon^2}{|s|^2}\frac{\mathrm{i}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\geqslant \frac{\varepsilon^2}{|s|^2}\mathrm{i}d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2},
\end{aligned}$$

thanks to Lagrange's inequality $\mathrm{i}\{D's,s\}\wedge\{s,D's\} \leqslant |s|^2\mathrm{i}\{D's,D's\}$. On the other hand, we have $d'\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon)d\sigma_\varepsilon$ with $1 \leqslant \chi_0'(\sigma_\varepsilon) \leqslant 2$, hence

$$\begin{aligned}
-\mathrm{i}d'd''\eta_\varepsilon &= \chi_0'(\sigma_\varepsilon)\mathrm{i}d'd''\sigma_\varepsilon + \chi_0''(\sigma_\varepsilon)\mathrm{i}d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon\\
&\geqslant \Big(\frac{1}{\chi_0'(\sigma_\varepsilon)}\frac{\varepsilon^2}{|s|^2} + \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\Big)\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon - \chi_0'(\sigma_\varepsilon)\frac{\{\mathrm{i}\Theta_E s,s\}}{|s|^2+\varepsilon^2}.
\end{aligned}$$

We consider the original metric of L multiplied by the weight $|s|^{-2r}$. In this way, we get a curvature form

$$\mathrm{i}\Theta_L + r\,\mathrm{i}d'd''\log|s|^2 \geqslant \frac{1}{2}\chi_0'(\sigma_\varepsilon)\alpha^{-1}\frac{\{\mathrm{i}\Theta_E s,s\}}{|s|^2+\varepsilon^2}$$

by hypothesis a), thanks to the semipositivity of the left hand side and the fact that $\frac{1}{2}\chi_0'(\sigma_\varepsilon)\frac{1}{|s|^2+\varepsilon^2} \leqslant \frac{1}{|s|^2}$. As $\eta_\varepsilon \geqslant 2\alpha$ on X_c for ε small, we infer

$$\eta_\varepsilon(\mathrm{i}\Theta_L + \mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon \geqslant \frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon$$

on X_c. Hence, if $\lambda_\varepsilon = \chi_0'(\sigma_\varepsilon)^2/\chi_0''(\sigma_\varepsilon)$, we obtain

$$\begin{aligned}
B_\varepsilon &:= \big[\eta_\varepsilon(\mathrm{i}\Theta_L + \mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \lambda_\varepsilon^{-1}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\big]\\
&\geqslant \Big[\frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\Big] = \frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star
\end{aligned}$$

as an operator on (n,q)-forms (see the proof of Lemma 13.1).

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\theta(t) = 1$ on $]-\infty, 1/2]$, $\mathrm{Supp}\,\theta \subset\ ]-\infty, 1[$ and $|\theta'| \leqslant 3$. For $\varepsilon > 0$ small, we consider the (n,q)-form $F_\varepsilon = \theta(\varepsilon^{-2}|s|^2)F_\infty$ and its D''-derivative

$$g_\varepsilon = D''F_\varepsilon = (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)d''\sigma_\varepsilon \wedge F_\infty + \theta(\varepsilon^{-2}|s|^2)D''F_\infty$$

[as is easily seen from the equality $1+\varepsilon^{-2}|s|^2 = \varepsilon^{-2}e^{\sigma_\varepsilon}$]. We observe that g_ε has its support contained in the tubular neighborhood $|s| < \varepsilon$; moreover, as $\varepsilon \to 0$, the second term in the right hand side converges uniformly to 0 on every compact set; it will therefore produce no contribution in the limit. On the other hand, the first term has the same order of magnitude as $d''\sigma_\varepsilon$ and $d''\eta_\varepsilon$, and can be controlled in terms of B_ε. In fact, for any (n,q)-form u and any $(n,q+1)$-form v we have

$$\begin{aligned}|\langle d''\eta_\varepsilon \wedge u, v\rangle|^2 &= |\langle u, (d''\eta_\varepsilon)^\star v\rangle|^2 \leqslant |u|^2|(d''\eta_\varepsilon)^\star v|^2 = |u|^2\langle (d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star v, v\rangle \\ &\leqslant \frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2\langle B_\varepsilon v, v\rangle.\end{aligned}$$

This implies

$$\langle B_\varepsilon^{-1}(d''\eta_\varepsilon \wedge u), (d''\eta_\varepsilon \wedge u)\rangle \leqslant \frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2.$$

The main term in g_ε can be written

$$g_\varepsilon^{(1)} := (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)\chi_0'(\sigma_\varepsilon)^{-1}d''\eta_\varepsilon \wedge F_\infty.$$

On $\mathrm{Supp}\, g_\varepsilon^{(1)} \subset \{|s| < \varepsilon\}$, since $\chi_0'(\sigma_\varepsilon) \geqslant 1$, we thus find

$$\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\rangle \leqslant (1+\varepsilon^{-2}|s|^2)^2\,\theta'(\varepsilon^{-2}|s|^2)^2|F_\infty|^2.$$

Instead of working on X itself, we will work rather on the relatively compact subset $X_c \smallsetminus Y_c$, where $Y_c = Y \cap X_c = Y \cap \{\psi < c\}$. We know that $X_c \smallsetminus Y_c$ is again complete Kähler by Lemma 11.9. In this way, we avoid the singularity of the weight $|s|^{-2r}$ along Y. We find

$$\int_{X_c\smallsetminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\rangle\,|s|^{-2r}dV_\omega \leqslant \int_{X_c\smallsetminus Y_c}|F_\infty|^2(1+\varepsilon^{-2}|s|^2)^2\theta'(\varepsilon^{-2}|s|^2)^2|s|^{-2r}dV_\omega.$$

Now, we let $\varepsilon \to 0$ and view s as "transverse local coordinates" around Y. As F_∞ coincides with f on Y, it is not hard to see that the right hand side converges to $c_r\int_{Y_c}|f|^2|\Lambda^r(ds)|^{-2}dV_{Y,\omega}$ where c_r is the "universal" constant

$$c_r = \int_{z\in\mathbb{C}^r,\,|z|\leqslant 1}(1+|z|^2)^2\theta'(|z|^2)^2\frac{\mathrm{i}^{r^2}\Lambda^r(dz)\wedge\Lambda^r(d\overline{z})}{|z|^{2r}} < +\infty$$

depending only on r. The second term

$$g_\varepsilon^{(2)} = \theta(\varepsilon^{-2}|s|^2)d''F_\infty$$

in g_ε satisfies $\mathrm{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|)$ (just look at the Taylor expansion of $d''F_\infty$ near Y). From this we easily conclude that

$$\int_{X_c\smallsetminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon^{(2)}, g_\varepsilon^{(2)}\rangle\,|s|^{-2r}dV_{X,\omega} = O(\varepsilon^2),$$

