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Deformations of locally complete intersections.

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Given a projective l.c.i. scheme, $X \subset \mathbb{P}^N$, we show that X has a smooth formal neighbourhood in which X is globally a complete intersection- that is, X is the intersection of codim(X) hypersurfaces.

1 Introduction

If $X \subset \mathbb{P}^N(k)$ (k being any field) is a projective local complete intersection scheme, then X is not necessarily a global complete intersection in projective space - that is, X is not necessarily embedded in $\mathbb{P}^N(k)$ as the vanishing locus of codim X polynomials. It seems natural to ask whether this is true for more general ambient varieties. In particular, given such an X, we may wonder whether it can be embedded in some smooth Y as a globally complete intersection; ie, the intersection of codim(X) hypersurfaces. The aim of this note is to answer this question in the affirmative, at least formally, by proving the following result:

Theorem 1.1 *Let $X \subset \mathbb{P}^N(k)$ be a projective local complete intersection scheme. Then there exists a smooth formal neighbourhood, X_∞ of X, a vector bundle, V on X_∞, and a section $\sigma : X_\infty \to V$,such that*

- *V is a direct sum of line bundles,*
- *rk (V)= codim$_{X_\infty} X$*
- *X is schematically the zero locus of σ.*

REMARK. I do not know whether or not this scheme X_∞ maybe chosen algebrisable.

1.1 Idea of the proof

We will embed $\mathbb{P}^N$ in a smooth space, W. The normal bundle of $\mathbb{P}^N$ in W will be highly negative, and $\mathbb{P}^N$ will be the zero locus of a section of a vector bundle, V, which is a direct sum of line bundles. We will consider the spaces $\mathbb{P}^N_n$, which will be cut out in W by $I^n_{\mathbb{P}^N}$, and will recursively construct an l.c.i. scheme X_n in $\mathbb{P}^N_n$ extending X_{n-1}. If V is negative enough, the construction of X_n will be unobstructed, and we may therefore continue this construction to ∞ to obtain a formal neighbourhood, X_∞ of X. We will also be able to impose that X_∞ is smooth. X_∞ will then satisfy all the requirements of the theorem.

2 Proof of Theorem 1

Let I_X be the ideal sheaf of X in $\mathbb{P}^N$. Since X is a l.c.i. subscheme of $\mathbb{P}^N$, the co-normal bundle I_X/I_X^2 is a locally free sheaf of $\mathcal{O}_X$ modules. We recall Serre's vanishing theorem, which may be found in [1]

Proposition 2.1 *Let F be a coherent sheaf on X, a projective scheme. There exists an i such that, for all $a \geq i$ and for all $j \geq 1$, we have:*

1. $H^j(X, F(a)) = 0$
2. *$F(a)$ is generated by its global sections.*

We may therefore, in particular, choose m such that

1. $H^1((I_X/I_X^2)^* \otimes \mathcal{O}_X(m))) = 0$
2. $(I_X/I_X^2)^* \otimes \mathcal{O}_X(m)$ is generated by its global sections.

We define:

- l, the dimension of $H^0((I_X/I_X^2)^* \otimes \mathcal{O}_X(m))$
- W, the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^N}(-m)^{\oplus l}$, in which $\mathbb{P}^N$ is naturally embedded as the zero section.
- π, the projection $\pi : W \to \mathbb{P}^N$.

We denote by V the bundle $\mathcal{O}_{\mathbb{P}^N}(m)^{\oplus l}$ and by $\mathbb{P}^N_n$ the n-th formal neighbourhood of $\mathbb{P}^N$ in W, that is, the subscheme defined by the ideal $I^n_{\mathbb{P}^N}$. Let c be the codimension of X in $\mathbb{P}^N$.

The following proposition allows us to recursively construct neighbourhoods of X in $\mathbb{P}^N_n$ in a compatible way:

Proposition 2.2 *If X_n is a l.c.i. subscheme of $\mathbb{P}^N_n$, such that $X_n \cap \mathbb{P}^{\mathbb{N}} = X$, and codim $X_n \subset \mathbb{P}^N_n = c$,then there exists X_{n+1}, an l.c.i. subscheme of $\mathbb{P}^N_{n+1}$ such that*

- *$X_{n+1} \cap \mathbb{P}^N_n = X_n$,*
- *codim $(X_{n+1}) = c$.*

If U is an open subscheme of $\mathbb{P}^N$, then we denote by U_n the open subscheme of $\mathbb{P}^N_n$ whose underlying geometric space is U. Let

- U^i be an affine open covers of $\mathbb{P}^N$, such that $X_n \cap U^i_n$ is a complete intersection,
- $f^i_1 \ldots f^i_c \subset H^0(U^i_n, I_{X^n \cap U^i_n})$ a regular sequence for the ideal sheaf of $X_n \cap U^i_n$.

