BUILDING INFINITESIMAL NEIGHBOURHOODS OF VARIETIES.

by Catriona Maclean

ABSTRACT. We develop a deformation-type tool for the study of embeddings of a singular variety X. Given a variety $X \subset Y$ there is a natural series of schemes $X = X_0 \subset X_1 \subset X_2 \ldots \subset Y$ of *infinitesimal neighbourhoods of* X defined as follows

$$X_n = \text{zero}(I^n_{X/Y}).$$

Under certain assumptions, we calculate the obstructions to the existence of infinitesimal neighbourhoods and classify them when they exist.

1. INTRODUCTION.

1.1 A MOTIVATING EXAMPLE.

Consider the following question. Let X be a complex normal crossing variety: for simplicity's sake we consider the case where X is the union of two smooth irreducible varieties X_1 and X_2 glued together along divisors $D_1 \subset X_1$ and $D_2 \subset X_2$ via an isomorphism $\phi : D_1 \to D_2$. The common image of D_1 and D_2 in X is denoted D.

It is natural to ask whether X can be embedded in a smooth variety Y as a normal crossing divisor. The following example (based on Friedman's paper [1]) shows that the answer is "no" in general.

Suppose that $X \subset Y$ is an inclusion of X as a normal crossing divisor in a smooth variety. Then there is an exact sequence

$$0 \to N^*_{X|Y} \to \Omega^1_Y \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

where $N^*_{X/Y}$ is the conormal bundle of X in Y, $N^*_{X/Y} = I_{X/Y}/I^2_{X/Y}$. We note that $I_{X_1|Y} \otimes \mathcal{O}_{X_1} = N^*_{X_1/Y}$ and hence

$$N^*_{X_1|Y} \otimes \mathcal{O}_D = I_{X_1|Y} \otimes \mathcal{O}_{X_1} \otimes \mathcal{O}_D = I_{X_1|Y} \otimes \mathcal{O}_D = I_{D|X_2} \otimes \mathcal{O}_D = N^*_{D|X_2}.$$

Likewise, $N^*_{X_2|Y} \otimes \mathcal{O}_D = N^*_{D|X_1}$, but since X is a normal crossing divisor in Y, $N^*_{X|Y}|_D = N^*_{X_1|Y}|_D \otimes N^*_{X_2|Y}|_D$ so $N^*_{X|Y}|_D = N^*_{D|X_1} \otimes N^*_{D|X_2}$. The point is that the right-hand side of this equation is the restriction to D of a line bundle defined on X, whereas the left-hand side does not depend on Y. We therefore have the following result.

PROPOSITION 1.1. *Let $X = X_1 \cup_D X_2$ be a normal crossing variety as above. If X can be embedded in a smooth variety Y as a normal crossing divisor then the line bundle $N^*_{D|X_1} \otimes N^*_{D|X_2}$ can be extended to a line bundle on X.*

Consider a pair (X, D), where D is a smooth divisor in X such that the restriction map $\text{Pic}(X) \to \text{Pic}(D)$ is not surjective. (This typically holds if X is a surface and D is a smooth curve of genus > 0.) Choose a line bundle L on D which is not the restriction of a line bundle on X and set $X_1 = \text{Proj}(L \oplus \mathcal{O}_D)$, $X_2 = X$, $D_1 = \text{Proj}(L)$ and $D_2 = D$.

We then have that $N^*_{D|X_1} = L$, $N^*_{D|X_2} = \mathcal{O}(-D)$ and by definition, $N^*_{D|X_1} \otimes N^*_{D|X_2}$ does not extend to a line bundle on X_2. It follows that X cannot be embedded in any smooth variety as a normal crossing divisor.

The obstruction given above to X being a normal crossing divisor is in fact *infinitesimal*: in other words, it is an obstruction not only to the existence of Y but also to the existence of the scheme $X_\epsilon = \text{zero}(I^2_{X|Y})$.

Indeed, suppose given a scheme supported on X, X_ϵ, such that
1. $I_{X|X_\epsilon}$ is a line bundle, L on X. (In particular, $I^2_{X|X_\epsilon} = 0$.)
2. The sheaf $\Omega^1_{X_\epsilon} \otimes \mathcal{O}_X$ is a locally free sheaf on X. (This condition means that X_ϵ is "potentially" the first infinitesimal neighbourhood of X in a *smooth* variety Y).

It then turns out that, as above, we can build an exact sequence

$$0 \to I_{X|X_\epsilon} \to \Omega^1_{X_\epsilon} \otimes \mathcal{O}_X \to \Omega^1_X \to 0.$$

Since $\Omega^1_{X_\epsilon} \otimes \mathcal{O}_X$ is assumed to be locally free, there is a surjective map $I^*_{X|X_\epsilon} \to \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)$. The sheaf $\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)$ is a line bundle on D which is proved in [1] (page 85) to be given by $N_{D|X_1} \otimes N_{D|X_2}$. The left-hand side is a line bundle on X: it follows that $I^*_{X|X_\epsilon}|_D \cong N^*_{D|X_1} \otimes N^*_{D|X_2}$ and hence $N^*_{D|X_1} \otimes N^*_{D|X_2}$ extends to a line bundle on X.

1.2 Definitions and statement of theorems.

Hopefully, the above example has convinced the reader that infinitesimal considerations can produce interesting information about embeddings of X and a systematic study of infinitesimal obstructions to the existence of embeddings can be useful.

Throughout what follows we work over k, an algebraically closed field of characteristic zero. Given an inclusion of k-schemes $X \subset Y$ there is an associated sequence $X = X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$ of nilpotent schemes supported on X given by $X_i = \text{zero}(I^{i+1}_{X|Y})$. Our aim is to classify these infinitesimal models under certain assumptions. We start by defining infinitesimal neighbourhoods, which should be thought of as nilpotent schemes whose underlying base scheme is X and which are potentially schemes of the form zero $(I^{n+1}_{X|Y})$ for "good" X and Y.

Definition 1.2. Let X be a reduced locally complete intersection k-variety, where k is an algebraically closed field. Let V be a vector bundle on X. An n-th order infinitesimal neighbourhood of X with normal bundle V, $\mathcal{X}_n$, is the data of a triple $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ such that
1. X_n is a k-scheme of finite type,
2. The map $i_{\mathcal{X}_n} : X \to X_n$ is an inclusion,
3. The ideal sheaf $I^{n+1}_{X|X_n} = 0$,
4. The map $\alpha_{\mathcal{X}_n} : V^* \to I_{X|X_n}/I^2_{X|X_n}$ is an isomorphism of $\mathcal{O}_X$-modules,
5. The multiplication map $\text{Sym}^n(\alpha_{\mathcal{X}_n}) : \text{Sym}^n(V^*) \to I^n_{X|X_n}$ is an isomorphism.

The bundle V^* is called the *conormal* bundle of the infinitesimal neighbourhood $\mathcal{X}_n$.

Remark 1.3. If $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ is a n-th order infinitesimal neighbourhood of X with normal bundle V then for any $1 \le i \le n$ there is an i-th order infinitesima l neighbourhood of X with normal bundle V, $\mathcal{X}_i = (X_i, i_{\mathcal{X}_i}, \alpha_{\mathcal{X}_i})$ which is defined as follows:

$$X_i = \text{zero } (I^{i+1}_{X|X_n}); \quad i_{\mathcal{X}_i} = i_{\mathcal{X}_n}|_{\mathcal{X}_i}; \quad \alpha_{\mathcal{X}_i} = \alpha_{\mathcal{X}_n}.$$

The infinitesimal neighbourhood $\mathcal{X}_i$ is called the i-th order truncation of $\mathcal{X}_n$.

We fix an open set $U \subset X$. We will need to know what we mean by the restictions of an n-th order infinitesimal neighbourhood of X to U. Note that if $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ then the nilpotent scheme X_n can be thought of as a sheaf of algebras on X. In particular, the restriction of X_n to U is well-defined. We denote the restriction of the nilpotent scheme X_n (resp. X^1_{n+1}, X^2_{n+1}) to U by U_n (resp U^1_{n+1}, U^2_{n+1}).

DEFINITION 1.4. Let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ be an n-th order infinitesimal neighbourhood of X with normal bundle V and let U be an open set in X. The restriction of $\mathcal{X}_n$ to U, denoted $\mathcal{U}_n$, is the n-th order infinitesimal neighbourhood of U with normal bundle $V|_U$ given by $\mathcal{U}_n = (U_n, i_{\mathcal{X}_n}|_U, \alpha_{\mathcal{X}_n}|_U)$.

DEFINITION 1.5. An n-th order infinitesimal neighbourhood $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ is said to be *potentially smooth* if the sheaf $\Omega^1_{X_n} \otimes \mathcal{O}_X$ is a locally free sheaf on X.

Consider an inclusion of schemes $i : X \to Y$ such that the conormal sheaf $V^* \stackrel{\text{def}}{=} I_{X|Y}/I^2_{X|Y}$ is a vector bundle on X. We can then form a series of triples

$$\mathcal{X}^Y_n = (X^Y_n, i_{\mathcal{X}^Y_n}, \alpha_{\mathcal{X}^Y_n})$$

given by

$$X^Y_n = \text{zero}(I^{n+1}_{X|Y}), \quad i_{\mathcal{X}^Y_n} = i|_{X^Y_n}, \quad \alpha_{\mathcal{X}^Y_n} = \text{id} : I_{X/X^Y_n}/I^2_{X/X^Y_n} \to I_{X/X^Y_n}/I^2_{X/X^Y_n} = I_{X/Y}/I^2_{X/Y} = V^*.$$

The above definitions are motivated by the following lemma.

LEMMA 1.6. *Let $i_Y : X \hookrightarrow Y$ be a closed embedding of X, a reduced l.c.i. k-variety. The space Y is then smooth in an open neighbourhood of X if and only if the triple $\mathcal{X}^Y_n$ is a potentially smooth infinitesimal neighbourhood of X for all $n \in \mathbf{N}$.*

Proof of Lemma 1.6.

We start by proving that if the triple $\mathcal{X}^Y_n$ is a potentially smooth infinitesimal neighbourhood for all $n \in \mathbf{N}$ then Y is smooth along X.

Let $x \in X$ be a point. The scheme Y is smooth at the point x if and only if $\Omega^1_Y \otimes k_x$ is a k_x-vector space of rank dimY (see [4] Theorem 8.15). We note that $\Omega^1_Y \otimes k_x = \Omega^1_{X^Y_1} \otimes k_x$. By assumption, $\Omega^1_{X^Y_1} \otimes \mathcal{O}_X$ is a vector bundle on X: it remains only to show that this vector bundle is of rank $\dim(Y)$. It will be enough to prove that $\Omega_{X^Y_1} \otimes k_x$ is a vector space of dimension dimY for any closed smooth point $x \in X$.

Let $f_1, \ldots, f_r$ be elements of the local ring $\mathcal{O}_Y(x)$ such that $df_1, \ldots, df_r$ form a basis of $\Omega^1_Y \otimes k_x$, $df_1, \ldots, df_m$ form a basis of $\Omega^1_X \otimes k_x$ and $f_{m+1}, \ldots, f_r$ are elements in $I_{X|Y}$. In particular, the classes $\overline{f}_{m+1}, \ldots, \overline{f_r}$ are then independent elements of the vector space $I_{X/Y}/I^2_{X/Y} \otimes k_x$. We wish to show that $r \leq \dim(Y)$ (we know that it is $\geq \dim(Y)$): to do this it will be enough to show that the elements $f_1, \ldots, f_r$ are algebraically independent over k. Suppose not: there is then an algebraic equation of the form $P(f_1, \ldots, f_r) = 0$, where P is a polynomial. We write

$$P(f_1, \ldots, f_r) = P_1(f_1, \ldots, f_r) + \ldots + P_D(f_1, \ldots, f_r)$$

where P_i is of total degree i with respect to the functions $f_{m+1}, \ldots, f_r$. (We note that there is no term P_0 because the functions $f_1|_X, \ldots, f_m|_X$ are algebraically independent). There is at least one $i \geq 1$ such that $P_i \neq 0$. Let d be the smallest non-zero integer such that $P_d \neq 0$: in a suitable Zariski neighbourhood of x we have that $0 = \overline{P_d(f_1, \ldots, f_m, f_{m+1}, \ldots f_r)} \in I^d_{X|Y}/I^{d+1}_{X|Y}$. Since X^Y_d is supposed to be an infinitesimal neighbourhood $P_d(f_1, \ldots, f_m, f_{m+1}, \ldots f_r) = 0$ is therefore identically 0 as a polynomial in variables $f_{m+1}, \ldots, f_r$ with coordinates in the local ring $\mathcal{O}_X(x)$. But now as $f_1|_X, \ldots, f_m|_X$ are algebraically independent this implies that $P_d = 0$, which is a contradiction.

Let us now prove that if Y is smooth then X^Y_n is an infinitesimal neighbourhood for all $n \in \mathbf{N}$. (It is then immediate that it is potentially smooth.) The only thing we have to prove is that if Y is smooth then the multiplication map

$$\mu_n : \text{Sym}^n(I_{X|Y}/I^2_{X|Y}) \to I^n_{X|Y}/I^{n+1}_{X|Y}$$

is an isomorphism. By Matsumura [9] p.121 we know that Y is Cohen Macaulay so by [9] p. 110 for any ideal I generated by a regular sequence $(f_1, \ldots, f_m)$ we know that $\mathrm{Sym}^n : I/I^2 \to I^n/I^{n+1}$ is an isomorphism. But now since X is a locally complete intersection $I_{X/Y}$ is generated by a regular series in any sufficiently small open set in X. This completes the proof of Lemma 1.6. □

We will also need in what follows to have a definition of an isomorphism of infinitesimal deformations.

DEFINITION 1.7. Let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ and $\mathcal{X}'_n = (X'_n, i_{\mathcal{X}'_n}, \alpha_{\mathcal{X}'_n})$ be two n-th order infinitesimal neighbourhoods of X with normal bundle V. An isomorphism of infinitesimal neighbourhoods between $\mathcal{X}_n$ and $\mathcal{X}'_n$ is an isomorphism of schemes $j_n : X_n \to X'_n$ such that $j_n \circ i_{\mathcal{X}_n} = i_{\mathcal{X}'_n}$ and $\alpha_{\mathcal{X}_n} \circ j_n^* = \alpha_{\mathcal{X}'_n}$.

Here, $j_n^* : I_{X|X'_n} \to I_{X|X_n}$ is the pull-back map. Note that if $j_n : X_n \to X'_n$ is an isomorphism of infinitesimal deformations of order n with normal bundle V then for any $1 \leq i \leq n$ the truncated morphism

$$j_i = j|_{X_i} : X_i \to X'_i$$

is an isomorphism of infinitesimal neighboourhoods of order i with normal bundle V between $\mathcal{X}_i$ and $\mathcal{X}'_i$.

We define an extension of an n-th order infinitesimal neighbourhood as follows.

DEFINITION 1.8. Let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. An extension of $\mathcal{X}_n$ is given by a pair $(\mathcal{X}'_{n+1}, j_n)$ where

1. $\mathcal{X}'_{n+1}$ is an $(n+1)$st order infinitesimal neighbourhood of X with normal bundle V and
2. $j_n : \mathcal{X}'_n \to \mathcal{X}_n$ is an isomorphism between $\mathcal{X}'_n$, the n-th order truncation of $\mathcal{X}'_{n+1}$, and $\mathcal{X}_n$.

By abuse of notation, if there is no risk of confusion we often denote the extension $(\mathcal{X}'_{n+1}, j_n)$ by $\mathcal{X}'_{n+1}$. We will also need to know what we mean by an isomorphism of extensions.

DEFINITION 1.9. Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V and let $(\mathcal{X}^1_{n+1}, j^1_n)$, $(\mathcal{X}^2_{n+1}, j^2_n)$ be two different extensions of $\mathcal{X}_n$. An isomorphism between the extensions $(\mathcal{X}^1_{n+1}, j^1_n)$ and $(\mathcal{X}^2_{n+1}, j^2_n)$ is an isomorphism of $(n+1)$-th order infinitesimal neighbourhoods with normal bundle V

$$J_{n+1} : \mathcal{X}^1_{n+1} \to \mathcal{X}^2_{n+1}$$

such that $j^2_n \circ J_n = j^1_n$

The aim of this article is to prove the following two theorems.

THEOREM 1.10. *Let n be an integer ≥ 1 and let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. Suppose that the set of extensions of $\mathcal{X}_n$ to $(n+1)$-st order is not empty. To any pair of extensions of $\mathcal{X}_n$, $(\mathcal{X}^1_{n+1}, j^1)$ and $(\mathcal{X}^2_{n+1}, j^2)$ we can then associate a difference*

$$\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_n} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*))$$

in such a way that

1. $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) = 0$ *if and only if* $(\mathcal{X}^1_{n+1}, j^1_{n+1})$ *and* $(\mathcal{X}^2_{n+1}, j^2_{n+1})$ *are isomorphic as extensions of* $\mathcal{X}_n$,
2. $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1}) = \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^3_{n+1})$ *for any triple of extensions* $(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1})$,
3. *Given any extension* $\mathcal{X}^1_{n+1}$ *and any element* $\omega \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_n} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*))$ *there is an extension* $\mathcal{X}^2_{n+1}$ *such that*

$$\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) = \omega.$$

THEOREM 1.11. *Given an n-th order infinitesimal neighbourhood $\mathcal{X}_n$ with normal bundle V we can assign to it an element*

$$\mathrm{ob}_{\mathcal{X}_n} \in \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_{X_n} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*))$$

such that $\mathcal{X}_n$ has an $(n+1)$st order extension if and only if $\mathrm{ob}_{\mathcal{X}_n} = 0$.

There are obvious analogies with the standard theorems of deformation theory (see [6], [12] and [11] for more details). The related problem of constructing infinitesimal neighbourhoods seems to have received relatively little attention: Illusie's book [6] proves a classification/obstruction result for extensions of sheaves of algebras by fixed sheaves (théorème fondamental, page 162) but, surprisingly, does not in general deal with the question of whether the algebras produced are infinitesimal neighbourhoods in the above sense. (See page 191 of [6] for a related result, which works for much more general choices of X and X_n, but which requires the presence of a base scheme Y_{n+1}. The aim of the present article is to deal with the many cases where no such base scheme can exist.)

REMARK 1.12. The sheaf $\Omega^1_{X_n} \otimes \mathcal{O}_X$ which appears in the above statements is isomorphic to $\Omega^1_{X_1} \otimes \mathcal{O}_X$. In particular it only depends on the first infinitesimal neighbourhood of X.

REMARK 1.13. If X_n is potentially smooth then $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_n} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*))$ is isomorphic to $H^1(\mathcal{H}om(\Omega^1_{X_n} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*)))$. Any two extensions are then locally isomorphic: this contrasts with the deformation theory of singular varieties.

We have the following immediate corollary of 1.10.

COROLLARY 1.14. *Let X be an l.c.i. reduced k-variety and let V be a vector bundle on X. Let $\mathcal{X}_1$ be a first-order infinitesimal neighbourhood of X with normal bundle V and suppose there is a number k such that $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X_1} \otimes \mathcal{O}_X, \mathrm{Sym}^{n+1}(V^*))$ vanishes for any $n \geq k+1$. Suppose given $\mathcal{X}^1_n$ and $\mathcal{X}^2_n$, two n-th order infinitesimal neighbourhoods of X, and isomorphisms $j^1_1 : \mathcal{X}^1_1 \to \mathcal{X}_1$ and $j^2_1 : \mathcal{X}^2_1 \to \mathcal{X}_1$. Suppose we also have an isomorphism $J_k : \mathcal{X}^1_k \to \mathcal{X}^2_k$ such that $J_1 = (j^2_1)^{-1} \circ j^1_1$. There is then an isomorphism of infinitesimal neighbourhoods with normal bundle V, $J_n : \mathcal{X}^1_n \to \mathcal{X}^2_n$, such that the k-th truncation of J_n is J_k.*

In the case where $k = \mathbf{C}$, V is a weakly negative vector bundle and X is smooth the above corollary can be seen as a weaker version of Grauert's theorem in [2].

THEOREM 1.15 (Grauert). *Let X and $\tilde{X}$ be two smooth complex varieties, and let A (respectively $\tilde{A}$) be a smooth codimension 1 subvariety of X (respectively $\tilde{X}$). Assume that the normal bundle of A in X is weakly negative. Then there is an integer ν_0 such that for any $\nu \geq \nu_0$ any isomorphism $A_\nu \cong \tilde{A}_\nu$ extends to an isomorphism of the ringed spaces $A^* = \mathcal{O}_X|_A$ and $\tilde{A}^* = \mathcal{O}_{\tilde{X}}|_{\tilde{A}}$.*

In [5], Hironaka and Rossi proved the following generalisation of the above result.

THEOREM 1.16 (Hironaka/Rossi). *Let A (resp. $\tilde{A}$) be a compact reduced complex subspace of a reduced complex space X (resp. $\tilde{X}$), such that $X - A$ (resp. $\tilde{X} - \tilde{A}$) is smooth. Assume that A is exceptional (i.e. it can be blown down to a point). Then there is an integer ν_0 such that for any $\nu \geq \nu_0$ any isomorphism $A_\nu \cong \tilde{A}_\nu$ extends to an isomorphism of the ringed spaces $A^* = \mathcal{O}_X|_A$ and $\tilde{A}^* = \mathcal{O}_{\tilde{X}}|_{\tilde{A}}$.*

REMARK 1.17. When $n = 0$ condition 5 of definition 1.2 is empty and the first order infinitesimal neighbourhoods of X with normal bundle V are simply algebra extensions of $\mathcal{O}_X$ by V^*. These have been classified by Illusie in [6] (page 162, théorème fondamental 1.2): there are no obstructions to the existence of such algebra extensions and they are classified by $\mathrm{Ext}^1(\Omega^1_X, V^*)$.

