VECTOR FIELDS AND GENUS IN DIMENSION 3

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Abstract. Given a vector field on a 3-dimensional rational homology sphere, we give a formula for the Euler characteristic of its transverse surfaces, in terms of boundary data only. This provides a formula for the genus of a transverse surface, and in particular, of a Birkhoff section. As an application, we show that for a right-handed flow with an ergodic invariant measure, the genus is an asymptotic invariant of order 2 proportional to helicity.

1. Introduction

In this paper we study topological properties of non-singular vector fields on 3-dimensional homology spheres. An important problem in this field is to obtain measure-preserving homeomorphism or diffeomorphism invariants, meaning that the value is the same for flows that are conjugated by a homeomorphism or a diffeomorphism that preserve a given measure. We provide a step towards the possible definition of such an invariant built from the genus of knots.

Introduced by Woltjer, Moreau and Moffatt, *helicity* is the main known invariant [Wol58, Mor61, Mof69]. For a divergence free vector field X on a closed Riemannian manifold M, it is defined by the formula $\mathrm{Hel}(X) = \int X \cdot Y$, where $Y = \mathrm{curl}^{-1}(X)$ is an arbitrary vector-potential of X. Arnold and Vogel proved that, on homology spheres, helicity coincides with the average asymptotic linking number [Arn73, Vog02]. More precisely, let ϕ^t_X for $t \in \mathbb{R}$ be the flow of X and denote by $k_X(p,t)$ the loop starting at the point p that follows the orbit until $\phi^t_X(p)$ and closes by an arbitrary segment of bounded length. The average asymptotic linking number is the double integral, with respect to an invariant measure, of $\lim_{t_1,t_2\to\infty} \frac{\mathrm{Lk}(k_X(p_1,t_1),k_X(p_2,t_2))}{t_1 t_2}$, where Lk is the linking number between the two loops.

In order to produce other asymptotic invariants, one is tempted to replace the linking number by another link or knot invariant. In this direction, Freedman and He constructed the asymptotic crossing number [FrH91], Gambaudo and Ghys constructed the asymptotic Ruelle invariant [GaG97] (see also Section 4.b), while the authors of this paper constructed the trunkenness based on the trunk of a knot [DeR15]. These are three examples of invariants that are not proportional to helicity. On the other hand, Gambaudo and Ghys considered ω-signatures of knots [GaG01], Baader considered linear saddle invariants [Baa11], and Baader and Marché considered Vassiliev's finite type invariants [BaM12]. All these constructions have the drawback that they do not yield any new invariant for ergodic vector fields, the obtained limits are all functions of the helicity. For an explanation for the ubiquity of helicity we refer to [Kud15, EPT16].

The genus of a knot k, denoted by $g(k)$ and defined as the minimal genus of an orientable surface spanned by k, is a fundamental invariant in knot-theory. An open problem in the context of vector fields is to prove that the limit of $\frac{1}{t^n} g(k_X(p,t))$ exists for some n. Actually $n = 2$ is the natural candidate as explained in [Deh15b, Question 5.4] and as a consequence of Theorem 1.1.

In this paper we study the genus of surfaces whose boundary is composed by one or several periodic orbits of the flow and whose interior is transverse to the vector field. We refer to such surfaces as **transverse surfaces**. Considering only such surfaces is a strong restriction, since among the collection of surfaces with fixed boundary, those of minimal genus need not be transverse to the vector field. But a specific hypothesis on the flow will ensure that it is the case. Theorem 1.1 stands for **right-handed vector fields**: these are vector fields on homology spheres all of whose invariant positive measures link positively [Ghy09].

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This hypothesis implies that any collection of periodic orbits of the flow bounds a transverse surface that intersects all the orbits of the flow.

Theorem 1.1. *Let M be a 3-manifold that is a rational homology sphere, X a non-singular right-handed vector field on M and μ a X-invariant measure. If $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of periodic orbits whose lengths $(t_n)_{n\in\mathbb{N}}$ tend to infinity and such that $(\frac{1}{t_n}\gamma_n)_{n\in\mathbb{N}}$ tends to μ in the weak-$*$ sense, then the sequence $(\frac{1}{t_n^2}g(\gamma_n))_{n\in\mathbb{N}}$ tends to half the helicity of (X,μ).*

Note that one can replace right-handedness by left-handedness, and only change the helicity by its absolute value. Right-, or left-, handedness is a strong restriction, but several important classes of vector fields have this property: for example the Lorenz vector field on $\mathbb{R}^3$ is (in a certain sense) right-handed [Ghy09], geodesic flows on positively curved surfaces are left-handed, as well as geodesic flows on hyperbolic triangular orbifolds [Deh16]. On the other hand many flows are neither right- nor left-handed, as for example the Ghrist flow which contains all types of knots as periodic orbits [Ghr97]. For such general flows, we also expect the genus to have a quadratic asymptotic behaviour, but we expect the asymptotic value to be strictly larger than half the absolute value of the helicity.