provided that B_ε remains locally uniformly bounded below near Y (this is the case for instance if we have strict inequalities in the curvature assumption a)). If this holds true, we apply Proposition 8.4 on $X_c \smallsetminus Y_c$ with the additional weight factor $|s|^{-2r}$. Otherwise, we use the modified estimate stated in Remark 8.5 in order to solve the approximate equation $D''u + \delta^{1/2}h = g_\varepsilon$ with $\delta > 0$ small. This yields sections $u = u_{c,\varepsilon,\delta}$, $h = h_{c,\varepsilon,\delta}$ such that

$$\begin{aligned}\int_{X_c\smallsetminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1}|u_{c,\varepsilon,\delta}|^2|s|^{-2r}\,dV_\omega + \int_{X_c\smallsetminus Y_c} |h_{c,\varepsilon,\delta}|^2|s|^{-2r}\,dV_\omega \\ \leqslant 2\int_{X_c\smallsetminus Y_c} \langle (B_\varepsilon + \delta I)^{-1}g_\varepsilon, g_\varepsilon\rangle |s|^{-2r}\,dV_\omega,\end{aligned}$$

and the right hand side is under control in all cases. The extra error term $\delta^{1/2}h$ can be removed at the end by letting δ tend to 0. Since there is essentially no additional difficulty involved in this process, we will assume for simplicity of exposition that we do have the required lower bound for B_ε and the estimates of $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ as above. For $\delta = 0$, the above estimate provides a solution $u_{c,\varepsilon}$ of the equation $D''u_{c,\varepsilon} = g_\varepsilon = D''F_\varepsilon$ on $X_c \smallsetminus Y_c$, such that

$$\begin{aligned}\int_{X_c\smallsetminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1}|u_{c,\varepsilon}|^2|s|^{-2r}dV_{X,\omega} &\leqslant 2\int_{X_c\smallsetminus Y_c} \langle B_\varepsilon^{-1}g_\varepsilon, g_\varepsilon\rangle\,|s|^{-2r}dV_{X,\omega} \\ &\leqslant 2\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega} + O(\varepsilon).\end{aligned}$$

Here we have

$$\begin{aligned}\sigma_\varepsilon &= \log(|s|^2+\varepsilon^2) \leqslant \log(e^{-2\alpha}+\varepsilon^2) \leqslant -2\alpha + O(\varepsilon^2) \leqslant -2 + O(\varepsilon^2),\\ \eta_\varepsilon &= \varepsilon - \chi_0(\sigma_\varepsilon) \leqslant (1+O(\varepsilon))\sigma_\varepsilon^2,\\ \lambda_\varepsilon &= \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)} = (1-\sigma_\varepsilon)^2 + (1-\sigma_\varepsilon) \leqslant (3+O(\varepsilon))\sigma_\varepsilon^2,\\ \eta_\varepsilon + \lambda_\varepsilon &\leqslant (4+O(\varepsilon))\sigma_\varepsilon^2 \leqslant (4+O(\varepsilon))\big(-\log(|s|^2+\varepsilon^2)\big)^2.\end{aligned}$$

As F_ε is uniformly bounded with support in $\{|s| < \varepsilon\}$, we conclude from an obvious volume estimate that

$$\int_{X_c}\frac{|F_\varepsilon|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \leqslant \frac{\mathrm{Const}}{(\log\varepsilon)^2}.$$

Therefore, thanks to the usual inequality $|t+u|^2 \leqslant (1+k)|t|^2 + (1+k^{-1})|u|^2$ applied to the sum $F_{c,\varepsilon} = \widetilde{f}_\varepsilon - u_{c,\varepsilon}$ with $k = |\log\varepsilon|$, we obtain from our previous estimates

$$\int_{X_c\smallsetminus Y_c}\frac{|F_{c,\varepsilon}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \leqslant 8\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega} + O(|\log\varepsilon|^{-1}).$$

In addition to this, we have $d''F_{c,\varepsilon} = 0$ by construction, and this equation extends from $X_c \smallsetminus Y_c$ to X_c by Lemma 11.10.

If $q = 0$, then $u_{c,\varepsilon}$ must be smooth also, and the non integrability of the weight $|s|^{-2r}$ along Y shows that $u_{c,\varepsilon}$ vanishes on Y, therefore

$$F_{c,\varepsilon\restriction Y} = F_{\varepsilon\restriction Y} = F_{\infty\restriction Y} = f.$$

The theorem and its final estimate are thus obtained by extracting weak limits, first as $\varepsilon \to 0$, and then as $c \to +\infty$. The initial assumption that $\Sigma = \{s = \Lambda^r(ds) = 0\}$ is empty can be easily removed in two steps: i) the result is true if X is Stein, since we can always find a complex hypersurface Z in X such that $\Sigma \subset \overline{Y} \cap Z \subsetneq \overline{Y}$, and then apply the extension theorem on the Stein manifold $X \smallsetminus Z$, in combination with Lemma 11.10 ; ii) the whole procedure still works when Σ is nowhere dense in $\overline{Y}$ (and possibly nonempty). Indeed local L^2 extensions $\widetilde{f}_j$ still exist by step i) applied on small coordinate balls U_j ; we then set $F_\infty = \sum \theta_j \widetilde{f}_j$ and observe that $|D''F_\infty|^2|s|^{-2r}$ is locally integrable, thanks to the estimate $\int_{U_j} |\widetilde{f}_j|^2|s|^{-2r}(\log|s|)^{-2}dV < +\infty$ and the fact that $|\sum d''\theta_j \wedge \widetilde{f}_j| = O(|s|^\delta)$ for suitable $\delta > 0$ [as follows from Hilbert's Nullstensatz applied to $\widetilde{f}_j - \widetilde{f}_k$ at singular points of $\overline{Y}$].