Our proof uses embeddings of infinitesimal neighbourhoods and their morphisms, which are defined below.

DEFINITION 1.18. Let X be a reduced locally complete intersection k-variety and let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. Let P be a smooth scheme and let W be a vector bundle on P. An embedding of $\mathcal{X}_n$ over P with normal bundle W is given by a pair $(\mathcal{P}_n, f_n)$ where

1. $\mathcal{P}_n$ is an n-th order infinitesimal neighbourhood of P with normal bundle W,

2. $f_n : X_n \to P_n$ is a closed inclusion of schemes such that $f_n^*(I_P) = I_X$ and the pull-back map

$$f_n^* : W^* = I_{P|P_n}/I^2_{P|P_n} \to I_{X|X_n}/I^2_{X|X_n} = V^*$$

is surjective.

Note that in the above definition we do not ask that f_n^* should be an isomorphism. As previously, we will abusively write $\mathcal{P}_n$ for the embedding $(\mathcal{P}_n, f_n)$ when there is no risk of confusion.

DEFINITION 1.19. Let X be a reduced locally complete intersection k-variety and let $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. Let P and Q be smooth schemes and let W_P and W_Q be vector bundles on P and Q, respectively. Let $(\mathcal{P}_n, f_n)$ and $(\mathcal{Q}_n, g_n)$ be embeddings of $\mathcal{X}_n$ over P and Q respectively. A map of embeddings from $\mathcal{P}_n$ to $\mathcal{Q}_n$ is a map

$$h : \mathcal{P}_n \to \mathcal{Q}_n$$

such that $h \circ f_n = g_n$ and $h^*(I_{Q|Q_n}) = I_{P|P_n}$.

1.3 Overview of the proof and notation.

We start by indicating a relatively elementary proof of standard deformation-theoretic results, and which parts do and don't work in our set-up.

1.4 Review of basic deformation theory.

Given a reduced locally complete intersection k-variety X, there is a well-developped theory of deformations of X. By a deformation of X over a local Artinian ring A, we mean a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

where X_A is flat over A. Intuitively, we think of such schemes as being “fattenings” of the base scheme X.

The first systematic study of deformations of structures of manifolds was carried out in the complex analytical category by Kodaira and Spencer in 1958 in [7]. The first comprehensive study of deformations in the algebraic category was completed by Schlessinger in [10] in the late 1960s: an exposition of this work is also contained in Grothendieck's E.G.A [3].

In the particular case where X_0 is a local complete intersection the theorem on extensions of deformations of X can be stated as follows. Let A' and A be Artinian rings, and consider an exact sequence

$$0 \to \mathfrak{a} \to A' \to A \to 0$$

where $\mathfrak{a}$ is an ideal of A' such that $\mathfrak{m}_{A'}.\mathfrak{a} = 0$. If $X_A \to \mathrm{Spec}(A)$ is a deformation of X over A and $X_{A'} \to \mathrm{Spec}(A')$ is a deformation of X over A' then we say that $X_{A'}$ is an extension of X_A if there is an isomorphism $X_A \cong X_{A'} \otimes_{A'} \mathrm{Spec}(A)$. Likewise, we say that two deformations of X over A, X_A^1 and X_A^2 are isomorphic as deformations of A if and only if they are isomorphic as A-schemes. We then have the following two theorems, which can be found in [12] or more generally in [6]. (See also the recent book [11]).

THEOREM 1.20. *To any ordered pair $(X^1_{A'}, X^2_{A'})$ of extensions of X_A over $\mathrm{Spec}(A')$, we can assign a difference $\mathcal{D}(X^1_{A'}, X^2_{A'}) \in \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X \otimes \mathfrak{a})$ in such a way that the following hold.*

1. *We have that $\mathcal{D}(X^1_{A'}, X^2_{A'}) = 0$ if and only if $X^1_{A'}$ and $X^2_{A'}$ are isomorphic as extensions of X_A.*
2. *For any triple of extensions over A', $X^1_{A'}$, $X^2_{A'}$ and $X^3_{A'}$ we have that $\mathcal{D}(X^1_{A'}, X^2_{A'}) + \mathcal{D}(X^2_{A'}, X^3_{A'}) = \mathcal{D}(X^1_{A'}, X^3_{A'})$.*
3. *If an extension $X^1_{A'}$ exists then for any $E \in \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X \otimes \mathfrak{a})$ there is an extension $X^2_{A'}$ such that $\mathcal{D}(X^1_{A'}, X^2_{A'}) = E$.*

THEOREM 1.21. *We can associate to X_A, a deformation of X over* $\mathrm{Spec}(A)$, *an element* $\mathrm{ob}_{X_A} \in \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a})$, *such that extensions of X_A over* $\mathrm{Spec}(A')$ *exist if and only if* $\mathrm{ob}_{X_A} = 0$.

We start by summarising an (elementary) proof of Theorem 1.20 (drawn from [12]) and indicating what doesn't work in our context. Our aim is to associate to a pair of extensions $X^1_{A'}$ and $X^2_{A'}$ of X_A a "difference" $\mathcal{D}(X^1_{A'}, X^2_{A'})$ in $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a})$.

1. Prove a classification theorem for embedded A- deformations. Given an embedding $X \in P$, we consider A-flat subschemes
$$X_A \subset P \times \mathrm{Spec}(A) = P_A$$
extending X. Extensions of embedded deformations of X_A to $X_{A'}$ form a torsor over $H^0(\mathfrak{a} \otimes N_{X|P})$.
2. To an element $h \in H^0(\mathfrak{a} \otimes N_{X|P})$ we associate the push-forward along the dual map h^* of the exact sequence
$$0 \to N^*_{X|P} \to \Omega^1_P \otimes \mathcal{O}_X \to \Omega^1_X \to 0.$$
3. We can therefore construct local extensions encoding the local difference between embeddings of X^1 and X^2. Given two different embeddings of the pair $X^1_{A'}, X^2_{A'}$ as embedded deformations in two different ambient spaces P and Q we can define canonical isomorphisms between the associated extensions, based on the the product diagram
$$\begin{array}{ccc} & P_A \times_A Q_A & \\ \swarrow & & \searrow \\ P_A & & Q_A \end{array}$$

The point which does not work directly for infinitesimal neighbourhoods is 3). The problem is the following: the construction of the gluing isomorphisms uses the product space $P_A \times_A Q_A$ in which X_A *remains transverse to the central fibre.* For infinitesimal neighbourhoods, there is no base scheme $\mathrm{Spec}(A)$. We can still define embeddings of deformations, $X_A \to P_n$, but P_n is no longer the product of a smooth space and the spec of an Artinian ring, but a simple kind of formal scheme. In particular, given two such objects, P and Q there is no canonical way to take a product in which X will be tranverse to the central fibre. For this reason, we are obliged to consider embeddings of deformations into formal thickenings of a smooth variety P in which the scheme X_n may not be transverse to the smooth scheme P. This is the fundamental reason for most of the technical problems that we will meet and deal with in this article. Globally, the proof follows closely the ideas and methods of Vistoli's article [12].

NOTATION

Throughout thi article k will be an algebraically closed field of characteristic zero.

X denotes a reduced locally complete intersection k-variety.

$\mathcal{X}_n$ denotes an n-th order infinitesimal neighbourhood of X with normal bundle V.

To simplify the notation, we will denote by Ω the sheaf $\Omega^1_{X_n} \otimes \mathcal{O}_X$ and we will denote by S the sheaf $\mathrm{Sym}^{n+1}(V^*_X)$.

For any open set U of X and any sheaf A on X we will denote by A_U the restriction of A to U. (In particular, the sheaf Ω_U is therefore not equal to Ω^1_U.)

Whenever X is a subscheme of a scheme Y we will denote by $I_{X|Y}$ the ideal sheaf of X in Y.

ORGANISATION

The article is organised as follows. In section 2 below we prove some preliminary lemmas. In section 3 we will define extensions associated to pairs of embedded extensions and show how to glue them together in order to create an extension encoding the "difference" between them. In section 4 we complete the proof of Theorem 1.10 by proving that this mapping turns the set of extensions of infinitesimal neighbourhoods into a torsor over $\mathrm{Ext}^1(\Omega, S)$ and in section 5 we prove Theorem 1.11.

2. Preliminary lemmas.

In this section we will prove some preliminary lemmas which will be useful in the rest of the proof.

Let X be a reduced l.c.i. k-variety. Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V and let $(\mathcal{P}_n, f_n)$ be an embedding of $\mathcal{X}_n$ with normal bundle W. Recall that we then have a surjective map

$$f_n^* : W^* \to V^*.$$

Definition 2.1. Let $\mathcal{X}_n$ and $(\mathcal{P}_n, f_n)$ be as above. We then denote by $L_{\mathcal{P}_n}$ the kernel of the pullback map $f_n^* : W^*|_X \to V^*$. We denote by $\mathcal{L}_{\mathcal{P}_n}$ the sheaf-theoretic kernel of $f_n^* : W^* \to V^*$, considered as a maps of sheaves. We note that $L_{\mathcal{P}_n}$ is a vector bundle on X.

We need to understand the structure of the ideal sheaf $I_{X_n|P_n}$. Note that $I_{P_{n-1}|P_n}$ is an $\mathcal{O}_P$-module; in particular we have that

$$I_{X|P} \cdot I_{P_{n-1}|P_n} = I_{X_n|P_n} . I_{P_{n-1}|P_n} \subset I_{X_n|P_n} \cap I_{P_{n-1}|P_n}.$$

We have an isomorphism

$$\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X : \mathrm{Sym}^n W^*|_X \to \frac{I_{P_{n-1}|P_n}}{I_{X|P} \cdot I_{P_{n-1}|P_n}}$$

We consider $\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X^{-1}\left(\frac{I_{X_n|P_n} \cap I_{P_{n-1}|P_n}}{I_{X|P} \cdot I_{P_{n-1}|P_n}}\right)$, the inverse image under $\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X$ of the subsheaf $\frac{I_{X_n|P_n} \cap I_{P_{n-1}|P_n}}{I_{X|P} \cdot I_{P_{n-1}|P_n}} \subset \frac{I_{X_n|P_n} \cdot I_{P_{n-1}|P_n}}{I_{X|P} \cdot I_{P_{n-1}|P_n}}$. The following lemma identifies this subsheaf explicitly.

Lemma 2.2. *Let X, $\mathcal{X}_n$ and $\mathcal{P}_n$ be as above. We then have that*

$$\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X^{-1}\left(\frac{I_{X_n|P_n} \cap I_{P_{n-1}|P_n}}{I_{X|P} \cdot I_{P_{n-1}|P_n}}\right) = [L_{\mathcal{P}_n} \cdot \mathrm{Sym}^{n-1}(W^*)|_X).$$

Proof of Lemma 2.2.

We consider the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sym}^n(W^*)|_X & \xrightarrow{\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X} & I_{P_{n-1}|P_n} \otimes \mathcal{O}_X \\ \downarrow{\scriptstyle \mathrm{Sym}^n(f_n^*)} & & \downarrow{\scriptstyle f_n^*} \\ \mathrm{Sym}^n(V^*) & \xrightarrow{\mathrm{Sym}^n(\alpha_{\mathcal{X}_n})} & I_{X_{n-1}|X_n} \end{array}$$

where on the right-hand side f_n^* is the pullback map $f_n^* : I_{P_{n-1}|P_n} \otimes \mathcal{O}_X \to I_{X_{n-1}|X_n}$ and on the left hand side f_n^* is the pullback map $f_n^* : W^* \to V^*$. The two horizontal maps are isomorphisms. On the right-hand side of the equation, $\mathrm{Ker}(f_n^*) = \frac{I_{P_{n-1}|P_n} \cap I_{X_n|P_n}}{(I_X \cdot I_{P_{n-1}|P_n})}$. It follows that

$$\frac{I_{P_{n-1}|P_n} \cap I_{X_n|P_n}}{(I_X \cdot I_{P_{n-1}|P_n})} = \mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X(\mathrm{Ker}(\mathrm{Sym}^n(f_n^*))$$

and hence

$$(\mathrm{Sym}^n(\alpha_{\mathcal{P}_n})|_X)^{-1}\left(\frac{I_{P_{n-1}|P_n} \cap I_{X_n|P_n}}{(I_X \cdot I_{P_{n-1}|P_n})}\right) = \mathrm{Ker}(\mathrm{Sym}^n(f_n^*)) = L_{\mathcal{P}_n} \cdot \mathrm{Sym}^{n-1}(W^*)|_X.$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of a reduced l.c.i. k-variety X with normal bundle V. The sequence $0 \to V^* \xrightarrow{d} \Omega^1_{\mathcal{X}_n} \otimes \mathcal{O}_X \xrightarrow{i^*_{\mathcal{X}_n}} \Omega^1_X \otimes \mathcal{O}_X \to 0$ is then exact.*

Proof of Lemma 2.3.

By [4], II.8.12 it will be enough to show that $d : V^* \to \Omega^1_{X_n} \otimes \mathcal{O}_X$ is injective. As the question is local, it will be enough to prove Lemma 2.3 on any small enough open set in X. We choose an affine open set $U \subset X$ such that $V|_U$ is trivial and U can be embedded in an affine space k^n as the zero locus of a regular series of functions $f_1, \ldots, f_m$. (This is possible because X is l.c.i..) We set $U = \mathrm{Spec}(A)$ where $A = k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m \rangle$. Choose elements $\epsilon_1, \ldots, \epsilon_k \in I_{U|U_n}$ such that the elements $\bar{\epsilon}_1, \ldots, \bar{\epsilon}_k \in I_{U|U_n} \otimes \mathcal{O}_U = V^*_U$ form a basis of sections of V^*_U. We can then write U_n in the form $U_n = \mathrm{Spec}(B)$ where

$$B = (k[x_1, \ldots, x_n, \epsilon_1, \ldots, \epsilon_k]/\tilde{I} \oplus J)$$

where I is an ideal of the form $\tilde{f}_1, \ldots, \tilde{f}_m$, where $\tilde{f}_i|_{\mathbf{A}^n} = f_i$ and $J \subset \langle \epsilon_1, \ldots, \epsilon_k \rangle$. We set $\mathfrak{m} = \langle \epsilon_1 \ldots, \epsilon_k \rangle$, the ideal generated in B by the ϵ_is: we have that $A = B/\mathfrak{m}$. Since the $\bar{\epsilon}_i$s form a basis of sections of $I_{U|U_n} \otimes \mathcal{O}_U$, we have that $J \subset \tilde{I} \oplus \mathfrak{m}^2$. Now, $\Omega^1_{U_n} \otimes \mathcal{O}_U = \Omega^1_{U_1} \otimes \mathcal{O}_U$ and $U_1 = \mathrm{Spec}(k[x_1, \ldots, x_n, \epsilon_1, \ldots, \epsilon_n]/\tilde{I} \oplus \mathfrak{m}^2$. It follows that the $\mathcal{O}_U$-module $\Omega^1_{U_1} \otimes \mathcal{O}_U$ is the sheafification of the A-module

$$\left(\frac{Bdx_1 \oplus \ldots \oplus Bdx_n \oplus Bd\epsilon_1 \ldots \oplus Bd\epsilon_k}{\langle d(\tilde{I}) \oplus d(\mathfrak{m}^2) \rangle} \right) \otimes_B A.$$

We note that $d(\mathfrak{m}^2) \subset \mathfrak{m} \cdot (Bdx_1 \oplus \ldots \oplus Bdx_n \oplus Bd\epsilon_1 \ldots \oplus Bd\epsilon_k)$ so $\Omega^1_{U_1} \otimes \mathcal{O}_U$ is the sheafification of the following A-module

$$\frac{Adx_1 \oplus \ldots \oplus Adx_n \oplus Ad\epsilon_1 \ldots \oplus Ad\epsilon_k}{\langle d\tilde{f}_1 \otimes A, \ldots, d\tilde{f}_m \otimes A \rangle}.$$

V^* is the sheafification of the A module $Ad\epsilon_1 \oplus \ldots \oplus Ad\epsilon_k$ so it remains to show that the map

$$Ad\epsilon_1 \oplus \ldots \oplus Ad\epsilon_k \to \frac{Adx_1 \oplus \ldots \oplus Adx_n \oplus Ad\epsilon_1 \ldots \oplus Ad\epsilon_k}{\langle d\tilde{f}_1 \otimes A, \ldots, d\tilde{f}_m \otimes A \rangle}$$

is injective. In other words, we must show that $\langle d\tilde{f}_1 \otimes A, \ldots, d\tilde{f}_m \otimes A \rangle \cap Ad\epsilon_1 \oplus \ldots \oplus Ad\epsilon_k = \{0\}$. Suppose now that there are elements $a_i \in A$ such that $\sum_i a_i d\tilde{f}_i \otimes A \subset \oplus_j Ad\epsilon_j$. We then have that $\sum_i a_i df_i = 0$ in $\Omega^1_{k_n} \otimes A$. But now since the f_is are a regular series for $I_{U|k^n}$ the df_is are independent over A and it follows that $a_i = 0$ for all i. It follows that the above maps are injective and hence $V^*|_U \xrightarrow{d} \Omega^1_{U_n} \otimes \mathcal{O}_U$ is also injective. This completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X and let U be an open set in X. Let $\mathcal{P}_n$ be an embedding of $\mathcal{U}_n$. Consider the map*

$$\tau_{\mathcal{P}_n} : I_{P|P_1} \cap I_{U_1|P_1} \to I_{U_1|P_1} \otimes \mathcal{O}_U.$$

Then we have that $\mathrm{Ker}(\tau_{\mathcal{P}_n}) = I_{U|P_1} \cdot I_{P|P_1}$. In particular, $I_{P|P_1} \cap I_{U_1|P_1}/\mathrm{Ker}(\tau_{\mathcal{P}_n}) = L_{\mathcal{P}_n}$.

Proof of Lemma 2.4.

It is immediate that $I_{U|P_1} \cdot I_{P|P_1} \subset \mathrm{Ker}(\tau_{\mathcal{P}_n})$. To prove the converse, we use the fact that U is a local complete intersection in P. Locally, we can choose a regular sequence of m elements $f_1, \ldots, f_m \in I_{U|P}$ such that if F denotes the vector space $kf_1 \oplus kf_2 \ldots \oplus kf_m$ then $I_{U|P}/I^2_{U|P} = F \otimes_k \mathcal{O}_U$ and more generally $I^n_U/I^{n+1}_U = \mathrm{Sym}^n(F) \otimes_k \mathcal{O}_U$.

We fix liftings $\tilde{f}_i$ of f_i to $I_{U_1|P_1}$. Choose an element $v \in (I_{U_1|P_1} \cap I_{P|P_1})$ such that $\tau_{\mathcal{P}_n}(v) = 0$: our aim is to show that $v \in I_{U|P_1} \cdot I_{U_1|P_1}$. We know that v can be written in the form $v = \sum_i j_i g_i$ where $j_i \in I_{U|P_1}$ and $g_i \in I_{U_1|P_1}$.

LEMMA 2.5. *Let U, P, U_1, P_1, $f_1 \dots, f_m$ and $\tilde{f}_1 \dots \tilde{f}_m$ be as above and let $v \in (I_{U_1|P_1} \cap I_{P|P_1})$ be an element of* $\mathrm{Ker}(\tau_{\mathcal{P}_n})$. *There is then a $w \in I_{U|P_1} \cdot I_{U_1|P_1}$ such that $v - w = \sum_i j'_i g'_i$ where for all i we have that $j'_i, g'_i \in \langle \tilde{f}_1, \tilde{f}_2 \dots \tilde{f}_m \rangle$.*

Proof of Lemma 2.5.

We can write $v = \sum_i j_i g_i$. We can write $g_i = g'_i + \epsilon_i$ where $g'_i \in \langle \tilde{f}_1, \tilde{f}_2 \dots \tilde{f}_m \rangle$ and $\epsilon_i \in I_{P|P_1}$. On setting $w_1 = \sum_i j_i \epsilon_i \in I_{U|P_1} \cdot I_{U_1|P_1}$ we see that $v - w_1 = \sum_i g'_i j_i$. We can write $j_i = j'_i + \nu_i$ where $j'_i \in I_{U_1|P_1}$ and $\nu_i \in I_{P|P_1}$. It follows that if we set $w_2 = \sum_i g'_i \nu_i \in I_{U|P_1} \cdot I_{U_1|P_1}$ then we have that $v - w_1 - w_2 = \sum_i g'_i j'_i$. And now we set $j'_i = j''_i + \epsilon'_i$, where $j''_i \in \langle \tilde{f}_1, \tilde{f}_2 \dots \tilde{f}_m \rangle$ and $\epsilon'_i \in I_{P|P_1}$. On setting $w_3 = \sum_i \epsilon'_i g'_i$ we see that $v - w_1 - w_2 - w_3 = \sum_i g'_i j''_i$. On taking $w = w_1 + w_2 + w_3$, this completes the proof of Lemma 2.5. □

LEMMA 2.6. *Let U, P, U_1, P_1, $f_1, \dots, f_m$ and $\tilde{f}_1, \dots, \tilde{f}_m$ be as above. Let $v \in (I_{U_1|P_1} \cap I_{P|P_1})$ be an element of* $\mathrm{Ker}(\tau_{\mathcal{P}_n})$ *which can be written in the form $\tilde{v} = \sum_i j_i g_i$, where $g_i, j_i \in \langle \tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_m \rangle$ for all i. Then for all integers n we have that*

$$v \in (I_{P|P_1} \cdot I_{U|P}) \oplus \mathrm{Sym}^n(\langle \tilde{f}_1, \dots \tilde{f}_m \rangle).$$

Proof of Lemma 2.6.

