Abstract and goals

• Recent work by Berman, Berndtsson, Boucksom, Eyssidieux, Guedj, Jonsson, Zeriahi (among others) leads to a new variational approach for the solution of Monge-Ampère equations on compact Kähler manifolds.

• The method can be made independent of the previous PDE technicalities of Yau's approach.

• It is based on the study of certain functionals (Ding-Tian, Mabuchi) on the space of Kähler metrics, and their geodesic convexity due to X.X. Chen and Berman-Berndtsson in its full generality.

• Applications include the existence and uniqueness of Kähler-Einstein metrics on Q-Fano varieties with log terminal singularities, and a new proof by Berman-Boucksom-Jonsson of a uniform version of the Yau-Tian-Donaldson conjecture solved around 2013 by Chen-Donaldson-Sun.
Kähler-Einstein metrics

To a Kähler metric on a compact complex \( n \) fold \( X \)

\[
\omega = i \sum_{1 \leq j, k \leq n} \omega_{jk}(z) \, dz_j \wedge d\bar{z}_k, \quad d\omega = 0
\]

one associates its Ricci curvature form

\[
\text{Ricci}(\omega) = \Theta_{\Lambda^n T_X, \Lambda^n \omega} = -dd^c \log \det(\omega_{jk})
\]

where \( d^c = \frac{1}{4i\pi}(\partial - \bar{\partial}), \) \( dd^c = \frac{i}{2\pi} \partial \bar{\partial}. \) The Kähler metric \( \omega \) is said to be Kähler-Einstein if

\[
(\ast) \quad \text{Ricci}(\omega) = \lambda \omega \quad \text{for some} \ \lambda \in \mathbb{R}.
\]

This requires \( \lambda \omega \in c_1(X), \) hence (\( \ast \)) can be solved only when \( c_1(X) \) is positive definite, negative definite or zero, and after rescaling \( \omega \) by a constant, one can always assume that \( \lambda \in \{0, 1, -1\}. \)

Kähler-Einstein \( \iff \) Monge-Ampère equation (1)

Fix a reference Kähler metric \( \omega_0 \) and put \( \omega = \omega_0 + dd^c \varphi. \) The KE condition (\( \ast \)) is equivalent to

\[
(\ast\ast) \quad (\omega_0 + dd^c \varphi)^n = e^{-\lambda \varphi + f} \omega_0^n.
\]

- When \( \lambda = -1 \) and \( c_1(X) < 0, \) i.e. \( c_1(K_Z) > 0, \) Aubin has shown in 1978 that there is always a unique solution, hence a unique Kähler metric \( \omega \in c_1(K_Z) \) such that

\[
\text{Ricci}(\omega) = -\omega.
\]

This is a very natural generalization of the existence of constant curvature metrics on complex algebraic curves, implied by Poincaré's uniformization theorem in dimension 1.
For \( \lambda = 0 \) and \( c_1(X) = 0 \), a celebrated result of Yau (solution of the Calabi conjecture, 1978) states that there exists a unique metric \( \omega = \omega_0 + dd^c\varphi \) in the given cohomology class \( \{\omega_0\} \) such that \( \text{Ricci}(\omega) = 0 \). Moreover, the Monge-Ampère equation

\[
(\omega_0 + dd^c\varphi)^n = e^{f}\omega_0^n
\]

has a unique solution whenever \( \int_X e^{f}\omega_0^n = \int_X \omega_0^n \). Equivalently, the Ricci curvature form can be prescribed to be equal any given smooth closed \((1,1)\)-form

\[
\text{Ricci}(\omega) = \rho,
\]

provided that \( \rho \in c_1(X) \).

The case of Fano manifolds

For \( \lambda = +1 \), the equation to solve is

\[
(\omega_0 + dd^c\varphi)^n = e^{-\varphi + f}\omega_0^n.
\]

This is possible only if \( -K_X \) (\( = \Lambda^n T_X \)) is ample. One then says that \( X \) is a Fano manifold. When solutions exist, it is known by Bando Mabuchi (1987) that they are unique up to the action of the identity component \( \text{Aut}^0(X) \) in the complex Lie group of biholomorphisms of \( X \).