We denote $U^i \cap U^j$ by $U^{i,j}$, $X_n \cap U^i_n$ by X^i_n and $X_n \cap U^{i,j}_n$ by $X^{i,j}_n$.
For every i, d, we choose $\tilde{f}^i_d \in H^0(U^i_{n+1}, \mathcal{O}_{\mathbb{P}^N_{n+1}})$ such that $\tilde{f}^i_d|_{\mathbb{P}^N_n} = f^i_d$. $\tilde{f}^i_d$ is then a regular sequence in $\mathcal{O}_{U^i_{n+1}}$. We denote by I^i_{n+1} the ideal sheaf of $\mathcal{O}_{U^i_{n+1}}$ generated by the $\tilde{f}^i_d$s.

I^i_{n+1} defines an l.c.i subscheme of U^i_{n+1}, which we denote by $\tilde{X}^i_{n+1}$. We will show that, after modification of the functions $\tilde{f}^i_d$, I^i_{n+1} will be equal to I^j_{n+1} on $U^{i,j}_{n+1}$ and therefore, the $\tilde{X}^i_{n+1}$'s may be glued together to form an l.c.i subscheme, $X_{n+1} \subset \mathbb{P}^N_{n+1}$, satisfying the requirements of the proposition.
Consider the following exact sequence of $\mathcal{O}_{\mathbb{P}^N_{n+1}}$ modules:

$$0 \to Sym^{n+1}(V) \to \mathcal{O}_{\mathbb{P}^N_{n+1}} \to \mathcal{O}_{\mathbb{P}^N_n} \to 0$$

$Sym^{n+1}(V)$ is here considered with its $\mathcal{O}_{\mathbb{P}^N}$-module structure. We will now define a Cech cocycle

$$h^{i,j} \in \Gamma((I_{X_n}/I^2_{X_n})^* \otimes_{\mathcal{O}_{X_n}} Sym^{n+1}(V|_X), U^{i,j}_n)$$

whose vanishing will be a sufficient condition for $\tilde{X}^i_{n+1}$ to be compatible.

Given $f \in \Gamma(I_{X_n}/I^2_{X_n}, U^{i,j}_n)$, we now construct $h(f) \in \Gamma(Sym^{n+1}(V|_X), U^{i,j})$. We choose

- $f' \in \Gamma(I_{X_n}, U^{i,j}_n)$, extending f. This is possible since, U^i_n and U^j_n, and therefore $U^{i,j}_n$ are affine.
- $f'^i \in \Gamma(I^i_{n+1}, U^{i,j}_{n+1})$ extending f'.
- $f'^j \in \Gamma(I^j_{n+1}, U^{i,j}_{n+1})$ extending f'.

Then, $f'^i - f'^j \in Sym^{n+1}(V)$ and $f'^i - f'^j|_X \in Sym^{n+1}(V)|_X$.

The difference $f'^i - f'^j|_X$ is independent of the choice of f'^i. Indeed, the choice of a different f'^i alters $f'^i - f'^j$ by an element of $I^i_{n+1} \cap Sym^{n+1}(V)$, which is equal to $Sym^{n+1}(V) \otimes I_X$, since $f^i_1 \dots f^i_c$ is a regular sequence. The same argument show that $f'^i - f'^j|_X$ is independent of the choice of f'^j

Likewise, $f'^i - f'^j|_X$ is independent of f'. If $f'' = f' + g_1 g_2$, $g_i \in I_{X_n}$, then we may choose

$$f''^i = f'^i + g^i_1 g^i_2 \quad f''^j = f'^j + g^j_1 g^i_2$$

and hence $f''^i - f''^j = (g^i_1 - g^j_1)g^i_2 + f'^i - f'^j$, whence $(f''^i - f''^j)|_X = (f'^i - f'^j)|_X$.

We may therefore define $h^{i,j}$ by $h^{i,j}(f) = f'^i - f'^j|_X$. We now need the following lemma:

Lemma 2.1 *If $h_{i,j} = 0$, then I^i_{n+1} and I^j_{n+1} are compatible on the intersection $U^{i,j}$.*

If $h^{i,j} = 0$, then, for every $f \in \Gamma(I^i_{n+1}, U^{i,j})$, there exists $g \in \Gamma(I^j_{n+1}, U^{i,j})$, such that

$$(g - f) \in I_X \otimes_{\mathcal{O}_X} Sym^{n+1}(V).$$

On $U^{i,j}$, we have $I_X \otimes_{\mathcal{O}_{\mathbb{P}^N}} Sym^{n+1}(V) = I^i_{n+1} \otimes_{\mathcal{O}_{\mathbb{P}^N_{n+1}}} Sym^{n+1}(V)$ and therefore $(g - f) \in I^j_{n+1}$ whence $f \in I^j_{n+1}$. We now finish the proof of the proposition. We alter the regular sequences $\tilde{f}^i_d$ so that $h^{i,j} = 0$. We note that

$$H^1((I_{X_n}/I^2_{X_n})^* \otimes_{\mathcal{O}_{X_n}} Sym^{n+1}(V|_X)) = H^1((I_X/I^2_X)^* \otimes_{\mathcal{O}_X} Sym^{n+1}(V|_X)) = 0,$$

and that, therefore, there exist elements $h_i \in \Gamma((I_{X_n}/I^2_{X_n})^* \otimes Sym^n(V|_X)), U_i)$, such that