We will prove the lemma by induction on n. The case $n = 2$ holds by definition. Assume that the induction hypothesis holds for $n - 1$. We then have $\tilde{v} = \epsilon + \sum_I \alpha_I \tilde{f}^I$ where $\epsilon \in I_{P|P_1} \cdot I_{U|P}$ and the sum is taken over all multi-indices I of degree $(n - 1)$. Since $f_1, \dots, f_m$ is a regular series for $I_{U|P}$, the map

$$\mathrm{Sym}^{n-1}(\langle f_1, \dots f_m \rangle) \otimes \mathcal{O}_U \to I^{n-1}_{U|P} / I^n_{U|P}$$

is an isomorphism. It follows that $\alpha_I|_U \in I_{U|P}$: in other words, we can write $v = \epsilon + \sum_I \delta_I \tilde{f}^I + \beta$ where $\beta \in \mathrm{Sym}^n(\langle f_1, \dots f_m \rangle)$ and $\delta_I \in I_{P|P_1}$. But we then have that $\epsilon + \sum_I \delta_I \tilde{f}^I \in I_{P|P_1} \cdot I_{U|P}$. This completes the proof of Lemma 2.6. □

We now show how Lemma 2.6 implies Lemma 2.4. It will be enough to show that for any point $x \in X$, the image of v in the local ring $\mathcal{O}_{P_1,x}$, which we denote by v_x, is contained in the localised ideal $(I_{P|P_1} \cdot I_{U|P})_x$.

We denote the ideal generated by $\tilde{f}_1 \dots \tilde{f}_m$ in $\mathcal{O}_{P_1,x}$ by $\mathfrak{a}$. We denote the localised ideal $(I_{P|P_1} \cap I_{U|P})_x$ by I_x. We note that by the Artin-Rees lemma the following sequence is exact

$$0 \to \widehat{I_x} \to \widehat{\mathcal{O}_{P_1,x}} \to \widehat{\mathcal{O}_{P_1,x}/I_x} \to 0$$

where $\hat{M}$ indicates completion of the module M with respect to $\mathfrak{a}$.

By Lemma 2.6, we know that the image of v_x in $\widehat{\mathcal{O}_{P_1,x}}$ is contained in $\widehat{I_x}$. In particular, we know that the image of v_x in $\widehat{\mathcal{O}_{P_1,x}/I_x}$ is 0. Krull's theorem says that there exists an element $f \in 1 + \mathfrak{a}$ such that $fv = 0$ in $\mathcal{O}_{P_1,x}/I_x$. But now the form of f implies that f is invertible since it cannot be an element of the maximal ideal of $\mathcal{O}_{X,x}$, so $v_x \in I_x$. This completes the proof of Lemma 2.4. □

LEMMA 2.7. *Consider the Artinian ring $A_n = k[\epsilon_1, \dots, \epsilon_m]/\mathfrak{m}^{n+1}$. Let $X_n \to \mathrm{Spec}(A_n)$ be an A_n-scheme whose central fibre $X = X_0$ is a reduced l.c.i. k-variety. Consider the triple $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$, where $i_{\mathcal{X}_n}$ is the inclusion of X in X_n and $\alpha_{\mathcal{X}_n}$ is the identification of $V = \mathcal{O}_X \epsilon_1 \oplus \dots \oplus \mathcal{O}_X \epsilon_n$ with $I_{X|X_n}/I^2_{X|X_n}$. Then the following are equivalent, a) $\mathcal{X}_n$ is an n-th order infinitesimal neighbourhood of X_0 and b) X_n is a flat A_n-scheme.*

Proof of Lemma 2.7.

We prove the lemma by induction on n. We may assume that X_n is affine, $X_n = \mathrm{Spec}(B_n)$, where B_n is an A_n-algebra. Throughout this section for any $1 \leq i \leq n$ we set $A_i = k[\epsilon_1, \dots, \epsilon_m]/\mathfrak{m}^{i+1}$, $B_i = B_n \otimes_{A_n} A_i$,

$X_i = \mathrm{Spec}(B_i)$.

In the case where $n = 1$ we have that $\mathfrak{m}^2 = 0$ so $\mathfrak{m}$ is simply a k-vector space and an ideal of A is the same thing as a k-subspace of $\mathfrak{m}$. Conditions 1), 2) and 3) of the definition of an n-th order infinitesimal neighbourhood are immediately satisfied, and conditions 4) and 5) are equivalent. We have that X_1 is flat if and only if for any subspace $\mathfrak{n} \subset \mathfrak{m}$ the map $\mathfrak{n} \otimes_{A_1} B_1 \to B_1$ is injective. Now, $\mathfrak{n} \otimes_{A_1} B_1 = \mathfrak{n} \otimes_k B_0$ and the above map is injective for any $\mathfrak{n}$ if and only if the map $\mathfrak{m} \otimes_k B_0 \to B_1$ is injective. But the sheafification of this morphism is the morphism $V \to \mathcal{O}_X$ and since the map $\mathfrak{m} \otimes_k B_0 \to I_{B_0}$ is surjective by definition, we see that X_1 is flat over A_1 if and only if $\alpha_{\mathcal{X}_n}$ is an isomorphism. This proves the lemma in the case where $n = 1$.

Now, let us consider the case where $n > 1$: we want to show that if one of the two conditions a) and b) hold then the other also holds. By the induction hypothesis we may assume that $\mathcal{X}_n$ has the property that $\mathcal{X}_{n-1}$ is flat over A_{n-1} and is an $n-1$-st order infinitesimal neighbourhood with normal bundle V. We note that $\mathcal{X}_n$ is then an infinitesimal neighbourhood if and only if the map $\mathrm{Sym}^n \alpha_{\mathcal{X}_n} : \mathfrak{m}^n \otimes_k \mathcal{O}_X \to \mathcal{O}_{X_n}$ is injective, or in other words if the map $\mathfrak{m}^n \otimes_k B_0 \to B_n$ is injective.

Now, let I be an ideal of A_n. Since B_{n-1} is flat over A_{n-1} we know that $0 \to B_n \otimes_{A_n} (I \cap \mathfrak{m}^n) \to B_n \otimes I \to B_n \otimes_{A_n} (I/I \cap \mathfrak{m}^n) \to 0$ is exact, since I, $I \cap \mathfrak{m}^n$ and $I/(I \cap \mathfrak{m}^n)$ are all A_{n-1}-modules. We know also that the map $B_{n-1} \otimes_{A_{n-1}} I/I \cap \mathfrak{m}^n \to B_{n-1}$ is injective, since B_{n-1} is A_{n-1}-flat. In particular, the map $B_n \otimes_{A_n} I \to B_n$ is injective for all ideals I if and only if the map $B_n \otimes_{A_n} (I \cap \mathfrak{m}^n) \to B_n$ is injective for all ideals I. In other words, B_n is flat over A_n if and only if for any subspace $\mathfrak{n} \subset \mathfrak{m}^n$ we have that $B_n \otimes_{A_n} \mathfrak{n} \to B_n$ is an injection into $\mathfrak{m}^n \cdot B_0$. Since we have that $B_n \otimes_{A_n} \mathfrak{n} = B_0 \otimes_k \mathfrak{n}$ this is the case if and only if $B_n \otimes_{A_n} \mathfrak{m}^n \to \mathfrak{m}^n \cdot B_n$ is an isomorphism. But this is the case precisely if $\mathrm{Sym}^n(\alpha_{\mathcal{X}_n})$ is an isomorphism, that is, if $\mathcal{X}_n$ is an infinitesimal neighbourhood. This completes the proof of Lemma 2.7. □

3. Construction of $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$.

Throughout this section, the following data will be fixed:

1. an n-th order infinitesimal neighbourhood of X, $\mathcal{X}_n = (X_n, i_{\mathcal{X}_n}, \alpha_{\mathcal{X}_n})$,
2. two extensions of $\mathcal{X}_n$, $(\mathcal{X}^1_{n+1}, j^1_{n+1})$ and $(\mathcal{X}^2_{n+1}, j^2_{n+1})$.

The aim of this section is to construct the extension $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$. We will now define categories $\mathcal{C}(U)$ and $\mathcal{E}(U)$ associated to the extensions $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$.

3.0.1 Definition of categories $\mathcal{C}(U)$ and $\mathcal{E}(U)$.

Definition 3.1. An element $\tilde{\mathcal{P}}_{n+1}$ of the category $\mathcal{C}(U)$ is a 4-tuple $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ where P is a smooth k-variety, $\mathcal{P}_{n+1} = (P_{n+1}, i_{\mathcal{P}_{n+1}}, \alpha_{\mathcal{P}_{n+1}})$ is an infinitesimal neighbourhood of order $(n+1)$ of P with normal bundle W and $f^1_{n+1} : \mathcal{U}^1_{n+1} \to \mathcal{P}_{n+1}$ and $f^2_{n+1} : \mathcal{U}^2_{n+1} \to \mathcal{P}_{n+1}$ are maps of schemes such that $(\mathcal{P}_{n+1}, f^i_{n+1})$ are embeddings of U^i_{n+1} and the truncated extension maps f^1_n and f^2_n have the property that

$$f^1_n \circ (j^1_n)^{-1} = f^2_n \circ (j^2_n)^{-1}.$$

(We recall that the maps j^1_n and j^2_n are the isomorphisms $j^i_n : U_n \to U^i_n$ from the n-th order truncation of U^i_{n+1} to U_n.)

If $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ is an element of $\mathcal{C}(U)$ then we will denote the map

$$f^1_n \circ (j^1_n)^{-1} = f^2_n \circ (j^2_n)^{-1} : U_n \to P_n$$

by f_n. The pair $(\mathcal{P}_n, f_n)$ is then an embedding of $\mathcal{U}_n$.

Definition 3.2. Let $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ and $\tilde{\mathcal{Q}}_{n+1} = (Q, \mathcal{Q}_{n+1}, g^1_{n+1}, g^2_{n+1})$ be two elements of $\mathcal{C}(U)$. A $\mathcal{C}(U)$-morphism from $\tilde{\mathcal{P}}_{n+1}$ to $\tilde{\mathcal{Q}}_{n+1}$ is a map of infinitesimal neighbourhoods $F : \mathcal{P}_{n+1} \to \mathcal{Q}_{n+1}$ such that $F \circ f^1_{n+1} = g^1_{n+1}$ and $F \circ f^2_{n+1} = g^2_{n+1}$.

(Recall that a map of infinitesimal neighbourhoods $F : \mathcal{P}_{n+1} \to \mathcal{Q}_{n+1}$ is a map of schemes $F : P_{n+1} \to Q_{n+1}$ such that $F^*(I_{Q|Q_{n+1}}) = I_{P|P_{n+1}}$.)

For any pair of open sets $V \subset U$ we define a restriction map $r_U^V : \mathcal{C}(U) \to \mathcal{C}(V)$.

DEFINITION 3.3. Let $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ be an element of $\mathcal{C}(U)$. We define $r_U^V(\tilde{\mathcal{P}}_{n+1}) \in \mathcal{C}(V)$ as follows:

$$r_U^V(\tilde{\mathcal{P}}_{n+1}) = (Z, \mathcal{Z}_{n+1}, f^1_{n+1}|_{V^1_{n+1}}, f^2_{n+1}|_{V^2_{n+1}})$$

where $Z = P \backslash \overline{(U \setminus V)}$.

We now define a category of extensions, $\mathcal{E}(U)$, in the following way.

DEFINITION 3.4. The members of $\mathcal{E}(U)$ are exact sequences of $\mathcal{O}_U$-modules.

$$0 \to S_U \xrightarrow{i_E} E \xrightarrow{\pi_E} \Omega_U \to 0.$$

Notation. Whenever dealing with an extension

$$0 \to F \to E \to G \to 0$$

we will denote the inclusion map $F \to E$ by i_E and the projection map $E \to G$ by π_E.

DEFINITION 3.5. Consider two elements of $\mathcal{E}(U)$,

$$0 \to S_U \xrightarrow{i_E} E \xrightarrow{\pi_E} \Omega_U \to 0,$$

$$0 \to S_U \xrightarrow{i_{E'}} E' \xrightarrow{\pi_{E'}} \Omega_U \to 0.$$

A $\mathcal{E}(U)$-morphism between E and E' is a map of $\mathcal{O}_U$-modules $f : E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_U & \xrightarrow{i_E} & E & & \\ & & & \searrow^{i_{E'}} & \downarrow f & \searrow^{\pi_E} & \\ & & & & E' & \xrightarrow{\pi_{E'}} & \Omega_U \longrightarrow 0 \end{array}.$$

Note that all maps are isomorphisms in this category. There is an obvious functor $r_U^V : \mathcal{E}(U) \to \mathcal{E}(V)$ given by restriction of extensions of $\mathcal{O}_U$-modules.

3.1 THE CONTRAVARIANT FUNCTOR $\mathcal{F}(U) : \mathcal{C}(U) \to \mathcal{E}(U)$: DEFINITION OF $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$.

We shall now construct a contravariant functor $\mathcal{F}(U) : \mathcal{C}(U) \to \mathcal{E}(U)$ which will be compatible with localisation (i.e. $r_U^V \circ \mathcal{F}(U) = \mathcal{F}(V) \circ r_U^V$.) This will be based on the conormal bundle which is constructed below.

3.1.1 CONSTRUCTION AND PROPERTIES OF THE CONORMAL BUNDLE $N^*_{\mathcal{P}_n}$.

DEFINITION 3.6. Let $(\mathcal{P}_n, f_n)$ be an embedding of an infinitesimal neighbourhood of U, $\mathcal{U}_n$, with normal bundle W. The conormal bundle of $(\mathcal{P}_n, f_n)$, $N^*_{\mathcal{P}_n}$, is defined by $N^*_{\mathcal{P}_n} = I_{U_n|P_n} \otimes \mathcal{O}_U$.

We will need a good understanding of $N^*_{\mathcal{P}_n}$ in what follows. We start with the following proposition.

LEMMA 3.7. *Let $(\mathcal{P}_n, f_n)$ be an embedding of an infinitesimal neighbourhood of U, $\mathcal{U}_n$, with normal bundle W. We then have that $N^*_{\mathcal{P}_n} = I_{U_1|P_1} \otimes \mathcal{O}_U$.*

Proof of Lemma 3.7.

We prove that for any $n \geq i \geq 1$ we have that $I_{U_i|P_i} \otimes_{\mathcal{O}_{P_n}} \mathcal{O}_U = I_{U_{i+1}|P_{i+1}} \otimes_{\mathcal{O}_{P_n}} \mathcal{O}_U$. We consider the surjective map $\pi_{i+1} : I_{U_{i+1}|P_{i+1}} \to I_{U_i|P_i}$. The kernel of π_{i+1} is $I_{U_{i+1}|P_{i+1}} \cap I_{P_i|P_{i+1}}$. It will be enough to show that the ideal sheaf $I_{U_{i+1}|P_{i+1}} \cap I_{P_i|P_{i+1}}$ is contained in $I_{U|P_{i+1}} \cdot I_{U_{i+1}|P_{i+1}}$. But now by Lemma 2.2 we know that

$$(I_{U_{i+1}|P_{i+1}} \cap I_{P_i|P_{i+1}})/(I_{P_i|P_{i+1}} \cdot I_{U|P_{n+1}}) = \mathrm{Sym}^{i+1}(\alpha_{\mathcal{P}_n})|_U(L_{\mathcal{P}_n} \cdot \mathrm{Sym}^i W^*|_U)$$

or in other words

$$\begin{aligned}(I_{U_{i+1}|P_{i+1}} \cap I_{P_i|P_{i+1}}) &= \mathrm{Sym}^{i+1}(\alpha_{\mathcal{P}_n})(\mathcal{L}_{\mathcal{P}_n} \cdot \mathrm{Sym}^i(W^*)) \\ &= \times(\mathcal{L}_{\mathcal{P}_n} \otimes I_{P_{i-1}|P_i})\end{aligned}$$

where here $\times : I_{P|P_1} \otimes_{\mathcal{O}_P} I_{P_{i-1}|P_i} \to I_{P_i|P_{i+1}}$ is the multiplication map. Let us consider $\times(a \otimes b)$ for some $a \in \mathcal{L}_{\mathcal{P}_n} = I_{U_1|P_1} \cap I_{P|P_1}$ and $b \in I_{P_{i-1}|P_i}$. Locally, there is an $a' \in I_{U_{i+1}|P_{i+1}} \cap I_{P|P_{i+1}}$ such that $a'|_{P_1} = a$ and a $b' \in I_{P_{i-1}|P_{i+1}}$ such that $b'|_{P_i} = b$. We have that $\times(a \otimes b) = a' \cdot b'$. Since $a' \in I_{U_{i+1}|P_{i+1}}$ and $b' \in I_{P_{i-1}|P_{i+1}} \subset I_{U|P_{i+1}}$ we have that $\times(a \otimes b) \in I_{U|P_{i+1}} \cdot I_{U_{i+1}|P_{i+1}}$. It follows that $(I_{U_{i+1}|P_{i+1}} \cap I_{P_i|P_{i+1}}) \subset I_{U|P_{i+1}} \cdot I_{U_{i+1}|P_{i+1}}$. This completes the proof of Lemma 3.7. □

We will now break $N^*_{\mathcal{P}_n}$ into $N^*_{U|P}$ and a part, $L_{\mathcal{P}_n}$, arising because the normal bundle of P in P_n may be larger than the normal bundle of U in U_n.

DEFINITION 3.8. Let $(\mathcal{P}_n, f_n)$ be an embedding of the infinitesimal neighbourhood $\mathcal{U}_n$. We define a map $r_{\mathcal{P}_n} : L_{\mathcal{P}_n} \to N^*_{\mathcal{P}_n}$ as follows. Let l be an element of $L_{\mathcal{P}_n}$: as by definition $L_{\mathcal{P}_n} = (I_{P|P_1} \cap I_{U_1|P_1})/(I_{P|P_1} \cdot I_{U|P})$ locally we can find an element $\widehat{l} \in I_{P|P_1} \cap I_{U_1|P_1}$ such that the class of $\widehat{l}$ in the quotient $L_{\mathcal{P}_n}$ is l. The element $r_{\mathcal{P}_n}(l)$ is then the class of $\widehat{l}$ in the quotient $I_{U_1|P_1} \otimes_{\mathcal{O}_{P_1}} \mathcal{O}_U$.

We note that $r_{\mathcal{P}_n}(l)$ is well-defined: if $\widehat{l}'$ is an alternative lifting of l then $(\widehat{l} - \widehat{l}') \in I_{P|P_1} \cdot I_{U_1|P_1} \subset I_{U|P_1} \cdot I_{U_1|P_1}$.

PROPOSITION 3.9. *Let $(\mathcal{P}_n, f_n)$ be an embedding of the infinitesimal neighbourhood $\mathcal{U}_n$. Let*

$$\pi_{\mathcal{P}_n} : N^*_{\mathcal{P}_n} = I_{U_n|P_n} \otimes \mathcal{O}_U \to N^*_{U|P} = I_{U|P} \otimes \mathcal{O}_U$$

be the map induced by the restriction map $i^*_{\mathcal{P}_n} : I_{U_n|P_n} \to I_{U|P}$. There is then an exact sequence

$$0 \to L_{\mathcal{P}_n} \stackrel{r_{\mathcal{P}_n}}{\to} N^*_{\mathcal{P}_n} \stackrel{\pi_{\mathcal{P}_n}}{\to} N^*_{U|P} \to 0.$$

where $r_{\mathcal{P}_n}$ is the map defined in Definition 3.8.

Proof of Proposition 3.9.

By Lemma 3.7, we know that $N^*_{\mathcal{P}_n} = I_{U_1|P_1} \otimes \mathcal{O}_U$. We consider the exact sequence of ideals

$$0 \to I_{P|P_1} \cap I_{U_1|P_1} \to I_{U_1|P_1} \stackrel{i^*_{\mathcal{P}_n}}{\to} I_{U|P} \to 0.$$

There is an induced exact sequence obtained by tensoring with the two right hand terms by $\mathcal{O}_U$

$$(I_{P|P_1} \cap I_{U_1|P_1}) \stackrel{\tau_{\mathcal{P}_n}}{\to} I_{U_1|P_1} \otimes \mathcal{O}_U \stackrel{\pi_{\mathcal{P}_n}}{\to} I_{U|P} \otimes \mathcal{O}_U \to 0,$$

from which it follows that the sequence

$$0 \to (I_{U_1|P_1} \cap I_{P|P_1})/\mathrm{Ker}(\tau_{\mathcal{P}_n}) \stackrel{\tau_{\mathcal{P}_n}}{\to} N^*_{\mathcal{P}_n} \stackrel{\pi_{\mathcal{P}_n}}{\to} N^*_{U|P} \to 0$$

is exact. But now by Lemma 2.4 we know that $\mathrm{Ker}(\tau_{\mathcal{P}_n}) = I_{U|P_1} \cdot I_{P|P_1}$ and $(I_{U_1|P_1} \cap I_{P|P_1})/\mathrm{Ker}(\tau_{\mathcal{P}_n}) = L_{\mathcal{P}_n}$. It follows from the definitions of $\tau_{\mathcal{P}_n}$ and $r_{\mathcal{P}_n}$ that $\tau_{\mathcal{P}_n} : L_{\mathcal{P}_n} \to N^*_{\mathcal{P}_n} = r_{\mathcal{P}_n}$. This completes the proof of Proposition 3.9. □

This summarises the results we need on the conormal bundle $N^*_{\mathcal{P}_n}$. Our next step will be to construct a map $f_{\tilde{\mathcal{P}}_{n+1}} : N^*_{\mathcal{P}_n} \to S_U$ associated to the data of $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$.

3.1.2 Construction of the map $f_{\tilde{\mathcal{P}}_{n+1}} : N^*_{\tilde{\mathcal{P}}_n} \to S_U$. Our aim is to define $f_{\tilde{\mathcal{P}}_{n+1}}(\beta)$ for any $\beta \in N^*_{\tilde{\mathcal{P}}_n} = I_{U_n|P_n} \otimes \mathcal{O}_U$. We will in fact define a map $\overline{f}_{\tilde{\mathcal{P}}_n} : I_{U_n|P_n} \to S_U$ which we will then tensorise with $\mathcal{O}_U$.