Berman-Boucksom-Jonsson 2015

Let \( X \) be a Fano manifold with finite automorphism group. Then \( X \) admits a Kähler-Einstein metric if and only if it is uniformly K-stable.

Recently, Chen, Donaldson and Sun got this result under the more general assumption that \( X \) is K-stable (Bourbaki/Ph. Eyssidieux, January 2015).
The case of log Fano varieties

Definition

A log Fano pair is a klt pair \((X, \Delta)\) such that \(X\) is projective and the \(\mathbb{Q}\)-divisor \(A = -(K_X + \Delta)\) is ample.

Here \(X\) is a normal compact complex variety \(X\) and \(\Delta\) an effective \(\mathbb{Q}\)-divisor such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. By Hironaka, there exists a log resolution \(\pi : \tilde{X} \to X\) of \((X, \Delta)\), i.e. a modification of \(X\) over the complement of the singular loci of \(X\) and \(\Delta\), such that the pull-back of \(\Delta\) and of \(X_{\text{sing}}\) consists of simple normal crossing (snc) divisors in \(\tilde{X}\). One writes

\[ \pi^*(K_X + \Delta) = K_{\tilde{X}} + E, \quad E = \sum_j a_j E_j \]

for some \(\mathbb{Q}\)-divisor \(E\) whose push-forward to \(X\) is \(\Delta\) (since \(X_{\text{sing}}\) has codimension 2, the components \(E_j\) that lie over \(X_{\text{sing}}\) yield \(\pi_* E_j = 0\)). The coefficient \(-a_j \in \mathbb{Q}\) is known as the discrepancy of \((X, \Delta)\) along \(E_j\).

The klt condition ("Kawamata Log Terminal")

Definition

\((X, \Delta)\) is klt if \(a_j < 1\) for all \(j\).

Let \(r\) be a positive integer such that \(r(K_X + \Delta)\) is Cartier, and \(\sigma\) a local generator of \(\mathcal{O}(r(K_X + \Delta))\) on some open set \(U \subset X\). Then the \((n, n)\) form

\[ |\sigma|^{2/r} := i^n \sigma^{1/r} \wedge \overline{\sigma^{1/r}} \]

is a volume form with poles along \(S = \text{Supp} \Delta \cup X_{\text{sing}}\).

By the change of variable formula, the local integrability can be checked by pulling back \(\sigma\) to \(\tilde{X}\), in which case it is easily seen that the integrability occurs if and only if \(a_j < 1\) for all \(j\), i.e. when \((X, \Delta)\) is klt.

In local coordinates

\[ |\sigma|^{2/r} \sim \frac{\text{volume form}}{\prod |z_j|^{2a_j}}. \]
Singular Monge-Ampère equation

By definition \( (X, \Delta) \log \text{Fano} \implies c_1(X, \Delta) \ni \omega_0 \text{ Kähler}. \)

Every form \( \omega = \omega_0 + dd^c \psi \in \{\omega_0\} \) can be seen as the curvature form of a smooth hermitian metric \( h \) on \( \mathcal{O}(-(K_X + \Delta)) \), whose weight is \( \phi = u_0 + \psi \) where \( u_0 \) is a local potential of \( \omega_0 \), hence

\[
\omega = \omega_0 + dd^c \psi = dd^c \phi
\]

where \( \phi \) is understood as the weight of a global metric formally denoted \( h = e^{-\phi} \) on the \( \mathbb{Q} \)-line bundle \( \mathcal{O}(-(K_X + \Delta)) \).