The proof of Theorem 1.1 mostly relies on an adaptation of results on global sections to flows to the case of surfaces with boundary, coupled with the classical fact in knot theory that fiber surfaces for knots are genus-minimizing. More precisely, a transverse surface is a **Birkhoff section** if it intersects all the orbits of the flow. If the boundary is empty, one speaks of a **global cross section**. When a flow admits a global cross section, up to changing the time-parameter (*i.e.* multiplying the vector field by a striclty positive function), the dynamics of the flow is described by the dynamics of the first-return map on the global cross section. The description of all global cross sections to a vector field is given by Schwartzman-Fuller-Sullivan-Fried Theory, which is purely homological [Sch57, Ful65, Sul76, Fri82]. Moreover, Thurston and Fried gave formulas for computing the genus of global cross sections [Thu86, Fri79].

Schwartzman-Fuller-Sullivan-Fried Theory may be extended to Birkhoff sections, but this has only been partially done [Fri82, Ghy09, Hry19]. What we do here is to push a bit further Fried's and Ghys' ideas. We provide, in Corollary 1.3, a formula for the genus of transverse surfaces with boundary, that depends only on data calculated along boundary components. The corollary is deduced from the following result that provides a formula for the Euler characteristic of these type of surfaces.

Theorem 1.2. *Assume that M is a 3-dimensional rational homology sphere and that X is a non-singular vector field on M. Let $\{\gamma_i\}_{1\leqslant i\leqslant m}$ be a finite collection of periodic orbits of X and $\{n_i\}_{1\leqslant i\leqslant m}$ a collection of integers. If S is a transverse surface to X with oriented boundary $\cup n_i\gamma_i$, then the Euler characteristic of S is given by*

$$\chi(S) = -\sum_{1\leqslant i<j\leqslant m}(n_i+n_j)\mathrm{Lk}(\gamma_i,\gamma_j) - \sum_{1\leqslant i\leqslant m} n_i \mathrm{Slk}^{\zeta_X}(\gamma_i),$$

where ζ_X denotes any vector field everywhere transverse to X and Slk^{ζ_X} the self-linking given by the framing ζ_X (see Definition 2.3).

We can easily deduce the formula for the genus of the surface.

Corollary 1.3. *Assume that M is a 3-dimensional rational homology sphere and that X is a non-singular vector field on M. Let $\{\gamma_i\}_{1\leqslant i\leqslant m}$ be a finite collection of periodic orbits of X and $\{n_i\}_{1\leqslant i\leqslant m}$ a collection of integers. If S is a transverse surface to X with oriented boundary $\cup n_i\gamma_i$, then the genus of S is given by*

$$g(S) = 1 + \frac{1}{2}\left(\sum_{1\leqslant i<j\leqslant m}(n_i+n_j)\mathrm{Lk}(\gamma_i,\gamma_j) + \sum_{1\leqslant i\leqslant m}\left(n_i\mathrm{Slk}^{\zeta_X}(\gamma_i) - \gcd(n_i, \sum_{j\neq i} n_j\,\mathrm{Lk}(\gamma_i,\gamma_j))\right)\right).$$

It is likely that Theorem 1.2 may be adapted to an arbitrary 3-manifold M. In this case, one would have to adapt the definition of linking number which is not anymore well-defined.

Theorem 1.2 and Corollary 1.3 also have an independent interest. Given a flow in a 3-manifold, it is a natural question to look for Birkhoff sections of minimal genus or minimal Euler characteristics. For example Fried asked whether every transitive Anosov flow with orientable invariant foliations admits a genus-one Birkhoff section [Fri83]. Similarly, Etnyre asked wether every contact structure can be defined by a contact form whose Reeb flow admits a genus-one Birkhoff section [Etn06]. Corollary 1.3 was implemented by the first author for the geodesic flows on some hyperbolic orbifolds, which led to answer positively Fried's question in these cases [Deh15a]. It was also used by Dehornoy and Shannon to numerically check that suspensions of linear automorphisms of $\mathbb{T}^2$ admit infinitely many genus-one Birkhoff sections [DeS19].

The paper is organized as follows. In Section 2 we give a proof of Theorem 1.2 and Corollary 1.3. In Section 3 we illustrate Theorem 1.2 with the example of the Hopf vector field. Theorem 1.1 is proved in Section 4.

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2. The Euler characteristic of a transverse surface

The aim of this section is to give a proof of Theorem 1.2, which is done in Section 2.e. We first recall in 2.a some classical results on global cross sections and we explain in 2.b how the genus of such a section can be obtained. Section 2.c introduces linking and self-linking, and 2.d contains the key-lemma for the proof of Theorem 1.2.

2.a. **Schwartzman-Fuller-Sullivan-Fried Theory.** We recall classical results concerning global cross sections to flows, but we state them in the more general context of a 3-manifold M with toric boundary and a non-singular vector field X tangent to ∂M. The original proofs extend verbatim to this case.

First we recall the definition of **asymptotic cycles**. The original one is in terms of almost-periodic orbits [Sch57]: for $p \in M$ and $t > 0$, we denote by $k_X(p,t)$ the closed curve obtained by connecting the arc of orbit $\phi_X^{[0,t]}(p)$ with a segment of bounded length (recall that M is compact). Assume now that μ is an ergodic invariant positive measure and that p is a quasi-regular point for μ. The asymptotic cycle a_μ determined by μ is the weak-$*$ limit $(\frac{1}{t_n}[k_X(p,t_n)])_{n\in\mathbb{N}}$, with $t_n \to \infty$. It is independent of p and t_n. The set $\mathcal{S}_X$ of all asymptotic cycles is the convex hull of those asymptotic cycles associated to ergodic invariant positive measures. It is a convex cone in $\mathrm{H}_1(M;\mathbb{R})$.