When $q \geqslant 1$, the arguments needed to get a smooth solution involve more delicate considerations, and we will only sketch the details. The main difficulty lies in the presence of the weight $|s|^{-2r}$, which creates trouble at points of Y when one tries to use an elliptic regularity argument (see Remark 8.5). If $r = 1$, however, the subvariety Y is a divisor; therefore, when we consider a D'' equation with values in the line bundle $\Lambda^n T^\star_X \otimes L$, a L^2 solution for the weight $|s|^{-2}$ can be interpreted as a L^2 solution with values in the twisted line bundle $\Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X(-Y)$, equipped with a smooth hermitian metric. Hence, if $r = 1$, the minimal L^2 solution $u_{c,\varepsilon}$ of the D'' equation considered earlier satisfies the equations

$$D''u_{c,\varepsilon} = g_\varepsilon = D''F_\varepsilon, \qquad D''^\star(|s|^{-2}u_{c,\varepsilon}) = 0 \quad \text{(minimality condition)}$$

on $X_c \smallsetminus Y_c$. These equations can be rewritten as

$$D''(s^{-1}u_{c,\varepsilon}) = s^{-1}D''F_\varepsilon, \qquad D''^\star(s^{-1}u_{c,\varepsilon}) = 0.$$

By Lemma 11.10, the latter equalities are valid on X_c and not only on $X_c \smallsetminus Y_c$, for $s^{-1}u_{c,\varepsilon}$ is locally L^2 and $s^{-1}D''F_\varepsilon$ is locally bounded. From this, we infer that $F_{c,\varepsilon} = F_\varepsilon - u_{c,\varepsilon}$ satisfies

$$\begin{aligned} D''(s^{-1}F_{c,\varepsilon}) &= D''(s^{-1}F_\varepsilon) - s^{-1}D''F_\varepsilon, \\ D''^\star(s^{-1}F_{c,\varepsilon}) &= D''^\star(s^{-1}F_\varepsilon) = D''^\star\big(\theta(\varepsilon^{-2}|s|^2)s^{-1}F_\infty\big). \end{aligned}$$

It is easy to show that $D''(s^{-1}\widetilde{f}) - s^{-1}D''\widetilde{f}$ is independent of the choice of the smooth extension $\widetilde{f}$ of f (whether $\widetilde{f}$ is D''-closed or not is irrelevant), and that it is equal to the current $D''(s^{-1}) \wedge \widetilde{f}$ with support in Y. On the other hand, $s^{-1}F_\infty$ is locally integrable, hence $\theta(\varepsilon^{-2}|s|^2)s^{-1}F_\infty$ converges weakly to 0 as $\varepsilon \to 0$. By extracting a weak limit $F_{c,\varepsilon} \to F$ in $L^2_{\rm loc}((|s|\log|s|)^{-2})$, we easily see that $s^{-1}F_{c,\varepsilon} \to s^{-1}F$ in the weak topology of distributions, therefore

$$D''(s^{-1}F) = D''(s^{-1}) \wedge \widetilde{f}, \qquad D''^\star(s^{-1}F) = 0$$

in the limit. In particular $s^{-1}F$ is Δ''-harmonic on $X \smallsetminus Y$, hence F is smooth on $X \smallsetminus Y$. Unfortunately, the above equations do not imply smoothness of the coefficients of F all over X, but only Hölder continuity near Y (for any Hölder exponent $\gamma < 1$). In fact, we can always choose a smooth local extension $\widetilde{f}$ such that $D''\widetilde{f}=0$ and $\nabla^{0,1}\overline{s}\,\lrcorner\,\widetilde{f}=0$ on Y (if the second condition is not satisfied, we replace $\widetilde{f}$ with $\widetilde{f}-D''(\overline{s}\,h)$, where h is a suitable smooth $(n,q-1)$-form on X ; the values taken by $\widetilde{f}$ on Y are then uniquely defined). We find

$$D''(s^{-1}(F-\widetilde{f}))=0,\qquad D''^\star(s^{-1}(F-\widetilde{f}))=-D''^\star(s^{-1}\widetilde{f}),$$

and the condition $\nabla^{0,1}\overline{s}\,\lrcorner\,\widetilde{f}=0$ on Y shows that the singularity of $D''^\star(s^{-1}\widetilde{f})$ along Y is at most $O(|s|^{-1})$. Our equations yield

$$\Delta''(s^{-1}(F-\widetilde{f}))=-D''D''^\star(s^{-1}\widetilde{f}),$$

hence

$$s^{-1}(F-\widetilde{f})=G^{n,q}\big(D''D''^\star(s^{-1}\widetilde{f})\big)=D''G^{n,q-1}\big(D''^\star(s^{-1}\widetilde{f})\big)\qquad \operatorname{mod} C^\infty$$

where $G^{p,q}$ is a (local) Green kernel for the Δ'' operator in bidegree (p,q). As the derivatives of order 1 of $G^{n,q-1}$ have singularity $|x-y|^{-(2n-1)}$ along the diagonal and $D''^\star(s^{-1}\widetilde{f})=O(|s|^{-1})$, we find $s^{-1}(F-\widetilde{f})=O(\log|s|)$ and the Hölder continuity of $F-\widetilde{f}$ (hence of F) follows, as well as the fact that $F_{\restriction Y}=\widetilde{f}_{\restriction Y}=f$. We claim that F can be corrected so as to obtain a smooth extension $\widetilde{F}$ with $|F-\widetilde{F}|$ small and decaying as rapidly as we wish at infinity; hence $\widetilde{F}$ will satisfy the desired global L^2 estimate. Indeed, there is a covering of Y by open sets U_j in X such that f admits a smooth D''-closed extension $\widetilde{f}_j$ on U_j, with the following additional properties: $\nabla^{0,1}\overline{s}\,\lrcorner\,\widetilde{f}_j=0$ on $Y\cap U_j$, and $s^{-1}(\widetilde{f}_j-\widetilde{f}_k)$ is smooth on $U_j\cap U_k$. Only the latter property needs to be checked. We show by induction on $\ell\geqslant 1$ that $\widetilde{f}_j$ can be chosen so that

$$\widetilde{f}_k-\widetilde{f}_j=s\,v_{jk}+\overline{s}^\ell w_{jk}$$

with suitable smooth (n,q)-forms v_{jk} and w_{jk}. This is true when $\ell=1$, since by uniqueness $\widetilde{f}_k-\widetilde{f}_j$ must vanish on $Y\cap U_j$. Now, the D''-closedness implies

$$0=s\,D''v_{jk}+\overline{s}^\ell D''w_{jk}+\ell\overline{s}^{\ell-1}\overline{D's}\wedge w_{jk},$$

and an identification of the coefficients of the Taylor expansion in s, $\overline{s}$ shows that $\overline{D's}\wedge w_{jk}=0$ on $Y\cap U_j\cap U_k$. This implies

$$w_{jk}=\overline{D's}\wedge w^{(1)}_{jk}+s\,w^{(2)}_{jk}+\overline{s}\,w^{(3)}_{jk}$$

with smooth forms $w^{(1)}_{jk}$, $w^{(2)}_{jk}$, $w^{(3)}_{jk}$. The $(n,q-1)$ form $w^{(1)}_{jk}$ is uniquely defined if we require the additional condition $\nabla^{0,1}\overline{s}\,\lrcorner\,w^{(1)}_{jk}=0$. Then $(w^{(1)}_{jk})$ satisfies the Čech cocycle condition and we can write $w^{(1)}_{jk}=w^{(1)}_k-w^{(1)}_j$ for some 0-cochain $(w^{(1)}_j)$. We conclude from these relations that $\widetilde{f}_j-(\ell+1)^{-1}D''(\overline{s}^{\ell+1}w^{(1)}_j)$ admits a Čech 1-coboundary

$$\begin{aligned}\widetilde{f}_k-\widetilde{f}_j-(\ell+1)^{-1}D''(\overline{s}^{\ell+1}w^{(1)}_{jk})&=\widetilde{f}_k-\widetilde{f}_j-\overline{s}^\ell\overline{D's}\wedge w^{(1)}_{jk}\quad \operatorname{mod}(\overline{s}^{\ell+1})\\&=s(v_{jk}+\overline{s}^\ell w^{(2)}_{jk})\quad \operatorname{mod}(\overline{s}^{\ell+1}),\end{aligned}$$

hence it satisfies the induction hypothesis at order $\ell+1$. By arranging the asympotic expansion up to infinite order, we infer that $\widetilde{f}_k-\widetilde{f}_j-s\,v_{jk}$ is flat along Y, hence that