The inverse \( e^\phi \) is a hermitian metric on \( \mathcal{O}(K_X + \Delta) \). If \( \sigma \) is a local generator of \( \mathcal{O}(r(K_X + \Delta)) \), the product \( |\sigma|^2/r e^\phi = e^{\psi + u_0} \) is (locally) a smooth positive function whenever \( \varphi \) is smooth, hence

\[
e^{-\phi} = |\sigma|^2/r e^{-(\psi + u_0)}
\]

is an integrable volume form on \( X \) with poles along \( S := \text{Supp} \Delta \cup \{\text{singularities}\} \). The KE condition can be rewritten

\[
(dd^c \phi)^n = c e^{-\phi} \quad \text{on} \quad X \setminus S \iff \text{Ricci}(\omega) = \omega + [\Delta].
\]

The space of Kähler metrics

Let \( A \in H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{R}) \) be a Kähler \( \partial \bar{\partial} \)-cohomology class, and let

\[
\omega_0 = \alpha + dd^c \psi_0 = dd^c \phi_0 \in A
\]

be a Kähler metric.

Here we are mostly interested in the Fano case \( A = -K_X \) and the log Fano case \( A = -(K_X + \Delta) \). Let \( V = \int_X \omega_0^n = A^n \) be the volume of \( \omega_0 \).

**Definition**

The space \( \mathcal{K}_A \) of Kähler metrics (resp. \( \mathcal{P}_A \) of Kähler potentials) is the set of Kähler metrics \( \omega \) (resp. functions \( \psi \)) such that

\[
\omega = \omega_0 + dd^c \psi > 0.
\]

Here \( \phi = u_0 + \psi \) is thought intrinsically as a hermitian metric \( h = e^{-\phi} \) on \( A \) with strictly plurisubharmonic (psh) weight \( \phi \).

Clearly \( \mathcal{K}_A \simeq \mathcal{P}_A / \mathbb{R} \).
The Riemannian structure on $\mathcal{P}_A$

The basic operator of interest on $\mathcal{P}_A$ is the Monge-Ampère operator

$$\mathcal{P}_A \to \mathcal{M}_+, \quad \text{MA}(\phi) = (dd^c \phi)^n = (\omega_0 + dd^c \psi)^n$$

According to Mabuchi the space $\mathcal{P}_A$ can be seen as some sort of infinite dimensional Riemannian manifold: a “tangent vector” to $\mathcal{P}_A$ is an infinitesimal variation $\delta \phi \in C^\infty(X, \mathbb{R})$ of $\phi$ (or $\psi$), and the infinitesimal Riemannian metric at a point $h = e^{-\phi}$ is given by

$$||\delta \phi||^2_2 = \frac{1}{V} \int_X (\delta \phi)^2 \text{MA}(\phi).$$

X.X. Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{P}_A$, showing in particular that it is a path metric space (a non trivial assertion in this infinite dimensional setting) of nonpositive curvature in the sense of Alexandrov. A key step has been to produce almost $C^{1,1}$-geodesics which minimize the geodesic distance.

Basic functionals (1)

Given $\phi_0, \phi \in \mathcal{P}_A$, one defines:

- The Monge-Ampère functional

$$E_{\phi_0}(\phi) = \frac{1}{(n+1)V} \sum_{j=0}^n \int_X (\phi - \phi_0)(dd^c \phi)^j \wedge (dd^c \phi_0)^{n-j}$$

$$(***) \quad = \frac{1}{(n+1)V} \sum_{j=0}^n \int_X \psi(\omega_0 + dd^c \psi)^j \wedge \omega_0^{n-j} \wedge \omega_0^{n-j}, \quad \psi = \phi - \phi_0.$$

It is a primitive of the Monge-Ampère operator in the sense that $dE_{\phi_0}(\phi) = \frac{1}{V} \text{MA}(\phi)$, i.e. for any path $[T, T'] \ni t \mapsto \phi_t$, one has

$$\frac{d}{dt} E_{\phi_0}(\phi_t) = \frac{1}{V} \int_X \dot{\phi}_t \text{MA}(\phi_t) \quad \text{where} \quad \dot{\phi}_t = \frac{d}{dt} \phi_t.$$

This is easily checked by a differentiation under the integral sign.
As a consequence $E$ satisfies the cocycle relation

$$E_{\phi_0}(\phi_1) + E_{\phi_1}(\phi_2) = E_{\phi_0}(\phi_2),$$

so its dependence on $\phi_0$ is only up to a constant.

