Basic results on Sheaves and Analytic Sets

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The goal of these notes is mainly to fix the notation and prove a few fundamental results concerning sheaves, sheaf cohomology, in relation with coherent sheaves and complex analytic sets. Since we adopt later an analytic point of view, it is enough to develope sheaf theory in the case of paracompact spaces. The reader is referred to (Godement 1958) for a more complete treatment of sheaf theory. There are also many good references dealing with analytic sets and coherent sheaves, e.g. (Cartan 1950), (Gunning-Rossi 1965), (Narasimhan 1966) and (Grauert-Remmert 1984).

1. Presheaves and Sheaves

1.A. Definitions

Sheaves have become a very important tool in analytic or algebraic geometry as well as in algebraic topology. They are especially useful when one wants to relate global properties of an object to its local properties (the latter being usually easier to establish). We first introduce presheaves and sheaves in full generality and give some basic examples.

(1.1) Definition. *Let X be a topological space. A presheaf $\mathcal{A}$ on X is a collection of non empty sets $\mathcal{A}(U)$ associated to every open set $U \subset X$ (the sets $\mathcal{A}(U)$ are called sets of sections of $\mathcal{A}$ over U), together with maps*

$$\rho_{U,V} : \mathcal{A}(V) \longrightarrow \mathcal{A}(U)$$

defined whenever $U \subset V$ and satisfying the transitivity property

$$\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W} \quad \text{for } U \subset V \subset W, \qquad \rho_{U,U} = \mathrm{Id}_U \quad \text{for every } U.$$

Moreover, the presheaf $\mathcal{A}$ is said to be a sheaf if it satisfies the following additional gluing axioms, where (U_α) and $U = \bigcup U_\alpha$ are arbitrary open subsets of X :

(i) *If $F_\alpha \in \mathcal{A}(U_\alpha)$ are such that $\rho_{U_\alpha \cap U_\beta, U_\alpha}(F_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(F_\beta)$ for all α, β, there exists $F \in \mathcal{A}(U)$ such that $\rho_{U_\alpha, U}(F) = F_\alpha$;*

(ii) *If $F, G \in \mathcal{A}(U)$ and $\rho_{U_\alpha, U}(F) = \rho_{U_\alpha, U}(G)$ for all α, then $F = G$;*

in other words, local sections over the sets U_α can be glued together if they coincide in the intersections and the resulting section on U is uniquely defined.

(1.2) Examples. Not all presheaves are sheaves. Let E be an arbitrary set with a distinguished element 0. The *constant presheaf* E_X on X is defined to be $E_X(U) = E$ for all $\emptyset \neq U \subset X$ and $E_X(\emptyset) = \{0\}$, with restriction maps $\rho_{U,V} = \mathrm{Id}_E$ for $\emptyset \neq U \subset V$. Then clearly axiom (i) is not satisfied if U is the union of two disjoint open sets U_1, U_2 (and if E contains at least two elements).

On the other hand, if we assign to each open set $U \subset X$ the set $\mathcal{C}(U)$ of all real valued continuous functions on U and let $\rho_{U,V}$ be the obvious restriction morphism $\mathcal{C}(V) \to \mathcal{C}(U)$, then $\mathcal{C}$ is a sheaf on X, because continuity is a purely local notion. Similarly if X is a differentiable (resp. complex analytic) manifold, there are well defined sheaves of rings $\mathcal{C}^k$ (resp. $\mathcal{O}$) of functions of class C^k (resp. of holomorphic functions) on X. Because of these examples, the maps $\rho_{U,V}$ in Def. 1.1 are often viewed intuitively as "restriction homomorphisms", although the sets $\mathcal{A}(U)$ are not necessarily sets of functions defined over U.

Most often, the presheaf $\mathcal{A}$ is supposed to carry an additional algebraic structure.

(1.3) Definition. *A presheaf $\mathcal{A}$ is said to be a presheaf of abelian groups (resp. rings, R-modules, algebras) if all sets $\mathcal{A}(U)$ are abelian groups (resp. rings, R-modules, algebras) and if the maps $\rho_{U,V}$ are morphisms of these algebraic structures. In this case, we always assume that $\mathcal{A}(\emptyset) = \{0\}$.*

If $\mathcal{A}$, $\mathcal{B}$ are presheaves of abelian groups (or of some other algebraic structure) on the same space X, a presheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ commuting with the restriction morphisms, i.e. such that $\rho^{\mathcal{B}}_{U,V} \circ \varphi_V = \varphi_U \circ \rho^{\mathcal{A}}_{U,V}$. We say that $\mathcal{A}$ is a subpresheaf of $\mathcal{B}$ in the case where $\varphi_U : \mathcal{A}(U) \subset \mathcal{B}(U)$ is the inclusion morphism; the commutation property then means that $\rho^{\mathcal{B}}_{U,V}(\mathcal{A}(V)) \subset \mathcal{A}(U)$ for all U, V, and that $\rho^{\mathcal{A}}_{U,V}$ coincides with $\rho^{\mathcal{B}}_{U,V}$ on $\mathcal{A}(V)$. If $\mathcal{A}$ is a subpresheaf of a sheaf $\mathcal{B}$ of abelian groups, there is a quotient presheaf $\mathcal{C} = \mathcal{B}/\mathcal{A}$ defined by $\mathcal{C}(U) = \mathcal{B}(U)/\mathcal{A}(U)$. In a similar way, one can define direct sums $\mathcal{A} \oplus \mathcal{B}$ of presheaves of abelian groups, tensor products $\mathcal{A} \otimes \mathcal{B}$ of presheaves of R-modules, etc.

(1.4) Definition. *Let X and $\mathcal{S}$ be topological spaces (not necessarily Hausdorff), and let $\pi : \mathcal{S} \longrightarrow X$ be a mapping such that*

a) *π maps $\mathcal{S}$ onto X ;*

b) *π is a local homeomorphism, that is, every point in $\mathcal{S}$ has an open neighborhood which is mapped homeomorphically by π onto an open subset of X.*

Then $\mathcal{S}$ is called a sheaf-space on X and π is called the projection of $\mathcal{S}$ on X. If $x \in X$, then $\mathcal{S}_x = \pi^{-1}(x)$ is called the stalk of $\mathcal{S}$ at x.

If Y is a subset of X, we denote by $\mathit{\Gamma}(Y, \mathcal{S})$ the set of sections of $\mathcal{S}$ on Y, i.e. the set of continuous functions $F : Y \to \mathcal{S}$ such that $\pi \circ F = \mathrm{Id}_Y$. It is clear that the presheaf defined by the collection of sets $\mathcal{S}'(U) := \mathit{\Gamma}(U, \mathcal{S})$ for all open sets $U \subset X$ together with the restriction maps $\rho_{U,V}$ satisfies axioms 1.1(i) and (ii). hence $\mathcal{S}'$ is a sheaf. If Y is a subset of X, we denote by $\Gamma(Y,\mathcal{S})$ the set of sections of $\mathcal{S}$ on Y, i.e. the set of continuous functions $F : Y \to \mathcal{S}$ such that $\pi \circ F = \mathrm{Id}_Y$. It is clear that the presheaf defined by the collection of sets $\mathcal{S}'(U) := \Gamma(U,\mathcal{S})$ for all open sets $U \subset X$ together with the restriction maps $\rho_{U,V}$ satisfies axioms 1.1(i) and (ii). hence $\mathcal{S}'$ is a sheaf.

Conversely, it is possible to assign in a natural way a sheaf-space to any presheaf. If $\mathcal{A}$ is a presheaf, we define the set $\widetilde{\mathcal{A}}_x$ of germs of $\mathcal{A}$ at a point $x \in X$ to be the abstract inductive limit

$$\widetilde{\mathcal{A}}_x = \varinjlim_{U\ni x} \big(\mathcal{A}(U), \rho_{U,V}\big). \tag{1.5}$$

For every $F \in \mathcal{A}(U)$, there is a natural map which assigns to F its germ $F_x \in \widetilde{\mathcal{A}}_x$. Now, the disjoint sum $\widetilde{\mathcal{A}} = \coprod_{x\in X} \widetilde{\mathcal{A}}_x$ can be equipped with a natural topology as follows: for all $F \in \mathcal{A}(U)$, we set

$$\Omega_{F,U} = \big\{F_x \ ;\ x \in U\big\}$$

and choose the $\Omega_{F,U}$ to be a basis of the topology of $\widetilde{\mathcal{A}}$; note that this family is stable by intersection: $\Omega_{F,U} \cap \Omega_{G,V} = \Omega_{H,W}$ where W is the (open) set of points $x \in U \cap V$ at which $F_x = G_x$ and $H = \rho_{W,U}(F)$. The obvious projection map $\pi : \widetilde{\mathcal{A}} \to X$ which sends $\widetilde{\mathcal{A}}_x$ to $\{x\}$ is then a local homeomorphism (it is actually a homeomorphism from $\Omega_{F,U}$ onto U). Hence $\widetilde{\mathcal{A}}$ is a sheaf-space.

If $\mathcal{S}$ is a sheaf-space, the sheaf-space associated to the presheaf $\mathcal{S}'(U) := \Gamma(U,\mathcal{S})$ is canonically isomorphic to $\mathcal{S}$, since

$$\varinjlim_{U\ni x} \Gamma(U,\mathcal{S}) = \mathcal{S}_x$$

thanks to the local homeomorphism property. Hence, we will usually not make any difference between a sheaf-space $\mathcal{S}$ and its associated sheaf $\mathcal{S}'$, both will be denoted by the same symbol $\mathcal{S}$.

Now, given a presheaf $\mathcal{A}$, there is an obvious presheaf morphism $\mathcal{A} \to \widetilde{\mathcal{A}}'$ defined by

$$\mathcal{A}(U) \longrightarrow \widetilde{\mathcal{A}}'(U) := \Gamma(U,\widetilde{\mathcal{A}}), \qquad F \longmapsto \widetilde{F} = (U \ni x \mapsto F_x). \tag{1.6}$$

This morphism is clearly injective if and only if $\mathcal{A}$ satisfies axiom 1.1(i) and it is not difficult to see that 1.1(i) and (ii) together imply surjectivity. Therefore $\mathcal{A} \to \widetilde{\mathcal{A}}'$ is an isomorphism if and only if $\mathcal{A}$ is a sheaf. According to what we said above, $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}'$ will be denoted by the same symbol $\widetilde{\mathcal{A}}$: $\widetilde{\mathcal{A}}$ is called *the sheaf associated to the presheaf* $\mathcal{A}$. It will be identified with $\mathcal{A}$ itself if $\mathcal{A}$ is a sheaf, but the notational difference will be kept if $\mathcal{A}$ is not a sheaf.

(1.7) Example. The sheaf associated to the constant presheaf of stalk E over X is the sheaf of locally constant functions $X \to E$. This sheaf will be denoted merely by E_X or E if there is no risk of confusion with the corresponding presheaf.

In the sequel, we usually work in the category of sheaves rather than in the category of presheaves themselves. For instance, the quotient $\mathcal{B}/\mathcal{A}$ of a sheaf $\mathcal{B}$ by a subsheaf $\mathcal{A}$ generally refers to the sheaf associated with the quotient presheaf: its stalks are equal to $\mathcal{B}_x/\mathcal{A}_x$, but a section G of $\mathcal{B}/\mathcal{A}$ over an open set U need not necessarily come from a global section of $\mathcal{B}(U)$; what can be only said is that there is a covering (U_α) of U and local sections $F_\alpha \in \mathcal{B}(U_\alpha)$ representing $G_{\restriction U_\alpha}$ such that $(F_\beta - F_\alpha)_{\restriction U_\alpha \cap U_\beta}$ belongs to $\mathcal{A}(U_\alpha \cap U_\beta)$. A sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be injective (resp. surjective) if the germ morphism $\varphi_x : \mathcal{A}_x \to \mathcal{B}_x$ is injective (resp. surjective) for every $x \in X$. Let us note again that a surjective sheaf morphism φ does not necessarily give rise to surjective morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$.