$s^{-1}(\widetilde{f}_k - \widetilde{f}_j)$ is smooth. Our claim is thus proved. Now, the Green kernel argument shows that F can be written as $F = \widetilde{f}_j + sD''h_j$ on U_j, where

$$h_j = G^{n,q-1}(D''^\star(s^{-1}\widetilde{f}_j)) \qquad \mathrm{mod}\, C^\infty$$

is smooth on $U_j \smallsetminus Y$ and has its first order derivatives bounded by $O(\log|s|)$ near Y. Furthermore, $h_k - h_j$ is smooth on $U_j \cap U_k$. Therefore, if we select sufficiently good approximations $h_j \star \rho_{\varepsilon_j}$ of h_j and a collection of smooth functions (θ_j) with $\mathrm{Supp}\,\theta_j \subset U_j$, $0 \leqslant \theta_j \leqslant 1$ and $\sum \theta_j = 1$ near Y, the (n,q)-form

$$\widetilde{F} = F - D''\Big(s\sum_j \theta_j(h_j - h_j \star \rho_{\varepsilon_j})\Big)$$

is smooth and satisfies all our requirements.

When $r > 1$, the above argument can no longer be applied directly; one possibility to overcome this difficulty is to blow-up Y so as to deal again with the case of a divisor. We may assume that $\Sigma = \emptyset$ (otherwise, we just replace X_c with $X_c \smallsetminus \Sigma$, which is again complete Kähler). Instead of working on $X_c \smallsetminus Y_c$ as we did earlier, we work on the blow-up $\widehat{X}_c$ of X_c along Y_c. If $\mu : \widehat{X}_c \to X_c$ is the blow-up map, $\widehat{Y}_c = \mu^{-1}(Y_c)$ the exceptional divisor and γ a positive constant, we equip $\widehat{X}_c$ with the smooth Kähler metric $\widehat{\omega}_\gamma = \mu^\star\omega + \gamma\,(\mathrm{i}d'd''\log|s|^2 + \frac{\mathrm{i}}{r}\Theta(L)) \geqslant \mu^\star\omega$. Then the minimal $L^2(\omega_\gamma)$ solution $u_{c,\varepsilon,\gamma}$ satisfies the equations

$$D''u_{c,\varepsilon,\gamma} = \mu^\star g_\varepsilon = D''(\mu^\star F_\varepsilon), \qquad D''^\star_{\omega_\gamma}(|s|^{-2r}u_{c,\varepsilon,\gamma}) = 0$$

on $\widehat{X}_c \smallsetminus \widehat{Y}_c$, and $\widehat{F}_{c,\varepsilon,\gamma} = F_\varepsilon - u_{c,\varepsilon,\gamma}$ satisfies the L^2 estimate

$$\int_{\widehat{X}_c} \frac{|\widehat{F}_{c,\varepsilon,\gamma}|^2}{(|\widehat{s}|^2+\varepsilon^2)^r(-\log(|\widehat{s}|^2+\varepsilon^2))^2} dV_{\widehat{X}_c,\omega_\gamma} \leqslant 8\,c_r \int_{Y_c} \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega} + \frac{\mathrm{Const}}{(\log\varepsilon)^2}$$

where $\widehat{s} = s \circ \mu$ (one can use monotonicity with respect to metrics and the fact that $\omega_\gamma \geqslant \mu^\star\omega$ to see that the right hand side always admits the ω-bound as an upper bound). We can view X_c as a submanifold of the projectivized bundle $P(E)$ of lines of E, and $\mathcal{O}_{\widehat{X}_c}(-\widehat{Y}_c)$ as the restriction to X_c of the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ on $P(E)$. We thus view $\widehat{s}$ as a section of $\mathcal{O}_{\widehat{Y}_c}(-\widehat{Y}_c)$ (actually, $\widehat{s}$ is a generator of that ideal sheaf). Since $|\widehat{s}|^{-2r}|u_{c,\gamma,\varepsilon}|^2$ is locally integrable by construction, we get

$$D''(\widehat{s}^{\,-r}u_{c,\varepsilon,\gamma}) = \widehat{s}^{\,-r}D''(\mu^\star F_\varepsilon), \qquad D''^\star_{\omega_\gamma}(\widehat{s}^{\,-r}u_{c,\varepsilon,\gamma}) = 0$$

on $\widehat{X}_c$. Passing to the limit as ε, γ tend to 0 and c tends to $+\infty$, we find a (n,q)-form $\widehat{F}$ with L^2_{loc} coefficients on $\widehat{X}$ such that

$$\int_{\widehat{X}} \frac{|\widehat{F}|^2}{|\widehat{s}|^{2r}(-\log|\widehat{s}|)^2} dV_{\widehat{X},\omega} \leqslant 8\,c_r \int_{Y_c} \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}$$

and

$$D''(\widehat{s}^{\,-r}\widehat{F}) = D''(\widehat{s}^{\,-r}\mu^\star\widetilde{f}) - \widehat{s}^{\,-r}D''(\mu^\star\widetilde{f}) = D''(\widehat{s}^{\,-1}) \wedge (\widehat{s}^{\,-(r-1)}\mu^\star\widetilde{f}),$$
$$D''^\star_{\mu^\star\omega}(\widehat{s}^{\,-r}\widehat{F}) = 0$$

in the sense of distributions (everywhere on $\widehat{X}$ and not only on $\widehat{X} \smallsetminus \widehat{Y}$); indeed, thanks to the equality

$$\mu^\star(\Lambda^n T^\star_{X_c}) = \Lambda^n T^\star_{\widehat{X}_c} \otimes \mathcal{O}_{\widehat{X}_c}(-(r-1)\widehat{Y}_c),$$

we see that $\mu^\star\widetilde{f}$ vanishes at order $r-1$ along $\widehat{Y}$. If we view $\widehat{F}$ as a $(0,q)$-form with values in $\mu^\star(\Lambda^n T^\star_X \otimes L)$ rather than as a (n,q)-form on $\widehat{X}$, we may consider philosophically that we cancel out a factor $\widehat{s}^{\,r-1}$ in the equations. This shows that we are essentially in the same situation as in the earlier case $r=1$, except that $D''^\star_{\mu^\star\omega}$ is computed with respect to a metric $\mu^\star\omega$ which is degenerate along $\widehat{Y}$. A finer analysis of the Green kernel of $\Delta''_{\mu^\star\omega}$ shows that $F := \mu_\star F$ is smooth on $X \smallsetminus Y$, that F still has continuous coefficients near Y, and that $F_{\restriction Y} = f$. We then produce the desired solution by taking a small perturbation of F as above. The details are rather tedious and will be left to the reader. □