Finally, if $\phi_t$ depends linearly on $t$, one has $\dddot{\phi}_t = \frac{d^2}{dt^2} \phi_t = 0$ and a further differentiation of (***) yields

$$\frac{d^2}{dt^2} E_{\phi_0}(\phi) = \frac{n}{V} \int_X \dddot{\phi}_t \ddc \dot{\phi} \wedge (\ddc \phi)^{n-1}$$

$$= - \frac{n}{V} \int_X d\dddot{\phi}_t \wedge dd^c \dddot{\phi} \wedge (\ddc \phi)^{n-1} \leq 0.$$

It follows that $E_{\phi_0}$ is concave on $P_A$.

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**The $J$ and $J^*$ functionals**

- The concavity of $E$ implies the nonnegativity of $J_{\phi_0}(\phi) := dE_{\phi_0}(\phi_0) \cdot (\phi - \phi_0) - E_{\phi_0}(\phi)$, This quantity is called the Aubin $J$-energy functional

$$J_{\phi_0}(\phi) = V^{-1} \int_X (\phi - \phi_0)(dd^c \phi_0)^n - E_{\phi_0}(\phi) \geq 0.$$

- By exchanging the roles of $\phi$, $\phi_0$ and putting $J_{\phi_0}^*(\phi) = J_{\phi_0}(\phi_0) \geq 0$, the cocycle relation for $E$ yields $E_{\phi}(-\phi_0) = -E_{\phi_0}(\phi)$. The transposed $J$-energy functional is

$$J_{\phi_0}^*(\phi) := E_{\phi_0}(\phi) - V^{-1} \int_X (\phi - \phi_0)(dd^c \phi)^n$$

$$= E_{\phi_0}(\phi) - V^{-1} \int_X \psi(\omega_0 + dd^c \psi)^n \geq 0, \; \psi = \phi - \phi_0.$$
The symmetric $I$ functional

- The $I$-functional is the symmetric functional defined by

$$I_{\phi_0}(\phi) = I_\phi(\phi_0) := -\frac{1}{V} \int_X (\phi - \phi_0)(\text{MA}(\phi) - \text{MA}(\phi_0))$$

$$= \sum_{j=0}^{n-1} V^{-1} \int_X d(\phi - \phi_0) \wedge d^c(\phi - \phi_0) \wedge (dd^c \phi)^j \wedge (dd^c \phi_0)^{n-1-j} \geq 0.$$  

In fact $I_{\phi_0}(\phi) = J_{\phi_0}(\phi) + J^*_{\phi_0}(\phi)$, and one can also write

$$I_{\phi_0}(\phi) = V^{-1} \left( \int_X \psi \omega_0^n - \int_X \psi(\omega_0 + dd^c \psi)^n \right).$$

It satisfies the quasi-triangle inequality: $\exists c_n > 0$ s.t.

$$I_{\phi_0}(\phi) \leq c_n (I_{\phi_0}(\phi_1) + I_{\phi_1}(\phi)). \quad \forall \phi_0, \phi_1, \phi \in \mathcal{P}_A.$$

The Ding and Mabuchi functionals (1)

- In the Fano or log Fano setting, the Ding functional is defined by

$$D_{\phi_0} = L - L(\phi_0) - E_{\phi_0}, \quad \text{where} \quad L(\phi) = -\log \int_X e^{-\phi}.$$  

Recall: $e^{-\phi}$ is integrable by the klt condition.

- Given probability measures $\mu, \nu$ on a space $X$, the relative entropy $\text{Entr}_\mu(\nu) \in [0, +\infty]$ of $\nu$ with respect to $\mu$ is defined as the integral

$$\text{Entr}_\mu(\nu) := \int_X \log \left( \frac{d\nu}{d\mu} \right) d\nu,$$

if $\nu$ is absolutely continuous w.r.t. $\mu$; $\text{Entr}_\mu(\nu) = +\infty$ otherwise.

