Kobayashi-Lübke inequalities for Chern classes of Hermite-Einstein vector bundles and Guggenheimer-Yau-Bogomolov-Miyaoka inequalities for Chern classes of Kähler-Einstein manifolds

Let \((X,\omega)\) be a compact Kähler manifold, \(n = \dim X\), and let \(E\) be a holomorphic vector bundle over \(X\), \(r = \text{rank } E\). We suppose that \(E\) is equipped with a hermitian metric, \(h\) and denote by \(D_{E,h}\) the Chern connection on \((E,h)\). The Chern curvature form is

\[
\Theta_h(E) = D^2_{E,h}.
\]

In a (local) orthonormal frame \((e_\alpha)_{1 \leq \alpha \leq r}\) of \(E\), we write

\[
\Theta_h(E) = (\Theta_{\alpha\beta})_{1 \leq \alpha, \beta \leq r}
\]

where the \(\Theta_{\alpha\beta}\) are complex valued \((1,1)\)-forms satisfying the hermitian condition \(\overline{\Theta_{\alpha\beta}} = \Theta_{\beta\alpha}\). We denote

\[
\Theta_{\alpha\beta} = i \sum_{1 \leq \alpha, \beta \leq r, 1 \leq j, k \leq n} \Theta_{\alpha\beta jk} dz_j \wedge dz_k.
\]

The hermitian symmetry condition can then be read \(\overline{\Theta_{\alpha\beta jk}} = \Theta_{\beta\alpha kj}\). If at some point \(x_0 \in X\) the coordinates \((z_j)\) are chosen so that \((dz_j(x_0))\) is an orthonormal basis of \(T_{X,x_0}\), we define

\[
\text{Tr}_\omega \Theta_h(E) = \left( \sum_j \Theta_{\alpha\beta jj} \right) \in C^\infty(X,\text{hom}(E,E)).
\]

**Definition.** — The hermitian vector bundle \((E,h)\) is said to be Hermite-Einstein with respect to the Kähler metric \(\omega\) if there is a constant \(\lambda > 0\) such that \(\text{Tr}_\omega \Theta_h(E) = \lambda \text{Id}_E\).

Recall that the Chern forms \(c_k(E)_h\) are defined by the formula

\[
\det \left( I + t\Theta_h(E) \right) = \det(\delta_{\alpha\beta} + t\Theta_{\alpha\beta}) = 1 + tc_1(E)_h + \ldots + t^rc_r(E)_h.
\]

This gives in particular the identities

\[
c_1(E)_h = \sum_\alpha \Theta_{\alpha\alpha},
\]

\[
c_2(E)_h = \sum_{\alpha < \beta} \Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} - \Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha} = \frac{1}{2} \sum_{\alpha, \beta} \Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} - \Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha}.
\]

The trace \(\text{Tr}_\omega \Theta_h(E)\) can be computed by the formula

\[
\Theta_h(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = \text{Tr}_\omega \Theta_h(E) \frac{\omega^n}{n!}.
\]
By taking the trace with respect to the indices $\alpha$ in $E$ and taking the Hermite-Einstein equation into account, we find
\[
c_1(E)_h \wedge \frac{\omega^{n-1}}{(n-1)!} = \lambda r \frac{\omega^n}{n!}.
\]
This implies that the number $\lambda$ in the definition of Hermite-Einstein metrics is a purely numerical invariant, namely
\[
\lambda = \frac{n}{r} \int_X c_1(E) \wedge \omega^{n-1} / \int_X \omega^n.
\]

Kobayashi-Lübke inequality. — If $E$ admits a Hermite-Einstein metric $h$ with respect to $\omega$, then
\[
[(r - 1)c_1(E)^2_h - 2r c_2(E)_h] \wedge \omega^{n-2} \leq 0
\]
at every point of $X$. Moreover, the equality holds if and only if
\[
\Theta_h(E) = \frac{1}{r} c_1(E)_h \otimes \text{Id}_E.
\]
Observe that the equality holds pointwise already if we have the numerical equality
\[
\int_X [(r - 1)c_1(E)^2 - 2r c_2(E)] \wedge \omega^{n-2} = 0.
\]
If we introduce the (formal) vector bundle $\tilde{E} = E \otimes (\det E)^{-1/r}$ ($\tilde{E}$ is the “normalized” vector bundle such that $\det \tilde{E} = \mathcal{O}$), then $c_1(\tilde{E})_h = 0$ and
\[
\Theta_h(\tilde{E}) = (\Theta_h(E) - \frac{1}{r} c_1(E)_h \otimes \text{Id}_E) \otimes \text{Id}_{(\det E)^{-1/r}}.
\]
By the formula for the chern classes of $E \otimes L$, the Kobayashi-Lübke inequality can be rewritten as
\[
c_2(\tilde{E})_h \wedge \omega^{n-2} \leq 0,
\]
with equality if and only if $\tilde{E}$ is unitary flat. In that case, we say that $E$ is projectively flat.

