# ON THE EXISTENCE OF SUPPORTING BROKEN BOOK DECOMPOSITIONS FOR CONTACT FORMS IN DIMENSION $3$

VINCENT COLIN, PIERRE DEHORNOY, AND ANA RECHTMAN

ABSTRACT. We prove that in dimension 3 every nondegenerate contact form is carried by a broken book decomposition. As an application we obtain that on a closed 3-manifold, every nondegenerate Reeb vector field has either two or infinitely many periodic orbits, and two periodic orbits are possible only on the tight sphere or on a tight lens space. Moreover we get that if $M$ is a closed oriented 3-manifold that is not a graph manifold, for example a hyperbolic manifold, then every nondegenerate Reeb vector field on $M$ has positive topological entropy.

## 1. INTRODUCTION

On a closed $3$-manifold $M$, the *Giroux correspondence* asserts that every contact structure $\xi$ is carried by some open book decomposition of $M$: there exists a Reeb vector field for $\xi$ transverse to the interior of the pages and tangent to the binding [Gir]. The dynamics of this specific Reeb vector field is then captured by its first-return map on a page, which is a flux zero area preserving diffeomorphism of a compact surface, a much simplified data. When one is interested in the dynamics of a *given* Reeb vector field this Giroux correspondence is quite unsatisfactory – though there are ways to transfer some properties of an adapted Reeb vector field to every other one through contact homology techniques [CH, ACH] – and the question one can ask is: Is every Reeb vector field adapted to some (rational) open book decomposition? Equivalently, does every Reeb vector field admit a Birkhoff section?

---

*Date*: This version: September 22, 2020.

2000 *Mathematics Subject Classification.* 58A30.

*Key words and phrases.* open book decomposition, broken book decomposition, Birkhoff section, Reeb vector field, entropy, periodic orbits.

V. Colin thanks the ANR Quantact for its support.

P. Dehornoy thanks the projects ANR-15-IDEX-02, ANR-11LABX-0025-01, and ANR-19-CE-40-007 Gromeov for their supports.

A. Rechtman thanks the IdEx Unistra, Investments for the future program of the French Government.

We give here a positive answer to these questions for the generic class of nondegenerate Reeb vector fields and the extended class of *broken book decompositions* (Definitions 2.1 and 2.4).

**Theorem 1.1.** *Every nondegenerate Reeb vector field is carried by a broken book decomposition.*

A contact form and the corresponding Reeb vector field are *nondegenerate* if all the periodic orbits of the Reeb vector field are nondegenerate, namely the eigenvalues of a Poincaré map are all different from one. The nondegeneracy condition is generic for Reeb vector fields, see for example [CH, Lemma 7.1]. In this case, periodic orbits are either elliptic or hyperbolic.

A *Birkhoff section* of a vector field $R$ is a surface with boundary whose interior is embedded and transverse to $R$, whose boundary is immersed and composed of periodic orbits. A Birkhoff section must intersect all orbits of $R$ within bounded time, so that there is a well-defined return map in the interior of the surface. The boundary will be called the *binding*. These surfaces are also known as global surfaces of section. A Birkhoff section induces a rational open book decomposition of the manifold.

Broken book decompositions are generalisations of Birkhoff sections and rational open book decompositions, reminiscent of finite energy foliations constructed by Hofer-Wyszocki-Zehnder for nondegenerate Reeb vector fields on $\mathbb{S}^3$ [HWZ2]. In a broken book decomposition we allow the binding to have broken components, in addition to *radial* ones modelled on the classical open book case. The complement of the binding is foliated by surfaces. A radial component of the binding has a tubular neighborhood in which the pages of the broken book induce a radial foliation. The foliation in a tubular neighborhood of a broken component has sectors that are radially foliated and sectors that are transversely foliated by hyperbolas. For the broken books we construct in this paper, each broken component has either two or four sectors foliated by hyperbolas.

A broken book decomposition *carries*, or *supports*, a Reeb vector field if the binding is composed of periodic orbits, while the other orbits are transverse to the foliation given on the complement of the binding by the interior of the pages (this foliation by relatively compact leaves is usually non trivial, as opposed to the genuine open book case). In the proof of Theorem 1.1, we construct a supporting broken book decomposition for any fixed nondegenerate Reeb vector field on a 3-manifold $M$ from a covering of $M$ by pseudo-holomorphic curves, given by the non-triviality of the $U$-map in embedded contact homology. The projected pseudo-holomorphic curves are then converted into surfaces with boundary whose interior is transverse to the Reeb vector field using a construction of Fried [Fri]. These surfaces give a complete system of transverse surfaces to the Reeb vector field, meaning that their union intersects every orbit.

Weinstein conjectured in 1979 that a Reeb vector field on a closed 3-manifold always has at least one periodic orbit [Wei]. The conjecture was proved in full generality by Taubes using Seiberg-Witten Floer homology [Tau]. It is also a consequence of the $U$-map property we use here, and it is no surprise that our result indeed implies the existence of the binding periodic orbits. Taubes' result was then improved by Cristofaro-Gardiner and Hutchings [C-GH], who proved that every Reeb vector field on a closed 3-manifold has at least two periodic orbits, following a work of Ginzburg, Hein, Hryniewicz and Macarini on $\mathbb{S}^3$ [GHHM]. It is now moreover conjectured that a Reeb vector field has either two or infinitely many periodic orbits. The existence of infinitely many periodic orbits has been established under some hypothesis (see the survey [GG]) and it is known to be generic [Iri]. Here we extend a recent result of Cristofaro-Gardiner, Hutchings and Pomerleano, originally obtained for *torsion contact structures* $\xi$ (with $c_1(\xi) \in \mathrm{Tor}(H^2(M,\mathbb{Z}))$) [C-GHP] and prove the conjecture for nondegenerate Reeb vector fields.

**Theorem 1.2.** *If $M$ is a closed oriented $3$-manifold that is not the sphere or a lens space, then every nondegenerate Reeb vector field on $M$ has infinitely many simple periodic orbits. In the case of the sphere or a lens space, there are either two or infinitely many periodic orbits.*

We point out that the cases where Reeb vector fields have exactly two nondegenerate periodic orbits are well-understood: they exist only on the sphere or on lens spaces, both periodic orbits are elliptic and are the core circles of a genus one Heegaard splitting of the manifold [HT1]. Also the contact structure has to be tight, since a nondegenerate Reeb vector field of an overtwisted contact structure always has a hyperbolic periodic orbit (see for example Theorem 8.9 in [HK]).

Beyond the number of periodic orbits, the study of the topological entropy of Reeb vector fields started with the works of Macarini and Schlenk [MS] and has been continued by Alves [ACH, Alv]. We recall that topological entropy measures the complexity of a flow by computing the growth of the number of "different" orbits. If this number grows exponentially then the entropy is positive. For flows in dimension 3, if the topological entropy is positive then the number of periodic orbits is infinite.

As an application of Theorem 1.1 we get a result on topological entropy

**Theorem 1.3.** *If $M$ is a closed oriented $3$-manifold that is not a graph manifold, then every nondegenerate Reeb vector field on $M$ has positive topological entropy.*

Theorems 1.2 and 1.3 are obtained by analysing the broken binding components of the broken book decomposition. Indeed, a broken component of the binding is a hyperbolic periodic orbit and we can prove that there are heteroclinic cycles between these periodic orbits. If there are no such broken components, then we have a rational open book decomposition and the results come from an analysis of its monodromy. In particular, we obtain

**Theorem 1.4.** *Every strongly nondegenerate Reeb vector field without homoclinic orbits is carried by a rational open book decomposition (where we drop the compatibility of orientations condition along the binding), or equivalently has a Birkhoff section.*

A *homoclinic* orbit is an orbit that is contained in a stable and an unstable manifold of the same hyperbolic periodic orbit. Equivalently, it is an orbit that is forward and backward asymptotic to the same hyperbolic periodic orbit. A vector field is *strongly nondegenerate if it is nondegenerate and the intersections of the stable and unstable manifolds of the hyperbolic orbits are transverse. A strongly nondegenerate vector field with a homoclinic orbit has positive topological entropy, thus Theorem 1.4 implies that a strongly nondegenerate Reeb vector field whose topological entropy is zero is carried by a rational open book decomposition.*

Our techniques, combined with Fried's construction [Fri], also allow to establish the existence of a supporting rational open book decomposition (where we drop the orientation assumption on the binding) when there is only one broken component in the binding. We refer to Theorem 4.4 for the details. Supported by these constructions, we make the optimistic Conjecture 4.5 that broken book decompositions can be transformed into rational open book decompositions (with no assumption on the orientation of the binding), and thus that nondegenerate Reeb vector fields always admit Birkhoff sections.

A broken book decomposition having broken components in the binding has a finite number of *rigid* pages (these are pages of the broken book decomposition that are not surrounded by similar pages). The union of the rigid pages intersects every orbit of the Reeb vector field, and for the orbits that are not in the binding, the intersection is transversal. Thus if we number the rigid pages, there should be some symbolic dynamical system associated to the intersection of the orbits with the rigid pages. There is a feature of the dynamics that one has to be careful about when developing this analysis: the broken components of the binding are hyperbolic orbits and hence have stable and unstable manifolds. The orbits in these manifolds do not behave as in a classical open book decomposition. That is, the first-return time to the rigid pages is not bounded everywhere, and the fact that there are orbits asymptotic to the binding means that the discrete dynamics on the rigid pages has to be modelled by a pseudo-group of local diffeomorphisms. In modelling the dynamics, passing from an open book to a broken book is analogous to pass from a diffeomorphism of a surface to a pseudo-group acting on a disjoint union of surfaces.

The paper is organised as follows. In Section 2 we define broken book decompositions and how they support a contact form or its Reeb vector field. The existence of broken book decompositions is established in Section 3, in particular we give a proof of Theorem 1.1. The applications of this theorem are discussed in Section 4.

**Acknowledgements:** We thank Oliver Edtmair, Michael Hutchings and Rohil Prasad for useful exchanges.

## 2. BROKEN BOOK DECOMPOSITIONS

Recall that a *rational open book decomposition* of a closed 3-manifold $M$ is a pair $(K, \mathcal{F})$ where $K$ is an oriented link called the *binding* of the open book and $M \setminus K$ fibers over $\mathbb{S}^1$, and near every component $k$ of $K$ the foliation is as in Figure 1. The fibers define the foliation $\mathcal{F}$ of $M \setminus K$ and a page of the open book is the closure of a leaf of $\mathcal{F}$ which is obtained by its union with $K$. The adjective *rational* is dropped when moreover each page is embedded. So in an *open book decomposition* each page appears exactly once along each component of the binding. In both cases we say that $k$ is radial with respect to $\mathcal{F}$.

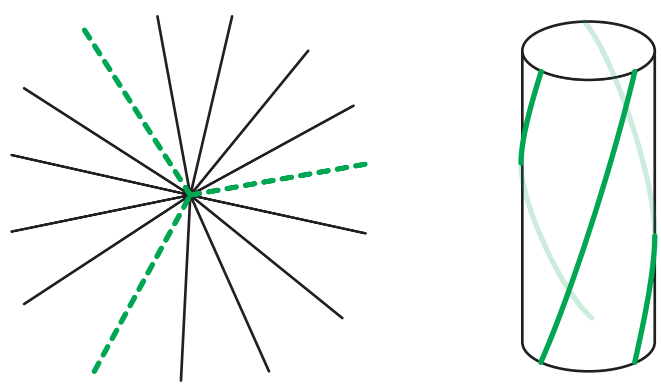

FIGURE 1. On the left, a transversal section of a radial component in a rational open book with a page drawn in green. On the right, the intersection of a page with the boundary of a tubular neighborhood of a component of $K$.