Pinsker inequality: for all proba measures $\mu, \nu$ one has

$$\text{Entr}_\mu(\nu) \geq \frac{1}{2} \|\mu - \nu\|^2 \geq 0.$$  

In particular, $\mu = \nu \iff \text{Entr}_\mu(\nu) = 0$.  

The Ding and Mabuchi functionals (2)

In the Fano or log Fano situation, the entropy functional $H_{\phi_0}(\phi)$ is defined to be the entropy of the probability measure $\frac{1}{V}(dd^c \phi)^n$ with respect to $e^{L(\phi_0)}e^{-\phi_0}$, namely

$$H_{\phi_0}(\phi) = \int_X \log \left( \frac{(dd^c \phi)^n}{e^{L(\phi_0)}e^{-\phi_0}} \right) \left( \frac{(dd^c \phi)^n}{V} \right) \geq 0.$$

- The Mabuchi functional is then defined by

$$M_{\phi_0} = H_{\phi_0} - J^*_{\phi_0}.$$

One gets the more explicit expression

$$M_{\phi_0}(\phi) = \int_X \log \left( \frac{e^{\phi}(dd^c \phi)^n}{V} \right) \left( \frac{dd^c \phi)^n}{V} \right) - E_{\phi_0}(\phi) - L(\phi_0).$$

Comparison properties

Observation

If $c$ is a constant, then

$$E_{\phi_0}(\phi + c) = E_{\phi_0}(\phi) + c \quad \text{and} \quad L(\phi + c) = L(\phi) + c.$$

On the other hand, the functionals $I_{\phi_0}, J_{\phi_0}, J^*_{\phi_0}, D_{\phi_0}, H_{\phi_0}, M_{\phi_0}$ are invariant by $\phi \mapsto \phi + c$ and therefore descend to the quotient space $\mathcal{K}_A = \mathcal{P}_A/\mathbb{R}$ of Kähler metrics $\omega = dd^c \phi \in A$.

Comparison between $I, J, J^*$

The functionals $I, J, J^*$ are essentially growth equivalent:

$$\frac{1}{n}J_\phi(\phi_0) \leq J_{\phi_0}(\phi) \leq \frac{n+1}{n}J_\phi(\phi_0) \leq I_{\phi_0}(\phi) \leq (n+1)J_{\phi_0}(\phi).$$
Comparison between Ding and Mabuchi functionals

**Proposition**

Let \((X, \Delta)\) be a log Fano manifold. Then \(M_{\phi_0}(\phi) \geq D_{\phi_0}(\phi)\) and, in case of equality, \(\phi\) must be Kähler-Einstein.

**Proof.** From the definitions one gets

\[
M - D = (H - J^*)(L - L(\phi_0) - E),
\]

\[
E_{\phi_0}(\phi) - J^*_{\phi_0}(\phi) = V^{-1} \int_X (\phi - \phi_0)(dd^c \phi)^n,
\]

\[
M_{\phi_0}(\phi) - D_{\phi_0}(\phi) = \int_X \left( \log \left( \frac{(dd^c \phi)^n / V}{e^{L(\phi_0)} e^{-\phi_0}} \right) + (\phi - \phi_0) \right) \frac{(dd^c \phi)^n}{V} + L(\phi_0) - L(\phi)
\]

\[
= \int_X \log \left( \frac{(dd^c \phi)^n / V}{e^{L(\phi)} e^{-\phi}} \right) \frac{(dd^c \phi)^n}{V} \geq 0.
\]

In case of equality, Pinsker implies KE condition: \(\frac{(dd^c \phi)^n}{V} = e^{L(\phi)} e^{-\phi}\)

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**Non pluripolar products**

- **Bedford-Taylor Monge-Ampère products**: for \(u_j \in L^\infty_{\text{loc}}\), one sets inductively

  \[dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k := dd^c (u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_k)\]

- **Non pluripolar products (Guedj-Zeriahi)**

  Let \(\mathcal{P}(X, \omega_0)\) be the set of \(\omega_0\)-psh potentials, i.e. \(\phi = \phi_0 + \psi\) such that \(dd^c \phi = \omega_0 + dd^c \psi \geq 0\).

