On deformations of l.c.i. schemes

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Deformations of locally complete intersections.

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14th December 2006

Given a projective l.c.i. scheme, $X \subset \mathbb{P}^N$, we show that $X$ has a smooth formal neighbourhood in which $X$ is globally a complete intersection - that is, $X$ is the intersection of codim($X$) hypersurfaces.

1 Introduction

If $X \subset \mathbb{P}^N(k)$ ($k$ being any field) is a projective local complete intersection scheme, then $X$ is not necessarily a global complete intersection in projective space - that is, $X$ is not necessarily embedded in $\mathbb{P}^N(k)$ as the vanishing locus of codim $X$ polynomials. It seems natural to ask whether this is true for more general ambient varieties. In particular, given such an $X$, we may wonder whether it can be embedded in some smooth $Y$ as a globally complete intersection; ie, the intersection of codim($X$) hypersurfaces. The aim of this note is to answer this question in the affirmative, at least formally, by proving the following result:

**Theorem 1.1** Let $X \subset \mathbb{P}^N(k)$ be a projective local complete intersection scheme. Then there exists a smooth formal neighbourhood, $X_\infty$ of $X$, a vector bundle, $V$ on $X_\infty$, and a section $\sigma : X_\infty \to V$, such that

- $V$ is a direct sum of line bundles,
- $rk(V) = \text{codim}_{X_\infty} X$
- $X$ is schematically the zero locus of $\sigma$.

**Remark.** I do not know whether or not this scheme $X_\infty$ maybe chosen algebrisable.

1.1 Idea of the proof

We will embed $\mathbb{P}^N$ in a smooth space, $W$. The normal bundle of $\mathbb{P}^N$ in $W$ will be highly negative, and $\mathbb{P}^N$ will be the zero locus of a section of a vector bundle, $V$, which is a direct sum of line bundles. We will consider the spaces $\mathbb{P}_n^N$, which will be cut out in $W$ by $I_{\mathbb{P}_n^N}$, and will recursively construct an l.c.i. scheme $X_n$ in $\mathbb{P}_n^N$ extending $X_{n-1}$. If $V$ is negative enough, the construction of $X_n$ will be unobstructed, and we may therefore continue this construction to $\infty$ to obtain a formal neighbourhood, $X_\infty$, of $X$. We will also be able to impose that $X_\infty$ is smooth. $X_\infty$ will then satisfy all the requirements of the theorem.
2 Proof of Theorem 1

Let \( I_X \) be the ideal sheaf of \( X \) in \( \mathbb{P}^N \). Since \( X \) is a l.c.i. subscheme of \( \mathbb{P}^N \), the co-normal bundle \( I_X/I_X^2 \) is a locally free sheaf of \( \mathcal{O}_X \) modules. We recall Serre’s vanishing theorem, which may be found in [1].

**Proposition 2.1** Let \( F \) be a coherent sheaf on \( X \), a projective scheme. There exists an \( i \) such that, for all \( a \geq i \) and for all \( j \geq 1 \), we have:

1. \( H^j(X, F(a)) = 0 \)
2. \( F(a) \) is generated by its global sections.

We may therefore, in particular, choose \( m \) such that

1. \( H^1((I_X/I_X^2)^* \otimes \mathcal{O}_X(m))) = 0 \)
2. \( (I_X/I_X^2)^* \otimes \mathcal{O}_X(m) \) is generated by its global sections.

We define:

- \( l \), the dimension of \( H^0((I_X/I_X^2)^* \otimes \mathcal{O}_X(m)) \)
- \( W \), the total space of the vector bundle \( \mathcal{O}_{\mathbb{P}^N}(-m)^\otimes l \), in which \( \mathbb{P}^N \) is naturally embedded as the zero section.
- \( \pi \), the projection \( \pi : W \to \mathbb{P}^N \).