We generalise this definition by allowing another behaviour in the binding, namely *broken* components. It coincides with the transverse foliations proposed by Hryniewicz and Salamão in [HS].

**Definition 2.1.** A *degenerate broken book decomposition* of a closed 3-manifold $M$ is a pair $(K, \mathcal{F})$ such that:

- $K$ is a link (with finitely many components).
- $\mathcal{F}$ is a cooriented foliation of $M \setminus K$ such that each leaf $L$ of $\mathcal{F}$ is properly embedded in $M \setminus K$ and admits an immersed compactification $\overline{L}$ in $M$ which is a compact surface, called a *page*, whose boundary is contained in $K$ .
- there is a disjoint decomposition $K = K_r \sqcup K_b$ into the radial and broken components respectively; a component $k_r$ of $K$ is *radial* if $\mathcal{F}$ foliates a neighborhood of $k_r$ by annuli all having exactly one boundary on $k_r$. The other components of $K$ are called *broken*.

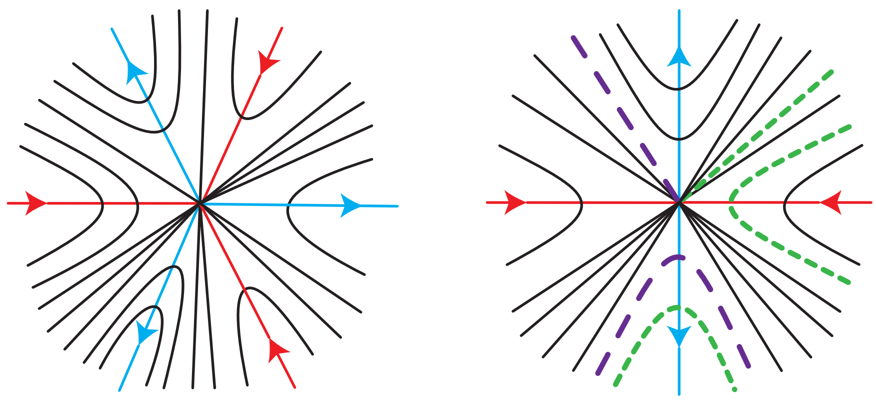

FIGURE 2. Transversal sections of broken components. On the left the broken component is degenerate, while on the right it is not. A rigid page is in purple, a regular page in green (the two or three segments of each color belong to the same page: in general it visits several times a neighborhood of a given broken component of the binding). An adapted Reeb vector field is also pictured. The leaves are positively transverse to it.

The set $K_r$ is never empty, and if $K_b$ is empty a degenerate broken book decomposition is a rational open book decomposition. When $K_b$ is non-empty, we can distinguish different types of leaves or pages. A leaf/page which belongs to the interior of a 1-parameter family of leaves/pages that are all diffeomorphic is *regular*. On the other hand, a leaf/page that is not in the interior of a 1-parameter family is *rigid*. A rigid page must have at least one boundary component in $K_b$. The complement of the rigid pages fibers over $\mathbb{R}$. Hence, when there are rigid pages, each connected component of the complement of the rigid pages can be thought of as a product of a leaf in it and $\mathbb{R}$.

**Definition 2.2.** A contact form $\lambda$ is *carried* by a degenerate broken book decomposition $(K, \mathcal{F})$ if its Reeb vector field $R_\lambda$ is tangent to $K$ and positively transverse to the leaves of $\mathcal{F}$.

Here we do not require the binding to be positively oriented by $R_\lambda$ for the orientation coming from the cooriented pages, as it is in the classical open book case.

The periodic orbits of a nondegenerate Reeb vector field $R_\lambda$ split into elliptic, positive hyperbolic and negative hyperbolic ones, when the linearized first-return map is respectively conjugated to a rotation, has positive eigenvalues, or negative eigenvalues.

*Remark* 2.3. We point out that a *radial component* of $K$ of a broken book supporting a contact form $\lambda$ can be an elliptic or hyperbolic periodic orbit of $R_\lambda$; while a broken component of $K$ is necessarily a hyperbolic periodic orbit of $R_\lambda$.

In a neighborhood of a broken component of the binding the foliation has locally 4 sectors transversally foliated by hyperbolas, separated by 4 sectors radially foliated (as in the right hand illustration of Figure 2). In this situation, there are only finitely many rigid pages since every rigid page must be somewhere in a boundary of one of the sectors foliated by the hyperbolas. Since a broken component of the binding is a hyperbolic periodic orbit, it can be positive or negative. If positive, the monodromy along this orbit is the identity; and if negative the monodromy is a $\pi$-rotation, implying that the 4 local sectors correspond to 2 global sectors.

**Definition 2.4.** A *broken book decomposition* is a degenerate broken book whose broken components of the binding locally transversally have 4 sectors transversally foliated by hyperbolas (and globally 2 or $4$), separated by 4 sectors radially foliated.

An immersed oriented compact surface whose boundary is made of periodic orbits and whose interior is embedded and positively transverse to the Reeb vector field $R_\lambda$ will be called an *$R_\lambda$-section*. Pages of supporting open book decompositions are examples of $R_\lambda$-sections, but an $R_\lambda$-section does not need to intersect all the orbits of the vector field. Given an $R_\lambda$-section $S$ (or a collection of $R_\lambda$-sections), an orbit $\gamma$ of $R_\lambda$ is *asymptotically linking* $S$ if for all $T > 0$ (resp. $T < 0$) the flow for time $t > T$ (resp. $t < -T$) intersects $S$.

Also if $\gamma$ is an orbit in the boundary of an $R_\lambda$-section $S$, its *asymptotic self-linking with* $S$ is the average intersection number of $\gamma$ pushed along $DR_\lambda$ with $S$. More precisely, one can blow-up $\gamma$ so that it is replaced with its unit normal bundle $\nu^1\gamma$, which is a 2-dimensional torus. The vector field $R_\lambda$ then extends to $\nu^1\gamma$. The boundary $\partial S$ induces a curve on $\nu^1\gamma$. The asymptotic self-linking with $S$ is defined as the rotation number of the extension of $R_\lambda$ to $\nu^1\gamma$, with respect to the 0-slope given by $\partial S$.

## 3. CONSTRUCTION OF SURFACES OF SECTION FROM EMBEDDED CONTACT HOMOLOGY THEORY

For an introduction to embedded contact homology, we refer to [Hu] and [C-GHP]. From now on we fix a contact form $\lambda$ whose Reeb vector field $R_\lambda$ is nondegenerate. The periodic orbits of $R_\lambda$ split into elliptic, positive hyperbolic and negative hyperbolic ones. The ECH chain complex $ECC(M,\lambda)$ is generated over $\mathbb{Z}_2$ (or $\mathbb{Z}$) by finite sets of simple periodic orbits together with multiplicities. Whenever a periodic orbit of an orbit set is hyperbolic, its multiplicity is taken to be $1$. This last condition is consistent with the way the ECH index $1$ or $2$ pseudo-holomorphic curves involved in the definition of the differential or in the $U$-map break, see [Hu, Section 5.4]. Recall that when considering an ECH holomorphic curve between orbit sets $\Gamma$ and $\Gamma'$, the multiplicity of an orbit $\gamma$ in $\Gamma$ or $\Gamma'$ is the number of times the curve asymptotically covers $\gamma$ at its positive or negative end, or alternatively the degree of the map from the positive or negative part of the boundary (going to $\pm\infty$ in the symplectization) of the compactified curve to the orbit. If a breaking involves a hyperbolic periodic orbit with multiplicity strictly larger than $1$, then there is an even number of ways to glue and these contributions algebraically cancel [Hu, Section 5.4]. The way the ECH index $1$ and $2$ curves approach their limit orbits is governed by the *partition conditions* [Hu, Section 3.9]. It is associated with usual SFT exponential convergence to multisections of the normal bundle of the orbit, see [HWZ1, Theorem 1.4] as well as to the asymptotic properties of [HWZ1, Theorem 2.8], [Sie1], [Sie2, Theorem 2.2], [HT2, Proposition 3.2], [We, Theorem 3.11]: the order 1 expansion of the planar coordinate $z(s,t)$ (transverse to the orbit in $M$) says that near its ends, an ECH curve is tangent (at the first order) to an annular half helix. In particular, near an elliptic periodic orbit, there is a well-defined germ of the bounded time first return map of the Reeb flow on the (projection to $M$ of the) corresponding cylindrical end. Near a hyperbolic limit periodic orbit, every orbit in its stable/unstable manifold has $0$ asymptotic linking number at, respectively, $+\infty/-\infty$ with respect to each of the corresponding cylindrical ends. Said differently, the asymptotic self-linking number of a hyperbolic limit periodic orbit with respect to the projection of the holomorphic curve is $0$.

Now, there exists a class $[\Gamma]$ in $ECH(M,\lambda)$ such that $U([\Gamma]) \neq 0$, where the map $U : ECC(M,\lambda) \to ECC(M,\lambda)$ is a degree $-2$ map counting pseudoholomorphic curves passing through a point $(0,z)$ of the symplectization $\mathbb{R} \times M$ of $M$, where $z$ does not sit on a periodic orbit of $R_\lambda$. This is established via the naturality of the isomorphism between Heegaard Floer homology and embedded contact homology with respect to the $U$-map [CGH0, CGH1, CGH2, CGH3] and the non-triviality of the $U$-map in Heegaard Floer homology [OS, Section 10], or via the isomorphism with Seiberg-Witten Floer homology, as explained in [C-GHP].

The class $[\Gamma]$ is the class of a finite sum of orbit sets $\Gamma = \Sigma_{i=1}^{k} \Gamma_i$. By the nondegeneracy assumption, there are only finitely many periodic orbits of action less than the action $\mathcal{A}(\Gamma)$ of $\Gamma$. Recall that the action of an orbit (or a portion of orbit) $\gamma$ of $R_\lambda$ is the integral $\int_\gamma \lambda$. If $\Gamma$ is a collection of orbits, its action is the sum of the actions of its elements, counted with multiplicities. We let $\mathcal{P}$ be the finite set of periodic orbits of the Reeb vector field $R_\lambda$ of action less than $\mathcal{A}(\Gamma)$.

The main input from ECH-holomorphic curve theory is the following.

**Lemma 3.1.** *For every $z$ in $M \setminus \mathcal{P}$, there exists an embedded pseudo-holomorphic curve $u : F \to \mathbb{R} \times M$ asymptotic to periodic orbits of $R_\lambda$ in $\mathcal{P}$ and whose projection to $M$ contains $z$ in its interior. If $z$ belongs to $\mathcal{P}$, it is either in the interior of the projection of a curve or in a boundary component of its closure.*

*Proof.* By definition of the $U$-map, for every generic $z \in M$, there is an ECH-index 2 embedded curve in $\mathbb{R} \times M$ from $\Gamma$ and passing through $(0,z)$. Now, if $z$ is fixed, it is the limit of a sequence of generic points $(z_n)_{n\in\mathbb{N}}$. Through $(0,z_n)$ passes a pseudo-holomorphic curve $u_n$ with $\Gamma$ as a positive end. By compactness for pseudo-holomorphic curves in the ECH context, including taking care of possibly unbounded genus and relative homology class, see [Hu, Sections 3.8 and 5.3], there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to a pseudo-holomorphic building, a component of which is an embedded pseudo-holomorphic curve through $(0,z)$. All the asymptotics of the limit curves are in $\mathcal{P}$, since they all have action less than $\mathcal{A}(\Gamma)$. In particular, when $z$ is in $M \setminus \mathcal{P}$ it is contained in the interior of the projection of the curve to $M$. If $z$ is contained in one of the orbits of $\mathcal{P}$, it might be in a limit end of the curve and thus in the boundary of the closure of the projection of the curve to $M$.

