An example of non-contracting weakly branch automaton group

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Abstract. We study some properties of a group defined by a non-initial three states automaton. We show that it is regular weakly branch, but non-contracting. This appears to be the first example of group with this property.

The concept of automata groups appears in the 60’s, and find new perspectives in more recent work of R. Grigorchuk, A. Žuk, V. Nekrashevich, L. Bartholdi and many other authors [1-5]. It appears to be rich in examples of groups enjoying various exotic properties, algebraic as well as geometric (for example, about Burnside problems, growth, amenability, random walks, just infiniteness...). Automata groups act on a rooted tree by automorphisms, and their study is now strongly based on the investigation of self-similar properties of these actions. This yields the definitions of fractal groups, branch groups, and contracting groups (see Part 1). The most pleasant case seems to be the case of contracting branch groups, or at least contracting regular weakly branch groups.

In this Note, we produce the first example of regular weakly branch group which is not contracting.

Theorem 1. The group defined by the non-initial automaton $A$ (see Figure 1) is fractal, regular weakly branch over its commutator subgroup, just non-solvable, and non-contracting.

We begin, in Part 1, with some definitions, of automata, groups defined by automata, their action on a rooted tree, and the properties of fractalness, branchness and contraction. This is a rather quick overview, we suggest [2], and [3] for more comprehensive developments of this theory.

In Part 2 we prove Theorem 1, in several steps. The non-contractivity is shown by proving that a certain element is not of finite order.

1. Definitions

An invertible non-initial automaton on a finite alphabet $\mathcal{A}$ is given by a finite set $Q$ of states, with transition and exit functions: $\phi: \mathcal{A} \times Q \to Q$, $\psi: \mathcal{A} \times Q \to \mathcal{A}$.

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The automaton is invertible if the restriction of the exit function to any state, is a
bijection from $A$ to itself: $(\psi_q : A \times \{q\} \to A) \in S_A$, where $S_A$ is the group of the
permutations of $A$.

**Example:** on the diagram 1 for the automaton $A$, the alphabet is $A = \{0, 1\}$,
the states are $a$, $b$, and $c$, the exit functions are given by the element of $S_2$ associated
to each state ($\varepsilon$ is the non trivial element of $S_2$), and the transition functions are
given by the arrows (labeled by elements of the alphabet).

![Diagram of the automaton $A$](image)

**Figure 1.** Diagram of the automaton $A$

An initial automaton is an automaton with some state $q$ declared as the initial
state. Such an automaton acts (on the right) on the set of finite and infinite (on
the left) sequences over $A$: the given sequence provides, by the transition function,
a path in the automaton, starting at the initial state, and along this path, the exit
function defines a new sequence over $A$. In the case of an invertible automaton,
this defines an automorphism of $\mathcal{A}_N$.

The group defined by an invertible automaton on an alphabet $A$, is by definition
the subgroup of $\text{Aut}(\mathcal{A}_N)$ generated by the automorphisms defined by each initial
automaton that can be obtained by choosing a initial state among the set of states.
It acts naturally on a rooted tree of degree $d = |A|$ whose vertices are identified
with the set of words over the alphabet $A$. It also acts naturally on each level of
this tree (the level $n$ of a rooted tree is the set of vertices at distance $n$ from the
root, hence identified with the set of words of length $n$). We denote by $\emptyset$ the empty
word, and also the root of the tree.

Let $G$ be a group defined by an invertible non-initial automaton. To every
element $g$ of $G$, one can associate an element of the symmetric group $\sigma \in S_d$,
and $d$ elements of $G$, $g_1, ..., g_d$ as follows: the permutation $\sigma$ is the restriction of $g$
to the first level of the tree, $\{v_1, ..., v_d\}$, and for every $i \leq d$, $g_i$ is the induced
automorphism on the subtree $T_{v_i} = \{v, v_i \in [\emptyset, v]\}$. As $G$ is a group defined by a
non-initial automaton, the $g_i$ are naturally identified with elements of $G$, after the
natural identification of $T_{v_i}$ to $T$.

In other words, for any $g \in G$, one can describe $g$ by $g = (g_1, ..., g_d)\sigma$ where
$\sigma \in S_d$ and $g_i \in G$. This notation correspond to an embedding of $G$ into the
wreath product of $G$ with $S_d$, where the multiplication is defined as follows:

$[(g_1, ..., g_d)\sigma](h_1, ..., h_d)\rho] = (g_1h_{\sigma(1)}, ..., g_dh_{\sigma(d)})\sigma\rho$.