Alternatively, for μ a X-invariant positive measure, one can consider the 1-current $c_\mu : \Omega^1(M) \to \mathbb{R}$ which maps a 1-form f to $\int_M f(X(p))\,d\mu(p)$. The invariance of μ implies that c_μ is closed (*i.e.*, it vanishes on closed forms), hence it determines a 1-cycle $[c_\mu] \in \mathrm{H}_1(M;\mathbb{R})$ in the sense of De Rham. The two notions actually coincide: when μ is an ergodic measure, a_μ and $[c_\mu]$ are equal under the identification of singular and current homologies [Sul76].

Schwartzman's criterion [Sch57] is the following.

Theorem 2.1 (Schwartzman)**.** *A class $\sigma \in \mathrm{H}^1(M;\mathbb{Z})$ is dual to a global cross section if, and only if, for every asymptotic cycle $c \in \mathcal{S}_X$ one has $c(\sigma) > 0$.*

2.b. **Genus of global cross sections.** In the context of the previous part, a standard argument shows that if two global cross sections to a vector field are homologous, then they are isotopic along the flow [Thu86]. Actually it shows more: a global cross section minimises the genus in its homology class. So one may wonder how to compute this genus. Thurston and Fried give a satisfying answer [Fri79]. Denote by $X^\perp$ the normal bundle to X (it is the 2-dimensional bundle $TM/\mathbb{R}X$), and by $e(X^\perp) \in \mathrm{H}^2(M, \partial M;\mathbb{Z})$ its Euler class.

Theorem 2.2 (Fried-Thurston)**.** *Assume that S is a surface transverse to X. Then one has $\chi(S) = e(X^\perp)([S])$.*

The argument is short: since S is transverse to X, the restricted bundle $X^\perp|_S$ is isomorphic to the tangent bundle TS. In particular one has $e(X^\perp)([S]) = e(TS)([S]) = \chi(S)$.

Said differently, if ζ is a vector field in generic position with respect to X, the set $L_{\zeta,X}$ where ζ is tangent to X is a 1-manifold. In order to consider its homology class, one has to orient $L_{\zeta,X}$ and to equip it with multiplicities. Here is how one can do it (see Figure 1): one takes a small disc D positively transverse to X and transverse to $L_{\zeta,X}$. The projection of ζ on D along X defines a vector field ζ_D on D with a singularity at the center. The index of the singularity may be positive or negative (it cannot be 0 for in this case X and ζ would not be in generic position) and thus the product of its sign with the orientation of D induces a new orientation on D. Since D is transverse to $L_{\zeta,X}$, this new orientation induces an orientation of $L_{\zeta,X}$. The multiplicity then comes from the absolute value of the index of the singularity. Observe that the multiplicity is locally constant by continuity and hence constant on each connected component of $L_{\zeta,X}$. The point is that these choices are independent of D. If one perturbes D by keeping it transverse to X and $L_{\zeta,X}$, by continuity and discreteness, the multiplicity and orientation do not change. If one perturbes D by keeping it transverse to X but change the relative position to respect to $TL_{\zeta,X}$, the induced orientation on $L_{\zeta,X}$ changes, but the index is also changed by its opposite. Hence the product is constant.