□

**Corollary 3.2.** *For every $z$ in $M$ there exists an $R_\lambda$-section $S$ with boundary in $\mathcal{P}$ passing through $z$. Moreover if $z$ is in $M \setminus \mathcal{P}$ then $z$ is contained in the interior of $S$. Every positive hyperbolic orbit $k$ in $\partial S$ with asymptotic* *self-linking number 0 has local multiplicity* 1*: every component of* $\partial S$ *maps to* $k$ *with degree* 1*. Similarily, every negative hyperbolic orbit* $k$ *in* $\partial S$ *with asymptotic self-linking number 0 has local multiplicity* 2*: every component of* $\partial S$ *maps to* $k$ *with degree* 2*.*

*Proof.* The pseudo-holomorphic curve from Lemma 3.1 passing through $(0, z)$ is embedded in $\mathbb{R} \times M$.

Hence for each $z$ we have a projected pseudo-holomorphic curve containing $z$. Given a point $z$, the pseudo-holomorphic curve through $z$ has a finite number of points where it is tangent to the holomorphic $\langle \partial_s, R_\lambda \rangle$-plane, where $s$ is the extra $\mathbb{R}$-coordinate. Indeed, close enough to its limit end orbits in $\mathcal{P}$, the pseudo-holomorphic curve is not tangent to this plane field by the asymptotic behaviour given by [Sie2, Theorem 2.2] and, by the isolated zero property for holomorphic maps, all these tangency points are isolated. These correspond exactly to the points where the projection of the curve to $M$ is not an immersion. We call these points the singular points of the projected curve. Everywhere else, the projection of the curve to $M$ is positively transverse to the Reeb vector field $R_\lambda$.

We now modify a projected pseudo-holomorphic curve $S$ away from its singular points. First we put it in general position by a generic perturbation. Then, we surround each singular point $x_i$, $i = 1, \ldots, p$, of the projected curve $S$ in $M$ by a small ball $B_i$ of the form of a flow box $D^2 \times [-1, 1]$, where the $[-1, 1]$-direction is tangent to $R_\lambda$, so that the singular point $x_i$ is at the center and the boundary discs $D^2 \times \{\pm 1\}$ are disjoint from $S$. Then $S$ only intersects $\partial B_i$ along its vertical boundary $(\partial D^2) \times [-1, 1]$. On $M \setminus (\cup_{i=1}^p B_i)$, the surface $S \setminus (\cup_{i=1}^p B_i)$ is immersed. It has a transversal given by $R_\lambda$ so that we can resolve its self-intersections coherently to get an embedded surface $S'$ in $M \setminus (\cup_{i=1}^p B_i)$, positively transversal to $R_\lambda$. In this operation, triple points of intersection, coming generically from the transverse intersections of two branches of double points, are not an issue, since we can locally resolve one branch after another in any order and extend this resolution away, see Figure 3. Also, the self-intersections along a line of double points ending in a boundary component is pictured in the two rightmost drawings of Figure 7: before and after the resolution of the self-intersections. We can also deal with self-intersections along a line of double points of two sheets all ending in the same periodic orbit: we delete a small solid torus around the orbit and resolve the intersections outside. We then extend the obtained surface in the solid torus either by annuli with a boundary on the orbit, or by meridian disks in case the slope of the obtained surface on the boundary torus is meridional.

The surface $S'$ is hence embedded in $M \setminus (\cup_{i=1}^p B_i)$ and intersects every sphere $\partial B_i$ along an embedded collection of circles contained in the vertical

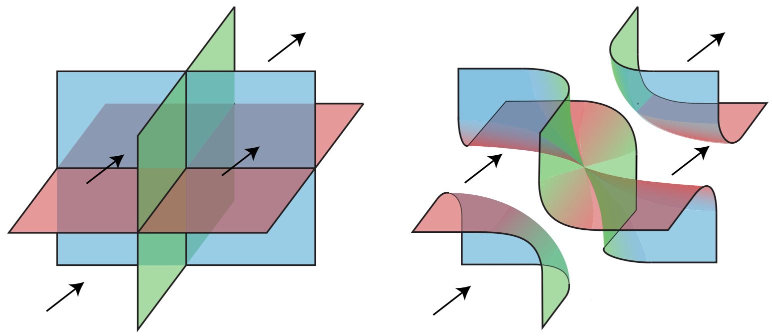

FIGURE 3. How to resolve a triple point of self-intersections. One other way to picture what happens is to first resolve the intersection of the union of two surfaces and then to add and resolve the intersections with the third one.

part $(\partial D^2) \times [-1, 1]$ and transverse to the $R_\lambda$, i.e. the $[-1, 1]$, direction. We can extend $S'$ inside the spheres $B_i$ by an embedded collection of disks transverse to $R_\lambda$. We get a surface $\overline{S}$ which is an $R_\lambda$-section. It is easy to perform these surgery operations to keep the constraint of passing through the point $z$.

Every positive or negative hyperbolic periodic orbit of $\partial\overline{S}$ having asymptotic self-linking number $0$ with $\overline{S}$ has local degree $1$ or $2$: this is the only slope possible for an approaching embedded surface of vanishing asymptotic self-linking number. □

Observe that the surface $\overline{S}$ might have several connected components, but each connected component is an $R_\lambda$-section with boundary (because a closed surface cannot be transverse to a Reeb vector field).

**Lemma 3.3.** *There exists a finite number of $R_\lambda$-sections with disjoint interiors, intersecting all orbits of $R_\lambda$, and such that if an orbit of $R_\lambda$ is not asymptotically linking this collection of sections, it has to converge to one of their boundary components, which is a hyperbolic periodic orbit of the flow with local multiplicity $1$ or $2$ (every boundary component of an $R_\lambda$-section mapping to this orbit is degree $1$ or $2$).*

*In this case, each one of the sectors transversally delimited by the stable and unstable manifolds of the hyperbolic periodic orbit is intersected by at least one $R_\lambda$-section having the orbit as a boundary component.*

*Proof.* The finite number of curves comes from a standard compactness argument and Corollary 3.2. Start with a finite covering of the complement of an open neighborhood of $\mathcal{P}$ by flow-boxes. Through every point in a flow-box, Corollary 3.2 provides an embedded surface with boundary in the orbits of $\mathcal{P}$. Since the closure of every flow-box is compact, there is a finite collection of surfaces intersecting every portion of orbit in the flow-box. Now again, we can make this collection of sections embedded by resolving intersections using the common transverse direction $R_\lambda$. The local degree $1$ or $2$ property for hyperbolic periodic orbits having asymptotic self-linking $0$ with a section (from Corollary 3.2) remains true because they cannot be touched by the desingularization for otherwise the self-linking value would change.

We now analyze what happens near a hyperbolic periodic orbit $k$ in $\mathcal{P}$ with asymptotic self-linking with $S$ equal to zero. The stable and unstable manifolds of the periodic orbit $k$ transversally delimit four sectors in a neighborhood of a point in the orbit. For each sector, we take a sequence of generic points contained in the sector that all limit to some point in $k$, together with a sequence of ECH index $2$ embedded holomorphic curves through these generic points. In the limit pseudo-holomorphic building, there is a pseudo-holomorphic curve whose projection to $M$ either:

- cuts $k$ transversally;
- is asymptotic to $k$ and intersects positively with orbits in the invariant manifolds of $k$ (this is in fact prohibited by the partition condition but we do not need this extra remark here);
- is asymptotic to $k$ and approaching from the fixed sector containing the sequence of generic points.

In the last case, the curve does not intersect the stable/unstable manifolds of $k$ and the limit curve approaches from the sector containing the sequence of generic points $z_n$ by compactness. In this case indeed the limit curves satisfy the asymptotic conditions of [HWZ1, Theorem 2.8], [Sie1] [Sie2, Theorem 2.2], [HT2, Proposition 3.2], [We, Theorem 3.11]: the order 1 expansion of the planar coordinate $z(s,t)$ (transverse to $k$ in $M$) says that along the boundary a limit curve is tangent (at the first order) to an annular half helix. The second argument is the transversality to the Reeb vector field that implies that the half helix has to stay in one sector. Indeed if it changes sector it can only change in the trigonometrical order. So in order to come back in the original sector after following $k$ for one longitude, it has to wrap around $k$: in this case the asymptotic linking number would be positive, a contradiction.

Now, there are two cases depending on wether $k$ is at an intermediate level of the limit building with non trivial incoming and outgoing ends, or if there is only a non trivial ingoing or outgoing end (possibly followed by connectors).

In the first case, for the limit building, the incoming and the outgoing ends at $k$ both have to follow a half-helix contained in a sector of $k$. If none of these sectors is the one containing the sequence $z_n$, then close to the breaking curve the curve has to enter and exit the sector containing $z_n$. We then look at its intersection with the stable and unstable manifolds delimiting the sector. This is a collection of circles that have to be positively transversal to the Reeb vector field foliating the stable/unstable manifolds. In particular, none of the circles are contractible in the stable/unstable manifolds. The non-contractible circles have a fixed co-orientation given by the Reeb vector field: when the curve enters a sector along a stable/unstable manifold, it cannot exit along the same stable/unstable manifold nor along the other unstable/stable manifold.

In the second case, the limit building has only one non trivial end to $k$ (though there can be connectors) defining a half-helix. The sequence of pseudo-holomorphic curves $u_n$ is converging to this half helix and $k$ must be an end of the $u_n$'s. Now we have by the same argument than in the first case that $z_n$ has to be either on the same sector than the half helix of $u_n$ or in the same sector than the one for the limit curve, otherwise we have a subsurface near the end of $u_n$ for $n$ large enough that is crossing a sector.

Again, we add these new curves to our previous collection of $R_\lambda$-sections and then resolve the intersections of this new family by an application of Corollary 3.2. □

Finally we have an $R_\lambda$-section $S$ (possibly disconnected), so that every orbit of $R_\lambda$ is either a boundary component or intersects $S$ strictly positively. We let $K = \partial S$ be the union of the boundary orbits of $S$. If every orbit is asymptotically linking $S$, we get a rational open book. However, we can have here boundary components where the orbits of $R_\lambda$ accumulate without intersecting the corresponding surface. That is, there are orbits of $R_\lambda$ that have asymptotically self-linking number with $S$ equal to $0$. These boundary components are necessarily hyperbolic periodic orbits and they all have local multiplicity one or two. In such a case, we obtain the existence of a broken book decomposition, as stated in Theorem 1.1.

*Proof of Theorem 1.1.* At this point, we have an $R_\lambda$-section $S$ intersecting every orbit of the flow, and we want to turn it into a broken book decomposition. Said differently, the $R_\lambda$-section $S$ forms a trivial lamination of $M \setminus K$, and we have to extend $S$ into a foliation of $M \setminus K$.

For convenience, we first double all the components of $S$ who have at least one boundary component on a hyperbolic periodic orbit and are not asymptotically linking the orbits in their stable/unstable manifolds. The two copies are separated in their interior by pushing along the flow of $R_\lambda$.

We keep the notation $S$ for this new $R_\lambda$-section. We then cut $M$ along $S$ and delete standard Morse type neighborhoods of $\partial S$ as in Figure 4.