Definition 3.10. Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ be as above; let U be an open set in X and let $\tilde{\mathcal{P}}_{n+1}$ be an element of $\mathcal{C}(U)$. Let β_n be a section of the sheaf $I_{U_n|P_n}$ over an affine open subset of U_n. There are surjective maps of coherent sheaves $\pi^1 : I_{U^1_{n+1}|P_{n+1}} \to I_{U_n|P_n}$ and $\pi^2 : I_{U^2_{n+1}|P_{n+1}} \to I_{U_n|P_n}$ and we let $\beta^1_{n+1} \in I_{U^1_{n+1}|P_{n+1}}$ and $\beta^2_{n+1} \in I_{U^2_{n+1}|P_{n+1}}$ be sections such that $\pi^i(\beta^i_{n+1}) = \beta_n$: we then have that $(\beta^1_{n+1} - \beta^2_{n+1}) \in I_{P_n|P_{n+1}}$. We define a map of sheaves $\overline{f}_{\tilde{\mathcal{P}}_{n+1}} : I_{U_n|P_n} \to \mathrm{Sym}^{n+1}(V^*)|_U = S_U$ by

$$\overline{f}_{\tilde{\mathcal{P}}_n}(\beta_n) = \mathrm{Sym}^{n+1} f^*_n((\mathrm{Sym}^{n+1}\alpha_{\mathcal{P}_{n+1}})^{-1}(\beta^1_{n+1} - \beta^2_{n+1}))|_U$$

where f^*_n here denotes the pullback map $f^*_n : W^*|_U \to V^*_U*$.

Lemma 3.11. *The map $\overline{f}_{\tilde{\mathcal{P}}_{n+1}}$ is well-defined.*

Proof of Lemma 3.11.

We have to prove that the element $\mathrm{Sym}^{n+1} f^*_n((\mathrm{Sym}^{n+1}\alpha_{\mathcal{P}_{n+1}})^{-1}(\beta^1_{n+1} - \beta^2_{n+1}))|_U$ is independent of the choice of β^1_{n+1} and β^2_{n+1}. Let $\beta^{1'}_{n+1}$ and $\beta^{2'}_{n+1}$ be another possible choice. We set $(\beta^1_{n+1} - \beta^{1'}_{n+1}) = \delta^1$ and $(\beta^2_{n+1} - \beta^{2'}_{n+1}) = \delta^2$: we then have that $\delta^i \in I_{P_n|P_{n+1}} \cap I_{U^i_{n+1}|P_{n+1}}$. By Lemma 2.2, if $\overline{\delta^i}$ is the class of δ^i in the quotient $I_{P_n|P_{n+1}}/I_{U|P} \cdot I_{P_n|P_{n+1}}$ then we have that $\overline{\delta^i} \in \mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})|_U((L_{\mathcal{P}_n} \otimes \mathrm{Sym}^n(W^*))|_U$. It follows in particular that $\mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}(\delta^i))|_U = 0$. In particular,

$$\begin{aligned} &\mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}(\beta^{1'}_{n+1} - \beta^{2'}_{n+1}))|_U \\ &= \mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}(\beta^1_{n+1} + \delta^1 - \beta^2_{n+1} - \delta^2))|_U \\ &= \mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}(\beta^1_{n+1} - \beta^2_{n+1}))|_U. \end{aligned}$$

This completes the proof of Lemma 3.11. □

Definition 3.12. Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ be as above. Let U be an open set in X and let $\tilde{\mathcal{P}}_{n+1}$ be an element of $\mathcal{C}(U)$. As S_U is an $\mathcal{O}_U$-module, we can define $f_{\tilde{\mathcal{P}}_{n+1}}$ to be the unique map of $\mathcal{O}_U$-modules $f_{\tilde{\mathcal{P}}_{n+1}} : I_{U_n|P_n} \otimes \mathcal{O}_U = N^*_{\tilde{\mathcal{P}}_{n+1}} \longrightarrow S_U$ such that $\overline{f}_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = f_{\tilde{\mathcal{P}}_{n+1}}(\overline{\sigma})$ for any section $\sigma \in I_{U_n|P_n}$. Here, $\overline{\sigma}$ denotes the class of σ in the quotient sheaf $I_{U_n|P_n} \otimes \mathcal{O}_U$.

3.1.3 Construction of the extension $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$. We now show how to associate to the embedding $(\mathcal{P}_n, f_n)$ of $\mathcal{U}_n$ a canonical exact short sequence of $\mathcal{O}_U$-modules.

Definition 3.13. Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V, let U be an open subset of X and let $(\mathcal{P}_n, f_n)$ be an embedding of $\mathcal{U}_n$. We denote by $E_{\mathcal{P}_n}$ the following short exact sequence of $\mathcal{O}_U$-modules.

$$0 \to N^*_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}_n}} \Omega^1_{P_n} \otimes \mathcal{O}_U \xrightarrow{f^*_n} \Omega_U \to 0. \quad (1)$$

where f^*_n is simply pull-back along the map of schemes $f_n : U_n \to P_n$ and $d_{\mathcal{P}_n}$ is the map defined below. (Of course, we have not yet established that this sequence is exact.)

Definition 3.14. Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ be as above. Let U be an open set in X and let $d : I_{U_n|P_n} \to \Omega^1_{P_n}$ be the map of sheaves given by derivation. Tensoring on the right by $\mathcal{O}_U$ we obtain a map

$$d \otimes \mathcal{O}_U : I_{U_n|P_n} \to \Omega^1_{P_n} \otimes \mathcal{O}_U.$$

Unlike d, $d \otimes \mathcal{O}_U$ is an $\mathcal{O}_{P_n}$-module map, since for any $u \in I_{U_n}$ and any $f \in \mathcal{O}_{P_n}$ we have that $d \otimes \mathcal{O}_U(fu) = fdu + udf = fdu$ because $udf = 0$ in $\Omega^1_{P_n} \otimes \mathcal{O}_U$. As $\Omega^1_{P_n} \otimes \mathcal{O}_U$ is an $\mathcal{O}_U$-module there is a unique $\mathcal{O}_U$-module map

$$d_{\mathcal{P}_n} : I_{U_n|P_n} \otimes \mathcal{O}_U = N^*_{\mathcal{P}_n} \to \Omega^1_{P_n} \otimes \mathcal{O}_U$$

such that for any section σ of $I_{U_n|P_n}$ $d \otimes \mathcal{O}_U(\sigma) = d_{\mathcal{P}_n}(\overline{\sigma})$, where $\overline{\sigma}$ is the class of σ in the quotient sheaf $I_{U_n|P_n} \otimes \mathcal{O}_U$.

It remains to be seen that the sequence $E_{\mathcal{P}_n}$ is exact.

PROPOSITION 3.15. *Let $(\mathcal{P}_n, f_n)$ be an embedding with normal bundle W of an n-th order infinitesimal neighbourhood $\mathcal{U}_n$ of U with normal bundle V. The exact sequence $E_{\mathcal{P}_n}$ defined above is then exact.*

Proof of Proposition 3.15.

It is only necessary to prove that the map $d_{\mathcal{P}_n}$ is injective. We consider the following commutative diagram, whose middle row is simply $E_{\mathcal{P}_n}$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{\mathcal{P}_n} & \longrightarrow & W^*|_U & \xrightarrow{f_n^*} & V^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r_{\mathcal{P}_n}} & & \downarrow{\scriptstyle d} & & \downarrow & & \\
0 & \longrightarrow & N^*_{\mathcal{P}_n} & \xrightarrow{d_{\mathcal{P}_n}} & \Omega^1_{P_n} \otimes \mathcal{O}_U & \xrightarrow{f_n^*} & \Omega^1_{U_n} \otimes \mathcal{O}_U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^*_{\mathcal{P}_n}} & & \downarrow{\scriptstyle i^*_{\mathcal{P}_n}} & & \downarrow{\scriptstyle i^*_{\mathcal{U}_n}} & & \\
0 & \longrightarrow & N^*_{U|P} & \xrightarrow{d} & \Omega^1_P \otimes \mathcal{O}_U & \xrightarrow{f^*} & \Omega^1_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i^*_{\mathcal{U}_n}} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

In the above diagram, f is the restriction of the map $f_n : U_n \to P_n$ to U. Suppose that σ is a section of $N^*_{\mathcal{P}_n}$ such that $d_{\mathcal{P}_n}(\sigma) = 0$. We then have that $i^*_{\mathcal{P}_n} \circ d_{\mathcal{P}_n}(\sigma) = 0$ so $d \circ i^*_{\mathcal{P}_n}(\sigma) = 0$. As the bottom row of the diagram is exact because U is a local complete intersection we have that $i^*_{\mathcal{P}_n}(\sigma) = 0$. As the left-hand column is exact by Proposition 3.9 there is a section $\mu \in L_{\mathcal{P}_n}$ such that $\sigma = r_{\mathcal{P}_n}(\mu)$. We consider μ as an element of W^*, which is possible because $L_{\mathcal{P}_n}$ is defined as a sub-bundle of W. We have that $d(\mu) = 0$ and it follows from Lemma 2.3 that $\mu = 0$. This completes the proof of Proposition 3.15. □

We now define the extension $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$.

DEFINITION 3.16. Let $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ be an element of $\mathcal{C}(U)$ and let $(\mathcal{P}_n, f_n)$ be the associated embedding of U_n. The extension $\mathcal{F}(\mathcal{P})$ is defined to be the pushforward along $f_{\tilde{\mathcal{P}}_{n+1}}$ of the extension $E_{\mathcal{P}_n}$ defined above.

We recall the definition of the pushforward because it will be important in what follows.

DEFINITION 3.17. In any abelian category, let $0 \to F \overset{i_E}{\to} E \overset{\pi_E}{\to} G \to 0$ be an extension of G by F. Let $f : F \to F'$ be a morphism from F to F'. We then define the pushforward of E by f to be the following extension

$$0 \to F' \overset{i_{E'}}{\to} E' \overset{\pi_{E'}}{\to} G \to 0$$

where $E' = \frac{F' \oplus E}{(f(\sigma),0)=(0,i_E(\sigma))\forall\sigma\in F}$, $i_{E'}(\mu) = [(\mu, 0)]$ for any $\mu \in F'$ and $\pi_{E'}[(\mu,\nu)] = \pi_E(\nu)$ for any $(\mu,\nu) \in F' \oplus E$. If E' is the pushforward of an extension E under a morphism $f : F \to F'$ then for any $e \in E$ and $f' \in F$ we denote the class of (e, f') in the quotient E' by $[e, f']_{E'}$.

In the particular case above, this means that $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$ is the extension

$$0 \to S_U \overset{i_{\mathcal{F}(\tilde{\mathcal{P}}_{n+1})}}{\to} \frac{S_U \oplus (\Omega^1_{P_n} \otimes \mathcal{O}_U)}{(f_{\tilde{\mathcal{P}}_{n+1}}(\sigma), 0) = (0, d_{\mathcal{P}_n}(\sigma))\forall\sigma \in N^*_{\mathcal{P}_n}} \overset{\pi_{\mathcal{F}(\tilde{\mathcal{P}}_{n+1})}}{\to} \Omega^1_{U_n} \otimes \mathcal{O}_U \to 0$$

where $\pi_{\mathcal{F}(\tilde{\mathcal{P}}_{n+1})}([s,\omega]) = \omega|_{U_n}$ and $i_{\mathcal{F}(\tilde{\mathcal{P}}_{n+1})}(s) = [s,0]$.

Definition 3.18. Let $\tilde{\mathcal{P}}_{n+1}$ be an element of $\mathcal{C}(U)$. For any choice of $s \in S_U$ and $\omega \in \Omega^1_{P_n} \otimes \mathcal{O}_U$ we denote by $[s,\omega]_{\tilde{\mathcal{P}}_{n+1}}$ the class of (s,ω) in the quotient $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$. The inclusion $S_U \to \frac{S_U \oplus (\Omega^1_{P_n} \otimes \mathcal{O}_U)}{(f_{\tilde{\mathcal{P}}_{n+1}}(s,0)=(0,d_{\mathcal{P}_n}(\sigma))\forall\sigma\in N^*_{\mathcal{P}_n}}$ given by $s \to [s,0]_{\tilde{\mathcal{P}}_{n+1}}$ will be denoted by $i_{\tilde{\mathcal{P}}_{n+1}}$ and the projection $\frac{S_U \oplus (\Omega^1_{P_n} \otimes \mathcal{O}_U)}{(f_{\tilde{\mathcal{P}}_{n+1}}(\sigma),0)=(0,d_{\mathcal{P}_n}(\sigma))\forall\sigma\in N^*_{\mathcal{P}_n}} \to \Omega_U$ given by $[s,\omega]_{\tilde{\mathcal{P}}_{n+1}} \to \omega|_{U_n}$ will be denoted by $\pi_{\tilde{\mathcal{P}}_{n+1}}$.

3.2 Construction of the functor $\mathcal{F} : \mathcal{C}(U) \to \mathcal{E}(U)$: construction of $\mathcal{F}(F)$.

Suppose that we have elements of $\mathcal{C}(U)$, $\tilde{\mathcal{P}} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ and $\tilde{\mathcal{Q}} = (Q, \mathcal{Q}_{n+1}, g^1_{n+1}, g^2_{n+1})$. Consider a map $F : \mathcal{P} \to \mathcal{Q}$ which is a $\mathcal{C}(U)$-morphism. (Recall that F is just a map $F : P_{n+1} \to Q_{n+1}$ satisfying various compatibility conditions.) There is an induced commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^*_{\mathcal{Q}_n} & \xrightarrow{d_{\mathcal{Q}_n}} & \Omega^1_{Q_n} \otimes \mathcal{O}_U & & & & \\
 & & \downarrow F^* & & \downarrow F^* & \searrow g^*_n & & & \\
0 & \longrightarrow & N^*_{\mathcal{P}_n} & \xrightarrow{d_{\mathcal{P}_n}} & \Omega^1_{P_n} \otimes \mathcal{O}_U & \xrightarrow{f^*_n} & \Omega_U & \longrightarrow & 0
\end{array}$$

The following lemma holds.

Lemma 3.19. *Let* $\tilde{\mathcal{P}} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ *and* $\tilde{\mathcal{Q}} = (Q, \mathcal{Q}_{n+1}, g^1_{n+1}, g^2_{n+1})$ *be elements of* $\mathcal{C}(U)$ *and let* $F : \mathcal{P} \to \mathcal{Q}$ *be a* $\mathcal{C}(U)$*-morphism. We have then have that* $f_{\tilde{\mathcal{Q}}_{n+1}} = f_{\tilde{\mathcal{P}}_{n+1}} \circ F^* : N^*_{\mathcal{Q}_n} \to S_U$.

Proof of Lemma 3.19.

We consider an element $\sigma \in N^*_{\mathcal{Q}_n} = I_{U_n|Q_n} \otimes \mathcal{O}_U = I_{U^1_{n+1}|Q_{n+1}} \otimes \mathcal{O}_U = I_{U^2_{n+1}|Q_{n+1}} \otimes \mathcal{O}_U$. Let $\tilde{\sigma}_1$ be a lifting of σ to $I_{U^1_{n+1}|Q_{n+1}}$ and let $\tilde{\sigma}_2$ be a lifting of σ to $I_{U^2_{n+1}|Q_{n+1}}$. We have then that $F^*(\tilde{\sigma}_1) \in I_{U^1_{n+1}|P_{n+1}}$ and $F^*(\tilde{\sigma}_2) \in I_{U^2_{n+1}|P_{n+1}}$ by definition of $\mathcal{C}(U)$-morphisms. By definition,

$$f_{\tilde{\mathcal{P}}_{n+1}}(F^*\sigma) = \mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}(F^*(\tilde{\sigma}_1) - F^*(\tilde{\sigma}_2)))|_U.$$

Since F is a map of infinitesimal neighbourhoods, we have that $F^* \circ \alpha_{\mathcal{Q}_{n+1}} = \alpha_{\mathcal{P}_{n+1}}$ so

$$f_{\tilde{\mathcal{P}}_{n+1}}(F^*\sigma) = \mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{Q}_{n+1}})^{-1}(\tilde{\sigma}_1 - \tilde{\sigma}_2))|_U = f_{\tilde{\mathcal{Q}}_{n+1}}(\sigma).$$

This completes the proof of Lemma 3.19. □

We are now in a position to define $\mathcal{F}(F)$.

Definition 3.20. Let $\tilde{\mathcal{P}}_{n+1}$ and $\tilde{\mathcal{Q}}_{n+1}$ be two elements of $\mathcal{C}(U)$, and let $F : P_{n+1} \to Q_{n+1}$ be a $\mathcal{C}(U)$-morphism from $\tilde{\mathcal{P}}_{n+1}$ to $\tilde{\mathcal{Q}}_{n+1}$. The map $\mathcal{F}(F) : \mathcal{F}(\tilde{\mathcal{Q}}_{n+1}) \to \mathcal{F}(\tilde{\mathcal{P}}_{n+1})$ is then defined by

$$\mathcal{F}(F)([s,\omega]_{\tilde{\mathcal{Q}}_{n+1}}) = [s, F^*(\omega)]_{\tilde{\mathcal{P}}_{n+1}}$$

for any $s \in S_U$ and any $\omega \in \Omega^1_{Q_n} \otimes \mathcal{O}_U$. (This map is well-defined on the quotient because $f_{\tilde{\mathcal{Q}}_{n+1}} = f_{\tilde{\mathcal{P}}_{n+1}} \circ F^*$.)

We note that the map $\mathcal{F} : \mathcal{C}(U) \to \mathcal{E}(U)$ is indeed a contravariant functor because if $F : \tilde{\mathcal{P}}_{n+1} \to \tilde{\mathcal{R}}_{n+1}$ and $G : \tilde{\mathcal{R}}_{n+1} \to \tilde{\mathcal{Q}}_{n+1}$ are $\mathcal{C}(U)$-morphisms then for any $s \in S_U$ and $\omega \in \Omega^1_{Q_n} \otimes \mathcal{O}_U$ we have that

$$\mathcal{F}(F \circ G)[s,\omega]_{\tilde{\mathcal{Q}}_{n+1}} = [s, (F \circ G)^*\omega]_{\tilde{\mathcal{P}}_{n+1}} = [s, G^*(F^*(\omega))]_{\tilde{\mathcal{P}}_{n+1}} = \mathcal{F}(G)[s, F^*(\omega)]_{\tilde{\mathcal{R}}_{n+1}} = \mathcal{F}(G) \circ \mathcal{F}(F)[s,\omega]_{\tilde{\mathcal{Q}}_{n+1}}.$$

Further, since the above construction is entirely local we have that $r^V_U \circ \mathcal{F} = \mathcal{F} \circ r^V_U$. This completes the construction of the functor $\mathcal{F} : \mathcal{C}(U) \to \mathcal{E}(U)$.

3.3 THE CANONICAL ISOMORPHISMS IN $\mathcal{C}(U)$.

Throughout this subsection we fix two elements of $\mathcal{C}(U)$, $\tilde{\mathcal{P}}_{n+1}$ and $\tilde{\mathcal{Q}}_{n+1}$. The aim of this subsection is to construct a canonical isomorphism between the extensions $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$ and $\mathcal{F}(\tilde{\mathcal{Q}}_{n+1})$. The first step will be to create products in the category $\mathcal{C}(U)$. To do this, we will need first of all products of infinitesimal neighbourhoods.

Notation.

If P and Q are k-varieties and $\mathcal{P}_n = (P_n, i_{\mathcal{P}_n}, \alpha_{\mathcal{P}_n})$ and $\mathcal{Q}_n = (Q_n, i_{\mathcal{Q}_n}, \alpha_{\mathcal{Q}_n})$ are n-th order infinitesimal neighbourhoods of P resp. Q then we denote by $(P \times Q)_n$ the subscheme of $P_n \times Q_n$ cut out by the ideal sheaf $I^{n+1}_{P\times Q|P_n\times Q_n}$.

We denote by π_{P_n} and π_{Q_n} the projection maps $\pi_{P_n} : (P \times Q)_n \to P_n$ and $\pi_{Q_n} : (P \times Q)_n \to Q_n$. We denote by π_P and π_Q the projection maps ${}_Pi_P : P \times Q \to P$ and $\pi_Q : P \times Q \to Q$.

DEFINITION 3.21. Let $\mathcal{P} = (P_n, i_{\mathcal{P}_n}, \alpha_{\mathcal{P}_n})$ and $\mathcal{Q} = (Q_n, i_{\mathcal{Q}_n}, \alpha_{\mathcal{Q}_n})$ be infinitesimal neighbourhoods of order $n \geq 1$ of varieties P and Q with normal bundles W_P and W_Q respectively. We define the product $(\mathcal{P} \times \mathcal{Q})_n$ to be the infinitesimal neighbourhood of order n of $P \times Q$ with normal bundle $\pi_P^*(W_P) \oplus \pi_Q^*(W_Q)$ given by $((P \times Q)_n, i_{\mathcal{P}_n} \times i_{\mathcal{Q}_n}, \phi \circ \pi^*_{P_n}(\alpha_{\mathcal{P}_n}) \oplus \pi^*_{Q_n}(\alpha_{\mathcal{Q}_n})$ where here ϕ is the isomorphism

$$\phi : \pi_P^*(I_{P|P_n}/I^2_{P|P_n}) \oplus \pi_Q^*(I_{Q|Q_n}/I^2_{Q|Q_n}) \to I_{P\times Q|P_n\times Q_n}/I^2_{P\times Q|P_n\times Q_n}$$

given by pullback along π_P and π_Q.

We can now define products in $\mathcal{C}(U)$.