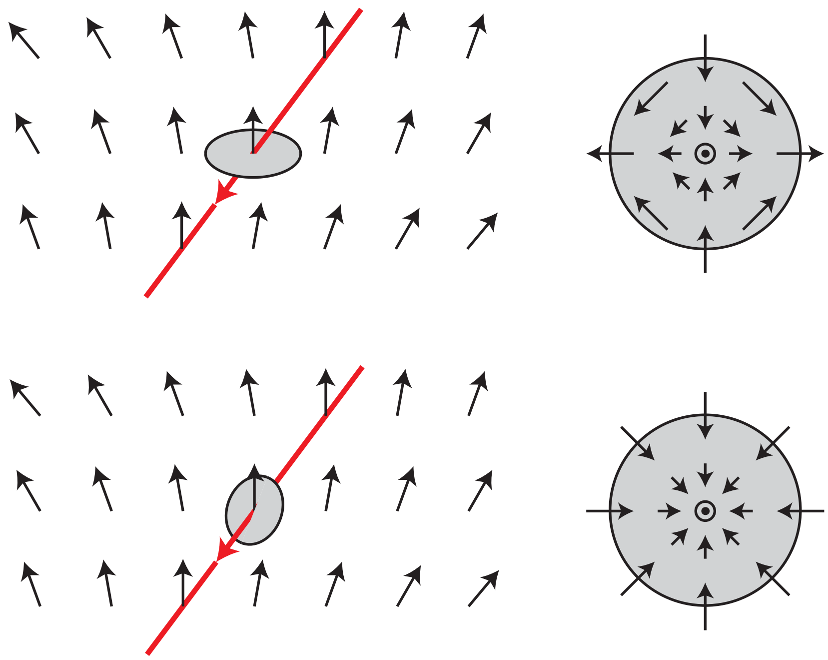

FIGURE 1. Orientation of the dual $L_{\zeta,X}$ of the Euler class $e(X^\perp)$. On this picture the vector field X is locally vertical. A vector field ζ in general position with respect to X is shown. It is tangent to X (*i.e.*, vertical) on a dimension 1-submanifold $L_{\zeta,X} := \{X \parallel \zeta\}$, in red. We consider an arbitrary disc D transverse to both X and $L_{\zeta,X}$. Projecting ζ on D along X we get a vector field ζ_D with one singularity at the center. The index of this vector field gives a multiplicity to the disc, and together with the orientation of D an orientation and a multiplicity to $L_{\zeta,X}$. One checks that changing D may change the index by its opposite (bottom), but also the orientation, so that their product is unchanged.

The class $[L_{\zeta,X}] \in \mathrm{H}_1(M;\mathbb{Z})$ is then Poincaré dual to $e(X^\perp)$, see [BoT82, Prop. 12.8]. Given a surface S transverse to X, projecting ζ on S along X yields a vector field ζ_S on S, which vanishes exactly when S intersects $L_{\zeta,X}$. The Euler characteristic of S can be computed with the Poincaré-Hopf formula for ζ_S. The crucial point [Thu86, Fri79] is that, thanks to the orientation and multiplicity of $L_{\zeta,X}$, each intersection point contributes with the right sign to the sum.

2.c. **Linking and self-linking on homology spheres.** We now assume that M is a rational homology sphere. Given two links L_1, L_2, their **linking number** $\mathrm{Lk}(L_1, L_2)$ is defined as $\langle L_1, S_2\rangle$, where S_2 is a rational 2-chain bounded by L_2. The existence of S_2 is guaranteed by the vanishing of $[L_2]$, whereas the vanishing of $[L_1]$ implies that the linking number is independent of the choice of S_2. Linking number is symmetric, although this is not obvious from the definition we just gave.

A **framing** f of a link L is a section of its unit normal bundle. It induces an isotopy class L^f in $M \setminus L$ obtained by pushing L off itself in the direction of f.

Definition 2.3. Given a link L and a framing f, the **self-linking** $\mathrm{Slk}^f(L)$ is defined as the linking number of L and L^f.

When M is an integral homology sphere, every individual curve (that is, the curve with multiplicity one) has a preferred framing corresponding to a self-linking number zero. By definition, this **zero-framing** is induced by any surface bounded by the considered curve. Therefore one can see the self-linking number with respect to a framing f as the algebraic intersection number between the framing f and a surface bounded by the considered curve.

When M is a rational non-integral homology sphere, there is not always a preferred framing in the above sense. One then has to consider **rational framings** which are multi-sections of the unit normal bundle. More precisely, the unit normal bundle to a curve γ is a torus $\gamma \times \mathbb{S}^1$. A rational framing is a homotopy class of a closed curve on that torus. Given such a rational framing f of a link L, we can extend the definition of self-linking to this context. Assume that f winds k_f times in the longitudinal direction along L, we define L^f as the link obtained by pushing the link L traveled k_f times off itself in the direction of f. Then $\mathrm{Slk}^f(L)$ is defined as $\frac{1}{k_f}\mathrm{Lk}(L, L^f)$. With this extended definition, there is always one (rational) zero-framing.

2.d. **Boundary slope.** We continue with the assumption that M is a rational homology sphere. Fix a link $L = K_1 \cup \cdots \cup K_m$ in M. The boundary operator ∂ realises an isomorphism $\mathrm{H}_2(M, L; \mathbb{R}) \simeq \mathrm{H}_1(L; \mathbb{R})$, as can be seen by writing the long exact sequence. Therefore a class in $\mathrm{H}_2(M, L; \mathbb{R})$ is determined by its boundary class. For σ in $\mathrm{H}_2(M, L; \mathbb{R})$, denote by $n_i(\sigma)$ its **longitudinal boundary coordinates**, that is, the real numbers such that $\partial\sigma = \sum n_i(\sigma)[K_i]$.

In this context, we denote by M_L the **normal compactification** of $M \setminus L$: the manifold obtained from M by replacing every point of L by the circle of those half-planes bounded by TL. The boundary ∂M_L is then isomorphic to $L \times \mathbb{S}^1$: it is a disjoint union of tori. The manifold M_L is actually isomorphic to $M \setminus \nu(L)$, where $\nu(L)$ is an open tubular neighbourhood of L.

By the excision theorem, there is an isomorphism $\mathrm{H}_2(M, L; \mathbb{R}) \simeq \mathrm{H}_2(M_L, \partial M_L; \mathbb{R})$, so that we can also see the class σ as an element of $\mathrm{H}_2(M_L, \partial M_L; \mathbb{R})$. There, its boundary is an element of $\mathrm{H}_1(\partial M_L; \mathbb{R})$, whose dimension is higher than the dimension of $\mathrm{H}_1(L; \mathbb{R})$. For distinction we then write $\partial^\bullet : \mathrm{H}_2(M, L; \mathbb{R}) \to \mathrm{H}_1(L; \mathbb{R})$ and $\partial^\circ : \mathrm{H}_2(M_L, \partial M_L; \mathbb{R}) \to \mathrm{H}_1(\partial M_L; \mathbb{R})$ for the two operators (see Figure 2). As we said the first one is an isomorphism, while the second one is only injective.