(13.7) Remarks.

a) When $q=0$, the estimates provided by Theorem 13.6 are independent of the Kähler metric ω. In fact, if f and F are holomorphic sections of $\Lambda^n T^\star_X \otimes L$ over Y (resp. X), viewed as $(n,0)$-forms with values in L, we can "divide" f by $\Lambda^r(ds) \in \Lambda^r(TX/TY)^\star \otimes \det E$ to get a section $f/\Lambda^r(ds)$ of $\Lambda^p T^\star_Y \otimes L \otimes (\det E)^{-1}$ over Y. We then find

$$|F|^2 dV_{X,\omega} = \mathrm{i}^{n^2}\{F,F\},$$

$$\frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega} = \mathrm{i}^{p^2}\{f/\Lambda^r(ds), f/\Lambda^r(ds)\},$$

where $\{\bullet,\bullet\}$ is the canonical bilinear pairing described in (6.3).

b) The hermitian structure on E is not really used in depth. In fact, one only needs E to be equipped with a *Finsler metric*, that is, a smooth complex homogeneous function of degree 2 on E [or equivalently, a smooth hermitian metric on the tautological bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $P(E)$]. The section s of E induces a section $[s]$ of $P(E)$ over $X \smallsetminus s^{-1}(0)$ and a corresponding section $\widetilde{s}$ of the pull-back line bundle $[s]^\star\mathcal{O}_{P(E)}(-1)$. A trivial check shows that Theorem 13.6 as well as its proof extend to the case of a Finsler metric on E, if we replace everywhere $\{\mathrm{i}\Theta(E)s, s\}$ by $\{\mathrm{i}\Theta([s]^\star\mathcal{O}_{P(E)}(-1))\widetilde{s}, \widetilde{s}\}$ (especially in hypothesis 13.6 b)). A minor issue is that $|\Lambda^r(ds)|$ is (a priori) no longer defined, since no obvious hermitian norm exists on $\det E$. A posteriori, we have the following ad hoc definition of a metric on $(\det E)^\star$ which makes the L^2 estimates work as before: for $x \in X$ and $\xi \in \Lambda^r E^\star_x$, we set

$$|\xi|^2_x = \frac{1}{c_r}\int_{z\in E_x} (1+|z|^2)^2\theta'(|z|^2)^2 \, \frac{\mathrm{i}^{r^2}\xi\wedge\overline{\xi}}{|z|^{2r}}$$

where $|z|$ is the Finsler norm on E_x [the constant c_r is there to make the result agree with the hermitian case; it is not hard to see that this metric does not depend on the choice of θ]. □

We now present a few interesting corollaries. The first one is a surjectivity theorem for restriction morphisms in Dolbeault cohomology.

(13.8) Corollary. *Let X be a projective algebraic manifold and E a holomorphic vector bundle of rank r over X, s a holomorphic section of E which is everywhere transverse to the zero section, $Y = s^{-1}(0)$, and let L be a holomorphic line bundle such that $F = L^{1/r} \otimes E^\star$ is Griffiths positive (we just mean formally that $\frac{1}{r}\mathrm{i}\Theta(L) \otimes \mathrm{Id}_E - \mathrm{i}\Theta(E) >_{\mathrm{Grif}} 0$). Then the restriction morphism*

$$H^{0,q}(X, \Lambda^n T_X^\star \otimes L) \to H^{0,q}(Y, \Lambda^n T_X^\star \otimes L)$$

is surjective for every $q \geqslant 0$.

Proof. A short computation gives

$$\begin{aligned}\mathrm{i}\, d'd'' \log |s|^2 &= \mathrm{i}\, d'\Big(\frac{\{s, D's\}}{|s|^2}\Big)\\ &= \mathrm{i}\Big(\frac{\{D's, D's\}}{|s|^2} - \frac{\{D's, s\} \wedge \{s, D's\}}{|s|^4} + \frac{\{s, \Theta(E)s\}}{|s|^2}\Big) \geqslant -\frac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}\end{aligned}$$

thanks to Lagrange's inequality and the fact that $\Theta(E)$ is antisymmetric. Hence, if δ is a small positive constant such that

$$-\mathrm{i}\Theta(E) + \frac{1}{r}\mathrm{i}\Theta(L) \otimes \mathrm{Id}_E \geqslant_{\mathrm{Grif}} \delta\, \omega \otimes \mathrm{Id}_E > 0,$$

we find

$$\mathrm{i}\Theta(L) + r\, \mathrm{i}\, d'd'' \log |s|^2 \geqslant r\delta\, \omega.$$

The compactness of X implies $\mathrm{i}\Theta(E) \leqslant C\omega \otimes \mathrm{Id}_E$ for some $C > 0$. Theorem 13.6 can thus be applied with $\alpha = r\delta/C$ and Corollary 13.8 follows. By remark 13.7 b), the above surjectivity property even holds if $L^{1/r} \otimes E^\star$ is just assumed to be ample (in the sense that the associated line bundle $\pi^\star L^{1/r} \otimes \mathcal{O}_{P(E)}(1)$ is positive on the projectivized bundle $\pi : P(E) \to X$ of lines of E). □

Another interesting corollary is the following special case, dealing with bounded pseudoconvex domains $\Omega \Subset \mathbb{C}^n$. Even this simple version retains highly interesting information on the behavior of holomorphic and plurisubharmonic functions.

(13.9) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \leqslant e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-\varphi} dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log |s|)^2} e^{-\varphi} dV_\Omega \leqslant C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-\varphi} dV_Y.$$

Proof. We apply essentially the same idea as for the previous corollary, in the special case when $L = \Omega \times \mathbb{C}$ is the trivial bundle equipped with a weight function $e^{-\varphi - A|z|^2}$.

