Bergman bundles and applications to the geometry of compact complex manifolds

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Virtual Conference in Complex Analysis and Geometry
hosted at Western University, London, Ontario
May 4 – 24, 2020

Projective vs Kähler vs non Kähler varieties

**Goal.** Investigate positivity for general compact manifolds/$\mathbb{C}$.
Obviously, non projective varieties do not carry any ample line bundle.
In the Kähler case, a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$, $\omega > 0$, may sometimes be used as a substitute for a polarization.
What for non Kähler compact complex manifolds?

**Surprising facts (?)**

- Every compact complex manifold $X$ carries a “very ample” complex Hilbert bundle, produced by means of a natural Bergman space construction.
- The curvature of this bundle is strongly positive in the sense of Nakano, and is given by a universal formula.

The aim of this lecture is to investigate further this construction and explain potential applications to analytic geometry (invariance of plurigenera, transcendental holomorphic Morse inequalities...
Let $X$ be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_X = \overline{\mathcal{O}_{\overline{X}}}$. The diagonal of $X \times \overline{X}$ is totally real, and by Grauert, we know that it possesses a fundamental system of Stein tubular neighborhoods. Assume that $X$ is equipped with a real analytic hermitian metric $\gamma$, and let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.

**Lemma**

Denote by $\text{exph}$ the “holomorphic” part of $\exp$, so that for $z \in X$ and $\xi \in T_{X,z}$

$$\exp_z(\xi) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha \beta}(z)\xi^\alpha \bar{\xi}^\beta,$$

$$\text{exph}_z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{0 \alpha}(z)\xi^\alpha.$$

Then $d_{\xi} \exp_z(\xi)_{\xi=0} = d_{\xi} \text{exph}_z(\xi)_{\xi=0} = \text{Id}_{T_X}$, and so $\text{exph}$ is a diffeomorphism from a neighborhood $V$ of the 0 section of $T_X$ to a neighborhood $V'$ of the diagonal in $X \times X$.

**Notation**

With the identification $\overline{X} \simeq_{\text{diff}} X$, let $\log_{\text{h}} : X \times \overline{X} \supset V' \to T_{\overline{X}}$ be the inverse diffeomorphism of $\text{exph}$ and

$$U_\varepsilon = \{(z, w) \in V' \subset X \times \overline{X}; |\log_{\text{h}}(w))_\gamma| < \varepsilon\}, \quad \varepsilon > 0.$$ 

Then, for $\varepsilon \ll 1$, $U_\varepsilon$ is Stein and $\text{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.
Such tubular neighborhoods are Stein.

In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n ; |z - w| < \varepsilon\}$ is of course Stein since

$$|z - w|^2 = |z|^2 + |w|^2 - 2 \text{Re} \sum z_j w_j$$

and $(z, w) \mapsto \text{Re} \sum z_j w_j$ is pluriharmonic.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma, \varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\text{pr}_1)|_{U_\varepsilon} : U_\varepsilon \rightarrow X, \quad \overline{p} = (\text{pr}_2)|_{U_\varepsilon} : U_\varepsilon \rightarrow \overline{X}$$

the natural projections.
Definition of the Bergman sheaf $\mathcal{B}_\varepsilon$

The Bergman sheaf $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is by definition the $L^2$ direct image

$$\mathcal{B}_\varepsilon = p_{\ast}^L (\bar{p}^\ast \mathcal{O}(K_X)),$$

i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V)$ = holomorphic sections $f$ of $\bar{p}^\ast \mathcal{O}(K_X)$ on $p^{-1}(V)$,

$$f(z, w) = f_1(z, w) \, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \subset V$:

$$\int_{p^{-1}(K)} i^n f(z, w) \wedge \bar{f}(z, w) \wedge \gamma(z)^n < +\infty, \quad \forall K \subset V.$$

(This $L^2$ condition is the reason we speak of “$L^2$ direct image”).

Clearly, $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module over $X$, but since it is a space of functions in $w$, it is of infinite rank.