DEFINITION 3.22. Let $\tilde{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1})$ and $\tilde{\mathcal{Q}}_{n+1} = (Q, \mathcal{Q}_{n+1}, g^1_{n+1}, g^2_{n+1})$ be two members of $\mathcal{C}(U)$. We then define the product $(\tilde{\mathcal{P}} \times \tilde{\mathcal{Q}})_{n+1}$ as follows:

$$(\tilde{\mathcal{P}} \times \tilde{\mathcal{Q}})_{n+1} = (P \times Q, (\mathcal{P} \times \mathcal{Q})_{n+1}, f^1_{n+1} \times g^1_{n+1}, f^2_{n+1} \times g^2_{n+1}).$$

For any pair $\tilde{\mathcal{P}}_{n+1}$ and $\tilde{\mathcal{Q}}_{n+1}$ the projection maps $\pi_{P_{n+1}} : (P\times Q)_{n+1} \to P_{n+1}$ and $\pi_{Q_{n+1}} : (P\times Q)_{n+1} \to Q_{n+1}$ are $\mathcal{C}(U)$-morphisms. It follows that there are induced contravariant maps $\mathcal{F}(\pi_{P_{n+1}}) : \mathcal{F}(\tilde{\mathcal{P}}_{n+1}) \to \mathcal{F}((\tilde{\mathcal{P}}\times\tilde{\mathcal{Q}})_{n+1})$ and $\mathcal{F}(\pi_{Q_{n+1}}) : \mathcal{F}(\tilde{\mathcal{Q}}_{n+1}) \to \mathcal{F}((\tilde{\mathcal{P}} \times \tilde{\mathcal{Q}})_{n+1})$.

DEFINITION 3.23. Let $\tilde{\mathcal{P}}_{n+1}$ and $\tilde{\mathcal{Q}}_{n+1}$ be two elements of $\mathcal{C}(U)$. There is then a canonical isomorphism $J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}} : \mathcal{F}(\tilde{\mathcal{P}}_{n+1}) \to \mathcal{F}(\tilde{\mathcal{Q}}_{n+1})$ defined by

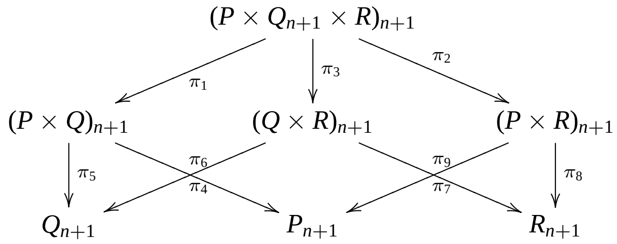
$$J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}} = \mathcal{F}(\pi_{Q_{n+1}})^{-1} \circ \mathcal{F}(\pi_{P_{n+1}}).$$

By construction, the isomorphism $J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}}$ is compatible with restriction to an open subset. We need the following proposition:

PROPOSITION 3.24. *Let $\tilde{\mathcal{P}}_{n+1}, \tilde{\mathcal{Q}}_{n+1}$ and $\tilde{\mathcal{R}}_{n+1}$ be elements of $\mathcal{C}(U)$. We then have that* $J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{R}}_{n+1}} \circ J^{\tilde{\mathcal{R}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}} = J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}}$.

Proof of Proposition 3.24.

Consider the following diagram:



We have that $\pi_4 \circ \pi_3 = \pi_5 \circ \pi_1$, $\pi_6 \circ \pi_1 = \pi_7 \circ \pi_2$ and $\pi_8 \circ \pi_2 = \pi_9 \circ \pi_3$. It follows that $\mathcal{F}(\pi_3) \circ \mathcal{F}(\pi_4) = \mathcal{F}(\pi_1) \circ \mathcal{F}(\pi_5)$, $\mathcal{F}(\pi_1) \circ \mathcal{F}(\pi_6) = \mathcal{F}(\pi_2) \circ \mathcal{F}(\pi_7)$ and $\mathcal{F}(\pi_2) \circ \mathcal{F}(\pi_8) = \mathcal{F}(\pi_3) \circ \mathcal{F}(\pi_9)$. Re-arranging, we see that

$$\mathcal{F}(\pi_1)^{-1} \circ \mathcal{F}(\pi_3) = \mathcal{F}(\pi_5) \circ \mathcal{F}(\pi_4)^{-1}$$

$$\mathcal{F}(\pi_3)^{-1} \circ \mathcal{F}(\pi_2) = \mathcal{F}(\pi_9) \circ \mathcal{F}(\pi_8)^{-1}$$

$$\mathcal{F}(\pi_2)^{-1} \circ \mathcal{F}(\pi_1) = \mathcal{F}(\pi_7) \circ \mathcal{F}(\pi_6)^{-1}.$$

Multiplying, we get that

$$\mathrm{Id} = \mathcal{F}(\pi_5) \circ \mathcal{F}(\pi_4)^{-1} \circ \mathcal{F}(\pi_9) \circ \mathcal{F}(\pi_8)^{-1} \circ \mathcal{F}(\pi_7) \circ \mathcal{F}(\pi_6)^{-1}$$

which we then write as

$$\mathcal{F}(\pi_5)^{-1} \circ \mathcal{F}(\pi_6) = \mathcal{F}(\pi_4)^{-1} \circ \mathcal{F}(\pi_9) \circ \mathcal{F}(\pi_8)^{-1} \circ \mathcal{F}(\pi_7)$$

or in other words $J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}} = J^{\tilde{\mathcal{Q}}_{n+1}}_{\tilde{\mathcal{R}}_{n+1}} \circ I^{\tilde{\mathcal{R}}_{n+1}}_{\tilde{\mathcal{P}}_{n+1}}$. This completes the proof of Proposition 3.24. □

3.4 Definition of $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$.

We start by checking that X can be covered by open sets such that $\mathcal{C}(U)$ is non-empty.

LEMMA 3.25. *Let U be an affine open set in X such that $V|_U$ is trivial. Then $\mathcal{C}(U)$ is not-empty.*

Proof of Lemma 3.25.

Let P be a smooth variety and let $f : U \to P$ be an inclusion of U in P. Let W be a trivial vector bundle on P whose rank is the same as V. Choose elements $\epsilon^1_1, \dots, \epsilon^1_r \in I_{U|U^1_{n+1}}$ such that $\{\bar{\epsilon}^1_i\} \subset I_{U|U^1_{n+1}}/I^2_{U|U^1_{n+1}}$ forms a basis of sections of $I_{U|U^1_{n+1}}/I^2_{U|U^1_{n+1}}$. Choose elements $\epsilon^2_1, \dots, \epsilon^2_r \in I_{U|U^2_{n+1}}$ such that $\epsilon^2_i|_{U_n} = \epsilon^1_i|_{U_n}$. These choices give rise to maps $\pi_i : U^i_{n+1} \to \mathrm{Spec}(A_{n+1})$ where $A_{n+1} = k[\epsilon_1, \dots, \epsilon_r]/\mathfrak{m}^{n+1}$ and by Lemma 2.7 the choice of the maps π_i turns U^i_{n+1} into a flat A_{n+1}-scheme. Moreover, by choice of the ϵ^i_js, $\pi_1|_{U_n} = \pi_2|_{U_n}$.

Now, let $f : U \to P$ be a closed immersion of U in a smooth k-variety P and consider $P_{n+1} = P \times \mathrm{Spec}(A_{n+1})$. By Lemma 2.7 $\mathcal{P}_{n+1} = (P_{n+1}, i_{\mathcal{P}_{n+1}}, \alpha_{\mathcal{P}_{n+1}})$ is an $n+1$-st infinitesimal neighbourhood of P_{n+1}. It will be enough to show that there are flat subschemes V^i_{n+1} of P_{n+1} such that there are isomorphisms of A_{n+1}-schemes $\phi_i : U^i_{n+1} \to V^i_{n+1}$ such that $\phi_1|_{U_n} = \phi_2|_{U_n}$ and $\phi_i|_U = f$. We start by recursively constructing flat subschemes $V_i \subset P_i = P \times \mathrm{Spec}(A_i)$ which are isomorphic to U_i for any $i \le n$. Suppose that V_{i-1} exists and is isomorphic to U_{i-1}. We then know by [12] (Thm 2.5, Thm 4.4 and Prop. 4.9) that flat subschemes of P_i extending V_{i-1} are a torsor over $\mathrm{Hom}(N^*_{U|P}, \mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes_k \mathcal{O}_U)$, that isomorphism classes of flat A_i-schemes extending V_{i-1} are a torsor over $\mathrm{Ext}^1(\Omega^1_U, \mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes_k \mathcal{O}_U)$ and that the forgetful map sending a flat subscheme of P_i extending V_{i-1} to its isomorphism class as a flat scheme is the boundary map $\delta : \mathrm{Hom}(N^*_{U|P}, \mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes_k \mathcal{O}_U) \to \mathrm{Ext}^1(\Omega^1_U, \mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes_k \mathcal{O}_U)$ associated to the exact sequence

$$0 \to N^*_{U|P} \to \Omega^1_P \otimes \mathcal{O}_U \to \Omega^1_U \to 0.$$

But now since P is smooth and U is affine we know that δ is a surjection. In particular, there is an A_i-flat subscheme of P_i, V_i, which extends V_{i-1} and which is isomorphic as an A_i-scheme to U_i. Iterating this procedure, we obtain a flat A_n-subscheme of P_n which is isomorphic to U_n. The same argument then also shows that there are flat subschemes V^1_{n+1} and V^2_{n+1} in P_{n+1} which are A_{n+1}-isomorphic to U^1_{n+1} and U^2_{n+1} respectively and which satisfy all the required conditions. This completes the proof of Lemma 3.25. □

We now choose an open affine covering U_i of X such that V_{U_i} is trivial for each i. For each i, we choose an element $\tilde{\mathcal{P}}^i_{n+1} = (P^i, \mathcal{P}^i_{n+1}, f^{1i}_{n+1}, f^{2i}_{n+1})$ of $\mathcal{C}(U_i)$. We denote by $\tilde{\mathcal{P}}^{ij}_{n+1}$ the element $r^{U_i \cap U_j}_{U_i}(\tilde{\mathcal{P}}^i_{n+1})$ in $\mathcal{C}(U_i \cap U_j)$.

Definition 3.26. Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. Let $\mathcal{X}^1_{n+1}$ and $\mathcal{X}_{n+1}$ be two extensions of X to $n+1$st order, let U_i be a covering of X by open affines such that $V|_{U_i}$ is trivial for each i. For each i, let $\tilde{\mathcal{P}}^i_{n+1}$ be an element of $\mathcal{C}(U_i)$. To the choice of elements $\{\tilde{\mathcal{P}}^i_{n+1}\}$ we associate the unique extension $\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1}) \in \mathcal{E}(X)$ such that:

1. There is an isomorphism $s_{\tilde{\mathcal{P}}^i_{n+1}} : \mathcal{D}(\tilde{\mathcal{P}}^i_{n+1}) \to \mathcal{F}(\tilde{\mathcal{P}}^i_{n+1})$,
2. The map $J^j_i : \mathcal{F}(\tilde{\mathcal{P}}^i_{n+1})|_{U_{ij}} \to \mathcal{F}(\tilde{\mathcal{P}}^j_{n+1})|_{U_{ij}}$ given by $J^j_i = s_{\tilde{\mathcal{P}}^j_{n+1}} \circ s^{-1}_{\tilde{\mathcal{P}}^i_{n+1}}$ satisfies $J^j_i = J^{\tilde{\mathcal{P}}^{ji}_{n+1}}_{\tilde{\mathcal{P}}^{ij}_{n+1}}$.

It follows from the various compatibilities proved above that

Proposition 3.27. *Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$ and U_i be as above. Let $(\tilde{\mathcal{P}}^i_{n+1})$ and $(\tilde{\mathcal{Q}}^i_{n+1})$ be two different choices of elements of $\mathcal{C}(U_i)$. There is then a unique isomorphism*

$$J^{\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})}_{\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})} : \mathcal{D}(\tilde{\mathcal{P}}^i_{n+1}) \to \mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})$$

such that over U_i we have that $s_{\tilde{\mathcal{Q}}^i_{n+1}} \circ J^{\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})}_{\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})} \circ s^{-1}_{\tilde{\mathcal{P}}^i_{n+1}} = J^{\tilde{\mathcal{Q}}^i_{n+1}}_{\tilde{\mathcal{P}}^i_{n+1}}$

Proof of Proposition 3.27. We define $J^{\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})}_{\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})}|_{U_i}$ by $J^{\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})}_{\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})}|_{U_i} = s^{-1}_{\tilde{\mathcal{Q}}^i_{n+1}} \circ J^{\tilde{\mathcal{Q}}^i_{n+1}}_{\tilde{\mathcal{P}}^i_{n+1}} \circ s_{\tilde{\mathcal{P}}^i_{n+1}}$. It will be enough to show that these definitions are compatible on the intersections U_{ij}, or in other words that

$$s^{-1}_{\tilde{\mathcal{Q}}^i_{n+1}} \circ J^{\tilde{\mathcal{Q}}^i_{n+1}}_{\tilde{\mathcal{P}}^i_{n+1}} \circ s_{\tilde{\mathcal{P}}^i_{n+1}} = s^{-1}_{\tilde{\mathcal{Q}}^j_{n+1}} \circ J^{\tilde{\mathcal{Q}}^j_{n+1}}_{\tilde{\mathcal{P}}^j_{n+1}} \circ s_{\tilde{\mathcal{P}}^j_{n+1}}$$

or in other words that

$$J^{\tilde{\mathcal{Q}}^j_{n+1}}_{\tilde{\mathcal{Q}}^i_{n+1}} \circ J^{\tilde{\mathcal{Q}}^i_{n+1}}_{\tilde{\mathcal{P}}^i_{n+1}} \circ J^{\tilde{\mathcal{P}}^i_{n+1}}_{\tilde{\mathcal{P}}^j_{n+1}} = J^{\tilde{\mathcal{Q}}^j_{n+1}}_{\tilde{\mathcal{P}}^j_{n+1}}.$$

But this has already been established in Proposition 3.24. This completes the proof of Proposition 3.27. □

Definition 3.28. Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$, $\mathcal{X}^1_{n+1}$ and U_i be as above. We identify any pair of extensions of the form $\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})$ and $\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})$ using the isomorphisms $J^{\mathcal{D}(\tilde{\mathcal{Q}}^i_{n+1})}_{\mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})}$. After this identification, we set $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) = \mathcal{D}(\tilde{\mathcal{P}}^i_{n+1})$ for any choice of elements $\tilde{\mathcal{P}}^i_{n+1} \in \mathcal{C}(U_i)$.

Throughout the rest of the paper, we denote by $s_{\tilde{\mathcal{P}}^i_{n+1}}$ the isomorphism

$$\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})|_U \to \mathcal{F}(\tilde{\mathcal{P}}^i_{n+1})$$

for any element $\tilde{\mathcal{P}}^i_{n+1} \in \mathcal{C}(U_i)$. Having thus constructed the element $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$, in the next section we will show that it has the required properties

4. Torsor character of $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$.

To complete the proof of Theorem 1.10, it remains to prove the following.

1. $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) \cong S \oplus \Omega$ as an extension if and only if $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ are isomorphic extensions of $\mathcal{X}_n$.
2. For any triple of extensions $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$ and $\mathcal{X}^3_{n+1}$ we have that $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^3_{n+1}) \cong \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1})$.
3. That if one extension $\mathcal{X}^1_{n+1}$ exists, then for any $E \in \mathcal{E}(X)$ there is an $\mathcal{X}^2_{n+1}$ such that $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) \cong E$.

We will begin by proving 1. In fact, we will prove something more, namely that there exists a canonical correspondence between splittings of $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$ and isomorphisms between $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$.

4.1 The canonical correspondence between splittings of $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$ and isomorphisms between $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$.

Throughout this section, $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ will be a pair of fixed extensions of $\mathcal{X}_n$. We start by setting up some notation.

Definition 4.1. Let $\mathcal{X}_n$ be an n-th order neighbourhood of X and let $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ be a pair of extensions of $\mathcal{X}_n$. For any open set in X, U, we let $R(U)$ be the set of splittings of $\mathcal{D}(\mathcal{U}^1_{n+1}, \mathcal{U}^2_{n+1})$ and we let $J(U)$ be the set of isomorphisms of extensions $j_{n+1} : \mathcal{U}^1_{n+1} \to \mathcal{U}^2_{n+1}$.

In this subsection we will prove the following proposition.

Proposition 4.2. *Let $\mathcal{X}_n$ be an n-th order neighbourhood of X with normal bundle V and let $\mathcal{X}^1_{n+1}$ and $\mathcal{X}^2_{n+1}$ be extensions of $\mathcal{X}_n$. Let U be an open set in X. There is then a canonical bijection of sets $b(U) : J(U) \to R(U)$ such that for any $V \subset U$ and $j \in J(U)$ we have that $b(V)(j|_V) = (b(U)(j))|_V$.*

Remark 4.3. Note that since the maps $U \to R(U)$ and $U \to J(U)$ define sheaves of sets, it will be enough to prove the existence of the map $b(U)$ for all sufficiently small open sets U.

Proof of Proposition 4.2.

By Remark 4.3, it will be enough to prove the existence of $b(U)$ for any U such that $\mathcal{C}(U)$ is not empty. Let U be such an open set in X and let $\tilde{\mathcal{P}}_{n+1}$ be an element of $\mathcal{C}(U)$. We introduce sets $T(\tilde{\mathcal{P}}_{n+1})$ and $R(\tilde{\mathcal{P}}_{n+1})$.

Definition 4.4. Let U be an open set in X and let $\tilde{\mathcal{P}}_{n+1}$ be an element of $\mathcal{C}(U)$. The set $T(\tilde{\mathcal{P}}_{n+1})$ is the set of all $\mathcal{O}_U$-linear maps $t : \Omega^1_{P_n} \otimes \mathcal{O}_U \to S_U$ such that $t \circ d_{\mathcal{P}_n} = f_{\tilde{\mathcal{P}}_{n+1}}$. The set $R(\tilde{\mathcal{P}}_{n+1})$ is the set of splittings of $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$.

There is a canonical isomorphism $s_{\tilde{\mathcal{P}}_{n+1}} : \mathcal{D}(\mathcal{U}^1_{n+1}, \mathcal{U}^2_{n+1}) \to \mathcal{F}(\tilde{\mathcal{P}}_{n+1})$. We have therefore a bijection $(s_{\tilde{\mathcal{P}}_{n+1}})^* : R(\tilde{\mathcal{P}}_{n+1}) \to R(U)$ We now construct a bijection $b(\tilde{\mathcal{P}}_{n+1}) : J(U) \to R(\tilde{\mathcal{P}}_{n+1})$. Recall that

$$\mathcal{F}(\tilde{\mathcal{P}}_{n+1}) = \frac{S_U \oplus (\Omega^1_{P_n} \otimes \mathcal{O}_U)}{(f_{\tilde{\mathcal{P}}_{n+1}}(\sigma), 0) = (0, d_{\mathcal{P}_n}(\sigma)) \ \forall \sigma \in N^*_{\mathcal{P}_n}}.$$

An element $r \in R(\tilde{\mathcal{P}}_{n+1})$ is therefore a map $r : S_U \oplus (\Omega^1_{P_n} \otimes \mathcal{O}_U) \to S_U$ such that
1. $r(s, 0) = s$ for any section $s \in S_U$.
2. for any section $\sigma \in N^*_{\mathcal{P}_n}$ we have that $f_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = r([0, d_{\mathcal{P}_n}(\sigma)]_{\tilde{\mathcal{P}}_{n+1}})$.

Note that by 1) the map r is characterised by the map $t_1(r) : \Omega^1_{P_n} \otimes \mathcal{O}_U \to S$ given by $t_1(r)(\omega) = r([0, \omega]_{\tilde{\mathcal{P}}_{n+1}})$ for any $\omega \in \Omega^1_{P_n} \otimes \mathcal{O}_U$. We note that 2) is equivalent to the fact that for any $\sigma \in N^*_{\mathcal{P}_n}$ we have that $f_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = t_1(r)(d_{\mathcal{P}_n}(\sigma))$. In other words, the map $t_1(r)$ is an element of $T(\tilde{\mathcal{P}}_{n+1})$. We now prove the following lemma.

Lemma 4.5. *The map $t_1 : R(\tilde{\mathcal{P}}_{n+1}) \to T(\tilde{\mathcal{P}}_{n+1})$ given by $r \to t_1(r)$ is a bijection.*

Proof of Lemma 4.5.

We construct an inverse map r_1 by letting $r_1(t)$ be the splitting given by $r_1(t)[s,\omega]_{\tilde{\mathcal{P}}_{n+1}} = s + t(\omega)$ for any $t \in T(\tilde{\mathcal{P}}_{n+1})$, any $\omega \in \Omega^1_{\tilde{\mathcal{P}}_{n+1}} \otimes \mathcal{O}_U$ and any $s \in S_U$. Note that for any section $\sigma \in N^*_{\tilde{\mathcal{P}}_n}$

$$r_1(t)[d_{\mathcal{P}_n}(\sigma), -f_{\tilde{\mathcal{P}}_{n+1}}(\sigma)]_{\tilde{\mathcal{P}}_{n+1}} = t(d_{\mathcal{P}_n}(\sigma)) - f_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = 0$$

by definition of $T(\tilde{\mathcal{P}}_{n+1})$ so $r_1(t)$ is a well defined splitting of $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$. It is immediate that $r_1(t_1(r)) = r$ and $t_1(r_1(t)) = t$. This completes the proof of Lemma 4.5. □

We now construct a bijection between $J(U)$ and $T(\tilde{\mathcal{P}}_{n+1})$. An element $j_{n+1} \in J(U)$ is determined by $j^*_{n+1} : \mathcal{O}_{U^2_{n+1}} \to \mathcal{O}_{U^1_{n+1}}$, the corresponding map of algebra sheaves. As $\mathcal{O}_{U^2_{n+1}}$ is a quotient algebra sheaf of $\mathcal{O}_{P_{n+1}}$ this can be seen as a map of algebra sheaves $j^*_{n+1} : \mathcal{O}_{P_{n+1}} \to \mathcal{O}^1_{U_{n+1}}$ such that $\mathrm{Ker}(j^*_{n+1}) = I_{U^2_{n+1}|P_{n+1}}$ and for any $f \in \mathcal{O}_{P_{n+1}}$ we have that $j^*_{n+1}(f)|_{U_n} = f|_{U_n}$.