Proof. By the above,
\[
(r - 1)c_1(E)_h^2 - 2r c_2(E)_h = \sum_{\alpha,\beta} -\Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} + r \Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha}.
\]
Taking the wedge product with $\omega^{n-2}/(n-2)!$ means taking the trace, i.e. the sum of coefficients of the terms $i d\bar{z}_j \wedge d\bar{z}_j \wedge i d\bar{z}_k \wedge d\bar{z}_k$ for all $j < k$. For this, we have to look at products of the type $(i d\bar{z}_j \wedge d\bar{z}_k) \wedge (i d\bar{z}_j \wedge d\bar{z}_k)$ or $(i d\bar{z}_j \wedge d\bar{z}_k) \wedge (i d\bar{z}_j \wedge d\bar{z}_j)$. This yields
\[
2[(r - 1)c_1(E)_h^2 - 2r c_2(E)_h] \wedge \frac{\omega^{n-2}}{(n-2)!} = \sum_{\alpha,\beta,j,k} - (\Theta_{\alpha\alpha} \Theta_{\beta\beta} \delta_{jk} - \Theta_{\alpha\alpha} \Theta_{\beta\beta} \delta_{jk}) + r (\Theta_{\alpha\beta} \Theta_{\beta\alpha} \delta_{jk} - \Theta_{\alpha\beta} \Theta_{\beta\alpha} \delta_{jk}).
\]
The initial factor 2 comes from the fact that the final sum is taken over all unordered indices $j, k$ (terms with $j = k$ cancel). The Hermite-Einstein condition yields $\sum_j \Theta_{\alpha \beta j j} = \lambda \delta_{\alpha \beta}$, so we get

$$\sum_{\alpha, \beta, j, k} -\Theta_{\alpha \alpha j j} \Theta_{\beta \beta k k} + r \Theta_{\alpha \beta j j} \Theta_{\beta \alpha k k} = \sum_{\alpha, \beta} -\lambda^2 \delta_{\alpha \alpha} \delta_{\beta \beta} + r \lambda^2 \delta_{\alpha \beta} \delta_{\beta \alpha}$$

$$= -r^2 \lambda^2 + r^2 \lambda^2 = 0.$$ 

Hence, using the hermitian symmetry of $\Theta_{\alpha \beta j k}$, we find

$$2 \left[ (r - 1) c_1(E) \right] \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$= \sum_{\alpha, \beta, j, k} \Theta_{\alpha \alpha j k} \Theta_{\beta \beta j k} - r \left| \Theta_{\alpha \beta j k} \right|^2$$

$$= -r \sum_{\alpha \neq \beta, j, k} \left| \Theta_{\alpha \beta j k} \right|^2 + \sum_{j, k} \left( \sum_{\alpha, \beta} \Theta_{\alpha \alpha j k} \Theta_{\beta \beta j k} - r \sum_{\alpha} \left| \Theta_{\alpha \alpha j k} \right|^2 \right)$$

$$= -r \sum_{\alpha \neq \beta, j, k} \left| \Theta_{\alpha \beta j k} \right|^2 - \frac{1}{2} \sum_{\alpha, \beta, j, k} \left| \Theta_{\alpha \alpha j k} - \Theta_{\beta \beta j k} \right|^2 \leq 0.$$

This proves the expected inequality. Moreover, the equality holds if and only if we have

$$\Theta_{\alpha \beta j k} = 0 \quad \text{for } \alpha \neq \beta, \quad \Theta_{\alpha \alpha j k} = \gamma_{j k} \quad \text{for all } \alpha.$$

where $\gamma = i \sum_{j, k} \gamma_{j k} dz_j \wedge d\bar{z}_k$ is a $(1, 1)$-form (take e.g., $\gamma_{j k} = \Theta_{11 j k}$). Hence $\Theta_h(E) = \gamma \otimes \text{Id}_E$. By taking the trace with respect to $E$ in this last equality, we get $c_1(E)_h = r \gamma$. Therefore the equality occurs if and only if

$$\Theta_h(E) = \frac{1}{r} c_1(E)_h \otimes \text{Id}_E.$$

\[ \square \]

**Corollary 1.** — Let $(E, h)$ be a Hermite-Einstein vector bundle with $c_1(E) = 0$ and $c_2(E) = 0$. Then $E$ is unitary flat for some hermitian metric $h' = h e^{-\psi}$.