Associated Bergman bundle and holom structure

Definition of the associated Bergman bundle $\mathcal{B}_\varepsilon$

We consider the vector bundle $\mathcal{B}_\varepsilon \to X$ whose fiber $\mathcal{B}_{\varepsilon, z_0}$ consists of all holomorphic functions $f$ on $p^{-1}(z_0) \subset U_\varepsilon$ such that

$$\| f(z_0) \|^2 = \int_{p^{-1}(z_0)} i^n f(z_0, w) \wedge \bar{f}(z_0, w) < +\infty.$$

Then $\mathcal{B}_\varepsilon$ is a real analytic locally trivial Hilbert bundle whose fiber $\mathcal{B}_{\varepsilon, z_0}$ is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0, \varepsilon))$ of $L^2$ holomorphic n-forms on $p^{-1}(z_0) \simeq B(0, \varepsilon) \subset \mathbb{C}^n$.

The Ohsawa-Takegoshi extension theorem implies that every $f \in \mathcal{B}_{\varepsilon, z_0}$ can be extended as a germ $\tilde{f}$ in the sheaf $\mathcal{B}_{\varepsilon, z_0}$.

Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $\mathcal{B}_{\varepsilon', z_0} \to \mathcal{B}_{\varepsilon, z_0}$ such that $\mathcal{B}_{\varepsilon, z_0}$ is the $L^2$ completion of $\mathcal{B}_{\varepsilon', z_0} / \mathcal{m}_{z_0} \mathcal{B}_{\varepsilon', z_0}$.

Question

Is there a “complex structure” on $\mathcal{B}_\varepsilon$ such that “$\mathcal{B}_\varepsilon = \mathcal{O}(\mathcal{B}_\varepsilon)$”?
Bergman Dolbeault complex

For this, consider the “Bergman Dolbeault” complex $\overline{\partial} : F^q_\varepsilon \to F^{q+1}_\varepsilon$ over $X$, with $F^q_\varepsilon (V) =$ smooth $(n, q)$-forms

$$f(z, w) = \sum_{|J|=q} f_J(z, w) \, dw_1 \wedge ... \wedge dw_n \wedge d\bar{z}_J, \quad (z, w) \in U_\varepsilon \cap (V \times \overline{X}),$$

such that $f_J(z, w)$ is holomorphic in $w$, and for all $K \subseteq V$ one has

$$f(z, w) \in L^2(p^{-1}(K)) \quad \text{and} \quad \overline{\partial} f(z, w) \in L^2(p^{-1}(K)).$$

An immediate consequence of this definition is:

**Proposition**

$\overline{\partial} = \overline{\partial} z$ yields a complex of sheaves $(F^\bullet_\varepsilon, \overline{\partial})$, and the kernel $\text{Ker} \overline{\partial} : F^0_\varepsilon \to F^1_\varepsilon$ coincides with $B_\varepsilon$.

If we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of $L^2_{\text{loc}}$ sections $f$ of $B_\varepsilon$ such that $\overline{\partial} f = 0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_\varepsilon) = B_\varepsilon$ as a sheaf.

Bergman sheaves are “very ample”

**Theorem**

Assume that $\varepsilon > 0$ is taken so small that $\psi(z, w) := |\log h_z (w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(F^\bullet_\varepsilon, \overline{\partial})$ is a resolution of $B_\varepsilon$ by soft sheaves over $X$ (actually, by $\mathcal{C}_X^\infty$-modules), and for every holomorphic vector bundle $E \to X$ we have

$$H^q(X, B_\varepsilon \otimes \mathcal{O}(E)) = H^q(\Gamma(X, F^\bullet_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}) = 0, \quad \forall q \geq 1.$$

Moreover the fibers $B_{\varepsilon, z} \otimes E_z$ are always generated by global sections of $H^0(X, B_{\varepsilon} \otimes \mathcal{O}(E))$.

In that sense, $B_\varepsilon$ is a “very ample holomorphic vector bundle” (as a Hilbert bundle of infinite dimension). The proof is a direct consequence of Hörmander’s $L^2$ estimates.