1.B. Direct and Inverse Images of Sheaves

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If $\mathcal{A}$ is a presheaf on X, the *direct image* $f_\star\mathcal{A}$ is the presheaf on Y defined by

$$f_\star\mathcal{A}(U) = \mathcal{A}\big(f^{-1}(U)\big) \tag{1.8}$$

for all open sets $U \subset Y$. When $\mathcal{A}$ is a sheaf, it is clear that $f_\star\mathcal{A}$ also satisfies axioms 1.1(i) and 1.1(ii), thus $f_\star\mathcal{A}$ is a sheaf. Its stalks are given by

$$(f_\star\mathcal{A})_y = \varinjlim_{V \ni y} \mathcal{A}\big(f^{-1}(V)\big) \tag{1.9}$$

where V runs over all open neighborhoods of $y \in Y$.

Now, let $\mathcal{B}$ be a sheaf on Y, viewed as a sheaf-space with projection map $\pi : \mathcal{B} \to Y$. We define the *inverse image* $f^{-1}\mathcal{B}$ by

$$f^{-1}\mathcal{B} = \mathcal{B} \times_Y X = \big\{(s,x) \in \mathcal{B} \times X\,;\ \pi(s) = f(x)\big\} \tag{1.10}$$

with the topology induced by the product topology on $\mathcal{B} \times X$. It is then easy to see that the projection $\pi' = \mathrm{pr}_2 : f^{-1}\mathcal{B} \to X$ is a local homeomorphism, therefore $f^{-1}\mathcal{B}$ is a sheaf on X. By construction, the stalks of $f^{-1}\mathcal{B}$ are

$$(f^{-1}\mathcal{B})_x = \mathcal{B}_{f(x)}, \tag{1.11}$$

and the sections $\sigma \in f^{-1}\mathcal{B}(U)$ can be considered as continuous mappings $s : U \to \mathcal{B}$ such that $\pi \circ \sigma = f$. In particular, any section $s \in \mathcal{B}(V)$ on an open set $V \subset Y$ has a *pull-back*

$$f^\star s = s \circ f \quad \in \quad f^{-1}\mathcal{B}\big(f^{-1}(V)\big). \tag{1.12}$$

There are always natural sheaf morphisms

$$f^{-1}f_\star\mathcal{A} \longrightarrow \mathcal{A}, \qquad \mathcal{B} \longrightarrow f_\star f^{-1}\mathcal{B} \tag{1.13}$$

defined as follows. A germ in $(f^{-1}f_\star\mathcal{A})_x = (f_\star\mathcal{A})_{f(x)}$ is defined by a local section $s \in (f_\star\mathcal{A})(V) = \mathcal{A}(f^{-1}(V))$ for some neighborhood V of $f(x)$; this section can be mapped to the germ $s_x \in \mathcal{A}_x$. In the opposite direction, the pull-back $f^\star s$ of a section $s \in \mathcal{B}(V)$ can be seen by (1.12) as a section of $f_\star f^{-1}\mathcal{B}(V)$. It is not difficult to see that these natural morphisms are not isomorphisms in general. For instance, if f is a finite covering map with q sheets and if we take $\mathcal{A} = E_X$, $\mathcal{B} = E_Y$ to be constant sheaves, then $f_\star E_X \simeq E_Y^q$ and $f^{-1}E_Y = E_X$, thus $f^{-1}f_\star E_X \simeq E_X^q$ and $f_\star f^{-1}E_Y \simeq E_Y^q$.

2. Coherent Sheaves

2.A. Locally Free Sheaves and Vector Bundles

Usually, the geometric structures introduced in analytic or algebraic geometry can be described in a convenient way by means of a structure sheaf $\mathcal{A}$ which, most often, is a sheaf of commutative rings. For instance, the holomorphic properties of a complex manifold X are well described by the sheaf $\mathcal{O}_X$ of holomorphic functions. Before introducing the more general notion of a coherent sheaf, we discuss the notion of locally free sheaves over a sheaf a ring. All rings occurring in the sequel are supposed to be commutative with unit (the non commutative case is also of considerable interest, e.g. in view of the theory of $\mathcal{D}$-modules, but this subject is beyond the scope of these notes).

(2.1) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally free of rank r over $\mathcal{A}$, if $\mathcal{S}$ is locally isomorphic to $\mathcal{A}^{\oplus r}$ on a neighborhood of every point, i.e. for every $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_r \in \mathcal{S}(\Omega)$ such that the sheaf homomorphism*

$$F : \mathcal{A}^{\oplus r}_{\restriction \Omega} \longrightarrow \mathcal{S}_{\restriction \Omega}, \qquad \mathcal{A}^{\oplus r}_x \ni (w_1, \ldots, w_r) \longmapsto \sum_{1 \le j \le r} w_j F_{j,x} \in \mathcal{S}_x$$

is an isomorphism.

By definition, if $\mathcal{S}$ is locally free, there is a covering $(U_\alpha)_{\alpha \in I}$ by open sets on which $\mathcal{S}$ admits free generators $F_\alpha^1, \ldots, F_\alpha^r \in \mathcal{S}(U_\alpha)$. Because the generators can be uniquely expressed in terms of any other system of independent generators, there is for each pair (α, β) a $r \times r$ matrix

$$G_{\alpha\beta} = (G_{\alpha\beta}^{jk})_{1 \le j,k \le r}, \qquad G_{\alpha\beta}^{jk} \in \mathcal{A}(U_\alpha \cap U_\beta),$$

such that

$$F_\beta^k = \sum_{1 \le j \le r} F_\alpha^j G_{\alpha\beta}^{jk} \qquad \text{on} \quad U_\alpha \cap U_\beta.$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^{\oplus r}_{\restriction U_\alpha \cap U_\beta} & \xrightarrow{F_\alpha} & \mathcal{S}_{\restriction U_\alpha \cap U_\beta} \\ G_{\alpha\beta} \uparrow & & \| \\ \mathcal{A}^{\oplus r}_{\restriction U_\alpha \cap U_\beta} & \xrightarrow[F_\beta]{} & \mathcal{S}_{\restriction U_\alpha \cap U_\beta} \end{array}$$

It follows easily from the equality $G_{\alpha\beta} = F_\alpha^{-1} \circ F_\beta$ that the *transition matrices* $G_{\alpha\beta}$ are invertible matrices satisfying the transition relation

$$(2.2) \qquad G_{\alpha\gamma} = G_{\alpha\beta} G_{\beta\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma$$

for all indices $\alpha, \beta, \gamma \in I$. In particular $G_{\alpha\alpha} = \mathrm{Id}$ on U_α and $G_{\alpha\beta}^{-1} = G_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.

Conversely, if we are given a system of invertible $r \times r$ matrices $G_{\alpha\beta}$ with coefficients in $\mathcal{A}(U_\alpha \cap U_\beta)$ satisfying the transition relation (2.2), we can define a locally free sheaf $\mathcal{S}$ of rank r over $\mathcal{A}$ by taking $\mathcal{S} \simeq \mathcal{A}^{\oplus r}$ over each U_α, the identification over $U_\alpha \cap U_\beta$ being given by the isomorphism $G_{\alpha\beta}$. A section H of $\mathcal{S}$ over an open set $\Omega \subset X$ can just be seen as a collection of sections $H_\alpha = (H_\alpha^1, \ldots, H_\alpha^r)$ of $\mathcal{A}^{\oplus r}(\Omega \cap U_\alpha)$ satisfying the transition relations $H_\alpha = G_{\alpha\beta} H_\beta$ over $\Omega \cap U_\alpha \cap U_\beta$.

The notion of locally free sheaf is closely related to another essential notion of differential geometry, namely the notion of vector bundle (resp. topological, differentiable, holomorphic ..., vector bundle). To describe the relation between these notions, we assume that the sheaf of rings $\mathcal{A}$ is a subsheaf of the sheaf $\mathcal{C}_\mathbb{K}$ of continous functions on X with values in the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, containing the sheaf of locally constant functions $X \to \mathbb{K}$. Then, for each $x \in X$, there is an evaluation map

$$\mathcal{A}_x \to \mathbb{K}, \qquad w \mapsto w(x)$$

whose kernel is a maximal ideal $\mathfrak{m}_x$ of $\mathcal{A}_x$, and $\mathcal{A}_x/\mathfrak{m}_x = \mathbb{K}$. Let $\mathcal{S}$ be a locally free sheaf of rank r over $\mathcal{A}$. To each $x \in X$, we can associate a $\mathbb{K}$-vector space $E_x = \mathcal{S}_x/\mathfrak{m}_x\mathcal{S}_x$: since $\mathcal{S}_x \simeq \mathcal{A}_x^{\oplus r}$, we have $E_x \simeq (\mathcal{A}_x/\mathfrak{m}_x)^{\oplus r} = \mathbb{K}^r$. The set $E = \coprod_{x \in X} E_x$ is equipped with a natural projection

$$\pi : E \to X, \qquad \xi \in E_x \mapsto \pi(\xi) := x,$$

and the fibers $E_x = \pi^{-1}(x)$ have a structure of r-dimensional $\mathbb{K}$-vector space: such a structure E is called a $\mathbb{K}$-*vector bundle of rank* r over X. Every section $s \in \mathcal{S}(U)$ gives rise to a *section* of E over U by setting $s(x) = s_x \bmod \mathfrak{m}_x$. We obtain a function (still denoted by the same symbol) $s : U \to E$ such that $s(x) \in E_x$ for every $x \in U$, i.e. $\pi \circ s = \mathrm{Id}_U$. It is clear that $\mathcal{S}(U)$ can be considered as a $\mathcal{A}(U)$-submodule of the $\mathbb{K}$-vector space of functions $U \to E$ mapping a point $x \in U$ to an element in the fiber E_x. Thus we get a subsheaf of the sheaf of E-valued sections, which is in a natural way a $\mathcal{A}$-module isomorphic to $\mathcal{S}$. This subsheaf will be denoted by $\mathcal{A}(E)$ and will be called the *sheaf of* $\mathcal{A}$-*sections* of E. If we are given a $\mathbb{K}$-vector bundle E over X and a subsheaf $\mathcal{S} = \mathcal{A}(E)$ of the sheaf of all sections of E which is in a natural way a locally free $\mathcal{A}$-module of rank r, we say that E (or more precisely the pair $(E, \mathcal{A}(E))$) is a $\mathcal{A}$-vector bundle of rank r over X.

(2.3) Example. In case $\mathcal{A} = \mathcal{C}_{X,\mathbb{K}}$ is the sheaf of all $\mathbb{K}$-valued continuous functions on X, we say that E is a *topological* vector bundle over X. When X is a manifold and $\mathcal{A} = \mathcal{C}_{X,\mathbb{K}}^p$, we say that E is a C^p-*differentiable* vector bundle; finally, when X is complex analytic and $\mathcal{A} = \mathcal{O}_X$, we say that E is a *holomorphic* vector bundle.