We denote by \( V \) the bundle \( \mathcal{O}_{\mathbb{P}^N}(m)^\otimes l \) and by \( \mathbb{P}^N \) the \( n \)-th formal neighbourhood of \( \mathbb{P}^N \) in \( W \), that is, the subscheme defined by the ideal \( I_{\mathbb{P}^N}^n \). Let \( e \) be the codimension of \( X \) in \( \mathbb{P}^N \).

The following proposition allows us to recursively construct neighbourhoods of \( X \) in \( \mathbb{P}^N \) in a compatible way:

**Proposition 2.2** If \( X_n \) is a l.c.i. subscheme of \( \mathbb{P}^N \), such that \( X_n \cap \mathbb{P}^N = X \), and codim \( X_n \subset \mathbb{P}^N = c \), then there exists \( X_{n+1} \), an l.c.i. subscheme of \( \mathbb{P}^N \) such that

- \( X_{n+1} \cap \mathbb{P}^N = X_n \),
- codim \( X_{n+1} \) = \( c \).

If \( U \) is an open subscheme of \( \mathbb{P}^N \), then we denote by \( U_n \) the open subscheme of \( \mathbb{P}^N \) whose underlying geometric space is \( U \). Let

- \( U^i \) be an affine open covers of \( \mathbb{P}^N \), such that \( X_n \cap U^i_n \) is a complete intersection,
- \( f_1^i \ldots f_c^i \subset H^0(U^i_n, I_{X_n \cap U^i_n}) \) a regular sequence for the ideal sheaf of \( X_n \cap U^i_n \).
We denote $U^i \cap U^j$ by $U^{i,j}$, $X_n \cap U^i$ by $X^i_n$ and $X_n \cap U^{i,j}$ by $X^{i,j}_n$. For every $i, d$, we choose $\tilde{f}^i_d \in H^0(\mathcal{O}_{\mathbb{P}^n_{|\mathbb{P}^n}}, \mathcal{O}_{\mathbb{P}^n})$ such that $\tilde{f}^i_d \equiv f^i_d$. $\tilde{f}^i_d$ is then a regular sequence in $\mathcal{O}_{U^i_{n+1}}$. We denote by $I^i_{n+1}$ the ideal sheaf of $\mathcal{O}_{U^i_{n+1}}$ generated by the $\tilde{f}^i_d$.

$I^i_{n+1}$ defines an l.c.i subscheme of $U^i_{n+1}$, which we denote by $\tilde{X}^i_{n+1}$. We will show that, after modification of the functions $\tilde{f}^i_d$, $I^i_{n+1}$ will be equal to $I^i_{n+1}$ on $U^{i,j}_{n+1}$ and therefore, the $\tilde{X}^i_{n+1}$’s may be glued together to form an l.c.i subscheme, $X_{n+1} \subset \mathbb{P}^N_{n+1}$, satisfying the requirements of the proposition.

Consider the following exact sequence of $\mathcal{O}_{\mathbb{P}^N}$ modules:

$$0 \to \text{Sym}^{n+1}(V) \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

$\text{Sym}^{n+1}(V)$ is here considered with its $\mathcal{O}_{\mathbb{P}^N}$-module structure. We will now define a Čech cocycle

$$h^{i,j} \in \Gamma((I_{X_n}/I^2_{X_n})^* \otimes O_{X_n} \text{Sym}^{n+1}(V|X), U^{i,j}_{n})$$

whose vanishing will be a sufficient condition for $\tilde{X}^i_{n+1}$ to be compatible.

Given $f \in \Gamma(I_{X_n}/I^2_{X_n}, U^{i,j}_{n})$, we now construct $h(f) \in \Gamma(\text{Sym}^{n+1}(V|X), U^{i,j}_{n})$. We choose

- $f' \in \Gamma(I_{X_n}, U^{i,j}_{n})$, extending $f$. This is possible since, $U^i_n$ and $U^j_n$, and therefore $U^{i,j}_n$ are affine.
- $f^{i'} \in \Gamma(I^i_{n+1}, U^{i,j}_{n+1})$ extending $f'$.
- $f^{j'} \in \Gamma(I^j_{n+1}, U^{i,j}_{n+1})$ extending $f'$.