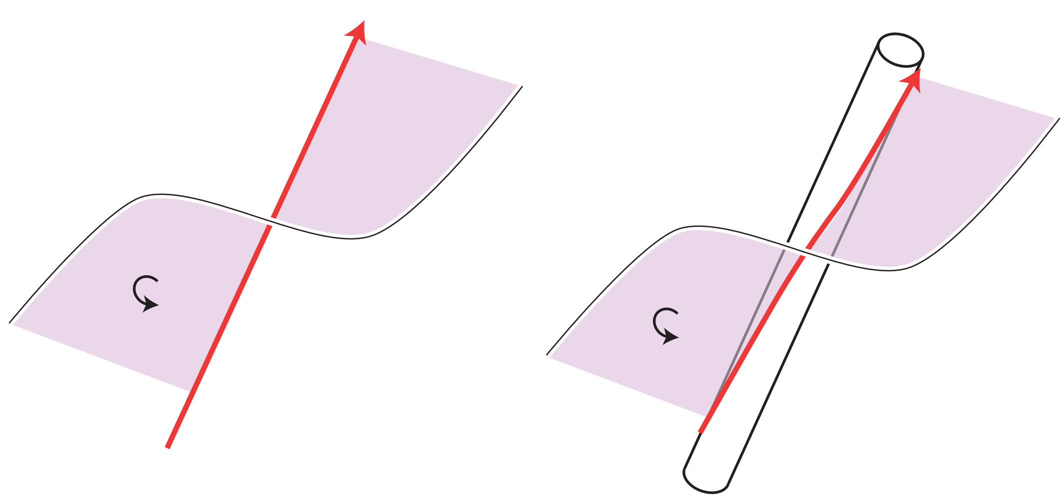

FIGURE 2. On the left, a link L (red) in a 3-manifold M and a surface S (purple) representing a class σ with $\partial^\bullet\sigma = L$. On the right, the corresponding manifold M_L with boundary $L \times \mathbb{S}^1$ and the corresponding surface S whose boundary $\partial^\circ S$ sits in $L \times \mathbb{S}^1$. The additional information given by the meridian coordinate of $\partial^\circ S$ tells how many times S wraps around L.

Let S be a surface representing the class σ above. In order to understand the image of ∂°, we have to understand the framing induced by S along every boundary component (that is, the slope of $S \cap (K_i \times \mathbb{S}^1)$ for every component K_i of L).

Lemma 2.4. *If M is a rational homology sphere and S is a surface with $\partial^\bullet S = \sum_i n_i K_i$, then the coordinates of $\partial^\circ S$ along K_i in the (meridian, 0-longitude)-basis are* $\left(-\sum_{j \neq i} n_j \,\mathrm{Lk}(K_i, K_j), n_i \right)$.

Proof. Since M is a homology sphere, all oriented surfaces with the same boundary induce the same framing on the boundary. An oriented surface realizing $[S]$ is obtained by desingularising the union $\cup_i n_i(\sigma) S_i$ where S_i is an oriented surface in M with boundary K_i. Observe that the intersection of S_i and S_j, for $i \neq j$, can be made either empty or transverse, and when non-empty it can be made of segments with one endpoint in K_i and the other in K_j.

If the intersection is non-empty, at each connected component of the intersection we have two choices of desingularisation, but only one that respects orientations. The desingularisation near the endpoints of each segment in the intersection is obtained by removing one meridian to K_i everytime S_j intersects K_i. This number is equal to $\mathrm{Lk}(K_i, K_j)$ and hence the total meridian contribution of all surfaces on K_i is $-\sum_{j \neq i} n_j \mathrm{Lk}(K_i, K_j)$. On the other hand the longitudinal coordinate is unchanged in this desingularisation process, thus it is n_i. The conclusion follows immediately. $\square$

2.e. **The Euler class of $X_\Gamma^\perp$.** Assume that M is a rational homology sphere with empty boundary. We are given a non-singular vector field X on M, a finite collection $\Gamma = \gamma_1 \cup \cdots \cup \gamma_m$ of periodic orbits of X and multiplicities $n_1, \ldots, n_m$ which are integers. In this context the existence of a Birkhoff section for X bounded by $\cup_{i=1}^m n_i \gamma_i$ is the same as the existence of a cross section $(S, \partial S)$ for the extension of X to the manifold M_Γ such that the longitudinal coordinates of $\partial^\circ S$ are $(n_1, \ldots, n_m)$.

Denote by X_Γ the extension of X to M_Γ. In order to understand the topology of the cross section, one wonders whether the class $e(X_\Gamma^\perp)$ may be easily represented.

Since the Euler class of $X^\perp$ vanishes, there exists non-singular vector fields on M everywhere transverse to X. Since M is a homology sphere, two such vector fields are homotopic through vector fields that are everywhere transverse to X. Indeed the first vector field gives an origin to the normal sphere bundle, so that the second vector field yields a function on the circle. Since M is a homology sphere, this function is homotopic to a constant function.