Likewise, any sheaf map $j^*_{n+1} : \mathcal{O}_{P_{n+1}} \to \mathcal{O}_{U^1_{n+1}}$ such that $\mathrm{Ker}(j^*_{n+1}) = I_{U^2_{n+1}|P_{n+1}}$ and for any $f \in \mathcal{O}_{P_{n+1}}$ we have that $j^*_{n+1}(f)|_{U_n} = f|_{U_n}$ gives rise to an element $j_{n+1} \in J(U)$. Given such a map j_{n+1}, we consider the map $d(j_{n+1}) : \mathcal{O}_{P_{n+1}} \to S_U$ given by

$$d(j_{n+1})(f) = (\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1}(-f|_{U^1_{n+1}} + j^*_{n+1}(f)).$$

Note that since $j^*_{n+1}(f)|_{U_n} = f|_{U_n}$ we do indeed have that $(-f|_{U^1_{n+1}} + j^*_{n+1}(f)) \in I_{U_n|U^1_{n+1}} \cong \mathrm{Sym}^{n+1}(V^*)$. The map $d(j_{n+1})$ is a derivation because for any sections $f, g \in \mathcal{O}_{P_{n+1}}$ we have that

$$d(j_{n+1})(fg) = (\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1}(-fg|_{U^1_{n+1}} + j^*_{n+1}(fg))$$

$$= f \cdot (\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1}(-g|_{U^1_{n+1}} + j^*_{n+1}(g)) + (\mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}^1_{n+1}})^{-1}(-f \cdot j^*_{n+1}(g) + j^*_{n+1}(f) \cdot j^*_{n+1}(g)))$$

$$= f \cdot d(j_{n+1})(g) + j^*_{n+1}(g) \cdot (\mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}^1_{n+1}})^{-1}((f + j^*_{n+1}(f)).$$

$$= f \cdot d(j_{n+1})(g) + j^*_{n+1}(g) \cdot d(j_{n+1})(f).$$

Since S_U is an $\mathcal{O}_U$-module, and $j^*_{n+1}(g)|_U = g|_U$, it follows that

$$d(j_{n+1})(fg) = f \cdot d(j_{n+1})(g) + g \cdot d(j_{n+1})(f).$$

By the universal property of derivations, it follows that there is a unique $\mathcal{O}_U$-linear map,

$$t_2(j_{n+1}) : \Omega^1_{P_{n+1}} \otimes \mathcal{O}_U \to S_U$$

such that for any $f \in \mathcal{O}_{P_{n+1}}$ we have that $t_2(j_{n+1})(df|_U) = d(j_{n+1})(f)$. We consider the map $t_2 : J(U) \to T(\tilde{\mathcal{P}}_{n+1})$ given by $t_2 : j_{n+1} \to t_2(j_{n+1})$. We now prove that $t_2(j_{n+1})$ is a member of $T(\tilde{\mathcal{P}}_{n+1})$. Let σ be a section of $N^*_{\tilde{\mathcal{P}}_{n+1}} = I_{U_n|P_n} \otimes \mathcal{O}_U = I_{U^1_{n+1}|P_{n+1}} \otimes \mathcal{O}_U = I_{U^2_{n+1}|P_{n+1}} \otimes \mathcal{O}_U$. Locally, we choose sections $\sigma_1 \in I_{U^1_{n+1}|P_{n+1}}$ and $\sigma_2 \in I_{U^2_{n+1}|P_{n+1}}$ which lift σ. We may assume that $\sigma_1|_{P_n} = \sigma_2|_{P_n}$. By definition, $t_2(j_{n+1})(d_{\mathcal{P}_n}\sigma) = d(j_{n+1})(\sigma_2)$. We know that $\sigma_2 \in \mathrm{Ker}(j^*_{n+1})$, and hence $d(j_{n+1})(\sigma_2) = -(\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1}\sigma_2|_{U^1_{n+1}}$. But now, by definition, $f_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = \mathrm{Sym}^{n+1}(f^*_n)((\mathrm{Sym}^{n+1}\alpha_{\mathcal{P}_{n+1}})^{-1}(\sigma_1 - \sigma_2))|_U$, where here f^*_n is the pull-back map $f^*_n : W^*|_U \to V^*|_U$. We know that $\mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}\alpha_{\mathcal{P}_{n+1}})^{-1}|_U = (\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1} \circ f^{1*}_{n+1}$, where here the map $f^{1*}_{n+1} : I_{P_n|P_{n+1}} \to I_{U_n|U^1_{n+1}}$ is the pull-back map. It follows that $f_{\tilde{\mathcal{P}}_{n+1}}(\sigma) = (\mathrm{Sym}^{n+1}\alpha_{\mathcal{U}^1_{n+1}})^{-1} \circ f^{1*}_{n+1}(-\sigma_2) = t_2(j_{n+1})(d_{\mathcal{P}_n}\sigma)$. We therefore have that $t_2(j_{n+1}) \in T(\tilde{\mathcal{P}}_{n+1})$.

LEMMA 4.6. *The map $t_2 : J(U) \to T(\tilde{\mathcal{P}}_{n+1})$ given by $j_{n+1} \to t_2(j_{n+1})$ is a bijection.*

Proof of Lemma 4.6.

We will do this by constructing an explicit inverse map $j_{n+1} : T(\tilde{\mathcal{P}}_{n+1}) \to J(U)$. Let $t : \Omega^1_{P_n} \otimes \mathcal{O}_U \to S_U$ be an element of $T(\tilde{\mathcal{P}}_{n+1})$. We let $j_{n+1}(t)$ be the map whose associated pull-back map is given by $j^*_{n+1}(t)(v) = v|_{U^1_{n+1}} + \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}^1_{n+1}})(t(dv))$ for any $v \in \mathcal{O}_{P_{n+1}}$. We need to show that j^*_{n+1} is indeed an algebra morphism, that $j_{n+1}^*(t)(f)|_{U_n} = f|_{U_n}$ and that $\mathrm{Ker}(j_{n+1}^*(t)) = I_{U_{n+1}^2|P_{n+1}}$.

The map $j_{n+1}^*(t)$ is an algebra morphism because

$$j_{n+1}^*(t)(vw) = v\cdot w + \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})t(d(vw)) = vw + v\cdot \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dw)) + w\cdot \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dv))$$

$$= j_{n+1}^*(t)(v)\cdot j_{n+1}^*(t)(w) - \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dv))\cdot \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dw)) = j_{n+1}^*(t)(v)\cdot j_{n+1}(t)(w)$$

where the last equality follows because $\mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dv))$ and $\mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dw))$ are both contained in $I_{U_n|U_{n+1}^1}$. Moreover, $j_{n+1}^*(t)(f)|_{U_n} = f|_{U_n}$ by definition. It remains only to show that $\mathrm{Ker}\ j_{n+1}^*(t) = I_{U_{n+1}^2|P_{n+1}}$. Suppose that $v|_{U_{n+1}^1} + \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dv)) = 0$ for some $v \in \mathcal{O}_{P_{n+1}}$; we then have that $v|_{U_n} = 0$. We choose $v_1 \in I_{U_{n+1}^1|P_{n+1}}$ and $v_2 \in I_{U_{n+1}^2|P_{n+1}}$ such that $v_1|_{P_n} = v_2|_{P_n} = v|_{P_n}$, so that $-v|_{U_{n+1}^1} = \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(t(dv)) = \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}_{n+1}^1})(f_{\tilde{\mathcal{P}}_{n+1}}(\overline{v})) = (v_1 - v_2)|_{U_{n+1}^1}$. (Here by $\overline{v}$ we mean the class of $v|_{P_n} \in I_{U_n|P_n}$ in $I_{U_n|P_n}\otimes \mathcal{O}_U$.) This implies that $(v - v_2)|_{U_{n+1}^1} = 0$. Since $v - v_2|_{P_n} = 0$, we have that $v - v_2 \in I_{P_n|P_{n+1}} \cap I_{U_{n+1}^1|P_{n+1}}$. But we know that

$$I_{P_n|P_{n+1}} \cap I_{U_{n+1}^1|P_{n+1}}/(I_{U|P_{n+1}}\cdot I_{P_n|P_{n+1}}) = \mathrm{Sym}^{n+1}\alpha_{\mathcal{P}_{n+1}}(L_{\mathcal{P}_n}\cdot \mathrm{Sym}^n(W|_U))$$

$$= I_{P_n|P_{n+1}} \cap I_{U_{n+1}^2|P_{n+1}}/(I_{U|P_{n+1}}\cdot I_{P_n|P_{n+1}}).$$

It follows that $(v - v_2) \in I_{U_{n+1}^2|P_{n+1}}$ and hence $v \in I_{U_{n+1}^2|P_{n+1}}$ since by definition $v_2 \in I_{U_{n+1}^2}$.

The maps t_2 and j_{n+1} are easily seen to be inverses. This completes the proof of Lemma 4.6. □

We therefore have a bijection $b(\tilde{\mathcal{P}}_{n+1}) : J(U) \to R(\tilde{\mathcal{P}}_{n+1})$ given by $b(\tilde{\mathcal{P}}_{n+1}) = r_1 \circ t_2$.

LEMMA 4.7. *For any elements $\tilde{\mathcal{P}}_{n+1}$ and $\tilde{\mathcal{Q}}_{n+1}$ in $\mathcal{C}(U)$ we have that $s_{\tilde{\mathcal{P}}_{n+1}}^* \circ b(\tilde{\mathcal{P}}_{n+1}) = s_{\tilde{\mathcal{Q}}_{n+1}}^* \circ b(\tilde{\mathcal{Q}}_{n+1})$*

Proof of Lemma 4.7.

We have to prove that $b(\tilde{\mathcal{P}}_{n+1}) = (s_{\tilde{\mathcal{P}}_{n+1}}^{-1})^* \circ s_{\tilde{\mathcal{Q}}_{n+1}}^* \circ b(\tilde{\mathcal{Q}}_{n+1})$ or alternatively $b(\tilde{\mathcal{P}}_{n+1}) = (s_{\tilde{\mathcal{Q}}_{n+1}} \circ s_{\tilde{\mathcal{P}}_{n+1}}^{-1})^* \circ b(\tilde{\mathcal{Q}}_{n+1})$. This can be re-written as

$$b(\tilde{\mathcal{P}}_{n+1}) = (J_{\tilde{\mathcal{P}}_{n+1}}^{\tilde{\mathcal{Q}}_{n+1}})^* \circ b(\tilde{\mathcal{Q}}_{n+1}).$$

In other words, we have to prove that for any $j_{n+1} \in J(U)$ we have that

$$b(\tilde{\mathcal{P}}_{n+1})(j_{n+1}) = b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1}) \circ J_{\tilde{\mathcal{P}}_{n+1}}^{\tilde{\mathcal{Q}}_{n+1}}$$

considered as maps from $\mathcal{F}(\tilde{\mathcal{P}}_{n+1})$ to S_U. We start by proving that if $F : \tilde{\mathcal{Q}}_{n+1} \to \tilde{\mathcal{P}}_{n+1}$ is any $\mathcal{C}(U)$-morphism then for any $j_{n+1} \in J(U)$ we have that $b(\tilde{\mathcal{P}}_{n+1})(j_{n+1}) = b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1}) \circ \mathcal{F}(F)$. We recall that for any $s \in S_U$ and $f \in \mathcal{O}_{P_{n+1}}$ we have that

$$b(\tilde{\mathcal{P}}_{n+1})(j_{n+1})[s, df \otimes \mathcal{O}_U]_{\tilde{\mathcal{P}}_{n+1}} = s + t_2(df|_U) = s + \mathrm{Sym}^{n+1} f_n^* \circ (\mathrm{Sym}^{n+1}\alpha_{\tilde{\mathcal{P}}_{n+1}})^{-1}(-f|_{U_{n+1}^1} + j_{n+1}^* f).$$

Likewise, for any $s \in S_U$ and $f \in \mathcal{O}_{P_{n+1}}$ we have that

$$b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1})\circ\mathcal{F}(F)[s, df\mathcal{O}_U]_{\tilde{\mathcal{P}}_{n+1}} = b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1})[s, dF^*f\otimes\mathcal{O}_U]_{\tilde{\mathcal{Q}}_{n+1}} = s + \mathrm{Sym}^{n+1}(f_n^*)(\mathrm{Sym}^{n+1}\alpha_{\mathcal{Q}_{n+1}})^{-1}(-F^*f|_{U_{n+1}^1} + j_{n+1}^* F^* f).$$

But by definition of $\mathcal{C}(U)$-morphisms we know that $F^*f|_{U_{n+1}^1} = f|_{U_{n+1}^1}$ and $F^*f|_{U_{n+1}^2} = f|_{U_{n+1}^2}$, which implies that $j_{n+1}^* F^* f = j_{n+1}^* f$. In particular it follows that

$$b(\tilde{\mathcal{P}}_{n+1})(j_{n+1}) = b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1}) \circ \mathcal{F}(F).$$

By definition $J_{\tilde{\mathcal{P}}_{n+1}}^{\tilde{\mathcal{Q}}_{n+1}} = \mathcal{F}(\pi_{Q_{n+1}})^{-1} \circ \mathcal{F}(\pi_{P_{n+1}})$ so it follows that for any pair $(\tilde{\mathcal{P}}_{n+1}, \tilde{\mathcal{Q}}_{n+1})$ of elements in $\mathcal{C}(U)$ we have that

$$b(\tilde{\mathcal{P}}_{n+1})(j_{n+1}) = b(\tilde{\mathcal{Q}}_{n+1})(j_{n+1}) \circ J_{\tilde{\mathcal{P}}_{n+1}}^{\tilde{\mathcal{Q}}_{n+1}}.$$

This completes the proof of Lemma 4.7 $\square$

We now set $b(U) = s^*_{\tilde{\mathcal{P}}_{n+1}} \circ b(\tilde{\mathcal{P}}_{n+1})$ for any U such that $\mathcal{C}(U)$ has an element $\tilde{\mathcal{P}}_{n+1}$. The local nature of this map follows from the local nature of all the constructions involved. This completes the proof of Proposition 4.2. $\square$

There is a special case of this isomorphism when $\mathcal{X}^1_{n+1} = \mathcal{X}^2_{n+1}$ and $j_{n+1} = \mathrm{Id}$.

DEFINITION 4.8. Let $\mathcal{X}^1_{n+1}$ be an extension of a n-th order infinitesimal neighbourhood of $\mathcal{X}_n$ of X, and consider the extension

$$0 \to S \overset{i_{\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^1_{n+1})}}{\longrightarrow} \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^1_{n+1}) \overset{\pi_{\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^1_{n+1})}}{\longrightarrow} \Omega \to 0.$$

We denote by r_{Id} the isomorphism $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^1_{n+1}) \overset{b(X)(\mathrm{Id}) \oplus \pi_{\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^1_{n+1})}}{\longrightarrow} S \oplus \Omega$.

Let now $\tilde{\mathcal{P}}_{n+1} \in \mathcal{C}(U)$ be of the form $(P, \mathcal{P}_{n+1}, f^1_{n+1}, f^1_{n+1})$. The map $r_{\mathrm{Id}} \circ s^{-1}_{\tilde{\mathcal{P}}_{n+1}} : \mathcal{F}(\tilde{\mathcal{P}}_{n+1}) \to S_U \oplus \Omega_U$ is then given by the formula $r_{\mathrm{Id}}([s, \omega]_{\tilde{\mathcal{P}}_{n+1}}) \to (s, \omega|_{U^1_{n+1}})$ for any $s \in S_U$ and $\omega \in \Omega^1_{P_n} \otimes \mathcal{O}_U$.

4.2 THE ISOMORPHISM $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^3_{n+1}) = \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1})$.

In this section we will construct a natural isomorphism between $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^3_{n+1})$ and $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1})$.

In what follows we will need a certain number of facts on sums and differences of extensions, which we now summarise.

4.2.1 SUM AND DIFFERENCE MAPS ON EXTENSIONS.

We place ourselves in an arbitrary abelian category C: let F and G be two elements of this category. Whenever $0 \to F \to E \to G \to 0$ is an extension in the category C the inclusion map $F \to E$ will be denoted by i_E and the projection map $E \to G$ will be denoted by π_E. Let E_1, E_2 be two extensions of F and G,

$$0 \to F \overset{i_{E_1}}{\to} E_1 \overset{\pi_{E_1}}{\to} G \to 0$$

$$0 \to F \overset{i_{E_2}}{\to} E_2 \overset{\pi_{E_2}}{\to} G \to 0$$

By definition, $E_1 + E_2$ is the space $U(E_1, E_2)/V(E_1, E_2)$ where $U(E_1, E_2)$ is defined by

$$U(E_1, E_2) = \{(e_1, e_2) \in E_1 \oplus E_2 | \pi_{E_1}(e_1) = \pi_{E_2}(e_2)\}$$

and $V(E_1, E_2)$ is defined by

$$V(E_1, E_2) = \{(i_{E_1}(f), -i_{E_2}(f)) | f \in F\}.$$

For any $e_1 \in E_1$, $e_2 \in E_2$ such that $\pi_{E_1}(e_1) = \pi_{E_2}(e_2)$ we write $[e_1, e_2]$ for the equivalence class in $E_1 + E_2$ of (e_1, e_2). There is an exact sequence

$$0 \to F \overset{i_{E_1+E_2}}{\to} E_1 + E_2 \overset{\pi_{E_1+E_2}}{\to} G \to 0$$

where by definition

$$i_{E_1+E_2}(f) = [i_{E_1}(f), 0] = [0, i_{E_2}(f)], \text{ and } \pi_{E_1+E_2}([e_1, e_2]) = \pi_{E_1}(e_1).$$

Note that if we have two extension maps $\phi_1 : E_1 \to E'_1$ and $\phi_2 : E_2 \to E'_2$ then the sum

$$\phi_1 \oplus \phi_2 : E_1 \oplus E_2 \to E'_1 \oplus E'_2$$

descends to an extension map

$$\phi_1 + \phi_2 : E_1 + E_2 \to E'_1 + E'_2.$$

We note further that if we consider the trivial extension $F \oplus G$, then there is a natural isomorphism $(F \oplus G) + E \to E$ given by

$$[(f,g),e] \to i_E(f)+e.$$

By abuse of notation, we will frequently identify the extensions $(F \oplus G)+E$ and E

More generally, given extensions $E_1, E_2, \ldots, E_n$, we have a multiple sum extension, $E_1+E_2+\ldots+E_n = U(E_1,\ldots E_n)/V(E_1 \ldots E_n)$ where by definition $U(E_1 \ldots E_n) \subset E_1 \oplus \ldots \oplus E_n$ is defined by $(e_1,\ldots,e_n) \in U(E_1 \ldots E_n)$ if and only if $\pi_{E_1}(e_1) = \pi_{E_2}(e_2) \ldots = \pi_{E_n}(e_n)$ and $V(E_1 \ldots E_n)$ is given by

$$V(E_1 \ldots E_n) = \{(i_{E_1}(f_1), \ldots, i_{E_n}(f_n)) | f_1, \ldots, f_n \in F, \sum_i f_i = 0\}.$$

We denote the equivalence class of $(e_1,\ldots,e_n)$ under this map by $[e_1,\ldots,e_n]$. We have an extension.

$$0 \to F \overset{i_{E_1+\ldots+E_n}}{\to} E_1+\ldots+E_n \overset{\pi_{E_1+\ldots+E_n}}{\to} G \to 0$$

given by $i_{E_1+\ldots+E_n}(f) = [i_{E_1}(f), 0, \ldots, 0]$ and $\pi_{E_1+\ldots+E_n}([e_1,\ldots,e_n]) = \pi_{E_i}(e_i)$ for all i. For any permutation σ of $[1,\ldots,n]$ there is a canonical isomorphism $E_1+\ldots+E_n \to E_{\sigma(1)}+\ldots+E_{\sigma(n)}$ given by $[e_1,\ldots,e_n] \to [e_{\sigma(1)},\ldots,e_{\sigma(n)}]$. We will therefore consider $E_1+\ldots+E_n$ and $E_{\sigma(1)}+\ldots+E_{\sigma(n)}$ to be equivalent. It is still the case that if $\phi_i : E_i \to E_i'$ is a map of extensions for any i then there is an induced map $\phi_1+\ldots+\phi_n : E_1+\ldots+E_n \to E_1'+\ldots+E_n'$. In the notation $\phi_1+\ldots+\phi_n$ we suppress any of the ϕ_is which are equal to the identity. Hence, for example, if $\phi : E_1 \to E_1'$ is a map of extensions then we will denote the map $\phi + \mathrm{Id} : E_1+E_2 \to E_1'+E_2$ simply by ϕ_1.

Given an extension

$$0 \to F \overset{i_E}{\to} E \overset{\pi_E}{\to} G \to 0$$

we can define an extension $-E$ in a similar way: $-E$ is equal to E as an element of the category C, the extension maps are given as follows

$$0 \to F \overset{i_{-E}=-i_E}{\longrightarrow} -E \overset{\pi_E}{\to} G \to 0.$$

We define the difference of extensions, $E_1 - E_2$, to be equal to $E_1 + (-E_2)$. Explicitly, this space can be constructed as follows: $E_1 - E_2$ is the space $U'(E_1,E_2)/V'(E_1,E_2)$ where $U'(E_1,E_2)$ is defined by

$$U'(E_1,E_2) = \{(e_1,e_2) \in E_1 \oplus E_2 | \pi_{E_1}(e_1) = \pi_{E_2}(e_2)\}$$

and $V'(E_1,E_2)$ is defined by

$$V'(E_1,E_2) = \{(i_{E_1}(f), i_{E_2}(f)) | f \in F\}.$$

We will write $[e_1,e_2]'$ for the equivalence class in $E_1 - E_2$ of (e_1,e_2). There is an exact sequence

$$0 \to F \overset{i_{E_1-E_2}}{\to} E_1 - E_2 \overset{\pi_{E_1-E_2}}{\to} G \to 0$$

where by definition

$$i_{E_1-E_2}(f) = [i_{E_1}(f), 0]' = [0, -i_{E_2}(f)]', \pi_{E_1-E_2}[e_1,e_2]' = \pi_{E_1}(e_1).$$

We will need the contraction maps in what follows.

