THE ISING MODEL
Exercises

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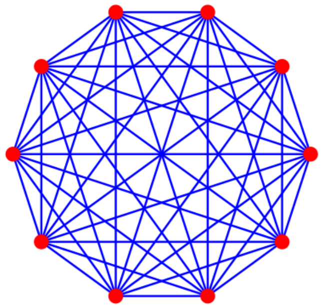
The mean field Ising model

Consider the complete graph with N vertices, denoted K_N. Let $\sigma_i \in \{-1,+1\}$, $i = 1 \dots N$ be the spin variables, with the notation : $\boldsymbol{\sigma} := (\sigma_i)_{i=1}^N$.



For $\beta \in \mathbb{R}^+$, and $h \in \mathbb{R}$, the Hamiltonian is:

$$\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma}) := -\frac{\beta}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j - h\sum_{i=1}^N \sigma_i$$

We consider the following probability measure on $\Omega_N := \{-1,+1\}^{K_N}$:

$$\mathbb{P}_{N,\beta,h}(\boldsymbol{\sigma}) := \frac{1}{Z_{N,\beta,h}}\exp\left(-\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma})\right) \quad \text{where} \quad Z_{N,\beta,h} = \sum_{\boldsymbol{\sigma}\in\Omega_N}\exp(-\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma})).$$

QUESTION 1. Calculate the distribution of the magnetization $m_N := \frac{1}{N}\sum_{i=1}^N \sigma_i$.
Hint : Express the Hamiltonien in terms of m_N : observe that $\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma}) = \mathcal{H}_{N,\beta,h}(m_N(\boldsymbol{\sigma}))$.

QUESTION 2.

1. Let $x_k := -1 + \frac{2k}{N}$ with $k = 0, 1, \dots, N$. Using the Stirling formula [1], show that there exists $c_1, c_2 \in (0,\infty)$ uniform in x_k such that

$$c_1 N^{-1/2} e^{Ns(x_k)} \leq \binom{N}{k} \leq c_2 N^{1/2} e^{Ns(x_k)}$$

with

$$s(x) := -\frac{1+x}{2}\log\left(\frac{1+x}{2}\right) - \frac{1-x}{2}\log\left(\frac{1-x}{2}\right)$$

[1] $N! = N^N e^{-N}\sqrt{2\pi N}(1 + O(1/N))$

2. Conclude that

$$c_1 N^{-1/2} e^{Nf(x_k)} \leq \sum_{\boldsymbol{\sigma}: m_N(\boldsymbol{\sigma}) = x_k} e^{-\mathcal{H}_{N,\beta,h}(m_N)} \leq c_2 N^{1/2} e^{Nf(x_k)} \tag{1}$$

with

$$f(x) = s(x) + \frac{\beta x^2}{2} + hx.$$

QUESTION 3.

1. Study the function f in terms of β and h.
2. Show that

$$N|\max_{x\in[-1,1]} f(x) - \max_{0\leq k\leq N} f(x_k)| \leq \text{const.}$$

3. Using (1), show that there exists $c_3 \in (0,\infty)$ such that

$$c_3 N^{-1/2} e^{N \max_{x\in[-1,1]} f(x)} \leq Z_{N,\beta,h} \leq (N+1) c_2 N^{1/2} e^{N \max_{x\in[-1,1]} f(x)}$$

QUESTION 4. Let $-1 \leq a < b \leq 1$. Show that

$$\left|\frac{1}{N}\log\left(\mathbb{P}(m_N \in [a,b])\right) - \left(\max_{x\in[a,b]} f(x) - \max_{y\in[-1,1]} f(y)\right)\right| = O\left(\frac{\log N}{N}\right)$$

QUESTION 5.

1. Let $\mathcal{M}(\beta,h)$ be the set of global maxima of f.
   Let $\mathcal{M}_\epsilon(\beta,h) := \{x \in [-1,1] : \min_{y\in\mathcal{M}(\beta,h)} |x-y| \leq \epsilon\}$.
   Show that for all $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{N,\beta,h}(m_N \notin \mathcal{M}_\epsilon(\beta,h)) < 0$$

2. Conclude that the law of large numbers for the magnetization is verified when $h \neq 0$ as well as when $h = 0$ and $\beta \leq 1$, but it is violated when $h = 0$ and $\beta > 1$.
3. Draw the graph of the magnetization as a function of h when $\beta \leq 1$ and when $\beta > 1$.