We claim that the resulting manifold is a sutured manifold, foliated by compact $R_\lambda$-intervals: it is an $I$-bundle with oriented fibers, thus a product and the conclusion follows. Observe that when $R_\lambda$ is asymptotically linking $S$ near an orbit $k$ of $\partial S$, the flow of $R_\lambda$ near $k$ has a well-defined first-return map on $S$. These orbits are then decomposed by $S$ into compact segments. When we are near a positive hyperbolic periodic orbit $k_b$ where the flow is not asymptotically linking with $S$, then $S$ intersects a Morse type tubular neighborhood of $k_b$ in at least $8$ annuli, two in each sector (because of the doubling operation). Between two annuli in the same sector, the orbits of $R_\lambda$ are locally going from one annulus to the other, thus an orbit is decomposed into compact intervals. If two consecutive annuli belong to different adjacent sectors, then they are cooriented in the same way by $R_\lambda$ and can be pushed in the direction of the invariant manifold of $k_b$ separating them and glued to form an annulus transverse to $R_\lambda$ and again every local orbit of $R_\lambda$ ends or starts in finite time on some (possibly glued) annulus.

To finish the proof, we just have to glue back the Morse-type neighbourhood of the $\partial S$, that we foliate with the local model of a broken book, as on the bottom of Figure 4. The construction near negative hyperbolic periodic orbits is similar. □

*Remark* 3.4. If $R_\lambda$ is supported by some open book decomposition, then every embedded holomorphic curve, not asymptotic to the binding, gives rise to a new rational open book decomposition for $R_\lambda$ by the constructions above. In particular, the abundance of embedded holomorphic curves given by the non triviality of the $U$-map in embedded contact homology, or the differential, typically furnishes many open books for the same Reeb vector field.

## 4. Applications

We first analyse the broken components of the binding of a supporting broken book using the Reeb property. Recall that the periodic orbits in $K_b$ are hyperbolic, the possible intersections of the stable and unstable manifolds of the periodic orbits in $K_b$, will play an important role. We will talk of heteroclinic orbit or intersection, even if it might be a homoclinic orbit or intersection, and reserve the use of homoclinic for when we can ensure it is a homoclinic orbit. We recall that a *heteroclinic* orbit is an orbit that lies in the intersection of a stable manifold of a hyperbolic periodic orbit and an unstable manifold of another hyperbolic periodic orbit.

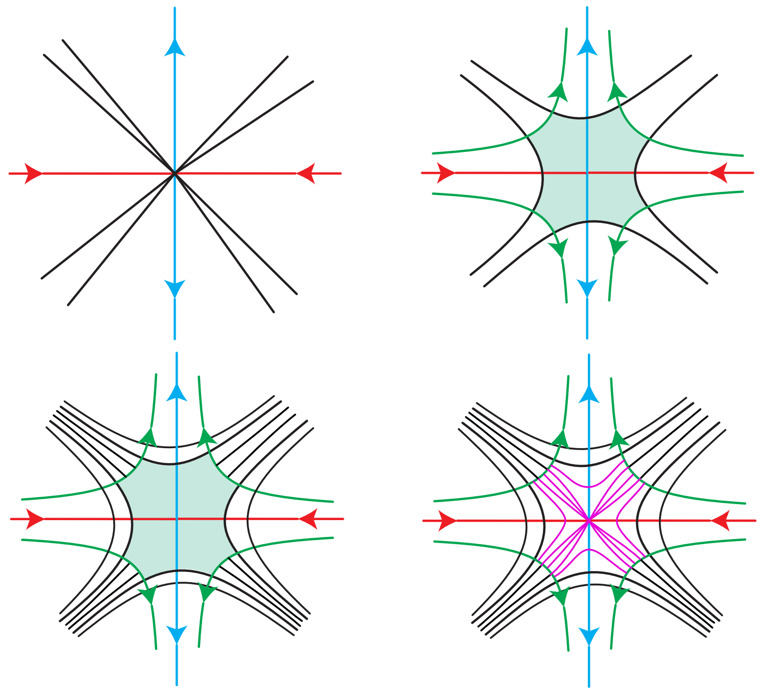

FIGURE 4. A broken component $k_b$ of the binding. For building the foliation of $M \setminus K$, one reconnects the part of an $R_\lambda$-section in the neighborhood of $k_b$, removes a smaller neighborhood (green), and cuts the resulting manifold along the (modified) $R_\lambda$-section. The result is a trivial $I$-bundle (top right). From the foliation of the trivial $I$-bundle (bottom left), one adds the neighbourhood of $k_b$ back, and foliated with the local model of a broken binding orbit (bottom right).

**Lemma 4.1.** *Let $R_\lambda$ be a Reeb vector field for a contact form $\lambda$ carried by a broken book decomposition $(K, \mathcal{F})$ and let $k_0$ be a broken component of the binding $K$. Then every unstable/stable manifold of $k_0$ contains a heteroclinic intersection with a broken component of $K_b$, i.e. each unstable/stable manifold of $k_0$ intersects the stable/unstable of some component of $K_b$.*

*Proof.* A component $k_0 \subset K_b$ has one or two unstable manifolds, each made of an $\mathbb{S}^1$-family of orbits of $R_\lambda$, asymptotic to $k_0$ at $-\infty$. Each $\mathbb{S}^1$-family of orbits is a cylinder in $M$ with a boundary component in $k_0$, that is injectively immersed in $M$ since its portion near $k_0$ for time $t < -T$ for $T$ large enough is embedded.

We now argue by contradiction. We consider the finite collection of all the rigid pages $\mathcal{R} = \{R_0, ..., R_k\}$ of the broken book decomposition. If no orbit in the $\mathbb{S}^1$-family limits to a broken component of $K$ at $+\infty$, then this

$\mathbb{S}^1$-family has a return map on $\mathcal{R}$ which is well-defined and intersects one of the pages of $\mathcal{R}$, say $R_0$, an infinite number of times. Since the $\mathbb{S}^1$-family is injectively immersed, the intersection with $R_0$ forms an infinite embedded collection $C_0$ of curves in $R_0$.

Observe that $d\lambda$ is an area form on $\mathcal{R}$. We claim that only a finite number of the curves in $C_0$ can be contractible in $R_0$. Two contractible components of $C_0$ bound disks $D$ and $D'$ in $R_0$, these disks have the same $d\lambda$-area. Indeed $D$ and $D'$ can be completed by an annular piece $A$ tangent to $R_\lambda$ to form a sphere, applying Stokes' theorem:

$$0 = \int_{D\cup A\cup D'} d\lambda = \int_D d\lambda + \int_A d\lambda - \int_{D'} d\lambda = \int_D d\lambda - \int_{D'} d\lambda, \tag{1}$$

because $d\lambda$ vanishes along $A$. Note that equation (1) implies also that $D$ is disjoint from $D'$, since $\partial D$ is disjoint from $\partial D'$ and $D$ and $D'$ have the same area. Since the total area of $R_0$ is bounded, there are only finitely many contractible curves in $C_0$, as we wanted to prove.

Thus infinitely many components of $C_0$ are not contractible in $R_0$, so at least two have to cobound an annulus $A'$ in $R_0$. The annulus $A'$ is transverse to $R_\lambda$ and its boundary components cobound by construction an annulus $A''$ tangent to the flow of $R_\lambda$. We now apply Stokes' theorem to this torus

$$0 = \int_{A'\cup A''} d\lambda = \int_{A'} d\lambda > 0,$$

a contradiction.

Hence each unstable/stable manifold of $k_0$ contains an orbit that is forward/backward asymptotic to a component of $K_b$. □

For two components $k_0$ and $k_1$ of $K_b$, a heteroclinic orbit from $k_0$ to $k_1$ is an orbit contained in the unstable manifold of $k_0$ and in the stable manifold of $k_1$.

**Lemma 4.2.** *There exists $k_0 \in K_b$ having two cyclic sequences of broken components*

$$\begin{aligned} A &= \{k_0, k_1, ..., k_{n-1}, k_n = k_0\} \\ B &= \{k_0, k'_1, ..., k'_{l-1}, k'_l = k_0\} \end{aligned}$$

*based at some $k_0$ so that there is a heteroclinic orbit $O_i$ of $R_\lambda$ from $k_i$ to $k_{i+1}$, $0 \le i \le n-1$ and a heteroclinic orbit $O'_i$ of $R_\lambda$ from $k'_i$ to $k'_{i+1}$, $0 \le i \le l-1$. If $k_0$ is positive, then $O_0$ and $O'_0$ are contained in each of the two unstable manifolds of $k_0$. If $k_0$ is negative, the two cycles can coincide.*

If $n > 1$ the sequence $A$ is a heteroclinic cycle, while if $n = 1$ we say that $A$ is a homoclinic intersection. To simplify the discussion, we call in both cases $A$ a heteroclinic cycle. The lemma then says that there is a component of $K_b$ with two heteroclinic cycles starting in the two possible unstable directions.

*Proof.* We know by Lemma 4.1 that if $k$ is a broken component of the binding, in each of its unstable manifolds, there is a heteroclinic orbit to some broken component of $K$. We argue by contradiction, assume that there is no such double heteroclinic cycle based at any $k_0$, starting from the unstable manifold if $k_0$ is a negative hyperbolic periodic orbit and in the two possible unstable direction if $k_0$ is a positive hyperbolic periodic orbit. We first assume that $k_0$ is a positive hyperbolic periodic orbit. Then from Lemma 4.1 we can build from a component $k_0 \in K_b$ two heteroclinic sequences $A_0$ and $B_0$, and at least one of them does not comes back to $k_0$.

Assume first that $A_0$ is a heteroclinic cycle, so it comes back to $k_0$ at some point. Consider the sequence $B_0$ starting at the other unstable manifold of $k_0$, that is not cyclic by assumption. Since $K_b$ is finite, there is a $k_1 \in B_0$ so that from $k_1$ the sequence $B_0$ is a heteroclinic cycle that comes back to $k_1$. This cyclic subsequence of $B_0$ cannot intersect $A_0$, because if it does, then $k_0$ admits two heteroclinic cycles starting in its two unstable directions. Hence $k_1$ admits one heteroclinic cycle starting in one of its unstable directions. If $k_1$ is a negative hyperbolic periodic orbit we are done, hence assume that $k_1$ is a positive hyperbolic periodic orbit. If in the other direction there is a cyclic sequence, we have a component as in the statement. If it is not cyclic, we can again consider this non cyclic sequence $B_1$ and find $k_2 \in B_1$ with a heteroclinic cycle $B_2 \subset B_1$ based at $k_2$. Observe that by assumption, $B_2$ is disjoint of $A_0 \cup B_0$ and we can again assume that $k_2$ is a positive hyperbolic periodic orbit. Since $K_b$ is finite, this process stops, implying that there is either a positive hyperbolic periodic orbit of $K_b$ with two different heteroclinic cycles or a negative hyperbolic periodic orbit of $K_b$ with a heteroclinic cycle.

Now assume that both sequences $A_0$ and $B_0$ starting in the two unstable directions of $k_0$ are not cyclic. We start following the direction $B_0$, since it is not cyclic there is a $k_1 \in B_0$ so that from $k_1$ the sequence $B_0$ is a heteroclinic cycle that comes back to $k_1$. We can repeat the argument above starting at $k_1$ to obtain either a positive hyperbolic periodic orbit of $K_b$ with two different heteroclinic cycles or a negative hyperbolic periodic orbit of $K_b$ with a heteroclinic cycle.

Observe that the arguments above imply also the result if we start with $k_0$ a negative hyperbolic periodic orbit in $K_b$. □

*Remark* 4.3. The interest of Lemma 4.2 is that one could try to apply the local construction of Fried [Fri] in the neighborhood of $(\cup_i k_i) \cup (\cup_i O_i) \cup (\cup_i k_i') \cup (\cup_i O_i')$ to get a surface of section $S_0$ that intersects transversally

$k_0$ in its interior, and thereby, changing the broken book along $S_0$, to decrease and finally get rid of all the broken components of its binding. This would construct a supporting (up to orientations of the binding) rational open book. Unfortunately, Lemma 4.2 does not seem sufficient to make sure that $S_0$ intersects $k_0$, since the two heteroclinic cycles might join two adjacent quadrants of $k_0$, like NW and SW, instead of opposite ones like NE and SW (see Figure 5). However this works if there is only one broken component in the binding.