The choice of a sufficiently large constant $A > 0$ guarantees that the curvature assumption 13.6 a) is satisfied (A just depends on the presupposed bound for the curvature tensor of E). □

(13.10) Remark. The special case when $Y = \{z_0\}$ is a point is especially interesting. In that case, we just take $s(z) = (e \operatorname{diam} \Omega)^{-1}(z - z_0)$, viewed as a section of the rank $r = n$ trivial vector bundle $\Omega \times \mathbb{C}^n$ with $|s| \leqslant e^{-1}$. We take $\alpha = 1$ and replace $|s|^{2n}(-\log|s|)^2$ in the denominator by $|s|^{2(n-\varepsilon)}$, using the inequality

$$-\log|s| = \frac{1}{\varepsilon}\log|s|^{-\varepsilon} \leqslant \frac{1}{\varepsilon}|s|^{-\varepsilon}, \qquad \forall \varepsilon > 0.$$

For any given value f_0, we then find a holomorphic function f such that $f(z_0) = f_0$ and

$$\int_\Omega \frac{|f(z)|^2}{|z - z_0|^{2(n-\varepsilon)}} e^{-\varphi(z)} dV_\Omega \leqslant \frac{C_n}{\varepsilon^2(\operatorname{diam}\Omega)^{2(n-\varepsilon)}}|f_0|^2 e^{-\varphi(z_0)}.$$

14. Approximation of psh functions by logarithms of holomorphic functions

We prove here, as an application of the Ohsawa-Takegoshi extension theorem, that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by functions of the form $c\log|f|$, where $c > 0$ and f is a holomorphic function. The main idea is taken from (Demailly 1992). For other applications to algebraic geometry, see (Demailly 1993b) and (Demailly-Kollár 1996). Recall that the Lelong number of a function $\varphi \in \operatorname{Psh}(\Omega)$ at a point x_0 is defined to be

$$\nu(\varphi, x_0) = \liminf_{z \to x_0} \frac{\log\varphi(z)}{\log|z - x_0|} = \lim_{r \to 0_+} \frac{\sup_{B(x_0,r)}\varphi}{\log r}.$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\operatorname{ord}_{x_0}(f) = \sup\{k \in \mathbb{N}\,;\, D^\alpha f(x_0) = 0,\ \forall|\alpha| < k\}$.

(14.1) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m}\log\sum|\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

a) $\varphi(z) - \dfrac{C_1}{m} \leqslant \varphi_m(z) \leqslant \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m}\log\dfrac{C_2}{r^n}$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

b) $\nu(\varphi, z) - \dfrac{n}{m} \leqslant \nu(\varphi_m, z) \leqslant \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. Note that $\sum |\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f \in B(1)} \frac{1}{m} \log |f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned} |f(z)|^2 &\leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{|\zeta - z| < r} |f(\zeta)|^2 d\lambda(\zeta) \\ &\leqslant \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|\zeta - z| < r} \varphi(\zeta)\Big) \int_\Omega |f|^2 e^{-2m\varphi} d\lambda. \end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \leqslant \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem (estimate 13.10) applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam} \Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \geqslant \frac{1}{m} \log |a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x - z| < r} \varphi_m(x) \leqslant \sup_{|\zeta - z| < 2r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}. \qquad \square$$

Theorem 14.1 implies in a straighforward manner a deep result of (Siu 1974) on the analyticity of the Lelong number sublevel sets.

(14.2) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number sublevel set*

$$E_c(\varphi) = \{z \in X\,;\ \nu(\varphi, z) \geqslant c\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in 14.1 b) imply that

$$E_c(\varphi) = \bigcap_{m \geqslant m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

15. Nadel vanishing theorem

In this final section, we prove a vanishing theorem due to (Nadel 1989), which has found recently many deep and important applications in complex differential geometry and in algebraic geometry. It contains as a special case the well-known Kawamata-Viehweg vanishing theorem (Kawamata 1982, Viehweg 1982), which can be seen as an algebraic version of the general vanishing theorem [here, the reader is assumed to have some knowledge of sheaf theory, namely coherent analytic sheaves, sheaf cohomology, etc]. We first introduce the concept of *multiplier ideal sheaf*, following (Nadel 1989). The main idea actually goes back to the fundamental works of (Bombieri 1970) and (Skoda 1972a).

(15.1) Definition. *Let φ be a psh function on an open subset $\Omega \subset X$; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.*

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is non integrable. Of course, such points occur only if φ has logarithmic poles. This is made precise as follows.

(15.2) Definition. *A psh function φ is said to have a logarithmic pole of coefficient γ at a point $x \in X$ if the Lelong number*

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$.

(15.3) Lemma (Skoda 1972a)**.** *Let φ be a psh function on an open set Ω and let $x \in \Omega$.*

a) *If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*

b) *If $\nu(\varphi, x) \geqslant n + s$ for some integer $s \geqslant 0$, then $e^{-2\varphi} \geqslant C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{\Omega,x}^{s+1}$, where $\mathfrak{m}_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

c) *The zero variety $V(\mathcal{I}(\varphi))$ of $\mathcal{I}(\varphi)$ satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X\,;\, \nu(\varphi, x) \geqslant c\}$ is the c-sublevel set of Lelong numbers of φ.

Proof. a) We use the following well-known facts about Lelong numbers, see (Lelong 1968) or (Demailly 1993a): set $d^c = \frac{1}{2\mathrm{i}\pi}(d' - d'')$, so that $dd^c = \frac{\mathrm{i}}{\pi} d'd''$, and put $\Theta = dd^c\varphi$; we then have the equality

$$\nu(\Theta, x, r) = \frac{1}{2^{n-1}r^{2n-2}} \int_{B(x,r)} \Theta \wedge (dd^c|z|^2)^{n-1} = \int_{B(x,r)} \Theta \wedge (dd^c \log|z-x|)^{n-1}\,;$$

it follows from the last integral that $\nu(\Theta, x, r)$ is an increasing function of r ; the limit $\nu(\Theta, x) := \lim_{r\to 0_+} \nu(\Theta, x, r)$ is equal to $\nu(\varphi, x)$. Now, let χ be a cut-off function will support in a small ball $B(x,r)$, equal to 1 in $B(x, r/2)$. As $(dd^c \log|z|)^n = \delta_0$, we get

$$\begin{aligned}\varphi(z) &= \int_{B(x,r)} \chi(\zeta)\varphi(\zeta)(dd^c \log|\zeta - z|)^n \\ &= \int_{B(x,r)} dd^c(\chi(\zeta)\varphi(\zeta)) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1}\end{aligned}$$

for $z \in B(x, r/2)$. Expanding $dd^c(\chi\varphi)$ and observing that $d\chi = dd^c\chi = 0$ on $B(x, r/2)$, we find

$$\varphi(z) = \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1} + \text{smooth terms}$$

on $B(x, r/2)$. Fix r so small that

$$\int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - x|)^{n-1} \leqslant \nu(\Theta, x, r) < 1.$$