  The functions \(\psi_\nu := \max\{\psi, -\nu\}\) are again \(\omega_0\)-psh and bounded for all \(\nu \in \mathbb{N}\). The Monge-Ampère measures \((\omega_0 + dd^c \psi_\nu)^n\) are therefore well-defined in the sense of Bedford-Taylor, and one defines for any bidegree \((p, p)\) a positive current

  \[T = (\omega_1 + dd^c \psi_1) \wedge \ldots \wedge (\omega_p + dd^c \psi_p) = \lim_{\nu \to +\infty} \mathbf{1}_{\{\psi_j > -\nu\}} (\omega_1 + dd^c \max\{\psi_1, -\nu\}) \wedge \ldots \wedge (\omega_p + dd^c \max\{\psi_p, -\nu\})\]

  **Basic fact**: \(T\) is still closed [Proof uses ideas of Skoda & Sibony].
Space of potentials of finite energy

One introduces for any $p \in [1, +\infty[$ the space

$$E^p(X, \omega_0) := \left\{ \phi = \phi_0 + \psi; \int_X |\psi|^p MA(\omega_0 + dd^c \psi) < +\infty \right\},$$

and $\int_X MA(\omega_0 + dd^c \psi) = \int_X \omega_0^p$ ("full non pluripolar mass"). One says that functions $\psi \in E^p(X, \omega_0)$ have finite $E^p$-energy. One also denotes by

$$T^p(X, \omega_0) \subset T^p_{\text{full}}(X, \omega_0)$$

the corresponding set of currents with finite $E^p$-energy, which can be identified with the quotient space

$$T^p(X, \omega_0) = E^p(X, \omega_0)/\mathbb{R} \quad \text{via} \quad \phi \mapsto dd^c \phi = \omega_0 + dd^c \psi.$$

It is important to note that $T^p(X, \omega_0)$ is not a closed subset of $T(X, \omega_0)$ for the weak topology.

Finite energy extension of the functionals

All functionals $E, L, I, J, J^*, D, H, M$ have a natural extension to arguments $\phi, \phi_0 \in E^1(X, \omega_0)$, and $I, J, J^*, D, H, M$ descend to $T^1(X, \omega_0) = E^1(X, \omega_0)/\mathbb{R}$.

Theorem (BBGZ)

The map $T = \omega_0 + dd^c \psi \mapsto V^{-1}(T^n)$ is a bijection between $T^1(X, \omega_0)$ and the space of probability measures $\mathcal{M}^1(X, \omega_0)$ of finite energy.

Here one uses the Legendre-Fenchel transform

$$E^*_0(\mu) := \sup_{\phi = \phi_0 + \psi \in E^1(X, \omega_0)} \left( E_0(\psi) - \int_X \psi \mu \right) \in [0, +\infty]$$

where $E_0(\psi) = E_{\phi_0}(\phi_0 + \psi)$, and $\mu$ has finite energy if $E^*_0(\mu) < +\infty$. 
Theorem (BBEGZ)

For a current $\omega = dd^c \phi \in \mathcal{T}^1(X, A)$, the following conditions are equivalent.

(i) $\omega$ is a Kähler-Einstein metric for $(X, \Delta)$.

(ii) The Ding functional reaches its infimum at $\phi$:
$$D_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X, A)/\mathbb{R}} D_{\phi_0}.$$

(iii) The Mabuchi functional reaches its infimum at $\phi$:
$$M_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X, A)/\mathbb{R}} M_{\phi_0}.$$

Corollary (BBEGZ)

Let $X$ be a $\mathbb{Q}$-Fano variety with log terminal singularities.

(i) The identity component $\text{Aut}^0(X)$ of the automorphism group of $X$ acts transitively on the set of KE metrics on $X$.

(ii) If the Mabuchi functional of $X$ is proper, then $\text{Aut}^0(X) = \{1\}$ and $X$ admits a unique Kähler-Einstein metric.