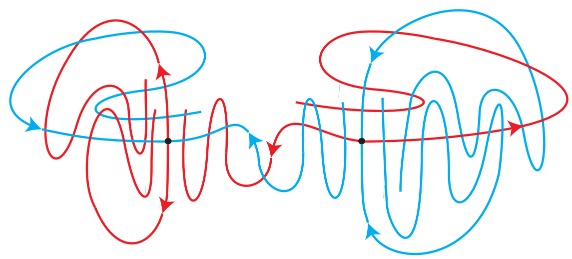

FIGURE 5. Two hyperbolic orbits and their stable/unstable manifolds at which one cannot directly apply Fried's construction.

Consider a nondegenerate Reeb vector field, if the invariant manifolds of the hyperbolic periodic orbits intersect transversally, we say that the vector field is *strongly nondegenerate*. Observe that this is a weaker hypothesis than being *Kupka-Smale*, since a Kupka-Smale vector field has in addition all its periodic orbits hyperbolic. The strongly nondegenerate condition is generic for vector fields due to [Kup, Sma] and also for Reeb vector fields, the proof of the genericity of the Kupka-Smale condition in [Pei] extends to give the strong nondegeneracy condition in the Reeb case.

**Theorem 4.4.** *Let $R_\lambda$ be a strongly nondegenerate Reeb vector field for a contact form $\lambda$ carried by a broken book decomposition $(K, \mathcal{F})$. Assume that $K$ contains at most one broken component. Then $R_\lambda$ has a Birkhoff section.*

*Proof.* Denote by $k_0$ the broken component in the binding $K$ and assume first that it is a positive hyperbolic periodic orbit. Thanks to Lemma 4.2, each of the two unstable manifolds of $k_0$ intersect at least one stable manifold of $k_0$, and each of the two stable manifolds intersect at least one unstable manifold. Therefore, up to a symmetry, there are two orbits $\gamma_a$ and $\gamma_b$ such that $\gamma_a$ belongs to both the east unstable manifold and the north stable manifold of $k_0$, and $\gamma_b$ belongs to both the west unstable manifold and the south stable manifold of $k_0$ (see Figure 6).

Consider a small local transverse section $D$ to $R_\lambda$ around $k_0$ and the induced first-return map $f$. By taking small transverse rectangles around $k_0$ and considering their images by $f$, one can find two periodic points $p_a$ in the $NE$-quadrant and $p_b$ in the $SW$-quadrant. Denote by $k_a, k_b$ the corresponding periodic orbits ot $R_\lambda$. For every word $w$ in the alphabet $\{a, b\}$, one can find a periodic point $p_w$ of $f$ that follows $k_a$, and $k_b$ in the order given by $w$. In particular one can consider the periodic orbit $k_{ab}$ through $p_{ab}$ and $p_{ba}$.

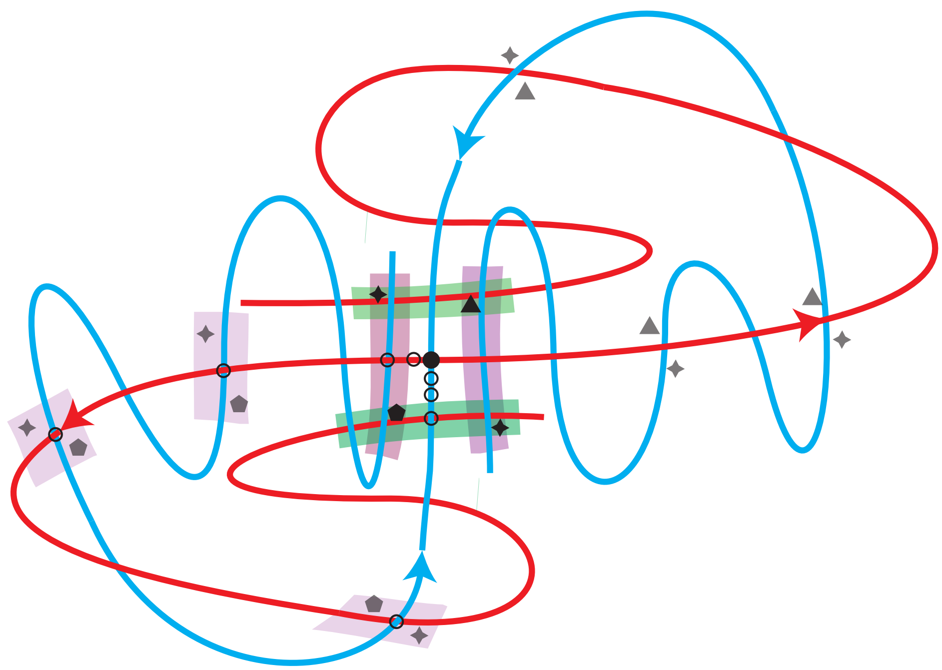

FIGURE 6. A transverse view of the orbit $k_0$ and its stable/unstable manifolds. Two small transverse rectangles $r_W$, $r_E$ in the W- and E-parts are shown in purple, together with their images by suitable iterates of the first-return map $f$, in green. In $r_W \cap f^{k_W}(r_W)$ lies a periodic point $p_a$ of $f$ of period $k_W$. Similarly in $r_E \cap f^{k_E}(r_E)$ lies a periodic point $p_b$ of $f$ of period $k_E$. Moreover, in $r_W \cap f^{k_E}(r_E)$ lies a periodic point $p_{ab}$ such that $p_{ba} := f^{k_W}(p_{ab})$ lies in $f^{k_W}(r_W) \cap r_E$ and $f^{k_E}(p_{ba}) = p_{ab}$, *i.e.*, $p_{ab}$ has period $k_W + k_E$. The rectangle $p_a p_{ab} p_b p_{ba}$ is then transverse to $k_0$.

Now consider an arc connecting $p_a$ to $p_{ab}$. When pushed by the flow, it describes a certain rectangle $R_1$ and comes back to an arc connecting $p_a$ to $p_{ba}$. Likewise an arc connecting $p_b$ to $p_{ba}$ describes a rectangle $R_2$ whose opposite side is an arc connecting $p_b$ to $p_{ab}$ (see Figure 7 left). Together these four arcs form a parallelogram $P$ in $D$ which contains $D \cap k_0$ in its interior. The union of $P$ and the two rectangles $R_1$ and $R_2$, forms an immersed topological pair of pants, which can be smoothed into a surface $S$ transverse to $R_\lambda$. The main properties of $S$ is that it is bounded by $k_a, k_b$ and $k_{ab}$, and it is transverse to $k_0$.

Now consider a page $F_0$ of the foliation $\mathcal{F}$, take the union $F_0 \cup S$, and use the flow direction $R_\lambda$ to desingularize the arcs and circles of intersection (see Figure 7 right and Figure 3). The obtained surface intersects any tubular neighborhood of $k_0$ along one (or several) meridian. Therefore $k_0$ is not anymore in the broken part of the binding, but is part of the boundary of the new surface. Also the surface $F_0$ intersects $k_a, k_b$, and $k_{ab}$, so that these periodic orbits link positively the union $F_0 \cup S$. The resulting surface is a then a genuine (rational) Birkhoff section for the Reeb vector field. We can thus obtain an open book decomposition from it adapted to the Reeb vector field. Observe that the orbits $k_0, k_a, k_b$, and $k_{ab}$ are boundary components of radial type with respect to the new foliation.

The case where $k_0$ is a negative hyperbolic periodic orbit is treated in the same manner. The difference is that now one needs to consider the second iterate of the return map to a local transversal to the periodic orbit in order to have Figure 6. □

**Conjecture 4.5.** *Every nondegenerate Reeb vector field has a Birkhoff section.*

It follows from the previous considerations that if a strongly nondegenerate Reeb vector field has no heteroclinic or homoclinic orbit, then any of its supporting broken book decomposition is in fact a rational open book, providing a proof of Theorem 1.4. Moreover, if a strongly nondegenerate Reeb vector field has at most one periodic orbit having a heteroclinic cycle then, by Lemma 4.1, it has a supporting broken book with at most one broken binding component, and by Theorem 4.4, a Birkhoff section.

*Remark* 4.6. Theorem 1.4 can be applied to the broken book decomposition (or system of transversal sections) constructed by de Paulo and Salomão in [dPS] to obtain an open book decomposition.

We now give a proof of Theorem 1.2 and postpone the proof of Theorem 1.3 to the end of the section. We first discuss what happens for strongly nondegenerate Reeb vector fields and then remove the strongly hypothesis.

In the strongly nondegenerate case, the theorem follows from the fact that if the broken book has broken components in its binding then Lemma 4.2

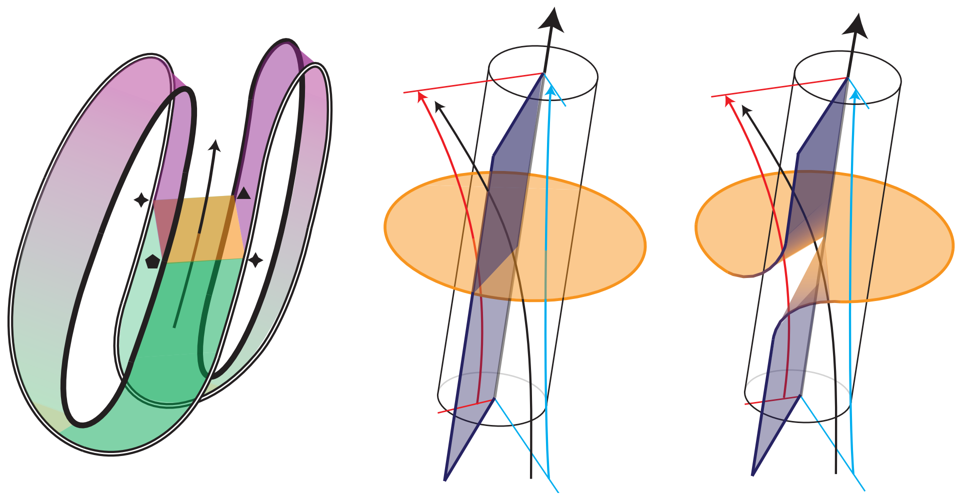

FIGURE 7. On the left the union of the rectangles $R_1$ and $R_2$ with the rectangle $p_a p_{ab} p_b p_{ba}$ yields a pair of pants. It can be smoothed into a surface transverse to the Reeb flow. On the center a page $F_0$ which contains $k_0$ (dark blue) and the rectangle $p_a p_{ab} p_b p_{ba}$ (orange), in a neighborhood of $k_0$. On the right the desingularization of their union (following Fried) yields a surface transverse to the flow, so that the local first-return time is now bounded by the period of $k_0$.

implies that there are heteroclinic cycles. The strongly nondegenerate hypothesis then gives a transverse homoclinic intersection, that implies the existence of infinitely many periodic orbits. If there are no broken components in the binding, the broken book is a rational open book. Then, whenever $M$ is not $\mathbb{S}^3$ or a lens space, the page $S$ is not a disk nor an annulus. The case when $S$ is a disk or an annulus was treated by Cristofaro-Gardiner, Hutchings and Pomerleano [C-GHP] and there are either 2 or infinitely many periodic orbits.

In the rest of the cases, the first-return map is a flux zero area preserving diffeomorphism of $S$ and we claim that it has infinitely many periodic points. To get this conclusion, we apply the following generalisation of a theorem of Franks and Handel [FH] originally stated for periodic points of Hamiltonian diffeomorphisms of surfaces.

While writing this paper, we learned about a more direct and general proof (for homeomorphisms, possibly degenerate) by Le Calvez and Sambarino [LS].