Let us introduce still a little more notation. Since $\mathcal{A}(E)$ is a locally free sheaf of rank r over any open set U_α in a suitable covering of X, a choice of generators

$(F^1_\alpha, \ldots, F^r_\alpha)$ for $\mathcal{A}(E)_{\restriction U_\alpha}$ yields corresponding generators $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of the fibers E_x over $\mathbb{K}$. Such a system of generators is called a *$\mathcal{A}$-admissible frame* of E over U_α. There is a corresponding isomorphism

$$\theta_\alpha : E_{\restriction U_\alpha} := \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^r \tag{2.4}$$

which to each $\xi \in E_x$ associates the pair $(x, (\xi^1_\alpha, \ldots, \xi^r_\alpha)) \in U_\alpha \times \mathbb{K}^r$ composed of x and of the components $(\xi^j_\alpha)_{1 \le j \le r}$ of ξ in the basis $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of E_x. The bundle E is said to be *trivial* if it is of the form $X \times \mathbb{K}^r$, which is the same as saying that $\mathcal{A}(E) = \mathcal{A}^{\oplus r}$. For this reason, the isomorphisms θ_α are called *trivializations* of E over U_α. The corresponding *transition automorphisms* are

$$\begin{aligned} &\theta_{\alpha\beta} := \theta_\alpha \circ \theta_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^r \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{K}^r, \\ &\theta_{\alpha\beta}(x, \xi) = (x, g_{\alpha\beta}(x) \cdot \xi), \qquad (x, \xi) \in (U_\alpha \cap U_\beta) \times \mathbb{K}^r, \end{aligned}$$

where $(g_{\alpha\beta}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ are the transition matrices already described (except that they are just seen as matrices with coefficients in $\mathbb{K}$ rather than with coefficients in a sheaf). Conversely, if we are given a collection of matrices $g_{\alpha\beta} = (g^{jk}_{\alpha\beta}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ satisfying the transition relation

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma,$$

we can define a $\mathcal{A}$-vector bundle

$$E = \Big(\coprod_{\alpha \in I} U_\alpha \times \mathbb{K}^r\Big)/\sim$$

by gluing the charts $U_\alpha \times \mathbb{K}^r$ via the identification $(x_\alpha, \xi_\alpha) \sim (x_\beta, \xi_\beta)$ if and only if $x_\alpha = x_\beta = x \in U_\alpha \cap U_\beta$ and $\xi_\alpha = g_{\alpha\beta}(x) \cdot \xi_\beta$.

(2.5) Example. When X is a real differentiable manifold, an interesting example of real vector bundle is the *tangent bundle* T_X ; if $\tau_\alpha : U_\alpha \to \mathbb{R}^n$ is a collection of coordinate charts on X, then $\theta_\alpha = \pi \times d\tau_\alpha : T_{X \restriction U_\alpha} \to U_\alpha \times \mathbb{R}^m$ define trivializations of T_X and the transition matrices are given by $g_{\alpha\beta}(x) = d\tau_{\alpha\beta}(x^\beta)$ where $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ and $x^\beta = \tau_\beta(x)$. The dual $T^\star_X$ of T_X is called the *cotangent bundle* of X. If X is complex analytic, then T_X has the structure of a holomorphic vector bundle.

We now briefly discuss the concept of sheaf and bundle morphisms. If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of $\mathcal{A}$-modules over a topological space X, then by a morphism $\varphi : \mathcal{S} \to \mathcal{S}'$ we just mean a $\mathcal{A}$-linear sheaf morphism. If $\mathcal{S} = \mathcal{A}(E)$ and $\mathcal{S}' = \mathcal{A}(E')$ are locally free sheaves, this is the same as a $\mathcal{A}$-linear bundle morphism, that is, a fiber preserving $\mathbb{K}$-linear morphism $\varphi(x) : E_x \to E'_x$ such that the matrix representing φ in any local $\mathcal{A}$-admissible frames of E and E' has coefficients in $\mathcal{A}$.

(2.6) Proposition. *Suppose that $\mathcal{A}$ is a sheaf of local rings, i.e. that a section of $\mathcal{A}$ is invertible in $\mathcal{A}$ if and only if it never takes the zero value in $\mathbb{K}$. Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a $\mathcal{A}$-morphism of locally free $\mathcal{A}$-modules of rank r, r'. If the rank of the $r' \times r$ matrix $\varphi(x) \in M_{r'r}(\mathbb{K})$ is constant for all $x \in X$, then* $\operatorname{Ker}\varphi$ *and* $\operatorname{Im}\varphi$ *are locally free subsheaves of $\mathcal{S}$, $\mathcal{S}'$ respectively, and* $\operatorname{Coker}\varphi = \mathcal{S}'/\operatorname{Im}\varphi$ *is locally free.*

Proof. This is just a consequence of elementary linear algebra, once we know that non zero determinants with coefficients in $\mathcal{A}$ can be inverted. □

Note that all three sheaves $\mathcal{C}_{X,\mathbb{K}}$, $\mathcal{C}^p_{X,\mathbb{K}}$, $\mathcal{O}_X$ are sheaves of local rings, so Prop. 2.6 applies to these cases. However, even if we work in the holomorphic category ($\mathcal{A} = \mathcal{O}_X$), a difficulty immediately appears that the kernel or cokernel of an arbitrary morphism of locally free sheaves is in general not locally free.

(2.7) Examples.

a) Take $X = \mathbb{C}$, let $\mathcal{S} = \mathcal{S}' = \mathcal{O}$ be the trivial sheaf, and let $\varphi : \mathcal{O} \to \mathcal{O}$ be the morphism $u(z) \mapsto z\,u(z)$. It is immediately seen that φ is injective as a sheaf morphism ($\mathcal{O}$ being an entire ring), and that $\mathrm{Coker}\,\varphi$ is the *skyscraper sheaf* $\mathbb{C}_0$ of stalk $\mathbb{C}$ at $z = 0$, having zero stalks at all other points $z \neq 0$. Thus $\mathrm{Coker}\,\varphi$ is not a locally free sheaf, although φ is everywhere injective (note however that the corresponding morphism $\varphi : E \to E'$, $(z, \xi) \mapsto (z, z\xi)$ of trivial rank 1 vector bundles $E = E' = \mathbb{C} \times \mathbb{C}$ is *not injective* on the zero fiber E_0).

b) Take $X = \mathbb{C}^3$, $\mathcal{S} = \mathcal{O}^{\oplus 3}$, $\mathcal{S}' = \mathcal{O}$ and

$$\varphi : \mathcal{O}^{\oplus 3} \to \mathcal{O}, \qquad (u_1, u_2, u_3) \mapsto \sum_{1 \le j \le 3} z_j u_j(z_1, z_2, z_3).$$

Since φ yields a surjective bundle morphism on $\mathbb{C}^3 \smallsetminus \{0\}$, one easily sees that $\mathrm{Ker}\,\varphi$ is locally free of rank 2 over $\mathbb{C}^3 \smallsetminus \{0\}$. However, by looking at the Taylor expansion of the u_j's at 0, it is not difficult to check that $\mathrm{Ker}\,\varphi$ is the $\mathcal{O}$-submodule of $\mathcal{O}^{\oplus 3}$ generated by the three sections $(-z_2, z_1, 0)$, $(-z_3, 0, z_1)$ and $(0, z_3, -z_2)$, and that any two of these three sections cannot generate the 0-stalk $(\mathrm{Ker}\,\varphi)_0$. Hence $\mathrm{Ker}\,\varphi$ is not locally free.

Since the category of locally free $\mathcal{O}$-modules is not stable by taking kernels or cokernels, one is led to introduce a more general category which will be stable under these operations. This leads to the notion of *coherent sheaves.*

2.B. Notion of Coherence

The notion of coherence again deals with sheaves of modules over a sheaf of rings. It is a semi-local property which says roughly that the sheaf of modules locally has a finite presentation in terms of generators and relations. We describe here some general properties of this notion, before concentrating ourselves on the case of coherent $\mathcal{O}_X$-modules.

(2.8) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally finitely generated if for every point $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_q \in \mathcal{S}(\Omega)$ such that for every $x \in \Omega$ the stalk $\mathcal{S}_x$ is generated by the germs $F_{1,x}, \ldots, F_{q,x}$ as an $\mathcal{A}_x$-module.*

(2.9) Lemma. *Let $\mathcal{S}$ be a locally finitely generated sheaf of $\mathcal{A}$-modules on X and $G_1, \ldots, G_N$ sections in $\mathcal{S}(U)$ such that $G_{1,x_0}, \ldots, G_{N,x_0}$ generate $\mathcal{S}_{x_0}$ at $x_0 \in U$. Then $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for x near x_0.*

Proof. Take $F_1, \ldots, F_q$ as in Def. 2.8. As $G_1, \ldots, G_N$ generate $\mathcal{S}_{x_0}$, one can find a neighborhood $\Omega' \subset \Omega$ of x_0 and $H_{jk} \in \mathcal{A}(\Omega')$ such that $F_j = \sum H_{jk} G_k$ on Ω'. Thus $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for all $x \in \Omega'$. □

If U is an open subset of X, we denote by $\mathcal{S}_{\restriction U}$ the restriction of $\mathcal{S}$ to U, i.e. the union of all stalks $\mathcal{S}_x$ for $x \in U$. If $F_1, \ldots, F_q \in \mathcal{S}(U)$, the kernel of the sheaf homomorphism $F : \mathcal{A}^{\oplus q}_{\restriction U} \longrightarrow \mathcal{S}_{\restriction U}$ defined by

$$\mathcal{A}^{\oplus q}_x \ni (g^1, \ldots, g^q) \longmapsto \sum_{1 \le j \le q} g^j F_{j,x} \in \mathcal{S}_x, \qquad x \in U \tag{2.10}$$

is a subsheaf $\mathcal{R}(F_1, \ldots, F_q)$ of $\mathcal{A}^{\oplus q}_{\restriction U}$, called the *sheaf of relations* between $F_1, \ldots, F_q$.

(2.11) Definition. *A sheaf $\mathcal{S}$ of $\mathcal{A}$-modules on X is said to be coherent if:*

a) *$\mathcal{S}$ is locally finitely generated*;

b) *for any open subset U of X and any $F_1, \ldots, F_q \in \mathcal{S}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is locally finitely generated.*

Assumption a) means that every point $x \in X$ has a neighborhood Ω such that there is a surjective sheaf morphism $F : \mathcal{A}^{\oplus q}_{\restriction \Omega} \longrightarrow \mathcal{S}_{\restriction \Omega}$, and assumption b) implies that the kernel of F is locally finitely generated. Thus, after shrinking Ω, we see that $\mathcal{S}$ admits over Ω a finite presentation under the form of an exact sequence

$$\mathcal{A}^{\oplus p}_{\restriction \Omega} \xrightarrow{\ G\ } \mathcal{A}^{\oplus q}_{\restriction \Omega} \xrightarrow{\ F\ } \mathcal{S}_{\restriction \Omega} \longrightarrow 0, \tag{2.12}$$

where G is given by a $q \times p$ matrix (G_{jk}) of sections of $\mathcal{A}(\Omega)$ whose columns $(G_{j1}), \ldots, (G_{jp})$ are generators of $\mathcal{R}(F_1, \ldots, F_q)$.

It is clear that every locally finitely generated subsheaf of a coherent sheaf is coherent. From this we easily infer:

(2.13) Theorem. *Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a $\mathcal{A}$-morphism of coherent sheaves. Then* $\operatorname{Im}\varphi$ *and* $\ker\varphi$ *are coherent.*

Proof. Clearly $\operatorname{Im}\varphi$ is a locally finitely generated subsheaf of $\mathcal{G}$, so it is coherent. Let $x_0 \in X$, let $F_1, \ldots, F_q \in \mathcal{F}(\Omega)$ be generators of $\mathcal{F}$ on a neighborhood Ω of x_0, and $G_1, \ldots, G_r \in \mathcal{A}(\Omega')^{\oplus q}$ be generators of $\mathcal{R}\big(\varphi(F_1), \ldots, \varphi(F_q)\big)$ on a neighborhood $\Omega' \subset \Omega$ of x_0. Then $\ker\varphi$ is generated over Ω' by the sections

$$H_j = \sum_{k=1}^{q} G_j^k F_k \in \mathcal{F}(\Omega'), \qquad 1 \le j \le r. \qquad □$$

(2.14) Theorem. *Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow 0$ be an exact sequence of $\mathcal{A}$-modules. If two of the sheaves $\mathcal{F}, \mathcal{S}, \mathcal{G}$ are coherent, then all three are coherent.*

Proof. If $\mathcal{S}$ and $\mathcal{G}$ are coherent, then $\mathcal{F} = \ker(\mathcal{S} \to \mathcal{G})$ is coherent by Th. 2.13. If $\mathcal{S}$ and $\mathcal{F}$ are coherent, then $\mathcal{G}$ is locally finitely generated; to prove the coherence, let $G_1, \ldots, G_q \in \mathcal{G}(U)$ and $x_0 \in U$. Then there is a neighborhood Ω of x_0 and sections $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega)$ which are mapped to $G_1, \ldots, G_q$ on Ω. After shrinking Ω, we may assume also that $\mathcal{F}_{\restriction\Omega}$ is generated by sections $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$. Then $\mathcal{R}(G_1, \ldots, G_q)$ is the projection on the last q-components of $\mathcal{R}(F_1, \ldots, F_p, \tilde{G}_1, \ldots, \tilde{G}_q) \subset \mathcal{A}^{p+q}$, which is finitely generated near x_0 by the coherence of $\mathcal{S}$. Hence $\mathcal{R}(G_1, \ldots, G_q)$ is finitely generated near x_0 and $\mathcal{G}$ is coherent.