By continuity, there exists $\delta, \varepsilon > 0$ such that

$$I(z) := \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1} \leqslant 1 - \delta$$

for all $z \in B(x, \varepsilon)$. Applying Jensen's convexity inequality to the probability measure

$$d\mu_z(\zeta) = I(z)^{-1}\chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1},$$

we find

$$\begin{aligned}-\varphi(z) &= \int_{B(x,r)} I(z) \log|\zeta - z|^{-1}\, d\mu_z(\zeta) + O(1) \quad\Longrightarrow \\ e^{-2\varphi(z)} &\leqslant C \int_{B(x,r)} |\zeta - z|^{-2I(z)}\, d\mu_z(\zeta).\end{aligned}$$

As

$$d\mu_z(\zeta) \leqslant C_1|\zeta - z|^{-(2n-2)}\Theta(\zeta) \wedge (dd^c|\zeta|^2)^{n-1} = C_2|\zeta - z|^{-(2n-2)}d\sigma_\Theta(\zeta),$$

we get

$$e^{-2\varphi(z)} \leqslant C_3 \int_{B(x,r)} |\zeta - z|^{-2(1-\delta)-(2n-2)}d\sigma_\Theta(\zeta),$$

and the Fubini theorem implies that $e^{-2\varphi(z)}$ is integrable on a neighborhood of x.

b) If $\nu(\varphi, x) = \gamma$, the convexity properties of psh functions, namely, the convexity of $\log r \mapsto \sup_{|z-x|=r} \varphi(z)$ implies that

$$\varphi(z) \leqslant \gamma \log |z - x|/r_0 + M,$$

where M is the supremum on $B(x, r_0)$. Hence there exists a constant $C > 0$ such that $e^{-2\varphi(z)} \geqslant C|z - x|^{-2\gamma}$ in a neighborhood of x. The desired result follows from the identity

$$\int_{B(0,r_0)} \frac{\left|\sum a_\alpha z^\alpha\right|^2}{|z|^{2\gamma}} dV(z) = \mathrm{Const} \int_0^{r_0} \Big(\sum |a_\alpha|^2 r^{2|\alpha|}\Big) r^{2n-1-2\gamma}\, dr,$$

which is an easy consequence of Parseval's formula. In fact, if γ has integral part $[\gamma] = n + s$, the integral converges if and only if $a_\alpha = 0$ for $|\alpha| \leqslant s$.

c) is just a simple formal consequence of a) and b). □

(15.4) Proposition (Nadel 1989). *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω.*

Proof. Since the result is local, we may assume that Ω is the unit ball in $\mathbb{C}^n$. Let E be the set of all holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi}\, d\lambda < +\infty$. By the strong noetherian property of coherent sheaves, the set E generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $d''u = g := d''(\theta f)$ by means of Hörmander's L^2 estimates 8.9, where E is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\widetilde{\varphi}(z) = \varphi(z) + (n + s)\log|z - x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z - x|^{-2(n+s)} d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in E$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves our contention. □

The multiplier ideal sheaves satisfy the following basic fonctoriality property with respect to direct images of sheaves by modifications.

(15.5) Proposition. *Let $\mu : X' \to X$ be a modification of non singular complex manifolds (i.e. a proper generically 1:1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star\big(\mathcal{O}(K_{X'})\otimes\mathcal{I}(\varphi\circ\mu)\big)=\mathcal{O}(K_X)\otimes\mathcal{I}(\varphi).$$

Proof. Let $n=\dim X=\dim X'$ and let $S\subset X$ be an analytic set such that $\mu : X'\smallsetminus S'\to X\smallsetminus S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X)\otimes\mathcal{I}(\varphi)$ is just the sheaf of holomorphic n-forms f on open sets $U\subset X$ such that $\mathrm{i}^{n^2}f\wedge\overline{f}\,e^{-2\varphi}\in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded from above, we may even consider forms f which are a priori defined only on $U\smallsetminus S$, because f will be in $L^2_{\mathrm{loc}}(U)$ and therefore will automatically extend through S. The change of variable formula yields

$$\int_U \mathrm{i}^{n^2}f\wedge\overline{f}\,e^{-2\varphi}=\int_{\mu^{-1}(U)}\mathrm{i}^{n^2}\mu^\star f\wedge\overline{\mu^\star f}\,e^{-2\varphi\circ\mu},$$

hence $f\in\Gamma(U,\mathcal{O}(K_X)\otimes\mathcal{I}(\varphi))$ if and only if $\mu^\star f\in\Gamma(\mu^{-1}(U),\mathcal{O}(K_{X'})\otimes\mathcal{I}(\varphi\circ\mu))$. Proposition 15.5 is proved. □

(15.6) Remark. If φ has "analytic singularities" the computation of $\mathcal{I}(\varphi)$ can be reduced to a purely algebraic problem.

The first observation is that $\mathcal{I}(\varphi)$ can be computed easily if φ has the form $\varphi=\sum\alpha_j\log|g_j|$ where $D_j=g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings. Then $\mathcal{I}(\varphi)$ is the sheaf of functions h on open sets $U\subset X$ such that

$$\int_U |h|^2\prod|g_j|^{-2\alpha_j}dV<+\infty.$$

Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1,\ldots,z_n)$, the condition is that h is divisible by $\prod g_j^{m_j}$ where $m_j-\alpha_j>-1$ for each j, i.e. $m_j\geqslant\lfloor\alpha_j\rfloor$ (integer part). Hence

$$\mathcal{I}(\varphi)=\mathcal{O}(-\lfloor D\rfloor)=\mathcal{O}(-\sum\lfloor\alpha_j\rfloor D_j)$$

where $\lfloor D\rfloor$ denotes the integral part of the $\mathbb{Q}$-divisor $D=\sum\alpha_jD_j$.

Now, consider the general case of analytic singularities, i.e., the case of a psh function such that

$$\varphi=\frac{\alpha}{2}\log\big(|f_1|^2+\cdots+|f_N|^2\big)+O(1)$$

near the poles. Let $\mathcal{J}$ be the (coherent) integrally closed ideal sheaf of holomorphic functions h such that $|h|\leqslant C\exp(\varphi/\alpha)$. In this case, the computation is made as follows. First, one computes a smooth modification $\mu:\widetilde{X}\to X$ of X such that $\mu^\star\mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D=\sum\lambda_jD_j$, where (D_j) are the components of the exceptional divisor of $\widetilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow up again by the Hironaka desingularization theorem (Hironaka 1964) to make X' smooth and D' have normal crossings). Now, we have $K_{\widetilde{X}}=\mu^\star K_X+R$ where $R=\sum\rho_jD_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By the direct image formula 15.5, we get

$$\mathcal{I}(\varphi)=\mu_\star\big(\mathcal{O}(K_{\widetilde{X}}-\mu^\star K_X)\otimes\mathcal{I}(\varphi\circ\mu)\big)=\mu_\star\big(\mathcal{O}(R)\otimes\mathcal{I}(\varphi\circ\mu)\big).$$