Test configurations

Definition

A test configuration $(\mathcal{X}, \mathcal{A})$ for a $(\mathbb{Q})$-polarized projective variety $(X, A)$ consists of the following data:

(i) a flat and proper morphism $\pi: \mathcal{X} \to \mathbb{C}$ of algebraic varieties; one denotes by $X_t = \pi^{-1}(t)$ the fiber over $t \in \mathbb{C}$.

(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;

(iii) an isomorphism $X_1 \simeq X$.

(iv) a $\mathbb{C}^*$-linearized ample line bundle $\mathcal{A}$ on $\mathcal{X}$; one puts $A_t = \mathcal{A}|_{X_t}$.

(v) an isomorphism $(X_1, A_1) \simeq (X, A)$ extending the one in (iii).

$K$ stability (and uniform $K$-stability) is defined in terms of certain numerical invariants attached to arbitrary test configurations.
Donaldson-Futaki invariants

Donaldson-Futaki invariant

Let \( N_m = h^0(X, mA) \) and \( w_m \in \mathbb{Z} \) be the weight of the \( \mathbb{C}^* \)-action on the determinant \( \text{det} H^0(X_0, mA_0) \). Then there is an asymptotic expansion

\[
\frac{w_m}{mN_m} = F_0 + m^{-1} F_1 + m^{-2} F_2 + \ldots .
\]

and one defines \( DF(X, A) := -2F_1 \).

Definition

The polarized variety \((X, A)\) is said K-stable if \( DF(X, A) \geq 0 \) for all normal test configurations, with equality iff \((X, A)\) is trivial.

Generalized Yau-Tian-Donaldson conjecture

Let \((X, A)\) be a polarized variety. Then \(X\) admits a cscK metric (short hand for Kähler metric with constant scalar curvature) \( \omega \in c_1(A) \) if and only if \((X, A)\) is K-stable.

Uniform K-stability

The Duistermaat-Heckman measure \( DH_{(X, A)} \) is the proba distribution measure of the \( \mathbb{C}^* \)-action weights:

\[
DH_{(X, A)} = \lim_{m \to \infty} \sum_{\lambda \in \mathbb{Z}} \frac{\dim H^0(X, mA)|_\lambda}{\dim H^0(X, mA)} \frac{\delta_{\lambda/m}}{\delta_p} = \text{Dirac at } p,
\]

where \( H^0(X, mA) = \bigoplus_{\lambda \in \mathbb{Z}} H^0(X, mA)|_\lambda \) is the weight space decomposition. For each \( p \in [1, \infty] \), the \( L^p \)-norm \( \| (X, A) \|_p \) of an ample test configuration \((X, A)\) is defined as the \( L^p \) norm

\[
\| (X, A) \|_p = \left( \int_{\mathbb{R}} |\lambda - b(\mu)|^p d\mu(\lambda) \right)^{1/p}, \quad b(\mu) = \int_{\mathbb{R}} \lambda d\mu(\lambda).
\]

Definition (Székelyhidi)

The polarized variety \((X, A)\) is said to be \( L^p \)-uniformly K-stable if there exists \( \delta > 0 \) such that \( DF(X, A) \geq \delta \| (X, A) \|_p \) for all normal test configurations. [Note: only possible if \( p < \frac{n}{n-1} \).]
Berman-Boucksom-Jonsson 2015

Let $X$ be a Fano manifold with finite automorphism group. Then $X$ admits a Kähler-Einstein metric if and only if it is uniformly K-stable (in a related and simpler “non archimedean” sense).

Let $A = -K_X$. A ray $(\phi_t)_{t \geq 0}$ in $\mathcal{P}_A$ corresponds to an $S^1$-invariant metric $\Phi$ on the pull-back of $-K_X$ to the product of $X$ with the punctured unit disc $\mathbb{D}^*$. The ray is called subgeodesic when $\Phi$ is plurisubharmonic (psh for short). Denoting by $F$ any of the functionals $M, D$ or $J$, there is a limit

$$\lim_{t \to +\infty} \frac{F(\phi_t)}{t} = F^{\text{NA}}(X, A)$$

Here $F^{\text{NA}}$ can be seen as the corresponding “non-Archimedean” functional.