Finally, assume that $\mathcal{F}$ and $\mathcal{G}$ are coherent. Let $x_0 \in X$ be any point, let $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$ and $G_1, \ldots, G_q \in \mathcal{G}(\Omega)$ be generators of $\mathcal{F}$, $\mathcal{G}$ on a neighborhood Ω of x_0. There is a neighborhood Ω' of x_0 such that $G_1, \ldots, G_q$ admit liftings $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega')$. Then $(F_1, \ldots, F_q, \tilde{G}_1, \ldots, \tilde{G}_q)$ generate $\mathcal{S}_{\restriction\Omega'}$, so $\mathcal{S}$ is locally finitely generated. Now, let $S_1, \ldots, S_q$ be arbitrary sections in $\mathcal{S}(U)$ and $\overline{S}_1, \ldots, \overline{S}_q$ their images in $\mathcal{G}(U)$. For any $x_0 \in U$, the sheaf of relations $\mathcal{R}(\overline{S}_1, \ldots, \overline{S}_q)$ is generated by sections $P_1, \ldots, P_s \in \mathcal{A}(\Omega)^{\oplus q}$ on a small neighborhood Ω of x_0. Set $P_j = (P_j^k)_{1 \le k \le q}$. Then $H_j = P_j^1 S_1 + \ldots + P_j^q S_q$, $1 \le j \le s$, are mapped to 0 in $\mathcal{G}$ so they can be seen as sections of $\mathcal{F}$. The coherence of $\mathcal{F}$ shows that $\mathcal{R}(H_1, \ldots, H_s)$ has generators $Q_1, \ldots, Q_t \in \mathcal{A}(\Omega')^s$ on a small neighborhood $\Omega' \subset \Omega$ of x_0. Then $\mathcal{R}(S_1, \ldots, S_q)$ is generated over Ω' by $R_j = \sum Q_j^k P_k \in \mathcal{A}(\Omega')$, $1 \le j \le t$, and $\mathcal{S}$ is coherent. □

(2.15) Corollary. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent subsheaves of a coherent analytic sheaf $\mathcal{S}$, the intersection $\mathcal{F} \cap \mathcal{G}$ is a coherent sheaf.*

Proof. Indeed, the intersection sheaf $\mathcal{F} \cap \mathcal{G}$ is the kernel of the composite morphism $\mathcal{F} \hookrightarrow \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{G}$, and $\mathcal{S}/\mathcal{G}$ is coherent. □

Case of a Coherent Sheaf of Rings.. A sheaf of rings $\mathcal{A}$ is said to be coherent if it is coherent as a module over itself. By Def. 2.11, this means that for any open set $U \subset X$ and any sections $F_j \in \mathcal{A}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is finitely generated. The above results then imply that all free modules $\mathcal{A}^{\oplus p}$ are coherent. As a consequence:

(2.16) Theorem. *If $\mathcal{A}$ is a coherent sheaf of rings, any locally finitely generated subsheaf of $\mathcal{A}^{\oplus p}$ is coherent. In particular, if $\mathcal{S}$ is a coherent $\mathcal{A}$-module and $F_1, \ldots, F_q \in \mathcal{S}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q) \subset \mathcal{A}^{\oplus q}$ is also coherent.*

Let $\mathcal{S}$ be a coherent sheaf of modules over a coherent sheaf of ring $\mathcal{A}$. By an iteration of construction (2.12), we see that for every integer $m \ge 0$ and every point $x \in X$ there is a neighborhood Ω of x on which there is an exact sequence of sheaves

$$\mathcal{A}_{\restriction\Omega}^{\oplus p_m} \xrightarrow{F_m} \mathcal{A}_{\restriction\Omega}^{\oplus p_{m-1}} \longrightarrow \cdots \longrightarrow \mathcal{A}_{\restriction\Omega}^{\oplus p_1} \xrightarrow{F_1} \mathcal{A}_{\restriction\Omega}^{\oplus p_0} \xrightarrow{F_0} \mathcal{S}_{\restriction\Omega} \longrightarrow 0, \tag{2.17}$$

where F_j is given by a $p_{j-1} \times p_j$ matrix of sections in $\mathcal{A}(\Omega)$.

2.C. Analytic Sheaves and the Oka Theorem

Many interesting properties of holomorphic functions can be expressed in terms of sheaves. Among them, analytic sheaves play a central role.

(2.18) Definition. *Let M be a n-dimensional complex analytic manifold and let $\mathcal{O}_M$ be the sheaf of germs of analytic functions on M. An analytic sheaf over M is by definition a sheaf $\mathcal{S}$ of modules over $\mathcal{O}_M$.*

(2.19) Coherence theorem of Oka (1950). *The sheaf of rings $\mathcal{O}_M$ is coherent for any complex manifold M.*

We refer e.g. to (Narasimhan 1965) for a proof of this theorem, which is certainly the most fundamental fact in the theory of analytic sheaves. This theorem can be seen as a semi-local version of the noetherianity property of rings $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions: more precisely, let $F_1, \ldots, F_q \in \mathcal{O}(U)$. Since $\mathcal{O}_{M,x}$ is Noetherian, we know that every stalk $\mathcal{R}(F_1, \ldots, F_q)_x \subset \mathcal{O}_{M,x}^{\oplus q}$ is finitely generated; the important new fact expressed by the theorem is that the sheaf of relations is locally finitely generated, namely that the "same" generators can be chosen to generate each stalk in a neighborhood of a given point. A by-product of the proof is the following:

(2.22) Strong Noetherian property. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold M and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an increasing sequence of coherent subsheaves of $\mathcal{F}$. Then the sequence $(\mathcal{F}_k)$ is stationary on every compact subset of M.*

3. Čech Cohomology

3.A. Definitions

There are many ways of introducing sheaf cohomology. The simplest way is probably the original definition of Čech cohomology used by (Leray 1950, Cartan 1950). Although Čech cohomology does not always work on arbitrary topological spaces, it is well behaved on the category of paracompact spaces and will be sufficient for our purposes. Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open covering of X. For the sake of simplicity, we denote

$$U_{\alpha_0 \alpha_1 \ldots \alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}.$$

The group $C^q(\mathcal{U}, \mathcal{A})$ of *Čech q-cochains* is the set of families

$$c = (c_{\alpha_0 \alpha_1 \ldots \alpha_q}) \in \prod_{(\alpha_0, \ldots, \alpha_q) \in I^{q+1}} \Gamma(U_{\alpha_0 \alpha_1 \ldots \alpha_q}, \mathcal{A}).$$

The group structure on $C^q(\mathcal{U}, \mathcal{A})$ is the obvious one deduced from the addition law on sections of $\mathcal{A}$. The *Čech differential* $\delta^q : C^q(\mathcal{U}, \mathcal{A}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{A})$ is defined by the formula

(3.1) $$(\delta^q c)_{\alpha_0\ldots\alpha_{q+1}} = \sum_{0\le j\le q+1} (-1)^j\, c_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{q+1}}\restriction_{U_{\alpha_0\ldots\alpha_{q+1}}},$$

and we set $C^q(\mathcal{U},\mathcal{A}) = 0$, $\delta^q = 0$ for $q < 0$. In degrees 0 and 1, we get for example

(3.2) $$q = 0,\quad c = (c_\alpha),\quad (\delta^0 c)_{\alpha\beta} = c_\beta - c_\alpha\restriction_{U_{\alpha\beta}},$$

(3.2′) $$q = 1,\quad c = (c_{\alpha\beta}),\quad (\delta^1 c)_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta}\restriction_{U_{\alpha\beta\gamma}}.$$

Easy verifications left to the reader show that $\delta^{q+1}\circ\delta^q = 0$. We get therefore a cochain complex $\big(C^\bullet(\mathcal{U},\mathcal{A}),\delta\big)$, called the *complex of Čech cochains* relative to the covering $\mathcal{U}$.

(3.3) Definition. *The Čech cohomology group of $\mathcal{A}$ relative to $\mathcal{U}$ is*

$$H^q(\mathcal{U},\mathcal{A}) = H^q\big(C^\bullet(\mathcal{U},\mathcal{A})\big).$$

Formula (3.2) shows that the set of Čech 0-cocycles is the set of families $(c_\alpha) \in \prod \mathcal{A}(U_\alpha)$ such that $c_\beta = c_\alpha$ on $U_\alpha \cap U_\beta$. Such a family defines in a unique way a global section $f \in \mathcal{A}(X)$ with $f_{\restriction U_\alpha} = c_\alpha$. Hence

(3.4) $$H^0(\mathcal{U},\mathcal{A}) = \Gamma(X,\mathcal{A}).$$

Now, let $\mathcal{V} = (V_\beta)_{\beta\in J}$ be another open covering of X that is finer than $\mathcal{U}$; this means that there exists a map $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for every $\beta \in J$. Then we can define a morphism $\rho^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{V},\mathcal{A})$ by

(3.5) $$(\rho^q c)_{\beta_0\ldots\beta_q} = c_{\rho(\beta_0)\ldots\rho(\beta_q)}\restriction_{V_{\beta_0\ldots\beta_q}}\ ;$$

the commutation property $\delta\rho^\bullet = \rho^\bullet\delta$ is immediate. If $\rho' : J \to I$ is another refinement map such that $V_\beta \subset U_{\rho'(\beta)}$ for all β, the morphisms $\rho^\bullet$, $\rho'^\bullet$ are homotopic. To see this, we define a map $h^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{V},\mathcal{A})$ by

$$(h^q c)_{\beta_0\ldots\beta_{q-1}} = \sum_{0\le j\le q-1} (-1)^j c_{\rho(\beta_0)\ldots\rho(\beta_j)\rho'(\beta_j)\ldots\rho'(\beta_{q-1})}\restriction_{V_{\beta_0\ldots\beta_{q-1}}}.$$

The homotopy identity $\delta^{q-1}\circ h^q + h^{q+1}\circ\delta^q = \rho'^q - \rho^q$ is easy to verify. Hence $\rho^\bullet$ and $\rho'^\bullet$ induce a map depending only on $\mathcal{U}$, $\mathcal{V}$:

(3.6) $$H^q(\rho^\bullet) = H^q(\rho'^\bullet)\ :\ H^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(\mathcal{V},\mathcal{A}).$$

Now, we want to define a *direct limit* $H^q(X,\mathcal{A})$ of the groups $H^q(\mathcal{U},\mathcal{A})$ by means of the refinement mappings (3.6). In order to avoid set theoretic difficulties, the coverings used in this definition will be considered as subsets of the power set $\mathcal{P}(X)$, so that the collection of all coverings becomes actually a set.