We denote by $\text{St}(n)$ the maximal subgroup stabilizing the level $n$ of the tree.
If $g$ belongs to $\text{St}(1)$, then one can associate to $g$ an element of $G \times G \times G$, by
considering, as before, its restriction on each subtree rooted in an element of the
first level. This defines canonical projections $\Psi_i : \text{St}(1) \to G$ on the $i$-th coordinate.
Definition 1. A group $G$ is called fractal if the image of each $\Psi_x$ is $G$ itself.

For each vertex $u$ of the tree, we consider $Rist(u)$ the subgroup of $G$ that acts trivially everywhere except on the subtree $T_u = \{v, u \in [0, v]\}$ associated to $u$, and for every integer $n$, $Rist(n)$ is the subgroup generated by all the groups $Rist(u)$ where the level of $u$ is $n$. A group acting transitively on each level is branch if $Rist(n)$ is a subgroup of $G$ of finite index, for all $n$. It is weakly branch if $|Rist(n)| = \infty$ for all $n$.

Definition 2. A group $G$ is regular weakly branch over a normal subgroup $K$ if $K \neq \{1\}$ and if $K \geq K \times K \times \ldots K$ (d factors of which acts on the corresponding subtree $T_u$, $|u| = 1$).

We already defined a projection of each element $g \in G$ on each vertices of the first level. By iterating this process, we can define the projection of $g$ on the vertices of any level. If $u$ is a vertex, let $g_u \in G$ be this projection. We will also write: $(u)g = (g_u)u'$ where $u'$ is the image of $u$ by the automorphism $g$.

Definition 3. A group is contracting if there is a constant $\lambda < 1$ and $C, L \in \mathbb{N}$, such that, for all vertex of level $l > L$, for all $g \in G$, $|g_u| < \lambda |g| + C$.

2. Proof of Theorem 1

The proof of each assertion will be given separately (Lemma 2, Proposition 1, Proposition 2, and Proposition 3).

In our case, the alphabet of the automaton is $\{0, 1\}$, so that $d = 2$. The tree is binary, and each of its vertices is well defined by a word in $0$ and $1$.

The three generators of $G$ are the following (written in the wreath product, with $\epsilon$ being the non trivial element of $S_2$):

\[ a = (c, a)\epsilon, \quad b = (b, a)\epsilon, \quad c = (c, b) \]

2.1. Generators of the stabilizer of the first level, $St(1)$. We use the classical method of Shreier: if we choose a set $N$ of generators of $G$, and a set $K$ of representatives of $G/St(1)$, then a set of generators of $St(1)$ is given by $\{S_{k,\bar{x}} = k\bar{x}(k\bar{x})^{-1}, \nu \in N, k \in K\}$, where $\bar{x}$ is the representative in $K$ of the coset of $x$. It is an easy exercise to prove this, we refer to [6] chapter 1. Here we choose $N = \{a, b, c\}$ and $K = \{1, a\}$, which gives (note that $S_{1,a} = 1$):

\begin{align*}
S_1 &= ba^{-1} = (bc^{-1}, 1) \\
S_2 &= ab = (ca, ab) \\
S_3 &= c = (c, b) \\
S_4 &= a^2 = (ca, ac) \\
S_5 &= aca^{-1} = (e, ca^{-1})
\end{align*}

After elementary Nielsen transformations on this system (see [6] Chapter 1), we find a new system of generators for $St(1)$ (first compute $W_5$ and $W_4$, then $W_2$ and $W_1$, then $W_3$ using $W_4$):

\begin{align*}
W_1 &= S_1S_3 = ba^{-1}c = (b, b) \\
W_2 &= S_5^{-1}S_2 = c^{-1}ab = (a, b^{-1}ab) \\
W_3 &= S_3S_4S_2^{-1} = ca^{2}b^{-1}a^{-1} = (c, bacb^{-1}a^{-1})
\end{align*}
\[ W_4 = S_4S_2^{-1} = a^2b^{-1}a^{-1} = (1, abc^{-1}a^{-1}) \]
\[ W_5 = S_5S_1^{-1}S_3^{-1} = abc^{-1}c^{-1} = (1, acac^{-1}b^{-1}) \]