Then, $f^{i'} - f^{j'} \in \text{Sym}^{n+1}(V)$ and $f^{i'} - f^{j'}|_X \in \text{Sym}^{n+1}(V)|_X$.

The difference $f^{i'} - f^{j'}|_X$ is independent of the choice of $f^{i'}$. Indeed, the choice of a different $f^{i'}$ alters $f^{i'} - f^{j'}$ by an element of $I^i_{n+1} \cap \text{Sym}^{n+1}(V)$, which is equal to $\text{Sym}^{n+1}(V) \otimes I_X$, since $f^1, \ldots, f^c$ is a regular sequence. The same argument show that $f^{i'} - f^{j'}|_X$ is independent of the choice of $f^{j'}$.

Likewise, $f^{i'} - f^{j'}|_X$ is independent of $f'$. If $f'' = f' + g_1 g_2$, $g_i \in I_{X_n}$, then we may choose

$$f'^{i'} = f'' + g^1 g^2, f^{j'} = f'' + g^1 g^2$$

and hence $f'' - f^{j'} = (g^1 - g^1) g^2 + f'' - f^{j'}$, whence $(f'' - f^{j'})|_X = (f^{i'} - f^{j'})|_X$.

We may therefore define $h^{i,j}$ by $h^{i,j}(f) = f^{i'} - f^{j'}|_X$. We now need the following lemma:

**Lemma 2.1** If $h^{i,j} = 0$, then $I^i_{n+1}$ and $I^j_{n+1}$ are compatible on the intersection $U^{i,j}_{n}$.  

If $h^{i,j} = 0$, then, for every $f \in \Gamma(I_{n+1}, U^{i,j})$, there exists $g \in \Gamma(I_{n+1}^{j+1}, U^{i,j})$, such that
\[(g - f) \in I_X \otimes_{O_X} Sym^{n+1}(V).\]

On $U^{i,j}$, we have $I_X \otimes_{O_X} Sym^{n+1}(V) = I_{n+1}^{j+1} \otimes_{O_{X_{n+1}}} Sym^{n+1}(V)$ and therefore
\[ (g - f) \in I_{n+1}^{j+1} \text{ whence } f \in I_{n+1}^{j+1}. \]

We now finish the proof of the proposition. We alter the regular sequences $\tilde{P}_d$ so that $h^{i,j} = 0$. We note that
\[ H^1((I_{X_n}/I_{X_n}^2)^* \otimes_{O_X} Sym^{n+1}(V|X)) = H^1((I_X/I_X^2)^* \otimes_{O_X} Sym^{n+1}(V|X)) = 0, \]
and that, therefore, there exist elements $h_i \in \Gamma((I_{X_n}/I_{X_n}^2)^* \otimes Sym^n(V|X)), U_i)$, such that
\[ h_{i,j} = h_i - h_j. \]

We now choose a new regular sequence $\tilde{T}_d$, in such a way that $\tilde{T}_d|_X = \tilde{P}_d|_X + h_i(f^X_d)$. These sequences generate new ideal sheaves $T_{n+1}$. It is immediate from the construction that the associated cocycle $h_{i,j}$ is 0, and hence, by the previous lemma, the sheaves $T_{n+1}$ are compatible on the intersections. Therefore, the $T_{n+1}$'s glue together to form a global ideal sheaf, $I_{n+1}$. $I_{n+1}$ defines a subscheme of $P_{n+1}^N$, which we denote by $X_{n+1}$. $X_{n+1}$ is l.c.i. and $\text{codim}(X_{n+1}) = c$ since $\tilde{T}_d$ is a regular sequence for $X_{n+1} \cap U^{i,j}$. $X_{n+1}$ satisfies, therefore, all the requirements of the proposition. The formal scheme $\lim_{n \to \infty} X_n = X_\infty$ is then a formal neighbourhood of $X$ in which $X$ is embedded as the zero locus of the tautological section of $\pi^*(\mathcal{O}_X(-m))^{\otimes l}$. It remains only to show that $X_\infty$ is smooth for some choice of $X_n$. The smoothness of $X_\infty$ depends only on the choice of $X_2$. All the results we now quote on Kähler differentials may be found in [1].