**Proof.** By the assumption $c_1(E) = 0$, we can write $c_1(E)_h = \frac{i}{2\pi} \partial \bar{\partial} \psi$ for some global function $\psi$ on $X$. The equality case of the Kobayashi-Lübke inequality yields

$$\Theta_{h e^{\psi r}}(E) = \Theta_h(E) - \frac{1}{r} \frac{i}{2\pi} \partial \bar{\partial} \psi \otimes \text{Id}_E = 0.$$ 

\[ \square \]

**Corollary 2.** — Let $X$ be a compact Kähler manifold with $c_1(X) = c_2(X) = 0$. Then $X$ is a finite unramified quotient of a torus.

**Proof.** By the Aubin-Calabi-Yau theorem, $X$ admits a Ricci-flat Kähler metric $\omega$. Since $\text{Ricci}(\omega) = \text{Tr}_\omega \Theta_\omega(T_X)$, we see that $(T_X, \omega)$ is a Hermite-Einstein vector bundle, and $c_1(T_X)_\omega = \text{Ricci}(\omega) = 0$. By the Kobayashi-Lübke inequality, we conclude that $(T_X, \omega)$ is unitary flat, given by a unitary representation...
Let \( \pi_1(X) \to U(n) \). Let \( \tilde{X} \) be the universal covering of \( X \) and \( \tilde{\omega} \) the induced metric. Then \( (T_X, \tilde{\omega}) \) is a trivial vector bundle equipped with a flat metric. Let \((\xi_1, \ldots, \xi_n)\) be an orthonormal parallel frame of \( T_X \). Since \( \nabla\xi_j = 0 \), we conclude that \( d\xi_j^* = 0 \), and it is easy to infer from this that \([\xi_j, \xi_k] = 0\). The flow of each vector field \( \sum \lambda_j \xi_j \) is defined for all times (this follows from the fact the length of a trajectory is proportional to the time, and \( \tilde{\omega} \) is complete). Hence we get an action of \( \Phi^n \) on \( \tilde{X} \), and it follows easily that \( (\tilde{X}, \tilde{\omega}) \simeq (\Phi^n, \text{can}) \). Now, \( \pi_1(X) \) acts by isometries on this \( \Phi^n \). The classification of subgroups of affine transformations acting freely (and with compact quotient) shows that \( \pi_1(X) \) must be a semi-direct product of a finite group of isometries by a group of translations associated to a lattice \( \Lambda \subset \Phi^n \). Hence there is an exact sequence

\[
0 \to \Lambda \to \pi_1(X) \to G \to 0
\]

where \( G \) is a finite group of isometries. It follows that there is a finite unramified covering map \( \Phi^n/\Lambda \to \tilde{X}/\pi_1(X) \simeq X \) of \( X \) by a torus. \( \square \)

We now discuss the special case of the tangent bundle \( T_X \) in case \((X, \omega)\) is a compact Kähler-Einstein manifold. The Kähler-Einstein condition means that \( \text{Ricci}(\omega) = \lambda \omega \) for some real constant \( \lambda \), i.e., \( \text{Tr}_\omega\Theta_\omega(T_X) = \lambda \text{Id}_{T_X} \). In particular, \((T_X, \omega)\) is a Hermite-Einstein vector bundle. Here, however, the coefficients \((\Theta_{\alpha\beta jk})_{1 \leq \alpha, \beta, j, k \leq n}\) of the curvature tensor \( \Theta_\omega(T_X) \) satisfy the additional symmetry relations

\[
(\star) \quad \Theta_{\alpha\beta jk} = \Theta_{j\beta\alpha k} = \Theta_{akj\beta} = \Theta_{jka\beta}.
\]

These relations follow easily from the identity \( \Theta_{\alpha\beta jk} = -\partial^2 \omega_{\alpha\beta}/\partial z_j \partial z_k \) in normal coordinates, when we apply the Kähler condition \( \partial\omega_{\alpha\beta}/\partial z_j = \partial\omega_{j\beta}/\partial z_\alpha \). It follows that the Chern forms satisfy a slightly stronger inequality than the general inequality valid for Hermite-Einstein bundles. In fact \((T_X, \omega)\) satisfies a similar inequality where the rank \( r = n \) is replaced by \( n + 1 \).