2.2. Fractalness, weakly branching.

**Lemma 1.** In the group \( G \), one has \( abc = bac = cba = 1 \).

The elements \( abc, bac \) and \( cba \) are conjugated to each other, thus it is enough to show that \( abc = 1 \). We will prove by induction on the level, that it is in the stabilizer of every level, hence it is the trivial element. One has \( abc = S_3S_2 \), therefore it is in the stabilizer of the first level. Assume that it is in the stabilizer of the level \( n \). Note that it is in the stabilizer of the level \( n + 1 \). This proves the lemma. □

**Lemma 2.** The group \( G \) is fractal.

The automorphisms of the tree defined on the first level by \( Z_1 = (a, a) \), \( Z_2 = (b, b) \) and \( Z_3 = (c, c) \) are elements of \( G \).

Indeed \( W_1 = (b, b) \in G \), and also \( W_1^{-1}W_5^{-1}W_2W_1 = (a, a) \in G \). By Lemma 1, \( W_3W_4^{-1} = (c, b) \), and \( W_5 = (1, b^{-1}c) \), and the product gives \( Z_3 \). From this (but also, more simply from the original system of generators), we see that the projections of the elements of \( St(1) \) on both subtrees of the first level generates \( G \).

**Proposition 1.** The group \( G \) is regular weakly branch over its commutator subgroup \( G' \).

First note that \( G' \neq 1 \) because it contains \( acac^{-1}b^{-1} = (cbee^{-2}, acac^{-1}b^{-1}) \), and that \( cbee^{-2} \neq 1 \) (it is not in \( St(1) \)).

We have to prove that \( G' \succeq G' \times G' \). With the use of the conjugation by \( a \notin St(1) \), it is enough to prove that \( G' \succeq \{1\} \times G' \).

The group \( G' \) is generated by the commutators \([a^{-1}, b^{-1}], [a, c^{-1}], [c^{-1}, b] \) and their conjugates. Thus, it suffices to show that the three automorphisms of the tree defined on the first level by \( (1, [a^{-1}, b^{-1}]), (1, [a, c^{-1}]), (1, [c^{-1}, b]) \) are in \( G' \).

Indeed we have
\[ (1, aba^{-1}b^{-1}) = W_5^{-1}W_5 = aba^{-1}cb^{-1}c^{-1} \in G'. \]

Note that from this, we have that \( W_4^{-1}W_5 \in G' \).

By conjugating \( W_5^{-1} \) by \( Z_1 \) (cf. Lemma 2), we get \( (1, a^{-1}bac^{-1}) \), and by conjugating \( W_4 \) by \((Z_1)^2\), we can get \((1, a^{-1}bc^{-1})\). Hence, we have (writing \( xy \) for the conjugate of \( x \) by \( y \)):
\[ W_4^{(Z_1)^2}(W_5^{-1})Z_1 = (1, a^{-1}ac^{-1}). \]

The sum of powers of \( a \) (resp. \( b, c \)) that appears in the word \( W_4^{(Z_1)^2}(W_5^{-1})Z_1 \) is 0, because it is the case for \( W_4W_5^{-1} \), as we already noticed. Hence \( W_4^{(Z_1)^2}(W_5^{-1})Z_1 \in G' \).

At last, we have \( W_4Z_1 = (1, cb^{-1}) \), and \((W_4^{-1})Z_1Z_2 = (1, c^{-1}b) \). We then see that \( W_4^{Z_1}(W_4^{-1})Z_1Z_2 = (1, cb^{-1}c^{-1}b) \).

Again, the sum of the powers of \( a \) (resp. \( b, c \)) appearing in the word on the left is 0. Therefore \( W_4^{Z_1}(W_4^{-1})Z_1Z_2 \in G' \). All this shows that \( G' \succeq \{1\} \times G' \). □
2.3. A corollary about just non-solvability.

**Proposition 2.** The group $G$ is just non-solvable, that is, $G$ is non solvable, whereas every proper quotient of $G$ is solvable.

We use, without proving it here, a general lemma appearing in [4] as Lemma 10 (see also [1] Proposition 3.7):

**Lemma 3.** Let $G$ be a regular weakly branch group over $K$ (i.e. $K$ is normal subgroup of $G$, and $K \geq K \times K \ldots K, K \neq \{1\}$), such that $G/K$ and $K/(K \times K \ldots K)$ are solvable. Then $G$ is just non-solvable.