Denote by ζ_X a vector field transverse to X. We use ζ_X to realise the Euler class $e(X_\Gamma^\perp)$. As in Fried and Thurston theorem (see Theorem 2.2), the Euler class of a vector bundle is represented by the intersection with a generic section (see [BoT82, Prop. 12.8]). Here one has to take into account the fact that M_Γ has boundary.

Proof. [Proof of Theorem 1.2] The vector field ζ_X is everywhere transverse to X on M but not tangent to ∂M_Γ. In order to make it tangent to the boundary of the manifold, we have to "rotate" ζ_X towards X around each component γ_i. This can be achieved by combining, near γ_i, the two vector fields ζ_X and X_Γ (see Figure 3).

We obtain a vector field $\zeta_{X,\Gamma}$ which is transverse to X_Γ on M_Γ, except at the boundary components, where it is tangent to X_Γ along two curves that correspond to the framings given by ζ_X and $-\zeta_X$. In particular the set $L_{\zeta_X,\Gamma} := \{p \in M_\Gamma \,|\, X_\Gamma(p) \parallel \zeta_{X,\Gamma}(p)\}$ is a collection of two longitudes $\gamma_i^{\mathrm{in}}, \gamma_i^{\mathrm{out}}$ for every boundary component $\gamma_i \times \mathbb{S}^1$ of ∂M_Γ.

Orienting $L_{\zeta_X,\Gamma}$ so that its class is dual to $e(X_\Gamma^\perp)$ can be done as in Section 2.b. Considering a component γ_i of Γ, one has to take a small disc D in M transverse to γ_i. The corresponding annulus D_Γ in M_Γ is automatically transverse to $L_{\zeta_X,\Gamma}$. The projection of $\zeta_{X,\Gamma}$ on D_Γ exhibits two singularities of index $-\frac{1}{2}$ (see the right-hand disc in Figure 3). Therefore, if γ_i^{in} and γ_i^{out} are oriented in the direction opposite to X, they both have multiplicty $\frac{1}{2}$.

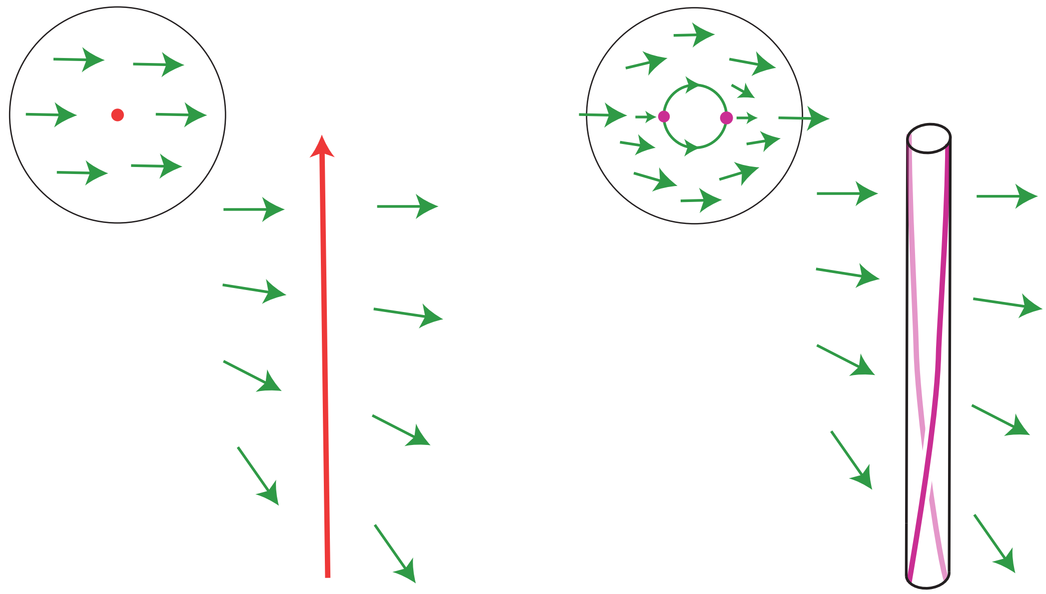

FIGURE 3. On the left, the vector field ζ_X (green) on M. Since it is transverse to X, is it transverse to the link Γ (red) made of periodic orbits of X. Seen from above, Γ is a point and ζ_X is a non-vanishing vector field. On the right, the modification of ζ_X into $\zeta_{X,\Gamma}$. Seen from above, one has to slow down ζ_X so that it has (transversal) speed 0 on Γ. The set $L_{\zeta_X,\Gamma}$ (pink) then consists of two longitudes per component of Γ.

Let S be the surface transverse to X in the statement of Theorem 1.2 and $\sigma \in \mathrm{H}_2(M_\Gamma, \partial M_\Gamma; \mathbb{Z})$ be its class, then $\chi(S) = e(X_\Gamma^\perp)(\sigma) = \langle L_{\zeta_X,\Gamma}, \sigma\rangle$. This intersection equals

$$\sum_{i=1}^{m} \langle \gamma_i^{\mathrm{in}} + \gamma_i^{\mathrm{out}}, (\partial^\circ\sigma)_i\rangle,$$

where $(\partial^\circ\sigma)_i$ denotes the part of $\partial^\circ\sigma$ on the component $\gamma_i \times \mathbb{S}^1$ of ∂M_Γ.