(3.7) Definition. *The Čech cohomology group $H^q(X,\mathcal{A})$ is the direct limit*

$$H^q(X,\mathcal{A}) = \varinjlim_{\mathcal{U}}\ H^q(\mathcal{U},\mathcal{A})$$

when $\mathcal{U}$ runs over the collection of all open coverings of X. Explicitly, this means that the elements of $H^q(X,\mathcal{A})$ are the equivalence classes in the disjoint union of the groups $H^q(\mathcal{U},\mathcal{A})$, with an element in $H^q(\mathcal{U},\mathcal{A})$ and another in $H^q(\mathcal{V},\mathcal{A})$ identified if their images in $H^q(\mathcal{W},\mathcal{A})$ coincide for some refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{V}$.

3.B. Long Exact Sequence of Cohomology Groups

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a sheaf morphism, we have an obvious induced morphism $\varphi^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{B})$, and therefore we find a morphism

$$H^q(\varphi^\bullet) : H^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(\mathcal{U},\mathcal{B}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. We have an exact sequence of groups

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q(\mathcal{U},\mathcal{C}), \tag{3.8}$$

but in general the last map is not surjective, because every section in $\mathcal{C}(U_{\alpha_0,\ldots,\alpha_q})$ need not have a lifting in $\mathcal{B}(U_{\alpha_0,\ldots,\alpha_q})$. The image of $C^\bullet(\mathcal{U},\mathcal{B})$ in $C^\bullet(\mathcal{U},\mathcal{C})$ will be denoted $C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C})$ and called the complex of *liftable cochains* of $\mathcal{C}$ in $\mathcal{B}$. By construction, the sequence

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow 0 \tag{3.9}$$

is exact, thus we get a corresponding long exact sequence of cohomology

$$H^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(\mathcal{U},\mathcal{B}) \longrightarrow H^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow H^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \cdots. \tag{3.10}$$

(3.11) Theorem. *Assume that X is paracompact. If*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is an exact sequence of sheaves, there is an exact sequence

$$\begin{aligned} 0 \longrightarrow & \varGamma(X,\mathcal{A}) \longrightarrow \varGamma(X,\mathcal{B}) \longrightarrow \varGamma(X,\mathcal{C}) \longrightarrow H^1(X,\mathcal{A}) \longrightarrow \cdots \\ & H^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{B}) \longrightarrow H^q(X,\mathcal{C}) \longrightarrow H^{q+1}(X,\mathcal{A}) \longrightarrow \cdots \end{aligned}$$

which is the direct limit of the exact sequences (3.10) *over all coverings $\mathcal{U}$.*

Proof. We have to show that the natural map

$$\varinjlim H^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow \varinjlim H^q(\mathcal{U},\mathcal{C})$$

is an isomorphism. This follows easily from the following lemma, which says essentially that every cochain in $\mathcal{C}$ becomes liftable in $\mathcal{B}$ after a refinement of the covering.

(3.12) Lifting lemma. *Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of X and $c \in C^q(\mathcal{U}, \mathcal{C})$. If X is paracompact, there exists a finer covering $\mathcal{V} = (V_\beta)_{\beta\in J}$ and a refinement map $\rho : J \to I$ such that $\rho^q c \in C^q_{\mathcal{B}}(\mathcal{V}, \mathcal{C})$.*

Proof. Since $\mathcal{U}$ admits a locally finite refinement, we may assume that $\mathcal{U}$ itself is locally finite. There exists an open covering $\mathcal{W} = (W_\alpha)_{\alpha\in I}$ of X such that $\overline{W}_\alpha \subset U_\alpha$. For every point $x \in X$, we can select an open neighborhood V_x of x with the following properties:

a) if $x \in W_\alpha$, then $V_x \subset W_\alpha$;

b) if $x \in U_\alpha$ or if $V_x \cap W_\alpha \neq \emptyset$, then $V_x \subset U_\alpha$;

c) if $x \in U_{\alpha_0\ldots\alpha_q}$, then $c_{\alpha_0\ldots\alpha_q} \in C^q(U_{\alpha_0\ldots\alpha_q}, \mathcal{C})$ admits a lifting in $\mathcal{B}(V_x)$.

Indeed, a) (resp. c)) can be achieved because x belongs to only finitely many sets W_α (resp. U_α), and so only finitely many sections of $\mathcal{C}$ have to be lifted in $\mathcal{B}$. b) can be achieved because x has a neighborhood V'_x that meets only finitely many sets U_α ; then we take

$$V_x \subset V'_x \cap \bigcap_{U_\alpha \ni x} U_\alpha \cap \bigcap_{U_\alpha \not\ni x} (V'_x \smallsetminus \overline{W}_\alpha).$$

Choose $\rho : X \to I$ such that $x \in W_{\rho(x)}$ for every x. Then a) implies $V_x \subset W_{\rho(x)}$, so $\mathcal{V} = (V_x)_{x\in X}$ is finer than $\mathcal{U}$, and ρ defines a refinement map. If $V_{x_0\ldots x_q} \neq \emptyset$, we have

$$V_{x_0} \cap W_{\rho(x_j)} \supset V_{x_0} \cap V_{x_j} \neq \emptyset \quad \text{for} \quad 0 \le j \le q,$$

thus $V_{x_0} \subset U_{\rho(x_0)\ldots\rho(x_q)}$ by b). Now, c) implies that the section $c_{\rho(x_0)\ldots\rho(x_q)}$ admits a lifting in $\mathcal{B}(V_{x_0})$, and in particular in $\mathcal{B}(V_{x_0\ldots x_q})$. Therefore $\rho^q c$ is liftable in $\mathcal{B}$. □

(3.13) Corollary. *Let $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ be an exact sequence of sheaves on X, and let U be a paracompact open subset. If $H^1(U, \mathcal{A}) = 0$, there is an exact sequence*

$$0 \longrightarrow \Gamma(U, \mathcal{A}) \longrightarrow \Gamma(U, \mathcal{B}) \longrightarrow \Gamma(U, \mathcal{C}) \longrightarrow 0.$$

(3.14) Proposition. *Let $\mathcal{A}$ be a sheaf on X. Assume that either*

a) *$\mathcal{A}$ is a sheaf of modules over a soft sheaf of rings $\mathcal{R}$ on X (by this we mean that for every covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X, there exists a $\mathcal{R}$-valued partition of unity $(\psi_\alpha)_{\alpha\in I}$ subordinate to $\mathcal{U}$).*

b) *$\mathcal{A}$ is flabby, that is, every section in $\mathcal{A}(U)$ extends to X, for any open set U.*

Then $H^q(\mathcal{U}, \mathcal{A}) = 0$ for every $q \ge 1$ and every open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X. In particular $H^q(X, \mathcal{U}) = 0$ for $q \ge 1$. Such a sheaf is said to be acyclic.

Proof. a) A typical example is the case where $\mathcal{R} = \mathcal{C}^\infty$ is the sheaf of C^∞ functions on a differentiable manifold. Under assumption a), we can define a map $h^q : C^q(\mathcal{U}, \mathcal{A}) \longrightarrow C^{q-1}(\mathcal{U}, \mathcal{A})$ by

$$(h^q c)_{\alpha_0 \ldots \alpha_{q-1}} = \sum_{\nu \in I} \psi_\nu \, c_{\nu\alpha_0 \ldots \alpha_{q-1}} \tag{3.15}$$

where $\psi_\nu \, c_{\nu\alpha_0 \ldots \alpha_{q-1}}$ is extended by 0 on $U_{\alpha_0 \ldots \alpha_{q-1}} \cap \complement U_\nu$. It is clear that

$$(\delta^{q-1} h^q c)_{\alpha_0 \ldots \alpha_q} = \sum_{\nu \in I} \psi_\nu \big(c_{\alpha_0 \ldots \alpha_q} - (\delta^q c)_{\nu\alpha_0 \ldots \alpha_q} \big),$$

i.e. $\delta^{q-1} h^q + h^{q+1} \delta^q = \mathrm{Id}$. Hence $\delta^q c = 0$ implies $\delta^{q-1} h^q c = c$ if $q \geq 1$.

b) Let $\mathcal{B}$ be the sheaf of sections of the sheaf-space $\widetilde{\mathcal{A}}$ which are not necessarily continuous, that is,

$$\mathcal{B}(U) = \prod_{x \in U} \widetilde{\mathcal{A}}_x.$$

Then $\mathcal{B}$ is a sheaf of modules over the sheaf of rings of arbitrary functions $U \to \mathbb{Z}$, which clearly has partitions of unity with respect to any covering of X. Hence by a) we have $H^q(\mathcal{U}, \mathcal{B}) = 0$ for $q \geq 1$. Moreover, there is a canonical injection $\mathcal{A} \to \mathcal{B}$ which allows us to consider the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{C} = \mathcal{B}/\mathcal{A}$. It is not difficult to check that $\mathcal{C}$ is also flabby and that every cochain with values in $\mathcal{C}$ is liftable to $\mathcal{B}$. Hence (3.10) yields an exact sequence

$$\begin{aligned} &\Gamma(X, \mathcal{B}) \longrightarrow \Gamma(X, \mathcal{C}) \longrightarrow H^1(\mathcal{U}, \mathcal{A}) \longrightarrow H^1(\mathcal{U}, \mathcal{B}) = 0, \\ &0 = H^q(\mathcal{U}, \mathcal{B}) \longrightarrow H^q(\mathcal{U}, \mathcal{C}) \longrightarrow H^{q+1}(\mathcal{U}, \mathcal{A}) \longrightarrow H^{q+1}(\mathcal{U}, \mathcal{B}) = 0. \end{aligned}$$

Clearly $\Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$ is surjective, thus $H^1(\mathcal{U}, \mathcal{A}) = 0$. Since $\mathcal{C}$ is flabby, we can argue by induction on q to get $H^{q+1}(\mathcal{U}, \mathcal{A}) \simeq H^q(\mathcal{U}, \mathcal{C}) = 0$ for $q \geq 1$. □

3.C. Leray's Theorem for Acyclic Coverings

By definition of Čech cohomology, for every exact sequence of sheaves $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ there is a commutative diagram

$$\begin{array}{ccccccccc} H^q(\mathcal{U}, \mathcal{A}) & \longrightarrow & H^q(\mathcal{U}, \mathcal{B}) & \longrightarrow & H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) & \longrightarrow & H^{q+1}(\mathcal{U}, \mathcal{A}) & \longrightarrow & H^{q+1}(\mathcal{U}, \mathcal{B}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q(X, \mathcal{A}) & \longrightarrow & H^q(X, \mathcal{B}) & \longrightarrow & H^q(X, \mathcal{C}) & \longrightarrow & H^{q+1}(X, \mathcal{A}) & \longrightarrow & H^{q+1}(X, \mathcal{B}). \end{array} \tag{3.16}$$

in which the vertical maps are the canonical arrows to the inductive limit.

(3.17) Theorem (Leray)**.** *Assume that*

$$H^s(U_{\alpha_0 \ldots \alpha_t}, \mathcal{A}) = 0$$

for all indices $\alpha_0, \ldots, \alpha_t$ and $s \geq 1$. Then $H^q(\mathcal{U}, \mathcal{A}) \simeq H^q(X, \mathcal{A})$.

We say that the covering $\mathcal{U}$ is *acyclic* (with respect to $\mathcal{A}$) if the hypothesis of Th. 3.17 is satisfied. Leray's theorem asserts that the cohomology groups of $\mathcal{A}$ on

X can be computed by means of an arbitrary acyclic covering (if such a covering exists), without using the direct limit procedure.