**Caution !!**

$B_\varepsilon$ is NOT a locally trivial holomorphic bundle.
Corollary of the very ampleness of Bergman sheaves

Let $X$ be an arbitrary compact complex manifold, $E \to X$ a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space $\mathbb{H} = H^0(X, B_\varepsilon \otimes O(E))$. Then one gets a “holomorphic embedding” into a Hilbert Grassmannian,

$$\Psi : X \to \text{Gr}(\mathbb{H}), \quad z \mapsto S_z,$$

mapping every point $z \in X$ to the infinite codimensional closed subspace $S_z$ consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_\varepsilon, z$, i.e. $f|_{\rho^{-1}(z)} = 0$.

The main problem with this “holomorphic embedding” is that the holomorphicity is to be understood in a weak sense, for instance the map $\Psi$ is not even continuous with respect to the strong metric topology of $\text{Gr}(\mathbb{H})$, given by $d(S, S') = \text{Hausdorff distance of the unit balls of } S, S'$.

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection on $B_\varepsilon$, and a natural hermitian metric as well, it follows from the usual formalism that $B_\varepsilon$ can be equipped with a unique Chern connection.

**Model case: $X = \mathbb{C}^n$, $\gamma = $ standard hermitian metric.**

Then one sees that an orthonormal frame of $B_\varepsilon$ is given by

$$e_\alpha(z, w) = \pi^{-n/2} \varepsilon^{-|\alpha| - n} \sqrt{\frac{(|\alpha| + n)!}{\alpha_1! \cdots \alpha_n!}} (w - z)^\alpha, \quad \alpha \in \mathbb{N}^n.$$

It is non holomorphic! The $(0, 1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

$$\nabla^{0,1} e_\alpha = \overline{\partial}_z e_\alpha(z, w) = \varepsilon^{-1} \sum_{1 \leq j \leq n} \sqrt{\alpha_j (|\alpha| + n)} \ d\bar{z}_j \otimes e_{\alpha - c_j}$$

where $c_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^n$. 
Curvature of Bergman bundles

Let $\Theta_{B_\varepsilon,h} = \nabla^2$ be the curvature tensor of $B_\varepsilon$ with its natural Hilbertian metric $h$. Remember that

$$\Theta_{B_\varepsilon,h} = \nabla^{1,0} \nabla^{0,1} + \nabla^{0,1} \nabla^{1,0} \in C^\infty(X, \Lambda^{1,1} T_X^* \otimes \text{Hom}(B_\varepsilon, B_\varepsilon)),$$

and that one gets an associated quadratic Hermitian form on $T_X \otimes B_\varepsilon$ such that

$$\tilde{\Theta}_\varepsilon(v \otimes \xi) = \langle \Theta_{B_\varepsilon,h} \sigma(v, Jv) \xi, \xi \rangle_h$$

for $v \in T_X$ and $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

**Definition**

One says that the curvature tensor is **Griffiths positive** if

$$\tilde{\Theta}_\varepsilon(v \otimes \xi) > 0, \quad \forall 0 \neq v \in T_X, \quad \forall 0 \neq \xi \in B_\varepsilon,$$

and **Nakano positive** if

$$\tilde{\Theta}_\varepsilon(\tau) > 0, \quad \forall 0 \neq \tau \in T_X \otimes B_\varepsilon.$$

**Calculation of the curvature tensor for $X = \mathbb{C}^n$**

A simple calculation of $\nabla^2$ in the orthonormal frame $(e_\alpha)$ leads to:

**Formula**

In the model case $X = \mathbb{C}^n$, the curvature tensor of the Bergman bundle $(B_\varepsilon,h)$ is given by

$$\tilde{\Theta}_\varepsilon(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \left( \left\| \sum_j \sqrt{\alpha_j} \xi_{\alpha - e_j} v_j \right\|^2 + \sum_j (|\alpha| + n) |\xi_\alpha|^2 |v_j|^2 \right).$$

**Consequence**

In $\mathbb{C}^n$, the curvature tensor $\Theta_\varepsilon(v \otimes \xi)$ is Nakano positive.

On should observe that $\tilde{\Theta}_\varepsilon(v \otimes \xi)$ is an **unbounded** quadratic form on $B_\varepsilon$ with respect to the standard metric $\|\xi\|^2 = \sum_\alpha |\xi_\alpha|^2$.