**GUGGENHEIMER-YAU INEQUALITY.** — Let \((T_X, \omega)\) be a compact \( n \)-dimensional Kähler-Einstein manifold, with constant \( \lambda \in \mathbb{R} \). If \( \lambda = 0 \), then \( c_2(T_X, \omega) \wedge \omega^{n-2} \geq 0 \). If \( \lambda \neq 0 \), we have the inequality

\[
[n c_1(X)^2 - (2n + 2)c_2(X)] \cdot (\lambda c_1(X))^{n-2} \leq 0,
\]

and the equality also holds pointwise if we replace the Chern classes by the Chern forms \( c_k(T_X, \omega) \). The equality occurs in the following cases:

(i) If \( \lambda = 0 \), then \((X, \omega)\) is a finite unramified quotient of a torus.

(ii) If \( \lambda > 0 \), then \((X, \omega) \simeq (\mathbb{P}^n, \text{Fubini Study}) \).

(iii) If \( \lambda < 0 \), then \((X, \omega) \simeq (\mathbb{B}_n/\Gamma, \text{Poincaré metric}) \), i.e. \( X \) is a compact unramified quotient of the ball in \( \Phi^n \).

**COROLLARY** (Bogomolov-Miyaoka-Yau). — Let \( X \) be a surface of general type with \( K_X \) ample. Then there is an inequality \( c_1(X)^2 \leq 3c_2(X) \), and the equality occurs if and only if \( X \) is a quotient of the ball \( \mathbb{B}_2 \)
Miyaoka has shown that the inequality holds in fact as soon as $X$ is a surface with general type.

**Proof.** As in the proof of the Kobayashi-Lübke inequality, we find

$$n c_1(T_X)^2 - (2n + 2) c_2(T_X)_\omega = \sum_{\alpha, \beta} -\Theta_{\alpha\alpha} \wedge \Theta_{\beta\beta} + (n + 1)\Theta_{\alpha\beta} \wedge \Theta_{\beta\alpha}.$$ 

Taking the wedge product with $\omega^{n-2}/(n-2)!$, we get

$$2[n c_1(T_X)^2 - (2n + 2) c_2(T_X)_\omega] \wedge \frac{\omega^{n-2}}{(n-2)!} = \sum_{\alpha, \beta, j, k} -\Theta_{\alpha\alpha jj} \Theta_{\beta\beta kk} - \Theta_{\alpha\alpha jj} \Theta_{\beta\beta kk} + (n + 1)(\Theta_{\alpha\beta jj} \Theta_{\beta\alpha kk} - \Theta_{\alpha\beta jj} \Theta_{\beta\alpha kk}).$$

If we had the factor $r = n$ instead of $(n + 1)$ in the right hand side, the terms in $jj$ and $kk$ would cancel (as they did before). Hence we find

$$2[n c_1(T_X)^2 - (2n + 2) c_2(T_X)_\omega] \wedge \frac{\omega^{n-2}}{(n-2)!} = \sum_{\alpha, \beta, j, k} 2\Theta_{\alpha\alpha jj} \Theta_{\beta\beta kk} - (n + 1)\Theta_{\alpha\beta jj} \Theta_{\beta\alpha kk}.$$ 

Here, the symmetry relation $(\star)$ was used in order to obtain the equality of the summation of the first two terms. Using also the hermitian symmetry relation, our sum $\Sigma$ can be rewritten as

$$\Sigma = - \sum_{\alpha, \beta, j, k} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta kk}|^2 - (n + 1)|\Theta_{\alpha\beta jj}|^2 + 2n \sum_{\alpha, j, k} |\Theta_{\alpha\alpha jj}|^2$$

$$= - \sum_{\alpha, \beta, j, k} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta kk}|^2 - (n + 1) \sum_{\alpha, \beta, j, k, \text{pairwise} \neq} |\Theta_{\alpha\beta jj}|^2$$