**Theorem 4.7.** *Let $S$ be a compact surface with boundary different from the disk or the annulus, and $\omega = d\beta$ an ideal Liouville form for $S$. If $h : S \to S$*

*is a nondegenerate area-preserving diffeomorphism with zero-flux then $h$ has infinitely many different periodic points.*

Here the flux condition holds on the kernel of the map

$$h_* - I : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z})$$

and $h$ is not assumed to be isotopic to the identity (nor Hamiltonian). It means that for every curve $\gamma$ whose homology class is in $\ker(h_* - I)$, then $\gamma$ and $h(\gamma)$ cobound a $d\beta$-area zero 2-chain.

*Proof.* The zero flux condition tells that $h$ can be realised by the first-return map of the flow of a Reeb vector field on a page of the mapping torus of $(S, h)$ [CHL]. If, in its Nielsen-Thurston decomposition, $h$ has a pseudo-Anosov component, then the conclusion of the theorem classically holds by Nielsen-Thurston theory even without any conservative hypothesis. Otherwise, all the pieces of $h$ in the decomposition are periodic. Up to taking a power of $h$, which does not change the problem, we can assume that, up to isotopy, $h$ is the identity on every piece. This means that $h$ is isotopic to a composition of Dehn twists on disjoint curves. If $h$ is isotopic to the identity, the conclusion is given by a theorem of Franks and Handel [FH], extended by Cristofaro-Gardiner, Hutchings and Pomerleano to fit exactly our case with boundary [C-GHP].

Otherwise, we can use the Nielsen-Thurston representative $h_0$ of $h$ given by a product of Dehn twists along disjoint annuli, one of them being non boundary parallel. We once again realise it as the first-return map of a Reeb flow $R_0$ on a fiber in the mapping torus of $(S, h)$. This Reeb flow has no contractible periodic orbits and thus can be used to compute cylindrical contact homology. In the mapping torus of a non-boundary parallel annulus where $h_0$ is the power of a Dehn twist, we have $\mathbb{S}^1$-families of periodic Reeb orbits realising infinitely many slopes in the suspended thickened torus. These all give generators in cylindrical contact homology, since the other orbits (corresponding to periodic points of $h_0$) belong to different Nielsen classes. The invariance of cylindrical contact homology suffices to conclude that the Reeb orbits given by the mapping torus of $h$ (that are also all non contractible in the mapping torus of $h$) are at least the number given by the rank of the cylindrical contact homology computed with $R_0$, i.e. in infinite number.

Note here that the first-return map on the page $S$ is well-defined on the interior of $S$. It might not extend smoothly to $\partial S$. In that case, we filter the cylindrical contact homology complex by the intersection number of orbits with the page, i.e. the period of the corresponding periodic points. We can then modify the monodromy $h$ near $\partial S$ to a zero-flux area preserving diffeomorphism $h_k$ so that $(1)$ the modified monodromy $h_k$ extends to $\partial S$,

$(2)$ the orbits of period less or equal to $k$ in the Nielsen classes not parallel to the boundary are not affected and $(3)$ $h_k$ is isotopic to $h$ and so has the same Nielsen-Thurston representative $h_0$. The arguments developed in the proof of Theorem 4.7 then apply to show the existence of periodic points of $h_k$ with period bounded above by $k$, these are also periodic points of $h$ with period bounded by $k$. $\square$

Note that the proof of Theorem 4.7 gives, if $h$ is nondegenerate, the existence of positive ($=$ even) hyperbolic periodic orbits, which are odd degree generators of cylindrical homology coming from the positive hyperbolic generators in the Morse-Bott families.

Thus we are able to answer positively to Question 1.8 of [C-GHP]:

**Corollary 4.8.** *If $M$ is not $\mathbb{S}^3$ nor a lens space and if $R_\lambda$ is a strongly nondegenerate Reeb vector field on $M$, then it has a positive hyperbolic periodic orbit.*

Indeed, if there were none, $R_\lambda$ would have vanishing topological entropy and would admit a supporting rational open book decomposition. Then Theorem 4.7 would give at least one positive hyperbolic periodic orbit (amongst infinitely many other ones) in case every piece of $h$ is periodic. In case there is a pseudo-Anosov piece, the existence of a Nielsen class with negative total Lefschetz index gives the same result, and leads to a contradiction.

We now prove Theorem 1.2 in the case where we drop the hypothesis strongly to obtain the result for nondegenerate Reeb vector fields.

Consider two hyperbolic periodic orbits (not necessarily different) with an heteroclinic orbit connecting them. We say that there is a homoclinic or heteroclinic *connection* if the corresponding stable and unstable manifolds coincide, otherwise it is a homoclinic or heteroclinic *intersection*. A homoclinic or heteroclinic intersection or orbit is said to be *one-sided* if the stable and unstable manifolds intersect and do not cross, where crossing is in the topological sense [BW]. In case they cross, we have a *crossing* intersection. Note that these definitions include the case where the stable and unstable manifolds intersect along an interval transverse to the flow and either cross or stay on one side at the boundary components.

We treat differently heteroclinic connections and one-sided intersections because in the Reeb context, a heteroclinic connection cannot be eliminated by a local perturbation of the Reeb vector field: one cannot displace a transverse circle form itself with a zero flux map close to the identity, whereas it is possible for a transverse interval, thus for eliminating a one-sided intersection (this is used in the proof of Proposition 4.10).

We start with the case when there are only complete connections.

**Lemma 4.9.** *Let $(K, \mathcal{F})$ be a broken book decomposition supporting a non-degenerate Reeb vector field $R_\lambda$. Assume that every broken component of the binding has its stable/unstable manifolds that coincide with unstable/stable manifolds of another broken component of the binding, i.e. all the homoclinic or heteroclinic intersections are connections. Then, if $M$ is different from the sphere or a lens space, the Reeb vector field $R_\lambda$ has infinitely many different simple periodic orbits.*

*Proof.* The idea is to cut $M$ along the stable/unstable manifolds of the broken components of the binding, to obtain a, possibly not connected, manifold with torus boundary. As a preparatory operation, we blow up every negative hyperbolic periodic orbit in $K_b$ to a 2-torus. That is, we replace each one of these periodic orbits by its normal transverse bundle producing a torus for each and changing $M$ by a manifold $M_b$ with a finite number of tori in its boundary. In $M_b$ we obtain a vector field tangent to the boundary and on each boundary torus, the vector field has two periodic orbits: one corresponding to the stable manifold and one to the unstable manifolds; that wrap twice in the longitudinal direction. These two periodic orbits are linked and the other orbits are forward and backward asymptotic to the periodic orbits. After this operation, we get a manifold $M_b$ with boundary, where all the hyperbolic orbits in $K_b \cap M_b$ are positive.

Now $M'$ is obtained as a metric completion of $M_b$ minus the stable/unstable manifolds of the broken components of the binding (including those that end in the boundary components of $M_b$). Hence $M'$ is a 3-manifold with boundary.

The boundary of $M'$ is made of copies of the stable and unstable manifolds of orbits in $K_b$ and annuli in the boundary components of $M_b$. It has corners along the copies of orbits of $K_b$, including here the periodic orbits produced by the blow-up operation. The Reeb vector field is tangent to the boundary and the foliation $\mathcal{F}$ is now transverse to the boundary and singular only along the radial components in $K_r$. This means that $M' \setminus K_r$ fibers over $\mathbb{S}^1$ and that the Reeb vector field has a first-return map defined on the interior of each page.

Whenever a page of the fibration is not a disk or an annulus, we have the conclusion by Theorem 4.7. Otherwise, since $K_r$ is not empty, there is a component $N$ of $M'$ where all pages are annuli, all having a boundary component on a radial component $k_r$ of $K_r$ in the interior of $N$ and the other on the boundary of $N$. Note that this implies that $N$ is a solid torus. The boundary of $N$ is decomposed into the annuli given by the homoclinic or heteroclinic connections, each annulus being bounded by two (not necessarily different) components of $K_b$. Observe that no annulus is foliated by Reeb components of $R_\lambda$, since $R_\lambda$ is geodesible ([Sul]): simply here, in the case of a Reeb component $d\lambda$ would be zero on the annulus, while the integral of $\lambda$ would be nonzero on the boundary, in contradiction with Stokes' theorem. Moreover, the periodic orbits in $K_b \cap \partial N$ are attracting on one side and repelling on the other side, since we are alternately passing from a stable manifold to an unstable manifold. Observe that this is also the case along the annuli of $\partial M_b$.

We now claim that we can change the fibration of $N \setminus k_r$ by another fibration by annuli, close to the previous one (in terms of their tangent plane fields), so that it is still transverse to $R_\lambda$ in the interior, but also at the boundary. Indeed, outside of a neighborhood of $\partial N$, the Reeb vector field is away from the tangent plane field of the fibration by a fixed factor, in particular near $k_r$ where the infinitesimal first-return map is a non trivial rotation. Near $\partial N$, the Reeb vector field gets close to the tangent field of the fibration and is tangent to it along $K_b \cap \partial N$, but with a fixed direction: we can tilt the fibration in the other direction to make it everywhere transverse. This operation changes the slope by which the fibration approaches $k_r$ and the boundary $\partial N$. If the slope was, say, $(1, 0)$ in some basis, it is now of the form $(P, \pm 1)$ for some $P \gg 1$.

Since the fibers are now everywhere transverse to $R_\lambda$, there is a well defined first-return map that extends to the boundary to give a diffeomorphism of a closed annulus. Observe that this annulus is not necessarily embedded along $k_r$, but the map is well defined. The boundary of any such annulus page intersects at least one component of $K_b$, so that the first-return map to this annulus has at least one periodic point in the boundary. A theorem of Franks implies that there are infinitely many periodic points (see Theorem 3.5 of [Fra]). □

We now prove that an unstable manifold of an orbit in $K_b$ that does not coincide with the stable manifold of some orbit of $K_b$ must have a crossing intersection with some unstable manifold of an orbit in $K_b$.

**Proposition 4.10.** *Let $(K, \mathcal{F})$ be a broken book decomposition supporting a nondegenerate Reeb vector field $R_\lambda$. If an unstable manifold $V^u(k)$ of some orbit $k \in K_b$ does not coincide with a stable manifold of an orbit in $K_b$, then it contains a crossing intersection.*

*Proof.* The proof of this result is not straightforward and will involve proving intermediate Lemmas 4.11 to 4.14. We know by Lemma 4.1 that $V^u(k)$ must intersect stable manifolds of other broken components of $K$. We argue by contradiction and assume that $V^u(k)$ has no crossing intersection. Then $V^u(k)$ must contain only one-sided intersections. We follow $V^u(k)$ from $k$. Consider the set $\mathcal{R}$ of rigid pages of the broken book decomposition. Then $M \setminus \mathcal{R}$ is formed of product-type components. The unstable manifold

$V^u(k)$ enters successively these components until it enters the first one $P$ that contains in its boundary an orbit $k' \in K_b$ such that $V^u(k)$ and $V^s(k')$ intersect. Before arriving near $k'$, the intersections of $V^u(k)$ with the regular pages of $(K, \mathcal{F})$ are along circles. We pick a regular page $S$ of $(K, \mathcal{F})$ in $P$. Then we have two components of $V^u(k) \cap S$ and of $V^s(k') \cap S$ which are circles $C(k)$ and $C(k')$. The circles $C(k)$ and $C(k')$ intersect into a nonempty compact set $\Delta$, containing only one-sided intersections. Indeed every point of $\Delta$ is located on an heteroclinic intersection from $k$ to $k'$, all of those being one-sided. The one-sided intersections can be on one side of $C(k)$ or the other, thus we further decompose $\Delta$ as the disjoint union of two compact sets $\Delta_+$ and $\Delta_-$, depending on the side of tangency.