However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, since then

$$\sum_\alpha (\varepsilon'/\varepsilon)^{2|\alpha|} |\xi_\alpha|^2 < +\infty.$$
Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

Let $X$ be a compact complex manifold equipped with a $C^\omega$ hermitian metric $\gamma$, and $B_\varepsilon = B_{\gamma, \varepsilon}$ the associated Bergman bundle. Then its curvature is given by an asymptotic expansion

$$\tilde{\Theta}_\varepsilon(z, v \otimes \xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v \otimes \xi), \quad v \in T_X, \quad \xi \in B_\varepsilon$$

where $Q_0(z, v \otimes \xi) = Q_0(v \otimes \xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v \otimes \xi) = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \left( \sum_j \sqrt{\alpha_j} \xi_{\alpha - c_j} v_j \right)^2 + \sum_j (|\alpha| + n) |\xi_{\alpha}|^2 |v_j|^2.$$  

The other terms $Q_p(z, v \otimes \xi)$ are real analytic; $Q_1$ and $Q_2$ depend respectively on the torsion and curvature tensor of $\gamma$.

In particular $Q_1 = 0$ is $\gamma$ is Kähler.

A consequence of the above formula is that $B_\varepsilon$ is strongly Nakano positive for $\varepsilon > 0$ small enough.

Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by Wang Xu, expressing the curvature of weighted Bergman bundles $\mathcal{H}_t$ attached to a smooth family $\{D_t\}$ of strongly pseudoconvex domains. Wang’s formula is however in integral form and not completely explicit.

Here, one simply uses the real analytic Taylor expansion of $\log h : X \times \overline{X} \to T_X$ (inverse diffeomorphism of $\exp h$)

$$\log h_z(w) = w - \bar{z} + \sum z_j a_j(w - \bar{z}) + \sum \bar{z}_j a'_j(w - \bar{z})$$

$$+ \sum z_j z_k b_{jk}(w - \bar{z}) + \sum \bar{z}_j \bar{z}_k b'_{jk}(w - \bar{z})$$

$$+ \sum z_j \bar{z}_k c_{jk}(w - \bar{z}) + O(|z|^3),$$

which is used to compute the difference with the model case $\mathbb{C}^n$, for which $\log h_z(w) = w - \bar{z}$. 


Conjecture

Let \( \pi : \mathcal{X} \to S \) be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base \( S \). Assume that the family admits a polarization, i.e. a closed smooth \((1,1)\)-form \( \omega \) such that \( \omega|_{X_t} \) is positive definite on each fiber \( X_t := \pi^{-1}(t) \). Then the plurigenera

\[
p_m(X_t) = h^0(X_t, mK_{X_t}) \text{ are independent of } t \text{ for all } m \geq 0.
\]

The conjecture is known to be true for a projective family \( \mathcal{X} \to S \):

- Siu and Kawamata (1998) in the case of varieties of general type

No algebraic proof is known in the latter case; one deeply uses the \( L^2 \) estimates of the Ohsawa-Takegoshi extension theorem.

Invariance of plurigenera: strategy of proof (1)

It is enough to consider the case of a family \( \mathcal{X} \to \Delta \) over the disc, such that there exists a relatively ample line bundle \( \mathcal{A} \) over \( \mathcal{X} \).

Given \( s \in H^0(X_0, mK_{X_0}) \), the point is to show that it extends into \( \tilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}}) \), and for this, one only needs to produce a hermitian metric \( h = e^{-\varphi} \) on \( K_{\mathcal{X}} \) such that:

- \( \Theta_h = i\partial\bar{\partial}\varphi \geq 0 \) in the sense of currents
- \( |s|_h^2 = |s|^2e^{-\varphi} \leq 1 \), i.e. \( \varphi \geq \log |s| \) on \( X_0 \).

The Ohsawa-Takegoshi theorem then implies the existence of \( \tilde{s} \).