Now the algebraic intersection of two curves on a 2-torus is the determinant of their coordinates in homology. Since γ_i^{in} and γ_i^{out} both have coordinates $(-\mathrm{Slk}^{\zeta_X}(\gamma_i), -1)$ in the (meridian, 0-longitude)-basis of $\gamma_i \times \mathbb{S}^1$, and since every intersection point contributes to $\frac{1}{2}$ to the intersection, thanks to Lemma 2.4, we have

$$\langle \gamma_i^{\mathrm{in}} + \gamma_i^{\mathrm{out}}, (\partial^\circ\sigma)_i\rangle = -n_i\mathrm{Slk}^{\zeta_X}(\gamma_i) - \sum_{j\neq i} n_j \,\mathrm{Lk}(K_i, K_j).$$

Summing over all boundary components, we get the desired formula. □

Proof. [Proof of Corollary 1.3] From the surface S bounded by $\cup n_i\gamma_i$ described above, we only have to cap all boundary components with discs for obtaining a closed surface. However counting how many discs we need is a bit subtle: along each orbit γ_i, the surface winds n_i times longitudinally and, thanks to Lemma 2.4, $-\sum_{j\neq i} n_j \,\mathrm{Lk}(\gamma_i, \gamma_j)$ meridionally. Therefore the number of boundary components along γ_i of the abstract surface S is the gcd of these numbers. We get

$$\begin{aligned} g(S) &= 1 - \frac{\chi(S) + \sum_i \gcd(n_i, \sum_{j\neq i} n_j \,\mathrm{Lk}(\gamma_i,\gamma_j))}{2} \\ &= 1 + \frac{1}{2}\left(\sum_{1\leqslant i<j\leqslant m} (n_i{+}n_j)\mathrm{Lk}(\gamma_i,\gamma_j) + \sum_{1\leqslant i\leqslant m}\left(n_i\mathrm{Slk}^{\zeta_X}(\gamma_i) - \gcd(n_i, \sum_{j\neq i} n_j \,\mathrm{Lk}(\gamma_i,\gamma_j))\right)\right). \end{aligned}$$

□

3. Examples: Birkhoff sections for the Hopf vector field

Let X_{Hopf} denote the Hopf vector field on $\mathbb{S}^3$ and ϕ_{Hopf} denote the associated flow. Every orbit γ of ϕ_{Hopf} is periodic and bounds a disc D_γ transverse to X_{Hopf}. Since any two orbits have linking number $+1$, the disc D_γ is a Birkhoff section for ϕ_{Hopf}. More generally, if $\cup n_i\gamma_i$ and $\cup n'_i\gamma'_i$ are two collections of disjoint periodic orbits with multiplicities, their linking number is given by $(\sum n_i)(\sum n'_i)$.

On the other hand, a vector field transverse to X_{Hopf} can be easily found by taking another Hopf fibration ζ_{Hopf} orthogonal to X_{Hopf}. One checks that for every periodic orbit γ of ϕ_{Hopf}, one has $\text{Slk}^{\zeta_{\text{Hopf}}}(\gamma) = -1$.

For a Birkhoff section that is a disc D_γ with one boundary component γ, Theorem 1.2 then yields

$$\chi(D_\gamma) = -(-1) = +1,$$

as expected.

Consider now a collection $\gamma_1, \dots, \gamma_m$ of m periodic orbits. The link $\Gamma := \sum_i \gamma_i$ links positively with any positive invariant measure. Hence it bounds a Birkhoff section, denoted by S_Γ. This section can be obtained from the union $D_{\gamma_1} \cup \dots \cup D_{\gamma_m}$ by desingularising along the segments where two discs intersect. As discussed in the proof of Lemma 2.4, for each segment there are two possible ways to resolve the intersection an obtain a (non-orientable) surface with the same boundary. But since here we want the resulting surface to be transverse to the vector field X_{Hopf}, there is only one way to desingularise.

The Euler characteristic of the resulting surface can be computed by hand, but Theorem 1.2 directly yields

$$\chi(S_\Gamma) = -m(m-1) - (-m) = -m(m-2).$$

Since S_Γ has m boundary components, we obtain

$$g(S_\Gamma) = 1 - \frac{\chi(S_\Gamma) + m}{2} = 1 + \frac{m(m-3)}{2}$$

which is the genus of a Hopf link with m components. One can generalise a bit more by considering a collection $\cup_i n_i\gamma_i$, where the n_i are integers. The condition for bounding a Birkhoff section then becomes $\sum_i n_i > 0$, since the linking number of $\cup_i n_i\gamma_i$ with any other orbit of the flow has to be positive. Denoting by $S_{\cup_i n_i\gamma_i}$ such a Birkhoff section, Theorem 1.2 yields

$$\chi(S_{\cup_i n_i\gamma_i}) = -\sum_{1\leqslant i<j\leqslant m}(n_i+n_j) + \sum_{1\leqslant i\leqslant m} n_i = \sum_{1\leqslant i\leqslant m}(1-m)n_i + \sum_{1\leqslant i\leqslant m} n_i = (2-m)\sum_{1\leqslant i\leqslant m} n_i.$$

4. Genus and the Ruelle invariant

In this section we prove Theorem 1.1. As discussed above, when M is a homology sphere and X a non-singular vector field on M, for every periodic orbit γ of X, we have presented two preferred framings, namely the zero-framing determined by a spanning surface, and the framing given by a vector field ζ_X everywhere transverse to X. By definition, the difference of these two framings along γ is $\pm\text{Slk}^{\zeta_X}(\gamma)$.