$$- (n + 1) \left[ 8 \sum_{\alpha \neq j < k \neq \alpha} |\Theta_{\alpha\alpha jj}|^2 + 4 \sum_{\alpha \neq j} |\Theta_{\alpha\alpha aj}|^2 + 4 \sum_{\alpha < j} |\Theta_{aaaj}|^2 + \sum_{a} |\Theta_{aaaa}|^2 \right]$$

$$+ 2n \left[ 2 \sum_{\alpha \neq j < k \neq \alpha} |\Theta_{\alpha\alpha jj}|^2 + 2 \sum_{\alpha \neq j} |\Theta_{\alpha\alpha aj}|^2 + 2 \sum_{\alpha < j} |\Theta_{aaaj}|^2 + \sum_{a} |\Theta_{aaaa}|^2 \right]$$

$$= - \sum_{\alpha \neq \beta, j, k} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta kk}|^2 - (n + 1) \sum_{\alpha, \beta, j, k, \text{pairwise} \neq} |\Theta_{\alpha\beta jj}|^2$$

$$- (4n + 8) \sum_{\alpha \neq j < k \neq \alpha} |\Theta_{\alpha\alpha jj}|^2 - 4 \sum_{\alpha \neq j} |\Theta_{\alpha\alpha aj}|^2 - 4 \sum_{\alpha < j} |\Theta_{aaaj}|^2$$

$$+ (n - 1) \sum_{\alpha} |\Theta_{aaaa}|^2.$$ 

All terms are negative except the last one. We try to absorb this term in the
summations involving the coefficients $\Theta_{\alpha\alpha jj}$. This gives
\[
\sum = - \sum_{\alpha \neq \beta, j \neq k} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - (n + 1) \sum_{\alpha, \beta, j, k, \text{pairwise } \neq} |\Theta_{\alpha\beta jj}|^2
- (4n + 8) \sum_{\alpha \neq j < k \neq \alpha} |\Theta_{\alpha\alpha jj}|^2 - 4 \sum_{\alpha \neq j} |\Theta_{\alpha\alpha oj}|^2
- \sum_{\alpha \neq \beta, j} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - 4 \sum_{\alpha < j} |\Theta_{\alpha\alpha jj}|^2 + (n - 1) \sum_{\alpha} |\Theta_{\alpha\alpha oo}|^2.
\]

The last line is equal to
\[
- \sum_{\alpha \neq \beta, j \neq \alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2
- (n - 1) \sum_{\alpha} |\Theta_{\alpha\alpha oo}|^2 - 8 \sum_{\alpha < \beta} |\Theta_{\alpha\alpha oo}|^2 + 2 \sum_{\alpha \neq \beta} \Theta_{\alpha\beta\beta\beta} T_{\alpha\beta\beta\beta} + \Theta_{\alpha\beta\beta\beta} \Theta_{\alpha\beta\beta\beta}
- \sum_{\alpha \neq \beta, j \neq \alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - \sum_{\alpha \neq \beta} |\Theta_{\alpha\alpha oo} - 2\Theta_{\alpha\beta\beta}|^2.
\]

Therefore we find
\[
\sum = - \sum_{\alpha \neq \beta, j \neq k} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - (n + 1) \sum_{\alpha, \beta, j, k, \text{pairwise } \neq} |\Theta_{\alpha\beta jj}|^2
- (4n + 8) \sum_{\alpha \neq j < k \neq \alpha} |\Theta_{\alpha\alpha jj}|^2 - 4 \sum_{\alpha \neq j} |\Theta_{\alpha\alpha oj}|^2
- \sum_{\alpha \neq \beta, j \neq \alpha} |\Theta_{\alpha\alpha jj} - \Theta_{\beta\beta jj}|^2 - \sum_{\alpha \neq \beta} |\Theta_{\alpha\alpha oo} - 2\Theta_{\alpha\beta\beta}|^2.
\]