To produce \( h = e^{-\varphi} \), one produces inductively (also by O-T !) sections of \( \sigma_{p,j} \) of \( \mathcal{L}_p := \mathcal{A} + pK_{\mathcal{X}} \) such that:

- \( (\sigma_{p,j}) \) generates \( \mathcal{L}_p \) for \( 0 \leq p < m \)
- \( \sigma_{p,j} \) extends \( (\sigma_{p-m,j})|_{X_0} \) to \( \mathcal{X} \) for \( p \geq m \)

- \( \int_{\mathcal{X}} \sum_j |\sigma_{p,j}|^2 \leq C \) for \( p \geq 1 \).
Invariance of plurigenera: strategy of proof (2)

By Hölder, the $L^2$ estimates imply $\int_X (\sum_j |\sigma_{p,j}|^2)^{1/p} \leq C$ for all $p$, and using the fact that $\lim \frac{1}{p} \Theta_\mathcal{A} = 0$, one can take

$$\varphi = \limsup_{p \to +\infty} \varphi_p, \quad \varphi_p := \frac{1}{p} \log \sum_j |\sigma_{p,j}|^2.$$  

**Idea.** In the polarized Kähler case, use the Bergman bundle $B_\varepsilon \to X$ instead of an ample line bundle $A \to X$. This amounts to applying the Ohsawa-Takegoshi $L^2$ extension on Stein tubular neighborhoods $U_\varepsilon \subset X \times \overline{X}$, with projections $\text{pr}_1 : U_\varepsilon \to X$ and $\pi : X \to \Delta$.

**Proposition**

In the polarized Kähler case $(X, \omega)$, shrinking from $U_\varepsilon$ to $U_{\rho \varepsilon}$ with $\rho < 1$, the $B_\varepsilon$ curvature estimate gives

$$\varphi_p := \frac{1}{p} \log \sum_j \|\sigma_{p,j}\|^2_{U_{\rho \varepsilon}} \Rightarrow \ i\partial\overline{\partial}\varphi_p \geq -\frac{C}{\varepsilon^2 \rho^2} (C' - \varphi_p) \omega.$$  

This implies that $\varphi = \limsup \varphi_p$ satisfies $\psi := -\log(C'' - \varphi)$ quasi-psh, but yields invariance of plurigenera only for $\varepsilon \to +\infty$.

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**Transcendental holomorphic Morse inequalities**

**Conjecture**

Let $X$ be a compact $n$-dimensional complex manifold and $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial \overline{\partial}$ exact forms. Set

$$\text{Vol}(\alpha) = \sup_{T = \alpha + i\partial\overline{\partial}\varphi \geq 0} \int_X T^n_{\alpha c}, \quad T \geq 0 \text{ current}.$$  

Then

$$\text{Vol}(\alpha) \geq \sup_{u \in \{\alpha\}, \ u \in C^\infty} \int_{X(u,0)} u^n$$  

where

$$X(u,0) = 0\text{-index set of } u = \{x \in X ; \ u(x) \text{ positive definite}\}.$$  

**Conjectural corollary (fundamental volume estimate)**

Let $X$ be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be nef classes. Then

$$\text{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$
The conjecture on Morse inequalities is known to be true when \( \alpha = c_1(L) \) is the class of a line bundle ([D-1985]), and the corollary can be derived from this when \( \alpha, \beta \) are integral classes (by [D-1993] and independently by [Trapani, 1993]).

Recently, the volume estimate for \( \alpha, \beta \) transcendental has been established by D. Witt-Nyström when \( X \) is projective, and Xiao-Popovici even proved in general that \( \text{Vol}(\alpha - \beta) > 0 \) if \( \alpha^n - n\alpha^{n-1} \cdot \beta > 0 \).

**Idea.** In the general case, one can find a sequence of non holomorphic hermitian line bundles \((L_m, h_m)\) such that

\[
m \alpha = \Theta_{L_m, h_m} + \gamma_m^{2,0} + \gamma_m^{0,2}, \quad \gamma_m \to 0.
\]

As \( U_\varepsilon \) is Stein, \( \gamma_m^{0,2} = \overline{\partial} \nu_m, \nu_m \to 0, \) and \( \text{pr}_1^* L_m \) becomes a holomorphic line bundle with curvature form \( \Theta_{\text{pr}_1^* L_m} \simeq m \text{pr}_1^* \alpha. \)

Then apply \( L^2 \) direct image \((\text{pr}_1)_* L_e^2\) and use Bergman estimates instead of dimension counts in Morse inequalities.

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**The end**

**Thank you for your attention**