Let $\pi$ be the projection $\pi : P_2^N \to P^n$. The sheaf of Kähler differentials $\Omega^1_{H}((P_2^N) \otimes_{O_{P^n}} \mathcal{O}_{P^n})$ is canonically isomorphic to $\pi^*\Omega^1_{H}(\mathcal{O}_{P^n}) \otimes V$. The universal derivative map: $d : I_{X_2}/I_{X_2}^2 \to \Omega^1_{H}(P_2^N) \otimes O_{X_2}$ is an $O_{X_2}$ linear map. After tensoring by $O_X$, we obtain an $O_X$-linear map:
\[ d_{X_2} : I_X/I_X^2 \to \Omega^1_{H}(P^n)|_X \otimes V|_X \]

$X_\infty$ is smooth at $x$ if $d_{X_2}(x)$ is injective. We now associate to any $\phi : I_X/I_X^2 \to V_X$ a candidate space for $X_2$, $X_\phi$, in such a way that $X_\phi$ will be smooth if $\phi(x)$ is injective for all $x$. $X_\phi \subset P_2^N$ is defined by the formula:
\[ f \in I_{X_\phi} \iff f|_{P^n} \in I_X \text{ and } (f - \pi^*(f|_{P^n}()))|_X = \phi(f) \]

For this $X_\phi$, we have that
\[ d_{X_\phi}(f) = \pi^*df + \phi(f), \]
whence $d_{X_\phi}(x)$ is injective for all $x$ if $\phi(x)$ is injective for all $x$. It remains only to find $\phi \in Hom(I_X/I_X^2, V)$ such that $\phi(x)$ is injective for every $x$. Now, $Hom(I_X/I_X^2, O_X(m))$ is globally generated. If $v_1 \ldots v_l$ is a basis for $H^0(Hom(I_X/I_X^2, O_X(m)))$, then
\[ \phi = \oplus_{b=1}^l v_b : I_X/I_X^2 \to V \]

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is injective on $I_X/I_X^2(x)$ for every $x$. The theorem follows.

**Remark.** One might wonder whether this work holds for quasi-projective $X$. We have used the fact that $X$ is projective only to invoke Serre’s vanishing theorem. Suppose that $X$ is a quasi projective variety- that is, $X$ is the complement in an projective variety, $V$, of an closed subvariety of $V$, $U$. The results of this paper will hold for $X$, provided that:

(A) For any coherent sheaf, $\mathcal{F}$ on $X$, there exists $k$ such that for any $m \geq k$ $H^1(\mathcal{F}(m)) = 0$ and $\mathcal{F}(m)$ is generated by its global sections.

If $U$ is of pure codimension 1, then $X$ is affine, and the condition (A) is immediately satisfied. Further, there is an exact sequence:

$$H^1(V, \mathcal{F}(m)) \to H^1(X, \mathcal{F}(m)) \to H^2_U(V, \mathcal{F}(m))$$

where $H^2_U(V, \mathcal{F}(m))$, the local cohomology of $F$ along $U$, vanishes if every component of $U$ has codimension $\geq 3$ (see [2] for more details.) It follows that the results of this note are, in fact, also valid for $X = V/U$, where $V$ is projective and $U$ is a closed subvariety containing no codimension 2 component. I would like to express my gratitude to my supervisor, Claire Voisin, for her unstinting help and guidance. I would also like to thank Yves Lazlo, who was always ready to answer my questions, and the referee, whose generous comments and help have much improved this work.

**References**