Now, $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi \circ \mu \sim \alpha \sum \lambda_j \log |g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha\lambda_j D_j$. We obtain $\mathcal{I}(\varphi \circ \mu) = \mathcal{O}(-\sum \lfloor \alpha\lambda_j \rfloor D_j)$, hence

$$\mathcal{I}(\varphi) = \mu_\star \mathcal{O}_{\widetilde{X}}\big(\sum (\rho_j - \lfloor \alpha\lambda_j \rfloor) D_j\big). \qquad \square$$

(15.7) Exercise. Compute the multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with $\varphi = \log(|z_1|^{\alpha_1} + \ldots + |z_p|^{\alpha_p})$ for arbitrary real numbers $\alpha_j > 0$.
Hint: using Parseval's formula and polar coordinates $z_j = r_j e^{\mathrm{i}\theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1, \ldots, \beta_p) \in \mathbb{N}^p$ the integral

$$\int_{[0,1]^p} \frac{r_1^{2\beta_1} \ldots r_p^{2\beta_p}\, r_1 dr_1 \ldots r_p dr_p}{r_1^{2\alpha_1} + \ldots + r_p^{2\alpha_p}} = \int_{[0,1]^p} \frac{t_1^{(\beta_1+1)/\alpha_1} \ldots t_p^{(\beta_p+1)/\alpha_p}}{t_1 + \ldots + t_p} \frac{dt_1}{t_1} \ldots \frac{dt_p}{t_p}$$

is convergent. Conclude from this that $\mathcal{I}(\varphi)$ is generated by the monomials $z_1^{\beta_1} \ldots z_p^{\beta_p}$ such that $\sum (\beta_p + 1)/\alpha_p > 1$. (This exercise shows that the analytic definition of $\mathcal{I}(\varphi)$ is sometimes also quite convenient for computations). $\square$

Let F be a line bundle over X with a singular metric h of curvature current $\Theta_h(F)$. If $e^{-2\varphi}$ is the weight representing the metric in an open set $\Omega \subset X$, the ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf $\mathcal{I}(h)$ on X. We will sometimes still write $\mathcal{I}(h) = \mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results of analytic and algebraic geometry (especially, it contains the Kawamata-Viehweg vanishing theorem as a special case).

(15.8) Nadel vanishing theorem (Nadel 1989, Demailly 1993b). *Let (X, ω) be a Kähler weakly pseudoconvex manifold, and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight $e^{-2\varphi}$. Assume that $\mathrm{i}\Theta_h(L) \geqslant \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(h)\big) = 0 \qquad \textit{for all } q \geqslant 1.$$

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of (n, q)-forms u with values in L and with measurable coefficients, such that both $|u|^2 e^{-2\varphi}$ and $|d''u|^2 e^{-2\varphi}$ are locally integrable. The d'' operator defines a complex of sheaves $(\mathcal{L}^\bullet, d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(\varphi)$: indeed, the kernel of d'' in degree 0 consists of all germs of holomorphic n-forms with values in L which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q \geqslant 1$ follows from Corollary 8.9 applied on arbitrary small balls. Each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, so $\mathcal{L}^\bullet$

is a resolution by acyclic sheaves. Let ψ be a smooth psh exhaustion function on X. Let us apply Corollary 8.9 globally on X, with the original metric of L multiplied by the factor $e^{-\chi\circ\psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor can be used to ensure the convergence of integrals at infinity. By Corollary 8.9, we conclude that $H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) = 0$ for $q \geqslant 1$. The theorem follows. $\square$

(15.9) Corollary. *Let (X,ω), L and φ be as in Theorem 15.8 and let $x_1,\ldots,x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X, K_X\otimes L) \longrightarrow\!\!\!\!\rightarrow \bigoplus_{1\leqslant j\leqslant N} \mathcal{O}(K_X\otimes L)_{x_j}\otimes\big(\mathcal{O}_X/\mathcal{I}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact sequence of cohomology associated to the short exact sequence $0\to\mathcal{I}(\varphi)\to\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}(\varphi)\to 0$ twisted by $\mathcal{O}(K_X\otimes L)$, and apply Theorem 15.8 to obtain the vanishing of the first H^1 group. The asserted surjectivity property follows. $\square$

(15.10) Corollary. *Let (X,ω), L and φ be as in Theorem 15.8 and suppose that the weight function φ is such that $\nu(\varphi,x)\geqslant n+s$ at some point $x\in X$ which is an isolated point of $E_1(\varphi)$. Then $H^0(X,K_X\otimes L)$ generates all s-jets at x.*

Proof. The assumption is that $\nu(\varphi,y)<1$ for y near x, $y\neq x$. By Skoda's lemma 15.3 b), we conclude that $e^{-2\varphi}$ is integrable at all such points y, hence $\mathcal{I}(\varphi)_y=\mathcal{O}_{X,y}$, whilst $\mathcal{I}(\varphi)_x\subset\mathfrak{m}_{X,x}^{s+1}$ by 15.3 a). Corollary 15.10 is thus a special case of 15.9. $\square$

The philosophy of these results (which can be seen as generalizations of the Hörmander-Bombieri-Skoda theorem (Bombieri 1970), (Skoda 1972a, 1975) is that the problem of constructing holomorphic sections of $K_X\otimes L$ can be solved by constructing suitable hermitian metrics on L such that the weight φ has isolated poles at given points x_j.

(15.11) Exercise. Assume that X is compact and that L is a positive line bundle on X. Let $\{x_1,\ldots,x_N\}$ be a finite set. Show that there are constants $a,b\geqslant 0$ depending only on L and N such that $H^0(X,L^{\otimes m})$ generates jets of any order s at all points x_j for $m\geqslant as+b$.
Hint: Apply Corollary 15.9 to $L'=K_X^{-1}\otimes L^{\otimes m}$, with a singular metric on L of the form $h=h_0e^{-\varepsilon\psi}$, where h_0 is smooth of positive curvature, $\varepsilon>0$ small and $\psi(z)\sim\log|z-x_j|$ in a neighborhood of x_j.

Recall that a line bundle L is said to be *very ample* if the sections of $H^0(X,L)$ generate any pair of $L_x\oplus L_y$ for distinct points $x\neq y$ in X, as well as 1-jets of L at any point $x\in X$. The line bundle L is said to be *ample* if some positive multiple $L^{\otimes m}$ is very ample. Then derive the Kodaira embedding theorem:

(15.12) Theorem (Kodaira 1954)**.** *If L is a line bundle on a compact complex manifold, then L is ample if and only if L is positive.*

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