If γ is the unique boundary component of a Birkhoff section, Corollary 1.3 says that the genus of this Birkhoff section is $1 + (\text{Slk}^{\zeta_X}(\gamma) - 1)/2$. One wonders whether this quantity has an asymptotic behaviour when γ tends to fill M. Two related quantities are known to have one, and we present them now. Both rely on a third framing on γ, given by the differential of the flow.

4.a. **Three framings on $\gamma \times \mathbb{S}^1$.** Recall that M_γ is the 3-manifold with boundary obtained from M by replacing every point of γ by its sphere normal bundle $S(TM/\mathbb{R}X)$ which is topologically a circle. If X is at least C^1, we can then extend X to $\gamma \times \mathbb{S}^1$ using the differential of the flow, and obtain a non-singular vector field X_γ on M_γ. Now we have three framings on $\gamma \times \mathbb{S}^1$, two integral ones (the zero-framing and the one induced by ζ_X) and one real (induced by DX).

The restriction of X_γ to ∂M_γ is a vector field on a torus whose first coordinate (in the X-direction) can be made constant. Hence it has a well-defined translation number: the Ruelle invariant $R^X(\gamma)$ is defined as the translation number of $X_\gamma|_{\gamma\times\mathbb{S}^1}$ with respect to the framing ζ_X [Rue85, GaG97]. On the other hand the rotation number of $X_\gamma|_{\gamma\times\mathbb{S}^1}$ with respect to the zero-framing is given by $\text{Slk}^{DX}(\gamma)$. Both these numbers are real (and not necessarily integers).

Since the quantities $R^X(\gamma), \mathrm{Slk}^{\zeta_X}(\gamma)$ and $\mathrm{Slk}^{DX}(\gamma)$ denote the respective difference between the three possible pairs of framings, we have

$$\mathrm{Slk}^{\zeta_X}(\gamma) = \mathrm{Slk}^{DX}(\gamma) - R^X(\gamma), \tag{1}$$

where only the term $\mathrm{Slk}^{\zeta_X}(\gamma)$ is always an integer.

The Ruelle invariant may be extended to any X-invariant measure using long arcs of orbits, but we do not need this here (see [GaG97]).

4.b. **Asymptotic genus for right-handed vector fields.** Assume that X is now a right-handed vector field on a rational homology sphere M, meaning that all X-invariant positive measures have positive linking number [Ghy09]. In this context Ghys proved that every periodic orbit bounds a Birkhoff section. It is also known that such a section is genus-minimizing (this is a folklore result among 3-dimensional topologists, a possible reference is [Thu86] although the statement is older). Therefore the genus of a periodic orbit γ is given by $1 + (\mathrm{Slk}^{\zeta_X}(\gamma) - 1)/2$.

Proof. [Proof of Theorem 1.1] Arnold and Vogel proved that if (γ_n) is a sequence a periodic orbits that tend to an invariant volume μ in the weak-$*$ sense, then writing t_n for the period of γ_n, the sequence $\frac{1}{t_n^2}\mathrm{Slk}^{DX}(\gamma_n)$ tends to the helicity $\mathrm{Hel}(X, \mu)$ [Arn73, Vog02]. Similarly, Gambaudo and Ghys proved that the sequence $\frac{1}{t_n}R^X(\gamma_n)$ tends to the Ruelle invariant $R^X(\mu)$ [GaG97].

Since one term grows quadratically and the other one linearly on t_n, in the right-hand side of Equation (1), the term $\frac{1}{t_n^2}R^X(\gamma_n)$ is negligible, and the asymptotic is dictated by $\mathrm{Slk}^{DX}(\gamma_n)$. In particular we have

$$\frac{1}{t_n^2}\chi_{min}(\gamma_n) = -\frac{1}{t_n^2}\mathrm{Slk}^{\zeta_X}(\gamma_n) \to -\mathrm{Hel}(X, \mu).$$

Then

$$\lim_{t_n\to\infty} \frac{1}{t_n^2} g(\gamma_n) = \lim_{t_n\to\infty} \frac{1}{t_n^2}\left(1 + \frac{\mathrm{Slk}^{\zeta_X}(\gamma_n) - 1}{2}\right) = \lim_{t_n\to\infty} \frac{\mathrm{Slk}^{\zeta_X}(\gamma_n)}{2t_n^2} = \frac{1}{2}\mathrm{Hel}(X, \mu).$$

□

In other words, the genus is an asymptotic invariant of order 2 for right-handed volume-preserving vector fields, and its asymptotic is half the helicity.

Remark that Baader proved that the slice genus (for arbitrary vector fields, not only right-handed) is an asymptotic invariant of order 2, and that it is also equal to half the helicity [Baa11]. So for right-handed vector fields, the long periodic orbits tend to have genus and slice genus of the same order.

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