The idea of the rest of the proof is to destroy, inductively, the one-sided intersections of $V^u(k)$, starting from those passing through $\Delta$ and to find a new Reeb vector field supported by the same broken book decomposition but such that $V^u(k)$ does not intersect any stable manifold up to a certain *length*. This will lead to a contradiction, by Lemma 4.1.

To follow this plan, we first need to define the length of a segment of orbit $\gamma$. It will be given by the number of components delimited in $\gamma$ by its intersections with the rigid pages. The length of a chain of heteroclinic connections will be the sum of the length of its components. Observe that the length of an orbit of $K$ is zero, while the length of a full orbit in a heteroclinic or homoclinic intersection is bounded.

We also want to consider convergence of sequences of orbits. For that we consider a small neighborhood $N(K_b)$ of $K_b$, made of the disjoint union of neighborhoods $N(k')$ of each $k' \in K_b$. These neighborhoods are taken to be a standard Morse type neighborhood. Hence any orbit that enters and exits $N(k')$ has to intersect a rigid page inside $N(k')$.

We have the following lemma whose first part is a tautology from the definition of length and the second part follows by compactness.

**Lemma 4.11.** *For every $L > 0$, there exists $N > 0$ such that every orbit $\gamma$ of length greater or equal to $L$ intersects $\mathcal{R}$ at least $L - 1$ times and the total action of $\gamma \setminus N(K_b)$ is less than $N$.*

Next observe:

**Lemma 4.12.** *Given $L > 0$, the set of heteroclinic intersections of length less than $L$ admits a natural compactification by chains of heteroclinic intersections of length less than $L$.*

*Proof.* Let $(\gamma_n)$ be a sequence of heteroclinic intersections of length bounded by $L$. Every orbit $\gamma_n$ passes through less than $L$ components of $N(K_b)$ and we can extract a subsequence such that the orbits in the subsequence have the same pattern of crossings with $N(K_b)$. Then the portions of orbits in the complement of $N(K_b)$ are segments of bounded action by Lemma 4.11 that are, up to extracting subsequences, converging to a collection of segments of orbits. Inside a component $N(k_1)$ of $N(K_b)$, they either converge to an orbit segment or to a sequence of one orbit in $V^s(k_1)$ followed by $k_1$ and then by an orbit in $V^u(k_1)$. This shows that a subsequence of $(\gamma_n)$ converges to a chain of heteroclinic intersections. It is then immediate that the chain has length less than $L$ and Lemma 4.12 follows. $\square$

Since each $N(k')$ is taken to be a standard Morse type neighborhood, the intersection $V^s(k') \cap N(k')$ has one connected component that contains $k'$ and which intersects $\partial N(k')$ along a circle $C^s(k')$.

We now explain how to eliminate the intersections from $\Delta_+$. They sit on one side of $V^s(k')$, so they determine locally transversely a quadrant $Q$ delimited by the component of $V^s(k')$ containing $\Delta_+$ and an unstable component $V^u(k')$ of $k'$.

**Lemma 4.13.** *If the component $V^u(k')$ is not a complete connection, i.e. it does not coincide with the stable manifold of an orbit $k'' \in K_b$, then we can slightly modify $R_\lambda$ to eliminate $\Delta_+$, without creating extra intersections of $V^u(k)$ of length less or equal to $L$.*

*Proof.* We first apply the following lemma:

**Lemma 4.14.** *If $V^u(k')$ is not a complete connection, then there is a point $p$ in $V^u(k')$ that is not on an heteroclinic intersection of length less than or equal to $L$.*

*Proof.* Assume by contradiction that every orbit is an intersection from $k'$ to some other orbit in $K_b$ of length less or equal to $L$. Then the set of intersections from $k'$, i.e. its entire unstable manifold $V^u(k')$, has a natural compactification by chains of intersections of length less than or equal to $L$ by Lemma 4.12.

To arrive to a contradiction to our assumption, as we did for $k$, we follow $V^u(k')$ in the product components of $M \setminus \mathcal{R}$. The heteroclinic intersections of $V^u(k')$ can be (partially) orderer by length, with respect to the integer valued length defined above. Let $C$ be the shortest length of an intersection and consider one intersection of length $C$, that goes to an orbit $k'' \in K_b$. Then one component $V^u(k'')$ of the unstable manifold of $k''$ is entirely contained in the closure of $V^u(k')$, as it can be seen locally in a neighborhood of $k''$. The compactness of the set implies then that every point of $V^u(k'')$ is itself contained in a intersection of length less or equal to $L - C$. If $V^u(k'')$ is not a complete connection to some other orbit, we replace $V^u(k')$ with $V^u(k'')$ and repeat the previous argument. If $V^u(k'')$ is a complete connection to some $k^{'''}$, we have that every orbit in a component of $V^u(k''')$ is also contained in limits of length less or equal to $L$ heteroclinic intersections from $k$ and we can continue with $V^u(k''')$ until we find an unstable manifold that is entirely made of heteroclinic intersections that are limits of those from $k$ and is not a complete connection (the process stops because at each step the length in $\mathbb{N}$ is strictly decreasing and we end with an unstable manifold that is not a complete connection because the orbit of $K_b$ in the sequence we end with has intersections from $k$ in its stable manifold). Up to reindexing, we call it $V^u(k'')$. We can apply the same argument we applied to $V^u(k')$ to $V^u(k'')$ and continue until we arrive at the $n$th step at an unstable manifold $V^u(k^{(n)})$ of some $k^{(n)}$ in $K_b$ which is only made of heteroclinic intersections to orbits in $K_b$, all having the same length which is also the minimal length. This means that $V^u(k^{(n)})$ is in fact a complete connection and no orbit could have come from $V^u(k'')$ to $k^{(n)}$, a contradiction. $\square$

Let $\gamma$ be the orbit of $R_\lambda$ through $p$ given by Lemma 4.14. We then claim that there exists a small neighborhood $N(p)$ of $p$ in $M$ that does not meet any intersection orbit from $k$ to some $k'' \in K_b$ of length less or equal to $L$.

Indeed, if every neighborhood of $p$ was having such an intersection, then we would have a sequence of orbits $\gamma_n$ from $k$ to some $k_n \in K_b$, where a point $p_n \in \gamma_n$ limits to $p$ and the length of $\gamma_n$ is bounded above by $L$. Using Lemma 4.12, we get that $p$ is on a heteroclinic intersection that is part of a chain to which a subsequence of $(\gamma_n)$ converges and the claimed is proved.

Next, we take a small arc $\delta$ in $N(p)$, starting at $p$ and going straight inside the quadrant $Q$ of $k'$ associated with $\Delta_+$. We push $\delta$ by the backward flow of $R_\lambda$ and look at the intersection generated by this half infinite strip $\delta \times (-\infty, 0]$ with the surface $S$. We recall that $S$ is the regular page of the broken book that was used to define the sets $\Delta_\pm$.

Since $\delta$ is anchored in $V^u(k')$, we get on $S$ a half infinite line $l$ spiralling to the circle $C(k')$ from the side containing $C(k)$ near $\Delta_+$, i.e. the side of the quadrant $Q$. This line $l$ crosses $C(k)$ near $\Delta_+$. The goal is now to eliminate $\Delta_+$ by replacing portions of $C(k)$ by portions of $l$. This will be done in a product neighborhood of $S$ by modifying the direction of $R_\lambda$ so that the circle $C(k)$ entering the neighborhood of $S$ will be mapped to the modified circle when exiting. The modification is performed in a neighborhood of $\Delta_+$ that does not meet a neighborhood of $\Delta_-$ (remember that $\Delta_-$ and $\Delta_+$ are disjoint compact sets in $S$).

Concretely, by a generic choice of $\delta$, we first make sure that $l$ is transverse to $C(k)$. Then between any two consecutive intersections of $C(k)$ with $l$, there is a segment of $l$ and a segment of $C(k)$. If the segment of $C(k)$ contains an intersection point of $\Delta_+$, we replace this segment of $C(k)$ with the segment of $l$. This procedure ends thanks to the compactness of $\Delta_+$.

Here there are two issues to address. First, we want our deformation of $C(k)$ to be smooth. We can perform the smoothing in the image of the neighborhood of $N(p)$ by the flow, which gives a neighborhood of $l$ in $S$.

More importantly, we need the $d\lambda$-area between $C(k)$ and its modification to be zero for being able to realise it with a modification of the Reeb vector field which gives a change of holonomy having zero flux, see [CHL]. When we replace a portion of $C(k)$ with a portion of $l$ we have a contribution to the flux which is the area between the two. Since $l$ spirals to $C(k')$, this can be taken to be small at will and with a $C^1$-small perturbation by taking the segments of $l$ close enough to $C(k')$. This total $\epsilon$ change of area can be compensated by another $C^1$-small perturbation near a fixed intersection of $C(k)$ with the image of the neighborhood of $N(p)$ by the flow, where we have a fixed area coming from $N(p)$ available. Note that we to perform this pertubation in a flow box based at a subdomain of $S$ whose length along the Reeb direction can be taken long at will, since the Reeb orbits from $N(p)$ to $S$ are getting longer as $N(p)$ is getting smaller. We have has much room as needed to perturb the Reeb flow and apply [CHL].

Doing so, we see that we can eliminate $\Delta_+$ and since the modification of $V^u(k)$ is contained in the orbits through $N(p)$, we do not create heteroclinic intersections of length less or equal to $L$. This proves Lemma 4.13. $\square$

Under the similar hypothesis, we can also eliminate $\Delta_-$.

We are left with the case in which the unstable component $V^u(k')$ is a complete connection to an orbit $k''$. We can repeat our argument of Lemma 4.13: either the corresponding unstable manifold of $k''$ is a complete connection, or we can eliminate $\Delta_+$, by using a segment $\delta$ anchored in $V^u(k'')$, whose image by the flow in $S$ is similar to the one of the previous case, i.e. spiralling to $C(k')$. Since by hypothesis not all the elements of $K_b$ have complete connections, this process stops and we can always eliminate $\Delta_\pm$. Arguing by induction, we eliminate successively all intersections from $k$ of length less or equal to $L$ (without creating new ones) and obtain a contradiction with Lemma 4.11 for the unstable manifold $V^u(k)$. This terminates the proof of Proposition 4.10. $\square$

We can now prove Theorems 1.2 and 1.3. In view of Lemma 4.9 and Proposition 4.10, to prove Theorem 1.2 we need to consider the case when there is at least one crossing intersection between the components of $K_b$.

**Lemma 4.15.** *Let $(K, \mathcal{F})$ be a broken book decomposition supporting a nondegenerate Reeb vector field $R_\lambda$. Assume that there is at least one heteroclinic intersection between broken components of the binding $K_b$, with* *a crossing of stable and unstable manifolds. Then there is at least one homoclinic intersection with a crossing of stable and unstable manifolds, and thus positive topological entropy and infinitely many periodic orbits.*

*Proof.* We consider the set $\mathcal{C}$ of complete connections between components of $K_b$ that belong to a cycle of such complete connections. We then as before cut $M$ along $\mathcal{C}$ to get a manifold $M'$ with boundary and corners. We let $K_b'$ be the collection of periodic orbits of $K_b$, when viewed in $M'$. The set $K_b'$ may contain several copies of the same orbit of $K_b$.

By hypothesis there is at least one heteroclinic orbit in $M'$ between elements of $K_b'$ along which there is a crossing intersection. Hence it is not in $\partial M'$. Note that for every component $T$ of $\partial M'$, the number of stable and unstable manifolds of orbits $k_b \in \partial T \subset M'$ that are not themselves contained in $\partial T$ is even, since there are as many stable than unstable manifolds in $\partial T$ (every heteroclinic or homoclinic connection in $\partial T$ involves a stable and an unstable manifold).