$$h_{i,j} = h_i - h_j.$$

We now choose a new regular sequence $\overline{f}^i_d$, in such a way that $\overline{f}^i_d|_X = \tilde{f}^i_d|_X + h_i(f^i_d)$. These sequences generate new ideal sheaves $\overline{I}^i_{n+1}$. It is immediate from the construction that the associated cocycle $\overline{h}_{i,j}$ is 0, and hence, by the previous lemma, the sheaves $\overline{I}^i_{n+1}$ are compatible on the intersections. Therefore, the $\overline{I}^i_{n+1}$'s glue together to form a global ideal sheaf, I_{n+1}. I_{n+1} defines a subscheme of $\mathbb{P}^N_{n+1}$, which we denote by X_{n+1}. X_{n+1} is l.c.i, and codim$(X_{n+1}) = c$ since $\overline{f}^i_d$ is a regular sequence for $X_{n+1} \cap U^i_{n+1}$. X_{n+1} satisfies, therefore, all the requirements of the proposition. The formal scheme $\lim_{n\to\infty} X_n = X_\infty$ is then a formal neighbourhood of X in which X is embedded as the zero locus of the tautological section of $\pi^*(\mathcal{O}_X(-m)^{\oplus l}$. It remains only to show that X_∞ is smooth for some choice of X_n. The smoothness of X_∞ depends only on the choice of X_2. All the results we now quote on Kähler differentials may be found in [1].

Let π be the projection $\pi : \mathbb{P}^N_2 \to \mathbb{P}^N$. The sheaf of Kähler differentials $\Omega^1_k(\mathbb{P}^N_2) \otimes \mathcal{O}_{\mathbb{P}^N}$ is canonically isomorphic to $\pi^*\Omega^1_k(\mathbb{P}^N) \oplus V$. The universal derivative map: $d : I_{X_2}/I^2_{X_2} \to \Omega^1_k(\mathbb{P}^N_2) \otimes \mathcal{O}_{X_2}$ is an $\mathcal{O}_{X_2}$ linear map. After tensoring by $\mathcal{O}_X$, we obtain an $\mathcal{O}_X$-linear map:

$$d_{X_2} : I_X/I^2_X \to \Omega^1_k(\mathbb{P}^N)|_X \oplus V|_X$$

X_∞ is smooth at x if $d_{X_2}(x)$ is injective. We now assocaite to any $\phi : I_X/I^2_X \to V_X$ a candidate space for X_2, X_ϕ, in such a way that X_ϕ will be smooth if $\phi(x)$ is injective for all x. $X_\phi \subset \mathbb{P}^N_2$ is defined by the formula:

$$f \in I_{X_\phi} \Leftrightarrow f|_{\mathbb{P}^N} \in I_X \text{ and } (f - \pi^*(f|_{\mathbb{P}^N}))|_X = \phi(f)$$

For this X_ϕ, we have that

$$d_{X_\phi}(f) = \pi^* df + \phi(f),$$

whence $d_{X_\phi}(x)$ is injective for all x if $\phi(x)$ is injective for all x. It remains only to find $\phi \in Hom(I_X/I^2_X, V)$ such that $\phi(x)$ is injective for every x. Now, $Hom(I_X/I^2_X, \mathcal{O}_X(m))$ is globally generated. If $v_1 \dots v_l$ is a basis for $H^0(Hom(I_X/I^2_X, \mathcal{O}_X(m)))$, then

$$\phi = \oplus^l_{b=1} v_b : I_X/I^2_X \to V$$

is injective on $I_X/I_X^2(x)$ for every x. The theorem follows.

REMARK. One might wonder whether this work holds for quasi-projective X. We have used the fact that X is projective only to invoke Serre's vanishing theorem. Suppose that X is a quasi projective variety- that is, X is the complement in an projective variety, V, of an closed subvariety of V, U. The results of this paper will hold for X, provided that:

(A) For any coherent sheaf, $\mathcal{F}$ on X, there exists k such that for any $m \geq k$ $H^1(\mathcal{F}(m)) = 0$ and $\mathcal{F}(m)$ is generated by its global sections.

If U is of pure codimension 1, then X is affine, and the condition (A) is immediately satisfied. Further, there is an exact sequence:

$$H^1(V, \mathcal{F}(m)) \to H^1(X, \mathcal{F}(m)) \to H^2_U(V, \mathcal{F}(m))$$

where $H^2_U(V, \mathcal{F}(m))$, the local cohomology of F along U, vanishes if every component of U has codimension ≥ 3 (see [2] for more details.) It follows that the results of this note are, in fact, also valid for $X = V/U$, where V is projective and U is a closed subvariety containing no codimension 2 component. I would like to express my gratitude to my supervisor, Claire Voisin, for her unstinting help and guidance. I would also like to thank Yves Lazlo, who was always ready to answer my questions, and the referee, whose generous comments and help have much improved this work.

References

[1] Algebraic Geometry, R. Hartshorne Springer, Graduate Texts in Mathematics, 52.

[2] S.G.A., 1962, Facsimile 1, exposes 1-3, Alexander Grothendieck, I.H.E.S. .