DEFINITION 4.9. Let E be an extension of F by G in an abelian category C. We then denote by c_E the contraction map $c_E : E - E \to F \oplus G$ given by

$$[e_1,e_2]' \to (i_E^{-1}(e_1-e_2), \pi_E(e_1)).$$

4.2.2 LOCAL CONSTRUCTION OF THE CANONICAL ISOMORPHISM. Throughout this section an n-th order infinitesimal neighbourhood of X, $\mathcal{X}_n$, and three extensions of $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$ and $\mathcal{X}^3_{n+1}$, are fixed. For any open set $U \subset X$ and any pair of distinct integers $i,j \in \{1,2,3\}$ we define a category $\mathcal{C}^{i,j}(U)$ as follows.

DEFINITION 4.10. An element $\tilde{\mathcal{P}}^{i,j}_{n+1}$ of $\mathcal{C}^{i,j}(U)$ is a quadruple $(P, \mathcal{P}_{n+1}, f^i_{n+1}, f^j_{n+1})$, where P is a smooth variety, $\mathcal{P}_{n+1}$ is an $(n+1)$-th order infinitesimal neighbourhood of P with normal bundle W and each $f^i_{n+1} : U^i_{n+1} \to P_{n+1}$ is a scheme morphism such that:

1. $(\mathcal{P}_{n+1}, f^k_{n+1})$ is an embedding of U^k_{n+1} for $k = i, j$.
2. $f^i_{n+1}|_{U_n} = f^j_{n+1}|_{U_n}$.

Given two elements of $\mathcal{C}^{i,j}(U)$, $\tilde{\mathcal{P}}^{i,j}_{n+1} = (P, \mathcal{P}_{n+1}, f^i_{n+1}, f^j_{n+1})$ and $\tilde{\mathcal{Q}}^{i,j}_{n+1} = (Q, \mathcal{Q}_{n+1}, g^i_{n+1}, g^j_{n+1})$, a $\mathcal{C}^{i,j}(U)$-morphism from $\tilde{\mathcal{P}}^{ij}_{n+1}$ to $\tilde{\mathcal{Q}}^{i,j}_{n+1}$ is a map $F : P_{n+1} \to Q_{n+1}$ such that F is a map of infinitesimal neighbourhoods and $F \circ f^k_{n+1} = g^k_{n+1}$ for $k = i$ or j.

We will prove the following proposition.

PROPOSITION 4.11. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood with normal bundle V and let $(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1})$ be three extensions of $\mathcal{X}_n$. There is then a canonical morphism*

$$\phi_{\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1}} : \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1}) \to \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^3_{n+1}).$$

Proof of Proposition 4.11. In order to construct this isomorphism, we will need an extra structure which we will call a triple.

DEFINITION 4.12. Let U be an open set of X. A triple $\hat{\mathcal{P}}_{n+1}$ over U is given by a data set $(P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1}, f^3_{n+1})$ where P is a smooth variety, $\mathcal{P}_{n+1}$ is an $(n+1)$-th infinitesimal neighbourhood of P and the maps $f^i_{n+1} : U^i_{n+1} \to P_{n+1}$ are maps of schemes such that for each pair i, j the 4-tuple $(P, \mathcal{P}_{n+1}, f^i_{n+1}, f^j_{n+1})$ is an element of $\mathcal{C}^{i,j}(U)$. We denote $(P, \mathcal{P}_{n+1}, f^i_{n+1}, f^j_{n+1})$ by $\tilde{\mathcal{P}}^{i,j}_{n+1}$.

DEFINITION 4.13. Let $\hat{\mathcal{P}}_{n+1} = (P, \mathcal{P}_{n+1}, f^1_{n+1}, f^2_{n+1}, f^3_{n+1})$ and $\hat{\mathcal{Q}} = (Q, \mathcal{Q}_{n+1}, g^1_{n+1}, g^2_{n+1}, g^3_{n+1})$ be triples over U. A map $F : P_{n+1} \to Q_{n+1}$ is said to be a map of triples if for all pairs $i, j \in \{1, 2, 3\}$ the map F is a $\mathcal{C}^{i,j}(U)$-morphism. The map F considered as a $\mathcal{C}^{i,j}(U)$-morphism will be denoted $F^{i,j}$.

Given a triple $\hat{\mathcal{P}}_{n+1}$ there are associated maps

$$f_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, f_{\tilde{\mathcal{P}}^{2,3}_{n+1}}, f_{\tilde{\mathcal{P}}^{1,3}_{n+1}} : \mathcal{N}^*_{\mathcal{P}_n} \to S_U$$

and associated extensions $\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1})$, $\mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1})$ and $\mathcal{F}(\mathcal{P}^{1,3}_{n+1})$. By definition of the maps $f_{\tilde{\mathcal{P}}^{i,j}_{n+1}}$ we have that $f_{\tilde{\mathcal{P}}^{1,3}_{n+1}} = f_{\tilde{\mathcal{P}}^{1,2}_{n+1}} + f_{\tilde{\mathcal{P}}^{2,3}_{n+1}}$. There is therefore an induced map of extensions

$$\phi_{\hat{\mathcal{P}}_{n+1}} : \mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1}) \to \mathcal{F}(\tilde{\mathcal{P}}^{1,3}_{n+1}).$$

given by

$$\phi_{\hat{\mathcal{P}}_{n+1}}([[s_1, \omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, [s_2, \omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}]) = ([s_1 + s_2, \omega]_{\tilde{\mathcal{P}}^{1,3}_{n+1}})$$

for any choice of $s_1, s_2 \in S_U$ and $\omega \in \Omega^1_{\mathcal{P}_n} \otimes \mathcal{O}_U$.

(We note that any element $[e_1, e_2]$ of $\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1})$, $e_1 = [s_1, \omega_1]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}$, $e_2 = [s_2, \omega_2]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}$, can be written in the above form because of the condition that $\pi_{\tilde{\mathcal{P}}^{1,2}_{n+1}}(e_1) = \pi_{\tilde{\mathcal{P}}^{2,3}_{n+1}}(e_2)$. There are, of course, several choices of s_1, s_2, and ω giving rise to different representations of the same element of $\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1})$: we leave it to the reader to prove that the above definition is independent of the choice of representation.)

DEFINITION 4.14. Let $\hat{\mathcal{P}}_{n+1}$ be a triple over U. We then let $\psi_{\hat{\mathcal{P}}_{n+1}}$ be the map

$$\psi_{\hat{\mathcal{P}}_{n+1}} : \mathcal{D}(\mathcal{U}^1_{n+1}, \mathcal{U}^2_{n+1}) + \mathcal{D}(\mathcal{U}^2_{n+1}, \mathcal{U}^3_{n+1}) \to \mathcal{D}(\mathcal{U}^1_{n+1}, \mathcal{D}^3_{n+1})$$

given by

$$\psi_{\hat{\mathcal{P}}_{n+1}} = s^{-1}_{\tilde{\mathcal{P}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{P}}_{n+1}} \circ (s_{\tilde{\mathcal{P}}^{1,2}_{n+1}} + s_{\tilde{\mathcal{P}}^{2,3}_{n+1}}).$$

4.2.3 GLOBALISATION OF THE CANONICAL ISOMORPHISM. To complete the proof of Proposition 4.11, it will be enough to prove the following proposition.

PROPOSITION 4.15. *Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$ and $\mathcal{X}^3_{n+1}$ be as above and let U be an open set in X. The map $\psi_{\hat{\mathcal{P}}_{n+1}}$ defined above is then independent of the choice of triple $\hat{\mathcal{P}}_{n+1} \in \mathcal{C}(U)$.*

Proof of Proposition 4.15.

We have that prove that for any pair of triples $\hat{\mathcal{P}}_{n+1}$ and $\hat{\mathcal{Q}}_{n+1}$ we have that

$$s^{-1}_{\tilde{\mathcal{P}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{P}}_{n+1}} \circ (s_{\tilde{\mathcal{P}}^{1,2}_{n+1}} + s_{\tilde{\mathcal{P}}^{2,3}_{n+1}}) = s^{-1}_{\tilde{\mathcal{Q}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{Q}}_{n+1}} \circ (s_{\tilde{\mathcal{Q}}^{1,2}_{n+1}} + s_{\tilde{\mathcal{Q}}^{2,3}_{n+1}})$$

which is equivalent to

$$\phi_{\hat{\mathcal{P}}_{n+1}} \circ (s_{\tilde{\mathcal{P}}^{1,2}_{n+1}} + s_{\tilde{\mathcal{P}}^{2,3}_{n+1}}) \circ (s_{\tilde{\mathcal{Q}}^{1,2}_{n+1}} + s_{\tilde{\mathcal{Q}}^{2,3}_{n+1}})^{-1} = s_{\tilde{\mathcal{P}}^{1,3}_{n+1}} \circ s^{-1}_{\tilde{\mathcal{Q}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{Q}}_{n+1}}$$

which we can also write as

$$\phi_{\hat{\mathcal{P}}_{n+1}} \circ (J^{\tilde{\mathcal{P}}^{1,2}_{n+1}}_{\tilde{\mathcal{Q}}^{1,2}_{n+1}} + J^{\tilde{\mathcal{P}}^{2,3}_{n+1}}_{\tilde{\mathcal{Q}}^{2,3}_{n+1}}) = J^{\tilde{\mathcal{P}}^{1,3}_{n+1}}_{\tilde{\mathcal{Q}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{Q}}_{n+1}}.$$

We start by proving that if $\hat{\mathcal{P}}_{n+1}$ and $\hat{\mathcal{Q}}_{n+1}$ are two triples over U and $F : P_{n+1} \to Q_{n+1}$ is a morphism of triples then

$$\phi_{\hat{\mathcal{P}}_{n+1}} \circ (\mathcal{F}(F^{1,2}) + \mathcal{F}(F^{2,3})) = \mathcal{F}(F^{1,3}) \circ \phi_{\hat{\mathcal{Q}}_{n+1}}$$

considered as maps from $\mathcal{F}(\tilde{\mathcal{Q}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{Q}}^{2,3}_{n+1})$ to $\mathcal{F}(\tilde{\mathcal{P}}^{1,3}_{n+1})$. Consider elements $e_1 \in \mathcal{F}(\tilde{\mathcal{Q}}^{1,2}_{n+1})$ and $e_2 \in \mathcal{F}(\tilde{\mathcal{Q}}^{2,3}_{n+1})$ such that $\pi_{\tilde{\mathcal{Q}}^{1,2}_{n+1}}(e_1) = \pi_{\tilde{\mathcal{Q}}^{2,3}_{n+1}}(e_2)$ and consider the element $[e_1, e_2] \in \mathcal{F}(\tilde{\mathcal{Q}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{Q}}^{2,3}_{n+1})$. We write $[e_1, e_2]$ in the form $[[s_1, w]_{\tilde{\mathcal{Q}}^{1,2}_{n+1}}, [s_2, w]_{\tilde{\mathcal{Q}}^{2,3}_{n+1}}]$. The map $\phi_{\hat{\mathcal{Q}}_{n+1}}$ is then given by $\phi_{\hat{\mathcal{Q}}_{n+1}}([e_1, e_2]) = [s_1 + s_2, w]_{\tilde{\mathcal{Q}}^{1,3}_{n+1}}$. We have that

$$\begin{aligned} \mathcal{F}(F^{1,3}) \circ \phi_{\hat{\mathcal{Q}}_{n+1}}([e_1, e_2]) &= \mathcal{F}(F^{1,3})([s_1 + s_2, w]_{\tilde{\mathcal{Q}}^{1,3}_{n+1}}) \\ &= [s_1 + s_2, F^*(w)]_{\tilde{\mathcal{P}}^{1,3}_{n+1}}. \end{aligned}$$

But on the other hand, we have that

$$\begin{aligned} &\phi_{\hat{\mathcal{P}}_{n+1}} \circ (\mathcal{F}(F^{1,2}) + \mathcal{F}(F^{2,3}))([[s_1, w]_{\tilde{\mathcal{Q}}^{1,2}_{n+1}}, [s_2, w]_{\tilde{\mathcal{Q}}^{2,3}_{n+1}}]) \\ &= \phi_{\hat{\mathcal{P}}_{n+1}}([[s_1, F^*(w)]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, [s_2, F^*(w)]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}]) = [s_1 + s_2, F^*(w)]_{\tilde{\mathcal{P}}^{1,3}_{n+1}}. \end{aligned}$$

It now follows that if $\hat{\mathcal{P}}_{n+1}$, $\hat{\mathcal{Q}}_{n+1}$ are two triples then $J^{\tilde{\mathcal{Q}}^{1,3}_{n+1}}_{\tilde{\mathcal{P}}^{1,3}_{n+1}} \circ \phi_{\hat{\mathcal{P}}_{n+1}} = \phi_{\hat{\mathcal{Q}}_{n+1}} \circ (J^{\tilde{\mathcal{Q}}^{1,2}_{n+1}}_{\tilde{\mathcal{P}}^{1,2}_{n+1}} + J^{\tilde{\mathcal{Q}}^{2,3}_{n+1}}_{\tilde{\mathcal{P}}^{2,3}_{n+1}})$. Indeed, if $\hat{\mathcal{P}}_{n+1}, \hat{\mathcal{Q}}_{n+1}$ are triples then $(P \times Q, (\mathcal{P} \times \mathcal{Q})_{n+1}, f^1_{n+1} \times g^1_{n+1}, f^2_{n+1} \times g^2_{n+1}, f^3_{n+1} \times g^3_{n+1})$ is again a triple, which we denote by $(\hat{\mathcal{P}} \times \hat{\mathcal{Q}})_{n+1}$. The projection maps $\pi_{P_{n+1}} : (P \times Q)_{n+1} \to P_{n+1}$ and $\pi_{Q_{n+1}} : (P \times Q)_{n+1} \to Q_{n+1}$ are then maps of triples, so

$$\phi_{(\hat{\mathcal{P}} \times \hat{\mathcal{Q}})_{n+1}} \circ (\mathcal{F}(\pi^{1,2}_{P_{n+1}}) + \mathcal{F}(\pi^{2,3}_{P_{n+1}})) = \mathcal{F}(\pi^{1,3}_{P_{n+1}})(\phi_{\hat{\mathcal{P}}_{n+1}}).$$

Likewise, we have that

$$\phi_{(\hat{\mathcal{P}} \times \hat{\mathcal{Q}})_{n+1}} \circ (\mathcal{F}(\pi^{1,2}_{Q_{n+1}}) + \mathcal{F}(\pi^{2,3}_{Q_{n+1}})) = \mathcal{F}(\pi^{1,3}_{Q_{n+1}})(\phi_{\hat{\mathcal{Q}}_{n+1}}).$$

Taking the inverse of the second equation multiplied by the first, we get that

$$(\mathcal{F}(\pi^{1,2}_{Q_{n+1}}) + \mathcal{F}(\pi^{2,3}_{Q_{n+1}}))^{-1} \circ (\mathcal{F}(\pi^{1,2}_{P_{n+1}}) + \mathcal{F}(\pi^{2,3}_{P_{n+1}})) = (\phi_{\hat{\mathcal{Q}}_{n+1}})^{-1} \mathcal{F}(\pi^{1,3}_{Q_{n+1}})^{-1} \mathcal{F}(\pi^{1,3}_{P_{n+1}})(\phi_{\hat{\mathcal{P}}_{n+1}})$$

or in other words $J^{\tilde{\mathcal{Q}}^{1,2}}_{\tilde{\mathcal{P}}^{1,2}} + J^{\tilde{\mathcal{Q}}^{2,3}}_{\tilde{\mathcal{P}}^{2,3}} = (\phi_{\hat{\mathcal{Q}}_{n+1}})^{-1} \circ J^{\tilde{\mathcal{Q}}^{1,3}}_{\tilde{\mathcal{P}}^{1,3}} \circ \phi_{\hat{\mathcal{P}}_{n+1}}$. This completes the proof of Proposition 4.15. □

DEFINITION 4.16. For any n-th order infinitesimal neighbourhood of X, $\mathcal{X}_n$ and any triple $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$ and $\mathcal{X}^3_{n+1}$ of extensions of $\mathcal{X}_n$ we set $\phi_{\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1}}|_U = \psi_{\hat{\mathcal{P}}_{n+1}}|_U$ for any triple $\hat{\mathcal{P}}_{n+1}$ defined over U.

Throughout the rest of this paper we will refer to the maps $\phi_{\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}, \mathcal{X}^3_{n+1}}$ as contraction maps.

In particular, this establishes condition 2). To prove the theorem it remains only to prove condition 3) (surjectivity): in the next section we will prove some results on the contraction maps that will be useful in what follows.

4.2.4 CALCULATIONS. In what follows, by abuse of notation, the subscript $n+1$ in expressions of the form $\mathcal{X}^{\alpha}_{n+1}$ will frequently be dropped in order to make the formulae manageable.

PROPOSITION 4.17. *Let $\mathcal{X}^1_{n+1}$, $\mathcal{X}^2_{n+1}$, $\mathcal{X}^3_{n+1}$, $\mathcal{X}^4_{n+1}$ be extensions of $\mathcal{X}_n$, an n-th order infinitesimal neighbourhood of X. We then have that*

$$\phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^4} \circ (\mathrm{Id}_{\mathcal{D}(\mathcal{X}^1,\mathcal{X}^2)} + \phi_{\mathcal{X}^2,\mathcal{X}^3,\mathcal{X}^4}) = \phi_{\mathcal{X}^1,\mathcal{X}^3,\mathcal{X}^4} \circ (\phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3} + \mathrm{Id}_{\mathcal{D}(\mathcal{X}^3,\mathcal{X}^4)})$$

as maps from $\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^2_{n+1}) + \mathcal{D}(\mathcal{X}^2_{n+1},\mathcal{X}^3_{n+1}) + \mathcal{D}(\mathcal{X}^3_{n+1},\mathcal{X}^4_{n+1})$ to $\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^4_{n+1})$.

Proof of Proposition 4.17.

Since the question is local on X, it will be enough to prove the proposition in any suitably small open set in X. Let U be an open set in X which is small enough that we can find a smooth variety P, an $(n+1)$-th order infinitesimal neighbourhood of P, $\mathcal{P}_{n+1}$, and maps $f^i_{n+1} : U^i_{n+1} \to P_{n+1}$ such that $(\mathcal{P}_{n+1}, f^i_{n+1})$ is an embedding of U^i_{n+1} for all i and $f^i_{n+1}|_{U_n} = f^j_{n+1}|_{U_n}$ for all i,j. Let $\hat{\mathcal{P}}^{i,j,k}_{n+1}$ be the triple $(P, \mathcal{P}_{n+1}, f^i_{n+1}, f^j_{n+1}, f^k_{n+1})$. Our aim is to show that

$$\phi_{\hat{\mathcal{P}}^{1,2,4}_{n+1}} \circ (\mathrm{Id}_{\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1})} + \phi_{\hat{\mathcal{P}}^{2,3,4}_{n+1}}) = \phi_{\hat{\mathcal{P}}^{1,3,4}_{n+1}} \circ (\phi_{\hat{\mathcal{P}}^{1,2,3}_{n+1}} + \mathrm{Id}_{\mathcal{F}(\tilde{\mathcal{P}}^{3,4}_{n+1})}).$$

as maps from $\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{3,4})$ to $\mathcal{F}(\tilde{\mathcal{P}}^{1,4}_{n+1})$. Let $[e_1, e_2, e_3]$ be an element of $\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1}) + \mathcal{F}(\tilde{\mathcal{P}}^{3,4}_{n+1})$, where $e_1 \in \mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1}), e_2 \in \mathcal{F}(\tilde{\mathcal{P}}^{2,3}_{n+1})$ and $e_3 \in \mathcal{F}(\tilde{\mathcal{P}}^{3,4}_{n+1})$. We write $[e_1, e_2, e_3]$ in the form $[[s_1, w]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, [s_2, w]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}, [s_3, w]_{\tilde{\mathcal{P}}^{3,4}_{n+1}}]$ for some choice of $s_1, s_2, s_3 \in S_U$ and $w \in \Omega^1_{P_n} \otimes \mathcal{O}_U$.

We have that

$$\phi_{\hat{\mathcal{P}}^{1,2,4}_{n+1}} \circ (\mathrm{Id}_{\mathcal{F}(\tilde{\mathcal{P}}^{1,2}_{n+1})} + \phi_{\hat{\mathcal{P}}^{2,3,4}_{n+1}})[e_1, e_2, e_3] = [s_1 + s_2 + s_3, w]_{\tilde{\mathcal{P}}^{1,4}_{n+1}}.$$

Likewise, we calculate that

$$\phi_{\hat{\mathcal{P}}^{1,3,4}_{n+1}} \circ (\phi_{\hat{\mathcal{P}}^{1,2,3}_{n+1}} + \mathrm{Id}_{\mathcal{F}(\tilde{\mathcal{P}}^{3,4}_{n+1})})[e_1, e_2, e_3] = [s_1 + s_2 + s_3, w]_{\tilde{\mathcal{P}}^{1,4}_{n+1}}.$$

This completes the proof of Proposition 4.17. □

In other words, the order of a sequence of contraction maps is not important.