Proof. By induction on q, the result being obvious for $q = 0$. Consider the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ where $\mathcal{B}$ is the sheaf of non necessarily continuous sections of $\widetilde{\mathcal{A}}$ and $\mathcal{C} = \mathcal{B}/\mathcal{A}$. As $\mathcal{B}$ is acyclic, the hypothesis on $\mathcal{A}$ and the long exact sequence of cohomology imply $H^s(U_{\alpha_0\ldots\alpha_t}, \mathcal{C}) = 0$ for $s \geq 1$, $t \geq 0$. Moreover $C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) = C^\bullet(\mathcal{U}, \mathcal{C})$ thanks to Cor. 3.13 applied on each open set $U_{\alpha_0\ldots\alpha_q}$. The induction hypothesis in degree q and diagram (3.16) give

$$\begin{array}{ccccccc} H^q(\mathcal{U},\mathcal{B}) & \longrightarrow & H^q(\mathcal{U},\mathcal{C}) & \longrightarrow & H^{q+1}(\mathcal{U},\mathcal{A}) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\ H^q(X,\mathcal{B}) & \longrightarrow & H^q(X,\mathcal{C}) & \longrightarrow & H^{q+1}(X,\mathcal{A}) & \longrightarrow & 0, \end{array}$$

hence $H^{q+1}(\mathcal{U}, \mathcal{A}) \longrightarrow H^{q+1}(X, \mathcal{A})$ is also an isomorphism. □

(3.18) Remark. The morphism $H^1(\mathcal{U}, \mathcal{A}) \longrightarrow H^1(X, \mathcal{A})$ is always injective. Indeed, (3.10) yields

$$\begin{array}{ccc} H^0_{\mathcal{B}}(\mathcal{U},\mathcal{C})/\operatorname{Im} H^0(\mathcal{U},\mathcal{B}) & \xrightarrow{\simeq} & H^1(\mathcal{U},\mathcal{A}) \\ \downarrow & & \downarrow \\ H^0(X,\mathcal{C})/\operatorname{Im} H^0(X,\mathcal{B}) & \xrightarrow{\simeq} & H^1(X,\mathcal{A}) \end{array}$$

and $H^0(\mathcal{U}, \mathcal{B}) = H^0(X, \mathcal{B}) = \Gamma(X, \mathcal{B})$, while $H^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \longrightarrow H^0(X, \mathcal{C})$ is an injection. As a consequence, the refinement mappings $H^1(\mathcal{U}, \mathcal{A}) \to H^1(\mathcal{V}, \mathcal{A})$ are also injective. □

3.D. Alternate Čech Cochains and Topological Dimension

For explicit calculations, it is sometimes useful to consider a slightly modified Čech complex which has the advantage of producing much smaller cochain groups. If $\mathcal{A}$ is a sheaf and $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open covering of X, we let $AC^q(\mathcal{U}, \mathcal{A}) \subset C^q(\mathcal{U}, \mathcal{A})$ be the subgroup of *alternate Čech cochains*, consisting of Čech cochains $c = (c_{\alpha_0\ldots\alpha_q})$ such that

$$(3.19) \qquad \begin{cases} c_{\alpha_0\ldots\alpha_q} = 0 & \text{if } \alpha_i = \alpha_j, \quad i \neq j, \\ c_{\alpha_{\sigma(0)}\ldots\alpha_{\sigma(q)}} = \varepsilon(\sigma)\, c_{\alpha_0\ldots\alpha_q} & \end{cases}$$

for any permutation σ of $\{1, \ldots, q\}$ of signature $\varepsilon(\sigma)$. Then the Čech differential (3.1) of an alternate cochain is still alternate, so $AC^\bullet(\mathcal{U}, \mathcal{A})$ is a subcomplex of $C^\bullet(\mathcal{U}, \mathcal{A})$.

Select a total ordering on the index set I. For each such ordering, we can define a projection $\pi^q : C^q(\mathcal{U}, \mathcal{A}) \longrightarrow AC^q(\mathcal{U}, \mathcal{A}) \subset C^q(\mathcal{U}, \mathcal{A})$ by

$$c \longmapsto \text{alternate } \widetilde{c} \ \text{ such that } \widetilde{c}_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} \ \text{ whenever } \ \alpha_0 < \ldots < \alpha_q.$$

One can check that $\pi^\bullet$ is a morphism of complexes and that $\pi^\bullet$ is homotopic to the identity on $C^\bullet(\mathcal{U}, \mathcal{A})$. Hence there is an isomorphism

$$H^q\big(AC^\bullet(\mathcal{U},\mathcal{A})\big) \simeq H^q\big(C^\bullet(\mathcal{U},\mathcal{A})\big) = H^q(\mathcal{U},\mathcal{A}). \tag{3.20}$$

(3.21) Corollary. *Assume that X is a paracompact space of topological dimension $\le n$, i.e. that X has arbitrarily fine open coverings $\mathcal{U} = (U_\alpha)$ such that more than $n+1$ distinct sets $U_{\alpha_0},\ldots,U_{\alpha_n}$ have empty intersection. Then $H^q(X,\mathcal{A}) = 0$ for $q > n$ and any sheaf $\mathcal{A}$ on X.*

3.E. The De Rham-Weil Isomorphism Theorem

Let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$, that is, an exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ j\ } \mathcal{L}^0 \xrightarrow{\ d^0\ } \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{\ d^q\ } \mathcal{L}^{q+1} \longrightarrow \cdots .$$

We assume in addition that all $\mathcal{L}^q$ are acyclic on X, i.e. $H^s(X,\mathcal{L}^q) = 0$ for all $q \ge 0$ and $s \ge 1$. Set $\mathcal{Z}^q = \ker d^q$. Then $\mathcal{Z}^0 = \mathcal{A}$ and for every $q \ge 1$ we get a short exact sequence

$$0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{L}^{q-1} \xrightarrow{\ d^{q-1}\ } \mathcal{Z}^q \longrightarrow 0.$$

Theorem 3.11 yields an exact sequence

$$H^s(X,\mathcal{L}^{q-1}) \xrightarrow{d^{q-1}} H^s(X,\mathcal{Z}^q) \xrightarrow{\partial^{s,q}} H^{s+1}(X,\mathcal{Z}^{q-1}) \to H^{s+1}(X,\mathcal{L}^{q-1}) = 0. \tag{3.22}$$

If $s \ge 1$, the first group is also zero and we get an isomorphism

$$\partial^{s,q} : H^s(X,\mathcal{Z}^q) \xrightarrow{\ \simeq\ } H^{s+1}(X,\mathcal{Z}^{q-1}).$$

For $s = 0$ we have $H^0(X,\mathcal{L}^{q-1}) = \Gamma(X,\mathcal{L}^{q-1})$ and $H^0(X,\mathcal{Z}^q) = \Gamma(X,\mathcal{Z}^q)$ is the q-cocycle group of $\mathcal{L}^\bullet(X)$, so the connecting map $\partial^{0,q}$ gives an isomorphism

$$H^q\big(\mathcal{L}^\bullet(X)\big) = \Gamma(X,\mathcal{Z}^q)/d^{q-1}\Gamma(X,\mathcal{L}^{q-1}) \xrightarrow{\ \widetilde{\partial}^{0,q}\ } H^1(X,\mathcal{Z}^{q-1}).$$

The composite map $\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q}$ therefore defines an isomorphism

$$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) \xrightarrow{\widetilde{\partial}^{0,q}} H^1(X,\mathcal{Z}^{q-1}) \xrightarrow{\partial^{1,q-1}} \cdots \xrightarrow{\partial^{q-1,1}} H^q(X,\mathcal{Z}^0) = H^q(X,\mathcal{A}). \tag{3.23}$$

This isomorphism behaves functorially with respect to morphisms of resolutions.

(3.24) De Rham-Weil isomorphism theorem. *If $(\mathcal{L}^\bullet, d)$ is a resolution of $\mathcal{A}$ by sheaves $\mathcal{L}^q$ which are acyclic on X, there is a functorial isomorphism*

$$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) \longrightarrow H^q(X,\mathcal{A}). \qquad \square$$

(3.25) Example: De Rham cohomology. Let X be a n-dimensional paracompact differential manifold. Consider the resolution

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{\ d\ } \mathcal{E}^1 \to \cdots \to \mathcal{E}^q \xrightarrow{\ d\ } \mathcal{E}^{q+1} \to \cdots \to \mathcal{E}^n \to 0$$

given by the exterior derivative d acting on the sheaves of germs of C^∞ differential q-forms. The *De Rham cohomology groups* of X are by definition

$$H^q_{\rm DR}(X,\mathbb{R}) = H^q\big(\Gamma(X,\mathcal{E}^\bullet)\big). \tag{3.26}$$

All sheaves $\mathcal{E}^q$ are $\mathcal{C}^\infty$-modules, so $\mathcal{E}^q$ is acyclic by Prop. (3.14 a). Therefore, we get an isomorphism

$$H^q_{\rm DR}(X,\mathbb{R}) \xrightarrow{\ \simeq\ } H^q(X,\mathbb{R}) \tag{3.27}$$

from the De Rham cohomology onto the cohomology with values in the constant sheaf $\mathbb{R}$. Instead of using C^∞ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents. Let $\mathcal{D}'^q$ be the sheaf of germs of currents of degree q (that is, differential forms of degree q with distribution coefficients). The Poincaré lemma still holds for currents, so we have a resolution

$$0 \to \mathbb{R} \to \mathcal{D}'^0 \xrightarrow{d} \mathcal{D}'^1 \to \cdots \to \mathcal{D}'^q \xrightarrow{d} \mathcal{D}'^{q+1} \to \cdots \to \mathcal{D}'^n \to 0.$$

Again, the sheaves $\mathcal{D}'^q$ are $\mathcal{C}^\infty$-modules, hence they are acyclic. Now, the inclusion $\mathcal{E}^q \subset \mathcal{D}'^q$ induces an isomorphism

$$H^q\big(\Gamma(X,\mathcal{E}^\bullet)\big) \simeq H^q\big(\Gamma(X,\mathcal{D}'^\bullet)\big), \tag{3.28}$$

both groups being isomorphic to $H^q(X,\mathbb{R})$. The isomorphism between cohomology of differential forms and singular cohomology (another topological invariant) was first established by (De Rham 1931). The above proof follows essentially the method given by (Weil 1952), in a more abstract setting.

3.F. Mayer-Vietoris Exact Sequence

Let U_1, U_2 be open subsets of X and $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. For any sheaf $\mathcal{A}$ on X, there exists a resolution

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ j\ } \mathcal{L}^0 \xrightarrow{\ d^0\ } \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{\ d^q\ } \mathcal{L}^{q+1} \longrightarrow \cdots$$

by flabby sheaves $\mathcal{L}^q$. In fact, it is enough to take for j the injection of $\mathcal{A}$ into the sheaf $\mathcal{L}^0$ of non necessarily continuous sections of $\widetilde{\mathcal{A}}$, and then inductively to embed $\mathcal{L}^0/j(\mathcal{A})$, $\mathcal{L}^1/d^0(\mathcal{L}^0)$, …, $\mathcal{L}^q/d^{q-1}(\mathcal{L}^{q-1})$ into flabby sheaves $\mathcal{L}^1$, $\mathcal{L}^2$, …, $\mathcal{L}^{q+1}$. For every q we get an exact sequence

$$0 \longrightarrow \mathcal{L}^q(U) \longrightarrow \mathcal{L}^q(U_1) \oplus \mathcal{L}^q(U_2) \longrightarrow \mathcal{L}^q(V) \longrightarrow 0$$

where the injection is given by $f \longmapsto (f_{\restriction U_1}, f_{\restriction U_2})$ and the surjection by $(g_1, g_2) \longmapsto g_{2\restriction V} - g_{1\restriction V}$; the surjectivity of this map follows immediately from the fact that $\mathcal{L}^q$ is flabby. The snake lemma and the De Rham-Weil isomorphism $H^q(X,\mathcal{A}) \simeq H^q(\Gamma(X,\mathcal{L}^\bullet))$ yield:

(3.29) Theorem. *For any sheaf $\mathcal{A}$ on X and any open sets $U_1, U_2 \subset X$, set $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. Then there is an exact sequence*

$$H^q(U,\mathcal{A}) \longrightarrow H^q(U_1,\mathcal{A}) \oplus H^q(U_2,\mathcal{A}) \longrightarrow H^q(V,\mathcal{A}) \longrightarrow H^{q+1}(U,\mathcal{A}) \cdots \qquad \square$$

4. Complex Analytic Sets and Complex Spaces

4.A. Definition. Irreducible Components

A complex analytic set is a set which can be defined locally by finitely many holomorphic equations; such a set has in general singular points, because no assumption is made on the differentials of the equations. For a detailed study of analytic set theory, we refer to H. Cartan's seminar (Cartan 1950), to the books of (Gunning-Rossi 1965), (Narasimhan 1966) or the recent book by (Grauert-Remmert 1984).