Consider the connected component $T$ of $M'$ that contains a crossing intersection, and let $k \in K_b$ be the orbit whose unstable manifold $V^u(k)$ is involved in this intersection. Following the heteroclinic intersections from $V^u(k)$, as in Lemma 4.2, we get a sequence of heteroclinic intersections. We claim that this sequence stays inside $T$. Indeed, if it arrives to a periodic orbit in $K_b \cap \partial T$ along a stable manifold, then the two components of the unstable manifold of this periodic orbit are in $T$. We can thus construct a sequence such that all the stable and unstable manifolds involved are in the interior of $T$. To see this, one can blow down every component of $\partial T$ to a generalized hyperbolic orbit with $2n$ stable/unstable manifolds. We now have a closed manifold together with a Reeb vector field without connections. Lemma 4.1 and Proposition 4.10 imply that there is a cycle with only crossing intersections.

Near this cycle, we obtain a crossing homoclinic intersection, which is also a homoclinic intersection in $M$. Positivity of topological entropy comes from [BW]. $\square$

We now prove Theorem 1.3 stating that on a 3-manifold that is not graphed, every nondegenerate Reeb vector field has positive topological entropy.

*Proof of Theorem 1.3.* A nondegenerate Reeb vector field is carried by some broken book decomposition. If there is no broken component in the binding, then the broken book is in fact a rational open book. If $M$ is not a graph manifold, then the monodromy of this rational open book must contain a pseudo-Anosov component in its Nielsen-Thurston decomposition. The first-return map of the Reeb vector field on a page is homotopic to the

Nielsen-Thurston monodromy, so its topological entropy is bounded from below by the latter one, that is positive.

If the binding of the broken book has broken components then all elements of $K_b$ that are not complete connections contain, by Proposition 4.10, a crossing intersection. This proves the positivity of the entropy in this case.

If all stable and unstable manifolds of elements of $K_b$ are complete connections, then as in Lemma 4.9, they decompose $M$ into partial open books and if $M$ is not graphed, then one of them must have some pseudo-Anosov monodromy piece in its Nielsen-Thurston decomposition and we obtain positive topological entropy. $\square$

## 5. Improving the statements

In this last section, we discuss the nondegeneracy hypothesis and observe that our Theorems 1.2, 1.3 and 1.4 hold for an open set of contact forms containing nondegenerate ones.

Indeed, once we have a supporting broken book decomposition, the arguments only use the fact that the periodic orbits in the binding are nondegenerate and do not care about nondegeneracy of "long" periodic orbits. The binding orbits are a subset of the set of orbits $\mathcal{P}$ for a nondegenerate contact form $\lambda$ that shows up in Lemma 3.1. The set $\mathcal{P}$ contains all periodic orbits of $R_\lambda$ whose actions are less than $\mathcal{A}(\Gamma)$, where $\Gamma$ is a representative of a homology class whose image by the $U$-map does not vanish. As we will see below, the action of an orbit set realizing a class which is not in the kernel of $U$ can be *a priori* bounded in a neighborhood of $\lambda$, a manifestation of the continuity of the spectral invariants.

Precisely, there exists a $C^2$-neighborhood $N(\lambda)$ of $\lambda$ such that for every nondegenerate contact form $\lambda'$ in $N(\lambda)$, there is a representative $\Gamma'$ for $\lambda'$ of a nonzero class in $ECH(M, \lambda')$ whose image under the $U$-map is nonzero and whose total action is *a priori* bounded by some $L > 0$, depending only on $N(\lambda)$. This is a consequence of the existence of the cobordism maps in $ECH$ (defined through Seiberg-Witten homology) given in [Hu2, Theorem 2.3]. We now shrink $N(\lambda)$ so that moreover for every form $\lambda'$ in $N(\lambda)$, the periodic Reeb orbits of $\lambda'$ of action less than $L$, forming a set $\mathcal{P}'$, are all nondegenerate – note this is an open condition. Now, if $\lambda'$ is a contact form in $N(\lambda)$, possibly degenerate, first its periodic Reeb orbits of action less than $L$ are nondegenerate and second it can be approximated by a sequence of nondegenerate contact forms $\lambda'_n$ in $N(\lambda)$ whose periodic orbits of actions less than $L$ coincide with those of $\lambda'$. Since $\lambda'_n$ is nondegenerate, the conclusion of Lemma 3.1 holds for $\lambda'_n$ and $\mathcal{P}'$: through every point $z$ in $M$, one can find a projected holomorphic curve through $z$ and with asymptotics in $\mathcal{P}'$. By compactness for pseudo-holomorphic curves in the

ECH context [Hu, Sections 3.8 and 5.3], this property also holds for $\lambda'$ and $\mathcal{P}'$. This is all we need to find a supporting broken book for $\lambda'$, with binding in the nondegenerate set $\mathcal{P}'$. The rest of the arguments to prove Theorems 1.2, 1.3 and 1.4 then carry over to $\lambda'$.

## REFERENCES


[Alv] M. Alves, *Legendrian contact homology and topological entropy*, J. Topol. Anal. **11** (2019), no. 1, 53–108.

[ACH] M. Alves, V. Colin and K. Honda, *Topological entropy for Reeb vector fields in dimension three via open book decompositions*, Jour. Ecole Polytechnique, Tome 6 (2019), 119–148.

[BW] K. Burns and H. Weiss, *A geometric criterion for positive topological entropy*, Comm. Math. Phys. **172** (1995), 95–118.

[CGH0] V. Colin, P. Ghiggini and K. Honda, *Embedded contact homology and open book decompositions*, preprint 2010. `ArXiv:1004.2942`.

[CGH1] V. Colin, P. Ghiggini and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions I*, preprint 2012. `ArXiv:1208.1074`.

[CGH2] V. Colin, P. Ghiggini and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions II*, preprint 2012. `ArXiv:1208.1077`.

[CGH3] V. Colin, P. Ghiggini and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions III: from hat to plus*, preprint 2012. `ArXiv:1208.1526`.

[CH] V. Colin and K. Honda, *Reeb vector fields and open book decompositions*, J. Eur. Math. Soc. **15** (2013), no. 2, 443–507.

[CHL] V. Colin, K. Honda, F. Laudenbach, *On the flux of pseudo-Anosov homeomorphisms*, Algebr. Geom. Topol. **8** (2008), 2147-2160.

[C-GH] D. Cristofaro-Gardiner and M. Hutchings, *From one Reeb orbit to two*, J. Diff. Geom. **102** (2016), 25–36.

[C-GHP] D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano, *Torsion contact forms in three dimensions have two or infinitely many Reeb orbits*, arXiv:1701.02262, to appear in Geom. Topol.

[Fra] J. Franks, *Geodesics on $\mathbb{S}^2$ and periodic points of annulus homeomorphisms*, Invent. Math. **108** (1992), no. 2, 403–418.

[FH] J. Franks and M. Handel, *Periodic points of Hamiltonian surface diffeomorphisms*, Geom. Topol. **7** (2003), 713–756.

[Fri] D. Fried, *Transitive Anosov flows and pseudo-Anosov maps*, Topology **22** (1983), 299–303.

[Ghy] E. Ghys, *Right-handed vector fields and the Lorenz attractor*, *Japan. J. Math.* **4** (2009), 47–61.

[GG] V. Ginzburg and B. Gürel, *The Conley conjecture and beyond*, Arnold Math. J. 1 **3** (2015), 299–337.

[GHHM] V. Ginzburg, D. Hein, U. Hryniewicz, and L. Macarini, *Closed Reeb orbits on the sphere and symplectically degenerate maxima*, Acta Math. Vietnam., **38** (2013) 55–78.

[Gir] E. Giroux, *Géométrie de contact : de la dimension trois vers les dimensions supérieures*, Proceedings of the ICM, Vol. II (Beijing 2002), Higher Ed. Press, Beijing (2002), 405–414.

[HK] H. Hofer and M. Kriener, *Holomorphic curves in contact dynamics*, Notes from 1997 ParkCity Mathematics Institute.

[HWZ1] H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectizations I: asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **13** (1996), 337–379.

[HWZ2] H. Hofer, K. Wysocki, E. Zehnder, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Ann. of Math. (2) **157** (2003), no. 1, 125–255.

[HS] U. Hryniewicz, P. Salamão, *Global surfaces of section for Reeb flows in dimension three and beyond*, `arXiv:1712.01925`.

[Hry] U. Hryniewicz, *A note on Schwartzman-Fried-Sullivan Theory, with an application*, `https://arxiv.org/abs/1904.12416`

[HT1] M. Hutchings and C. H. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, Geom. Topol. **13** (2009), 901–941.

[HT2] M. Hutchings and C. H. Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders II*, J. Symplectic Geom. 7 (2009), 29–133.

[Hu2] M. Hutchings, Quantitative embedded contact homology, Jour. Diff. Geom **88 (2)** (2011), 231–266.

[Hu] M. Hutchings, *Lecture notes on embedded contact homology*, Contact and symplectic topology, Bolyai Soc. Math. Stud. **26** (2014), 389–484.

[Iri] K. Irie, *Dense existence of periodic Reeb orbits and ECH spectral invariants*, J. Mod. Dyn. **9** (20125), 357–363.

[Kat] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 137–173.

[Kup] I. Kupka, *Contributions à la théorie des champs génériques*, Contrib. Diff. Equations **2** (1963), 457–484; ibid. **3** (1964), 411–420.

[LS] P. Le Calvez, M. Sambarino, *Homoclinic orbits for area preserving difffeomorphisms of surfaces*, `arXiv:1912.06792`.

[MS] L. Macarini and F. Schlenk, *Positive topological entropy of Reeb flows on spherizations*, Math. Proc. Cambridge Philos. Soc. **151** (2011), 103–128.

[dPS] N. V. de Paulo and P. A. S. Salomão, *Systems of transversal sections near critical energy levels of Hamiltonian systems in $\mathbb{R}^4$*, Mem. Amer. Math. Soc. 252 (2018), no. 1202, v+105 pp.

[OS] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), 1159–1245.

[Pei] M. Peixoto, *On an approximation Theorem of Kupka and Smale*, J. Differential Equations **3** (1966), 214–227.

[Sie1] R. Siefring, *Intersection theory of finite energy surfaces*, Ph.D. Thesis, New York University, 2005.

[Sie2] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. 61(2008), no. 12, 1631–1684.

[Sma] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Normale Superiore Pisa **18** (1963), 97–116.

[Sul] D. Sullivan, *A foliation of geodesics is characterized by having no "tangent homologies"*, J. Pure Appl. Algebra **13** (1978), no. 1, 101–104.

[Tau] C. H. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. **11** (2007), 2117–2202.

[Wei] A. Weinstein, *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Differential Equations **33** (1979), 353–358.

[We] C. Wendl, *Contact* 3*-manifolds, holomorphic curves and intersection theory*, arxiv:1706.05540

V. COLIN, UNIVERSITÉ DE NANTES, 2 RUE DE LA HOUSSINIÈRE, 44322 NANTES, FRANCE

*Email address*: vincent.colin@univ-nantes.fr

*URL*: https://www.math.sciences.univ-nantes.fr/~vcolin/

P. DEHORNOY, UNIV. GRENOBLE ALPES, CNRS, INSTITUT FOURIER, F-38000 GRENOBLE, FRANCE

*Email address*: pierre.dehornoy@univ-grenoble-alpes.fr

*URL*: http://www-fourier.ujf-grenoble.fr/~dehornop/

A. RECHTMAN, INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG, FRANCE

*Email address*: rechtman@math.unistra.fr

*URL*: https://irma.math.unistra.fr/~rechtman/