This proves the expected inequality $\sum \leq 0$. Moreover, we have $\sum = 0$ if and only if there is a scalar $\mu$ such that
\[
\Theta_{\alpha\beta\beta} = \Theta_{\alpha\beta\alpha} = \Theta_{\alpha\beta\alpha} = \mu \quad \text{for } \alpha \neq \beta, \quad \Theta_{\alpha\alpha\alpha\alpha} = 2\mu,
\]
and all other coefficients $\Theta_{\alpha\beta jj}$ are zero. By taking the trace $\sum \Theta_{\alpha\alpha jj}$, we get $\lambda = (n + 1)\mu$. We thus obtain that the hermitian form associated to the curvature tensor is
\[
\langle \Theta_{\omega}(T_X)(\xi \otimes \eta), \xi \otimes \eta \rangle_\omega = \frac{\lambda}{n + 1} \sum_{\alpha, \beta} \xi_\alpha \eta_\beta \bar{\xi}_\alpha \bar{\eta}_\beta + \xi_\alpha \eta_\beta \bar{\xi}_\beta \bar{\eta}_\alpha
\]
(\star) \quad = \frac{\lambda}{n + 1} (|\xi|^2 |\eta|^2 + |\langle \xi, \eta \rangle|^2)
\]
for all $\xi, \eta \in T_X$. When $\lambda = 0$, the curvature tensor vanishes identically and we have already seen that $X$ is a finite unramified quotient of a torus. Assume from
now on that $\lambda \neq 0$. The formula (⋆⋆) shows that the curvature tensor is constant and coincides with the curvature tensor of $\mathbb{P}^n$ (case $\lambda > 0$), or of the ball $\mathbb{B}_n$ (case $\lambda < 0$), relatively to the canonical metrics on these spaces. By a well-known result from the theory of hermitian symmetric spaces⋆, it follows that $(X, \omega)$ is locally isometric to $\mathbb{P}^n$ (resp. $\mathbb{B}_n$). Since the universal covering $\tilde{X}$ is a complete and locally symmetric hermitian manifold, we conclude that $\tilde{X} \simeq \mathbb{P}^n$, resp. $\tilde{X} \simeq \mathbb{B}_n$. In the case $\lambda > 0$, $X$ is a Fano manifold, thus $X$ is simply connected and $X = \tilde{X}$. The proof is complete.

To compute the curvature of $\mathbb{P}^n$ and $\mathbb{B}^n$, we use the fact that the canonical metric is

$$\omega = \frac{i}{2\pi} \partial \overline{\partial} \log(1 + |z|^2) \text{ on } \mathbb{P}^n, \text{ resp. } \omega = -\frac{i}{2\pi} \partial \overline{\partial} \log(1 - |z|^2) \text{ on } \mathbb{B}^n,$$

with respect to the non homogeneous coordinates on $\mathbb{P}^n$. We thus get

$$\omega = \frac{i}{2\pi} \left( \frac{dz \otimes d\overline{z}}{1 + |z|^2} - \frac{|\langle dz, z \rangle|^2}{(1 + |z|^2)^2} \right), \text{ resp. } \omega = \frac{i}{2\pi} \left( \frac{dz \otimes d\overline{z}}{1 - |z|^2} + \frac{|\langle dz, z \rangle|^2}{(1 - |z|^2)^2} \right).$$

By computing the derivatives $\Theta_{\alpha\beta jk} = -\partial^2 \omega_{\alpha\beta} / \partial z_j \partial \overline{z}_k$ at $z = 0$, we easily see that the curvature is given by (⋆⋆) with $\lambda = \pm (n + 1)$. The equality also holds at any other point by the homogeneity of $\mathbb{P}^n$ and $\mathbb{B}^n$.

Observe that the riemannian exponential map $\exp : T_{X,0} \rightarrow X$ at the origin of $\mathbb{Q}^n \subset \mathbb{P}^n$ or $\mathbb{B}^n$ is unitary invariant. It follows that the holomorphic part $h$ of the Taylor expansion of $\exp$ at 0 is unitary invariant. This invariance forces $h$ to coincide with the identity map in the standard coordinates of $\mathbb{P}^n$ and $\mathbb{B}^n$. From this observation, it is not difficult to justify intuitively the local isometry statement used above. In fact let $X$ be a hermitian manifold whose curvature tensor is given by (⋆⋆), $\lambda \neq 0$. Then $\omega$ is proportional to $c_1(T_X)\omega$ and so $\omega$ is a Kähler-Einstein metric. Since the Kähler-Einstein equation is (nonlinear) elliptic with real analytic coefficients in terms of any real analytic Kähler form, it follows that $\omega$ is real analytic, and so is the exponential map. Fix a point $x_0 \in X$ and let $h : \mathbb{Q}^n \simeq T_{X,x_0} \rightarrow X$ be the holomorphic part of the Taylor expansion of $\exp$ at the origin. Then $h$ must provide the holomorphic coordinates we are looking for, i.e. $h^* \omega$ must coincide with the metric of $\mathbb{P}^n$ (resp. $\mathbb{B}^n$) in a neighborhood of the origin.