(4.1) Definition. *Let M be a complex analytic manifold. A subset $A \subset M$ is said to be an analytic subset of M if A is closed and if for every point $x_0 \in A$ there exist a neighborhood U of x_0 and holomorphic functions $g_1, \ldots, g_n$ in $\mathcal{O}(U)$ such that*

$$A \cap U = \{z \in U \; ; \; g_1(z) = \ldots = g_N(z) = 0\}.$$

Then $g_1, \ldots, g_N$ are said to be (*local*) *equations of A in U.*

It is easy to see that a finite union or intersection of analytic sets is analytic: if (g'_j), (g''_k) are equations of A', A'' in the open set U, then the family of all products $(g'_j g''_k)$ and the family $(g'_j)\cup(g''_k)$ define equations of $A' \cup A''$ and $A' \cap A''$ respectively.

(4.2) Remark. Assume that M is connected. The analytic continuation theorem shows that either $A = M$ or A has no interior point. In the latter case, the Riemann extension theorem show that every function $f \in \mathcal{O}(M \smallsetminus A)$ that is locally bounded near A can be extended to a function $\tilde{f} \in \mathcal{O}(M)$. Moreover $M \smallsetminus A$ is connected. $\square$

We now focus our attention on local properties of analytic sets. If $A \subset M$ is an analytic set, we denote by (A, x) the germ of A at any given point $x \in M$, and by $\mathcal{I}_{A,x}$ the ideal of germs $f \in \mathcal{O}_{M,x}$ which vanish on (A, x). Conversely, if $\mathcal{J} = (g_1, \ldots, g_N)$ is an ideal of $\mathcal{O}_{M,x}$, we denote by $\big(V(\mathcal{J}), x\big)$ the germ at x of the zero variety $V(\mathcal{J}) = \{z \in U \; ; \; g_1(z) = \ldots = g_N(z) = 0\}$, where U is a neighborhood of x such that $g_j \in \mathcal{O}(U)$. It is clear that

(4.3′) for every ideal $\mathcal{J}$ in the ring $\mathcal{O}_{M,x}$, $\quad \mathcal{I}_{V(\mathcal{J}),x} \supset \mathcal{J}$,

(4.3″) for every germ of analytic set (A, x), $\quad \big(V(\mathcal{I}_{A,x}), x\big) = (A, x)$.

(4.4) Definition. *A germ (A, x) is said to be irreducible if it has no decomposition $(A, x) = (A_1, x) \cup (A_2, x)$ with analytic sets $(A_j, x) \neq (A, x)$, $j = 1, 2$.*

(4.5) Proposition. *A germ (A, x) is irreducible if and only if $\mathcal{I}_{A,x}$ is a prime ideal of the ring $\mathcal{O}_{M,x}$.*

Proof. Let us recall that an ideal $\mathcal{J}$ is said to be *prime* if $fg \in \mathcal{J}$ implies $f \in \mathcal{J}$ or $g \in \mathcal{J}$. Assume that (A,x) is irreducible and that $fg \in \mathcal{I}_{A,x}$. As we can write $(A,x) = (A_1,x) \cup (A_2,x)$ with $A_1 = A \cap f^{-1}(0)$ and $A_2 = A \cap g^{-1}(0)$, we must have for example $(A_1,x) = (A,x)$; thus $f \in \mathcal{I}_{A,x}$ and $\mathcal{I}_{A,x}$ is prime. Conversely, if $(A,x) = (A_1,x) \cup (A_2,x)$ with $(A_j,x) \neq (A,x)$, there exist $f \in \mathcal{I}_{A_1,x}$, $g \in \mathcal{I}_{A_2,x}$ such that $f, g \notin \mathcal{I}_{A,x}$. However $fg \in \mathcal{I}_{A,x}$, thus $\mathcal{I}_{A,x}$ is not prime. □

(4.6) Theorem. *Every decreasing sequence of germs of analytic sets (A_k,x) is stationary.*

Proof. In fact, the corresponding sequence of ideals $\mathcal{J}_k = \mathcal{I}_{A_k,x}$ is increasing, thus $\mathcal{J}_k = \mathcal{J}_{k_0}$ for $k \geq k_0$ large enough by the Noetherian property of $\mathcal{O}_{M,x}$. Hence $(A_k,x) = \big(V(\mathcal{J}_k),x\big)$ is constant for $k \geq k_0$. This result has the following straightforward consequence: □

(4.7) Theorem. *Every analytic germ (A,x) has a finite decomposition*

$$(A,x) = \bigcup_{1\leq k\leq N} (A_k,x)$$

where the germs (A_j,x) are irreducible and $(A_j,x) \not\subset (A_k,x)$ for $j \neq k$. The decomposition is unique apart from the ordering.

4.B. Local Structure of a Germ of Analytic Set

We are going to describe the local structure of a germ of analytic set, both from the holomorphic and topological points of view. By the above decomposition theorem, we may restrict ourselves to the case of irreducible germs. Let $\mathcal{J}$ be a prime ideal in $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0}$ and let $A = V(\mathcal{J})$ be its zero variety.

(4.8) Local parametrization theorem. *Let $\mathcal{J}$ be a prime ideal of $\mathcal{O}_n$ and let $A = V(\mathcal{J})$. There is an integer d, called the dimension of A, such that after rotating the coordinates by a generic linear transformation of $\mathbb{C}^n$, the coordinates*

$$(z'\,;z'') = (z_1,\ldots,z_d\,;z_{d+1},\ldots,z_n)$$

satisfy the following properties.

a) *The ring morphism*

$$\mathcal{O}_d = \mathbb{C}\{z_1,\ldots,z_d\} \longrightarrow \mathcal{O}_n/\mathcal{J} = \mathbb{C}\{z_1,\ldots,z_n\}/\mathcal{J}$$

defines a finite integral extension of $\mathcal{O}_d$.

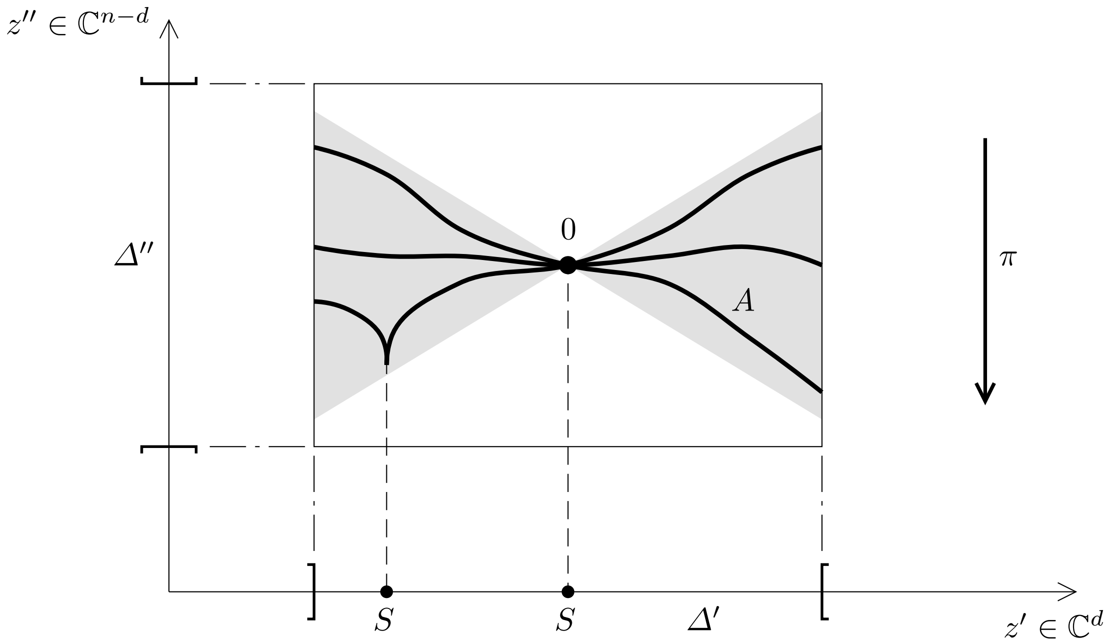
Let q be the degree of the extension and let $\delta(z') \in \mathcal{O}_d$ be the discriminant of the irreducible polynomial of a primitive element $u(z'') = \sum_{k>d} c_k z_k$. If Δ', Δ'' are polydisks of sufficiently small radii r', r'' and if $r' \leq r''/C$ with C large, the projection map $\pi : A\cap(\Delta'\times\Delta'') \longrightarrow \Delta'$ is a ramified covering with q sheets, whose ramification locus is contained in $S = \{z' \in \Delta'; \delta(z') = 0\}$. This means that:

b) *the open subset* $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$ *is a smooth d-dimensional manifold, dense in* $A \cap (\Delta' \times \Delta'')$;

c) $\pi : A_S \longrightarrow \Delta' \smallsetminus S$ *is a covering* ;

d) *the fibers* $\pi^{-1}(z')$ *have exactly* q *elements if* $z' \notin S$ *and at most* q *if* $z' \in S$.

Moreover, A_S *is a connected covering of* $\Delta' \smallsetminus S$, *and* $A \cap (\Delta' \times \Delta'')$ *is contained in a cone* $|z''| \le C|z'|$ *(see Fig. 1).*



Ramified covering from A to $\Delta' \subset \mathbb{C}^d$.

We refer to the references already mentioned for a proof of these properties. Another fundamental result is:

(4.9) Hilbert's Nullstellensatz. *For every ideal* $\mathcal{J} \subset \mathcal{O}_n$

$$\mathcal{I}_{V(\mathcal{J}),0} = \sqrt{\mathcal{J}},$$

where $\sqrt{\mathcal{J}}$ *is the radical of* $\mathcal{J}$, *i.e. the set of germs* $f \in \mathcal{O}_n$ *such that some power* f^k *lies in* $\mathcal{J}$.

In other words, if a germ $(B, 0)$ is defined by an arbitrary ideal $\mathcal{J} \subset \mathcal{O}_n$ and if $f \in \mathcal{O}_n$ vanishes on $(B, 0)$, then some power f^k lies in $\mathcal{J}$.

4.C. Regular and Singular Points. Dimension

The above powerful results enable us to investigate the structure of singularities of an analytic set. We first give a few definitions.

(4.10) Definition. *Let $A \subset M$ be an analytic set and $x \in A$. We say that $x \in A$ is a regular point of A if $A \cap \Omega$ is a $\mathbb{C}$-analytic submanifold of Ω for some neighborhood Ω of x. Otherwise x is said to be singular. The corresponding subsets of A will be denoted respectively $A_{\rm reg}$ and $A_{\rm sing}$.*

It is clear from the definition that $A_{\rm reg}$ is an open subset of A (thus $A_{\rm sing}$ is closed), and that the connected components of $A_{\rm reg}$ are $\mathbb{C}$-analytic submanifolds of M (non necessarily closed).