As $G/G'$ is abelian, we only need to check that $G'/(G' \times G')$ is solvable. Here is the expression of a system of generators, up to conjugation, of $G'$.

$$aba^{-1}b^{-1} = (ab^{-1}, abc^{-1}a^{-1})$$

$$aca^{-1}c^{-1} = (bca^{-1}, acc^{-1}b^{-1})$$

$$bcb^{-1}c^{-1} = (b^{-1}, acc^{-1}b^{-1})$$

By elementary Nielsen transformations, we easily get this other system of generators, up to conjugation:

$$x_1 = (ab^{-1}, abc^{-1}a^{-1})$$
$$x_2 = (bca^{-1}, abc^{-1}b^{-1}a^{-1}c^{-1}) \in G' \times G'$$
$$x_3 = (1, acc^{-1}b^{-1}a^{-1}c^{-1}) \in \{1\} \times G'$$

We see that the set of the powers of $x_1$ is a set of representatives for $G'/(G' \times G')$. Therefore, $G'/(G' \times G')$ is cyclic, hence solvable. This, with Lemma 3, proves the Proposition 2. □

2.4. $G$ is not contracting. We now prove the last part of Theorem 1.

**Proposition 3.** The group $G$ is not contracting.

We argue by contradiction. Assume that it is contracting. Then, there exists a finite set $F \subset G$ such that for all $g \in G$, there exists $m$ such that for all word $w$ on the alphabet $\{0,1\}$, of length $|w| \geq m$, one has (see notations before Definition 3) $w^m = hw$, where $h \in F$.

Consider $g = e^m = (e^m, b^n)$, and $w_m = 00\ldots0$ of length $m$. We have for all $m$, $w_m e^m = 00\ldots0e^m = e^m00\ldots0$

This implies that $e^m \in F$ for all $n$. But, since $F$ is finite, we can find $n_1 \neq n_2$ such that $e^{n_1} = e^{n_2}$, and therefore, $e$ is an element of finite order. But, since $e^n = (e^n, b^n)$, $b$ is also an element of finite order. As it acts on a binary tree, its order must be a power of 2. To get a contradiction, it is enough to prove Lemma 4, which provides, for each $n$, a word in 0 and 1 that is not fixed by $b^n$.

**Lemma 4.** Let $X_n$ be the word $X_n = 1010\ldots101$ of length $2n + 2$. One has $(X_n)b^{2n} = 01X_{n-1}$.

More precisely, if $n$ is even, the projection (see notations before Definition 3) gives $(X_n)b^{2n} = a(01X_{n-1})$ and $(01X_{n-1})b^{2n} = (c)X_n$. If $n$ is odd, $(X_n)b^{2n} = (ba)01X_{n-1}$ and $(01X_{n-1})b^{2n} = (eb)X_n$.

We argue by induction on $n$. It is easily verified for $n = 0, n = 1$. Suppose $n > 1$ is odd. Apply $b^{2n-1}$ to $X_n$. The statement for $(n-1)$ gives: $(X_n)b^{2n-1} = (10X_{n-1})b^{2n} = (10)a(01X_{n-2})$. Apply again $b^{2n-1}$: $(01X_{n-2})b^{2n-1} = (c)X_{n-1}$, and
then, since $(10)ac = (ba)01$, we have $(X_n)b^{2^n} = ba(01X_{n-1})$. Similarly, one can compute $(01X_{n-1})b^{2^n}$, by noticing that $(01)ac = (cb)10$. One finds $(01X_{n-1})b^{2^n} = (cb)X_n$, as expected.

If $n > 1$ is even, one has $(X_n)b^{2^n-1} = (10X_{n-1})b^{2^n} = (10)ba(01X_{n-2})$, and $(01X_{n-2})b^{2^n-1} = (cb)X_{n-1}$. By Lemma 1, $(10)bacb = (10)b = a(01)$, hence, $(X_n)b^{2^n} = a(01X_{n-1})$. Again, similarly, we find $(01)b = c(10)$, so that $(01X_{n-1})b^{2^n} = (c)X_n$.

This proves Lemma 4, and also Proposition 3. □

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References