(4.11) Definition. *The dimension of an irreducible germ of analytic set (A, x) is defined by $\dim(A, x) = \dim(A_{\rm reg}, x)$. If (A, x) has several irreducible components (A_l, x), we set*

$$\dim(A, x) = \max\{\dim(A_l, x)\}, \qquad \mathrm{codim}(A, x) = n - \dim(A, x).$$

4.D. Coherence of Ideal Sheaves

Let A be an analytic set in a complex manifold M. The *sheaf of ideals* $\mathcal{I}_A$ is the subsheaf of $\mathcal{O}_M$ consisting of germs of holomorphic functions on M which vanish on A. Its stalks are the ideals $\mathcal{I}_{A,x}$ already considered; note that $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ if $x \notin A$. If $x \in A$, we let $\mathcal{O}_{A,x}$ be the ring of germs of functions on (A, x) which can be extended as germs of holomorphic functions on (M, x). By definition, there is a surjective morphism $\mathcal{O}_{M,x} \longrightarrow \mathcal{O}_{A,x}$ whose kernel is $\mathcal{I}_{A,x}$, thus

$$\mathcal{O}_{A,x} = \mathcal{O}_{M,x}/\mathcal{I}_{A,x}, \qquad \forall x \in A,$$

i.e. $\mathcal{O}_A = (\mathcal{O}_M/\mathcal{I}_A)_{\restriction A}$. Since $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ for $x \notin A$, the quotient sheaf $\mathcal{O}_M/\mathcal{I}_A$ is zero on $M \smallsetminus A$.

(4.12) Theorem (Cartan 1950). *For any analytic set $A \subset M$, the sheaf of ideals $\mathcal{I}_A$ is a coherent analytic sheaf.*

(4.13) Corollary. *$A_{\rm sing}$ is an analytic subset of A.*

Proof. The statement is local. Assume first that $(A, 0)$ is an irreducible germ in $\mathbb{C}^n$. Let $g_1, \ldots, g_N$ be generators of the sheaf $\mathcal{I}_A$ on a neighborhood Ω of 0. Set $d = \dim A$. In a neighborhood of every point $x \in A_{\rm reg} \cap \Omega$, A can be defined by holomorphic equations $u_1(z) = \ldots = u_{n-d}(z) = 0$ such that $du_1, \ldots, du_{n-d}$ are linearly independant. As $u_1, \ldots, u_{n-d}$ are generated by $g_1, \ldots, g_N$, one can extract a subfamily $g_{j_1}, \ldots, g_{j_{n-d}}$ that has at least one non zero Jacobian determinant of rank $n - d$ at x. Therefore $A_{\rm sing} \cap \Omega$ is defined by the equations

$$\det\left(\frac{\partial g_j}{\partial z_k}\right)_{\substack{j\in J\\ k\in K}} = 0, \quad J\subset\{1,\ldots,N\},\ K\subset\{1,\ldots,n\},\ |J|=|K|=n-d.$$

Assume now that $(A,0)=\bigcup(A_l,0)$ with $(A_l,0)$ irreducible. The germ of an analytic set at a regular point is irreducible, thus every point which belongs simultaneously to at least two components is singular. Hence

$$(A_{\rm sing},0)=\bigcup(A_{l,{\rm sing}},0)\cup\bigcup_{k\neq l}(A_k\cap A_l,0),$$

and $A_{\rm sing}$ is analytic. □

4.E. Complex Spaces

A complex space is a space locally isomorphic to an analytic set A in an open subset $\Omega\subset\mathbb{C}^n$, together with a sheaf of rings $\mathcal{O}_A=\mathcal{O}_\Omega/\mathcal{J}$ associated to a coherent sheaf of ideals $\mathcal{J}$ on Ω such that $V(\mathcal{J})=A$. This idea is made precise by the following definitions.

(4.14) Definition. *A ringed space is a pair $(X,\mathcal{R}_X)$ consisting of a topological space X and of a sheaf of rings $\mathcal{R}_X$ on X, called the structure sheaf. A morphism*

$$F:(X,\mathcal{R}_X)\longrightarrow(Y,\mathcal{R}_Y)$$

of ringed spaces is a pair $(f,F^\star)$ where $f:X\longrightarrow Y$ is a continuous map and

$$F^\star\ :\ f^{-1}\mathcal{R}_Y\longrightarrow\mathcal{R}_X,\qquad F^\star_x\ :\ (\mathcal{R}_Y)_{f(x)}\longrightarrow(\mathcal{R}_X)_x$$

a homomorphism of sheaves of rings on X, called the comorphism of F.

If $F:(X,\mathcal{R}_X)\longrightarrow(Y,\mathcal{R}_Y)$ and $G:(Y,\mathcal{R}_Y)\longrightarrow(Z,\mathcal{R}_Z)$ are morphisms of ringed spaces, the composite $G\circ F$ is the pair consisting of the map $g\circ f:X\longrightarrow Z$ and of the comorphism $(G\circ F)^\star=F^\star\circ f^{-1}G^\star$:

$$\begin{aligned}F^\star\circ f^{-1}G^\star\ &:\ f^{-1}g^{-1}\mathcal{R}_Z\xrightarrow{f^{-1}G^\star}f^{-1}\mathcal{R}_Y\xrightarrow{F^\star}\mathcal{R}_X,\\ F^\star_x\circ G^\star_{f(x)}\ &:\ (\mathcal{R}_Z)_{g\circ f(x)}\longrightarrow(\mathcal{R}_Y)_{f(x)}\longrightarrow(\mathcal{R}_X)_x.\end{aligned}\tag{4.15}$$

We begin by a description of what will be the local model of an analytic complex space. Let $\Omega\subset\mathbb{C}^n$ be an open subset, $\mathcal{J}\subset\mathcal{O}_\Omega$ a coherent sheaf of ideals and $A=V(\mathcal{J})$ the analytic set in Ω defined locally as the zero set of a system of generators of $\mathcal{J}$. By Hilbert's Nullstellensatz 4.9 we have $\mathcal{I}_A=\sqrt{\mathcal{J}}$, but $\mathcal{I}_A$ differs in general from $\mathcal{J}$. The sheaf of rings $\mathcal{O}_\Omega/\mathcal{J}$ is supported on A, i.e. $(\mathcal{O}_\Omega/\mathcal{J})_x=0$ if $x\notin A$. Ringed spaces of the type $(A,\mathcal{O}_\Omega/\mathcal{J})$ will be used as the local models of analytic complex spaces.

(4.16) Definition. *A morphism*

$$F=(f,F^\star):(A,\mathcal{O}_\Omega/\mathcal{J}_{\restriction A})\longrightarrow(A',\mathcal{O}_{\Omega'}/\mathcal{J}'_{\restriction A'})$$

is said to be analytic if for every point $x \in A$ there exists a neighborhood W_x of x in Ω and a holomorphic function $\Phi : W_x \longrightarrow \Omega'$ such that $f_{\restriction A\cap W_x} = \Phi_{\restriction A\cap W_x}$ and such that the comorphism

$$F_x^\star : (\mathcal{O}_{\Omega'}/\mathcal{J}')_{f(x)} \longrightarrow (\mathcal{O}_\Omega/\mathcal{J})_x$$

is induced by $\Phi^\star : \mathcal{O}_{\Omega',f(x)} \ni u \longmapsto u\circ\Phi \in \mathcal{O}_{\Omega,x}$ with $\Phi^\star\mathcal{J}' \subset \mathcal{J}$.

(4.17) Example. Take $\Omega = \mathbb{C}^n$ and $\mathcal{J} = (z_n^2)$. Then A is the hyperplane $\mathbb{C}^{n-1}\times\{0\}$, and the sheaf $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$ can be identified with the sheaf of rings of functions $u + z_n u'$, $u, u' \in \mathcal{O}_{\mathbb{C}^{n-1}}$, with the relation $z_n^2 = 0$. In particular, z_n is a nilpotent element of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$. A morphism F of $(A, \mathcal{O}_{\mathbb{C}^n}/\mathcal{J})$ into itself is induced (at least locally) by a holomorphic map $\Phi = (\widetilde{\Phi}, \Phi_n)$ defined on a neighborhood of A in $\mathbb{C}^n$ with values in $\mathbb{C}^n$, such that $\Phi(A) \subset A$, i.e. $\Phi_{n\restriction A} = 0$. We see that F is completely determined by the data

$$\begin{aligned} f(z_1,\ldots,z_{n-1}) &= \widetilde{\Phi}(z_1,\ldots,z_{n-1},0), & f &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1},\\ f'(z_1,\ldots,z_{n-1}) &= \tfrac{\partial\Phi}{\partial z_n}(z_1,\ldots,z_{n-1},0), & f' &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n}, \end{aligned}$$

which can be chosen arbitrarily.

(4.18) Definition. *A complex space is a ringed space $(X, \mathcal{O}_X)$ over a separable Hausdorff topological space X, satisfying the following property: there exist an open covering (U_λ) of X and isomorphisms of ringed spaces*

$$G_\lambda : (U_\lambda, \mathcal{O}_{X\restriction U_\lambda}) \longrightarrow (A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$$

where A_λ is the zero set of a coherent sheaf of ideals $\mathcal{J}_\lambda$ on an open subset $\Omega_\lambda \subset \mathbb{C}^{N_\lambda}$, such that every transition morphism $G_\lambda\circ G_\mu^{-1}$ is a holomorphic isomorphism from $g_\mu(U_\lambda\cap U_\mu) \subset A_\mu$ onto $g_\lambda(U_\lambda\cap U_\mu) \subset A_\lambda$, equipped with the respective structure sheaves $\mathcal{O}_{\Omega_\mu}/\mathcal{J}_{\mu\restriction A_\mu}$, $\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda}$.

We shall often consider the maps G_λ as identifications and write simply $U_\lambda = A_\lambda$. A morphism $F : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of analytic complex spaces obtained by gluing patches $(A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$ and $(A'_\mu, \mathcal{O}_{\Omega'_\mu}/\mathcal{J}'_{\mu\, A'_\mu})$, respectively, is a morphism F of ringed spaces such that for each pair (λ,μ), the restriction of F from $A_\lambda\cap f^{-1}(A'_\mu) \subset X$ to $A'_\mu \subset Y$ is holomorphic in the sense of Def. 4.16.

Nilpotent Elements and Reduced Complex Spaces.. Let $(X, \mathcal{O}_X)$ be an analytic complex space. The set of *nilpotent elements* is the sheaf of ideals of $\mathcal{O}_X$ defined by

$$\mathcal{N}_X = \{u \in \mathcal{O}_X\,;\ u^k = 0 \ \text{ for some } k \in \mathbb{N}\}. \tag{4.19}$$

Locally, we have $\mathcal{O}_{X\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda}$, thus

$$\mathcal{N}_{X\restriction A_\lambda} = (\sqrt{\mathcal{J}_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda}, \tag{4.20}$$

$$(\mathcal{O}_X/\mathcal{N}_X)_{\restriction A_\lambda} \simeq (\mathcal{O}_{\Omega_\lambda}/\sqrt{\mathcal{J}_\lambda})_{\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{I}_{A_\lambda})_{\restriction A_\lambda} = \mathcal{O}_{A_\lambda}. \tag{4.21}$$

The complex space $(X, \mathcal{O}_X)$ is said to be *reduced* if $\mathcal{N}_X = 0$. The associated ringed space $(X, \mathcal{O}_X/\mathcal{N}_X)$ is reduced by construction; it is called the *reduced complex space* of $(X, \mathcal{O}_X)$. We shall often denote the original complex space by the letter X merely, the associated reduced complex space by X_{red}, and let $\mathcal{O}_{X,\mathrm{red}} = \mathcal{O}_X/\mathcal{N}_X$. There is a canonical morphism $X_{\mathrm{red}} \to X$ given by the comorphism

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\mathrm{red}}(U) = (\mathcal{O}_X/\mathcal{N}_X)(U) \tag{4.22}$$

for every open set U in X ; the kernel of (4.22) is the set of locally nilpotent sections of $\mathcal{O}_X$ on U. It is easy to see that a morphism F of reduced complex spaces X, Y is completely determined by the underlying set-theoretic map $f : X \to Y$.