DEFINITION 4.18. Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood. Let $\mathcal{X}^1_{n+1} \dots \mathcal{X}^m_{n+1}$ be an ordered sequence of extensions of $\mathcal{X}_n$. We define the contraction map

$$\phi_{\mathcal{X}^1 \dots \mathcal{X}^m} : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \dots + \mathcal{D}(\mathcal{X}^{m-1}, \mathcal{X}^m) \to \mathcal{D}(\mathcal{X}^1, \mathcal{X}^m)$$

by

$$\phi_{\mathcal{X}^1,\dots,\mathcal{X}^m} = \phi_{\mathcal{X}^1,\mathcal{X}^{m-1},\mathcal{X}^m} \circ \dots \circ \phi_{\mathcal{X}^1,\mathcal{X}^3,\mathcal{X}^4} \circ \phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3}.$$

We can in fact introduce a more general version of these isomorphisms: to do so we introduce some notation.

DEFINITION 4.19. A chain of extensions of $\mathcal{X}_n$ is a finite ordered set $I = (i_1, i_2, \dots, i_k)$ together with a choice of extension $\mathcal{X}^{i_m}_{n+1}$ of $\mathcal{X}_n$ for each index i_m.

DEFINITION 4.20. Given a chain of extensions of $\mathcal{X}_n$, $C = (\mathcal{X}^{i_1}_{n+1}, \mathcal{X}^{i_2}_{n+1}, \dots, \mathcal{X}^{i_k}_{n+1})$, we set

$$\mathcal{D}(C) = \mathcal{D}(\mathcal{X}^{i_1}_{n+1}, \mathcal{X}^{i_2}_{n+1}) + \mathcal{D}(\mathcal{X}^{i_2}_{n+1}, \mathcal{X}^{i_3}_{n+1}) + \dots + \mathcal{D}(\mathcal{X}^{i_{k-1}}_{n+1}, \mathcal{X}^{i_k}_{n+1})$$

DEFINITION 4.21. Let $C = (\mathcal{X}^{i_1}_{n+1}, \dots, \mathcal{X}^{i_k}_{n+1})$ be a chain of extensions of $\mathcal{X}_n$ indexed by an ordered set $I = (i_1, \dots, i_k)$. A subchain of C is a chain of the form $C' = (\mathcal{X}^{i'_1}_{n+1}, \dots, \mathcal{X}^{i'_{k'}}_{n+1}))$ where $I' = (i'_1, \dots, i'_{k'})$ is a sub-ordered set of I which contains i_1 and i_k.

Suppose that $C = (\mathcal{X}^{i_1}_{n+1}, \ldots, \mathcal{X}^{i_k}_{n+1})$ is a chain and $C' = (\mathcal{X}^{i_1}_{n+1}, \ldots, \not{\mathcal{X}}^{i_l}_{n+1}, \ldots, \mathcal{X}^{i_k}_{n+1})\}$ is a chain obtained from C by removing $\mathcal{X}^{i_l}_{n+1}$. We then define a contraction map $\phi_C^{C'} : \mathcal{D}(C) \to \mathcal{D}(C')$ by

$$\phi_C^{C'} = \mathrm{Id}_{\mathcal{D}(C_1)} + \phi_{\mathcal{X}^{i_{l-1}}, \mathcal{X}^{i_l}, \mathcal{X}^{i_{l+1}}} + \mathrm{Id}_{\mathcal{D}(C_2)}$$

where C_1 is the chain $(\mathcal{X}^{i_1}_{n+1}, \ldots, \mathcal{X}^{i_{l-1}}_{n+1})$ and C_2 is the chain $(\mathcal{X}^{i_{l+1}}_{n+1}, \ldots, \mathcal{X}^{i_k}_{n+1})$

DEFINITION 4.22. Let C be a chain of extensions and let C' be a subchain of C. Let $C_1, C_2, \ldots, C_l$ be a sequence of subchains of C such that C_1 is C, C_l is C' and C_i is obtained from C_{i-1} by deleting one element for all $i \in [2, \ldots, l]$. We then define a map $\phi_C^{C'} : \mathcal{D}(C) \to \mathcal{D}(C')$ by

$$\phi_C^{C'} = \phi_{C_{l-1}}^{C_l} \circ \ldots \circ \phi_{C_1}^{C_2}.$$

REMARK 4.23. Proposition 4.17 implies that the map $\phi_C^{C'}$ does not depend on the choice of intermediate subchains $C_2 \ldots C_{l-1}$. We note further that if C is a chain of extensions, C' is a subchain of C and C'' is a subchain of C' then $\phi_C^{C''} = \phi_{C'}^{C''} \circ \phi_C^{C'}$. If C is the concatenation of C_1 and C_2 (that is, $C_1 = (\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^i)$, $C_2 = (\mathcal{X}^i, \mathcal{X}^{i+1}, \ldots, \mathcal{X}^n)$ and $C = (\mathcal{X}^1, \ldots, \mathcal{X}^{i-1}, \mathcal{X}^i, \mathcal{X}^{i+1}, \ldots, \mathcal{X}^n)$) then

$$\mathcal{D}(C) = \mathcal{D}(C_1) + \mathcal{D}(C_2)$$

and if C_1' and C_2' are subchains of C_1 and C_2 whose concatenation is C' then $\phi_C^{C'} = \phi_{C_1}^{C_1'} + \phi_{C_2}^{C_2'}$.

DEFINITION 4.24. Let C be a chain of extensions of $\mathcal{X}_n$ and let C' be a sub-chain of C. We then denote the inverse map $(\phi_C^{C'})^{-1}$ by $\phi_{C'}^{C}$.

In the special case where C contains a pair of identical neighbours (ie. $C = (\ldots, \mathcal{X}^{i_m}, \mathcal{X}^{i_{m+1}}, \ldots)$ with $\mathcal{X}^{i_m} = \mathcal{X}^{i_{m+1}}$ then there are two (a priori distinct) ways of contracting $\mathcal{D}(C)$ to $\mathcal{D}(C')$, where C' is the chain $(\ldots \mathcal{X}^{i_m}, \mathcal{X}^{i_{m+2}}, \ldots)$. We can either use the map $\phi_C^{C'}$ or we can use the map

$$\mathrm{Id}_{\mathcal{D}(C_1)} + r_{\mathrm{Id}} + \mathrm{Id}_{\mathcal{D}(C_2)} : (\mathcal{D}(C_1) + \mathcal{D}(\mathcal{X}^{i_m}, \mathcal{X}^{i_{m+1}}) + \mathcal{D}(C_2)) \to (\mathcal{D}(C_1) + S \oplus \Omega + \mathcal{D}(C_2)) = \mathcal{D}(C_1) + \mathcal{D}(C_2) = \mathcal{D}(C').$$

where C_1 is the chain $(\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_m})$, C_2 is the chain $(\mathcal{X}^{i_{m+1}}, \ldots, \mathcal{X}^{i_k})$ and r_{Id} is the map constructed at the end of section 4.1.

LEMMA 4.25. *Let $C = (\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_k})$ be a chain of extensions of $\mathcal{X}_n$ and suppose that $\mathcal{X}^{i_m} = \mathcal{X}^{i_{m+1}}$. Let C' be the subchain obtained from C by deleting $\mathcal{X}^{i_{m+1}}$. Then we have that $\phi_C^{C'} = \mathrm{Id}_{\mathcal{D}(C_1)} + r_{\mathrm{Id}} + \mathrm{Id}_{\mathcal{D}(C_2)}$ where C_1 is the chain $(\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_m})$ and C_2 is the chain $(\mathcal{X}^{i_{m+1}}, \ldots, \mathcal{X}^{i_k})$.*

Proof of Lemma 4.25. By remark 4.23, it will be enough to deal with the case where $C = (\mathcal{X}^1, \mathcal{X}^1, \mathcal{X}^2)$ or $C = (\mathcal{X}^2, \mathcal{X}^1, \mathcal{X}^1)$. We treat the case where $C = (\mathcal{X}^1, \mathcal{X}^1, \mathcal{X}^2)$ below: the same argument works for $C = (\mathcal{X}^2, \mathcal{X}^1, \mathcal{X}^1)$.

Since the problem is local on X, we may assume there is a a triple $\hat{\mathcal{P}}_{n+1}$ over X of the form $(P, \mathcal{P}_{n+1}, f^1_{n+1}, f^1_{n+1}, f^2_{n+1})$. We have that $\phi_C^{C'}([[s_1, \omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, [s_2, \omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}]) = [s_1 + s_2, \omega]_{\tilde{\mathcal{P}}^{1,3}_{n+1}}$ for any $s_1, s_2 \in S_U$ and $\omega \in \Omega^1_{P_n} \otimes \mathcal{O}_U$. Further,

$$r_{\mathrm{Id}}([[s_1, \omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}, [s_2, \omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}]) = [(s_1, \omega|_{U_n}), [s_2, w]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}] = [s_1 + s_2, \omega]_{\tilde{\mathcal{P}}^{1,3}_{n+1}}.$$

This completes the proof of Lemma 4.25. □

We now consider the particular case where $C = (\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^1)$ and $C' = (\mathcal{X}^1, \mathcal{X}^1)$. In this case we obtain a contraction map

$$\phi_C^{C'} : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \mathcal{D}(\mathcal{X}^2, \mathcal{X}^1) \to \mathcal{D}(\mathcal{X}^1, \mathcal{X}^1).$$

We have a canonical isomorphism $r_{\mathrm{Id}} : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^1) \to S \oplus \Omega$ and hence in particular, we have a map

$$(r_{\mathrm{Id}} \circ \phi_C^{C'}) : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \mathcal{D}(\mathcal{X}^2, \mathcal{X}^1) \to S \oplus \Omega.$$

DEFINITION 4.26. For any pair of extensions $\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}$ of an n-th order neighbourhood $\mathcal{X}_n$ we denote the map $(r_{\mathrm{Id}} \circ \phi_C^{C'}) : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \mathcal{D}(\mathcal{X}^2, \mathcal{X}^1) \to S \oplus \Omega$ by $\tau_{\mathcal{X}^1,\mathcal{X}^2}$.

This gives rise to a map

$$\tau_{\mathcal{X}^1,\mathcal{X}^2} : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \mathcal{D}(\mathcal{X}^2, \mathcal{X}^1) + (-\mathcal{D}(\mathcal{X}^2, \mathcal{X}^1)) \to -\mathcal{D}(\mathcal{X}^2, \mathcal{X}^1)$$

and since we have for any E a canonical isomorphism $c_E : E + (-E) \to S \oplus \Omega$ this gives rise to a map which we denote by τ'_{X^1,X^2},

$$\tau'_{\mathcal{X}^1,\mathcal{X}^2} = r_{\mathrm{Id}(X^1)} \circ \phi_C^{C'}) \circ +c^{-1}_{\mathcal{D}(\mathcal{X}^2,\mathcal{X}^1)}) : \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) \to -\mathcal{D}(\mathcal{X}^2, \mathcal{X}^1).$$

We can restate Lemma 4.25 in the following form.

LEMMA 4.27. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X and let $\mathcal{X}^1$ and $\mathcal{X}^2$ be two extensions of X_n. Let $C = (\ldots \mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^1 \ldots)$ be a chain of extensions of $\mathcal{X}_n$ and let C' be the sub-chain of C obtained on suppressing the $\mathcal{X}^2$ term and the second $\mathcal{X}^1$ term. The following diagram then commutes.*

$$\begin{array}{ccc}
(\ldots + \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + \mathcal{D}(\mathcal{X}^2, \mathcal{X}^1) + \mathcal{D}(\mathcal{X}^1, \mathcal{X}^3)\ldots) & \xrightarrow{\phi_C^{C'}} & (\ldots + \mathcal{D}(\mathcal{X}^1, \mathcal{X}^3) + \ldots) \\
\downarrow \tau'_{\mathcal{X}^1,\mathcal{X}^2} & \nearrow c_{\mathcal{D}(\mathcal{X}^1,\mathcal{X}^2)} & \\
(\ldots + \mathcal{D}(\mathcal{X}^1, \mathcal{X}^2) + (-\mathcal{D}(\mathcal{X}^1, \mathcal{X}^2)) + \mathcal{D}(\mathcal{X}^1, \mathcal{X}^3)\ldots) & &
\end{array}$$

4.3 SURJECTIVITY.

In this paragraph, we will assume that at least one extension of $\mathcal{X}_n$ exists. We shall denote this "base" extension by $\mathcal{X}^1_{n+1}$. Our aim is to prove the following proposition.

PROPOSITION 4.28. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of a reduced l.c.i. k-variety X and suppose there is at least one extension of $\mathcal{X}_n$ to $n+1$st order, $\mathcal{X}^1_{n+1}$. Then for any extension E of Ω and S there exists $\mathcal{X}^2_{n+1}$, an extension to $n+1$st order of $\mathcal{X}_n$, such that $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) = E$.*

Let E be an extension $0 \to S \xrightarrow{i_E} E \xrightarrow{\pi_E} \Omega \to 0$. If U_i is an open cover of X then we denote $E|_{U_i}$ by E_i. We denote the intersection $U_{i_1} \cap, \ldots \cap U_{i_n}$ by $U_{i_1 \ldots i_n}$ and the restriction $E|_{U_{i_1 \ldots i_n}}$ by $E_{i_1 \ldots i_n}$.

PROPOSITION 4.29. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of a reduced l.c.i. k-variety X with normal bundle V. Let $\mathcal{X}^1_{n+1}$ be an extension of $\mathcal{X}_n$ and let E be an element of $\mathcal{E}(X)$. Suppose that there exists an open covering of X by sets U_i such that E_i is trivial for each i. Then there exists an extension of $\mathcal{X}_n$, $\mathcal{X}^2_{n+1}$, such that $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1}) = E$.*

Proof of Proposition 4.29.

For each i we choose an isomorphism $r_i : E_i \to S_{U_i} \oplus \Omega_{U_i}$. This choice of isomorphism gives rise over each $U_{i,j}$ to an map

$$\tilde{\phi}_{i,j} : S_{U_{i,j}} \oplus \Omega_{U_{i,j}} \to S_{U_{i,j}} \oplus \Omega_{U_{i,j}}.$$

given by $\tilde{\phi}_{i,j} = r_j \circ \mathrm{Id}_E \circ r_i^{-1}$. Since $\tilde{\phi}_{i,j}$ is a map of extensions, it is of the following form

$$\tilde{\phi}_{i,j}(s, w) = (s + \phi_{i,j}(w), w)$$

where $\phi_{i,j} : \Omega_{U_{i,j}} \to S_{U_{i,j}}$ is a linear map. It is immediate from the definition of $\tilde{\phi}_{i,j}$ that $\tilde{\phi}_{j,k} \circ \tilde{\phi}_{i,j} = \tilde{\phi}_{i,k}$. and it follows that $\phi_{i,j} + \phi_{j,k} = \phi_{i,k}$. Over $U_{i,j} = U_i \cap U_j$, there is an automorphism of algebra sheaves $A_{i,j} : \mathcal{O}_{X^1_{n+1}} \to \mathcal{O}_{X^1_{n+1}}$ given by $A_{i,j}(f) = f - \mathrm{Sym}^{n+1}(\alpha_{\mathcal{X}_n})(\phi_{i,j}(df|_U))$. Let $\mathcal{X}^2_{n+1}$ be the unique extension of $\mathcal{X}_n$ up to isomorphism that satisfies the following properties.

1. For every i there is an isomorphism $t_i : \mathcal{U}^2_{i,n+1} \to \mathcal{U}^1_{i,n+1}$, where $\mathcal{U}^2_{i,n+1}$ is the restriction of $\mathcal{X}^2_{n+1}$ to U_i, such that $t_i|_{U_{i,n}} = \mathrm{Id}$.

2. The isomorphism, $h_{i,j} : \mathcal{U}^1_{i,n+1} \to \mathcal{U}^1_{j,n+1}$ defined by $h_{i,j} = t_j \circ \mathrm{Id}_{U^2_{i,j,n+1}} \circ t_i^{-1}$ is given by the dual formula
$$h^*_{i,j}(f) = A_{i,j}(f) \text{ for all } f \in \mathcal{O}_{U^1_{i,j,n+1}}.$$

Since $\phi_{i,j} + \phi_{j,k} = \phi_{i,k}$ we have that $h_{j,k} \circ h_{i,j} = h_{i,k}$ so $\mathcal{X}^2_{n+1}$ does indeed exist.

PROPOSITION 4.30. *The element $\mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})$ is isomorphic to E.*

Proof of Proposition 4.30.

For each i we choose an element $\tilde{\mathcal{P}}_{i,n+1} \in \mathcal{C}(U_i)$ of the form $(P, \mathcal{P}_{n+1}, f^1_{n+1}, f^1_{n+1} \circ t_i)$. We let $\tilde{\mathcal{P}}_{i,n+1}$ have normal bundle W_{P_i}. Recall that for any pair i, j we denote by $\pi_{P_{i,n+1}}$ the projection map $\pi_{P_{i,n+1}} : (P_i \times P_j)_{n+1} \to P_{i,n+1}$. It is immediate from the definition of $f_{\tilde{\mathcal{P}}_{i,n+1}}$ that $f_{\tilde{\mathcal{P}}_{i,n+1}} = 0$. It follows that for every i we have a map
$$R_i : \mathcal{F}(\tilde{\mathcal{P}}_{i,n+1}) \to S_{U_i} \oplus \Omega^1_{U_i}$$
given by $R_i([s, w]_{\tilde{\mathcal{P}}_{i,n+1}}) = (s, w|_{U_i})$ for any where $s \in S_{U_i}$ and $w \in \Omega^1_{\mathcal{P}_{i,n}} \otimes \mathcal{O}_U$. We now seek to prove the following equation.

$$R_j \circ J^{\mathcal{P}_{j,n+1}}_{\tilde{\mathcal{P}}_{i,n+1}} \circ R_i^{-1} = \tilde{\phi}_{i,j}. \tag{2}$$

Throughout the proof of this equation we will tacitly assume ourselves to be working on a small enough neighbourhood for all necessary constructions. This is possible because of the local nature of all our constructions. We consider the following diagram

$$\begin{array}{ccc} U^1_{i,j,n+1} & \xrightarrow{f^1_{i,n+1} \times f^1_{j,n+1}} & (P_i \times P_j)_{n+1} \\ & & \uparrow{\scriptstyle (f^1_{i,n+1} \circ t_i) \times (f^1_{j,n+1} \circ t_j)} \\ & & U^2_{i,j,n+1} \end{array}$$

We will now attempt to determine the element $J^{\tilde{\mathcal{P}}_{j,n+1}}_{\tilde{\mathcal{P}}_{i,n+1}}([s, w]_{\tilde{\mathcal{P}}_{i,n+1}})$ for any $s \in S_{U_i}$ and $w \in \Omega^1_{\mathcal{P}_{i,n+1}} \otimes \mathcal{O}_{U_i}$. It will be enough to establish equation (2) for any w of the form dg_i, where $g_i \in \mathcal{O}_{\mathcal{P}_{i,n+1}}$. We fix an element $g_i \in \mathcal{O}_{\mathcal{P}_{i,n+1}}$ and denote its restriction to $U^1_{i,n+1}$ by g. We choose an element $g_j \in \mathcal{O}_{\mathcal{P}_{j,n+1}}$ such that $g_j|_{U^1_{j,n+1}} = g$. We note that
$$\pi^*_{P_{i,n+1}}(g_i) - \pi^*_{P_{j,n+1}}(g_j) \in I_{U^1_{n+1}|(P_i \times P_j)_{n+1}}.$$
We denote the function $\pi^*_{P_{i,n+1}}(g_i) - \pi^*_{P_{j,n+1}}(g_j)$ on $(P_i \times P_j)_{n+1}$ by h and we let $\overline{h}$ be the equivalence class of h in $N^*_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}$. We now show that $f_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}(\overline{h}) = \phi_{i,j}(dg)$. We have that $h|_{U^2_{n+1}} = (\pi^*_{P_{i,n+1}}(g_i) - \pi^*_{P_{j,n+1}}(g_j))|_{U^2_{n+1}} = g \circ t_i - g \circ t_j = g \circ (\mathrm{Id} - t_j \circ t_i^{-1}) \circ t_i$. By definition, $g \circ t_j \circ t_i^{-1} = A_{i,j}(g)$ so $h|_{U^2_{n+1}} = (g - A_{i,j}(g)) \circ t_i = \mathrm{Sym}^{n+1} \alpha_{\mathcal{U}^1_{n+1}}(\phi_{i,j}(dg)) \circ t_i = \mathrm{Sym}^{n+1}(\alpha_{\mathcal{U}^2_{n+1}})(\phi_{i,j}(dg))$. It follows that if $\Phi_{i,j}(dg)$ is an arbitrary lifting of $\phi_{i,j}(dg) \in \mathrm{Sym}^{n+1}(V^*_U)$ to $\mathrm{Sym}^{n+1}(W_{P_i} \oplus W_{P_j})$ then $h' = h - \alpha_{(\mathcal{P}_i \times \mathcal{P}_j)_{n+1}}(\Phi_{i,j}(dg))$ is a lifting to $I_{U^2_{n+1}|(P_i \times P_j)_{n+1}}$ of $g|_{U_n}$. By definition of $f_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}$ we therefore have that
$$f_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}(\overline{h}) = \mathrm{Sym}^{n+1}(f^*_n)\mathrm{Sym}^{n+1}(\alpha_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}})^{-1}(h - (h - \mathrm{Sym}^{n+1}(\alpha_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}} \Phi_{i,j}(dg)))$$
where here $\mathrm{Sym}^{n+1}(f^*_n)$ is the surjective pull-back map
$$\mathrm{Sym}^{n+1}(W^*_{P_i} \oplus W^*_{P_j}) \to \mathrm{Sym}^{n+1}(V^*_{U_{i,j}})$$
and hence
$$f_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}(\overline{h}) = \mathrm{Sym}^{n+1}(f^*_n)(\Phi_{i,j}(dg)) = \phi_{i,j}(dg).$$
By definition of $\mathcal{F}(\pi_{P_{i,n+1}})$ we have that for any $s \in S$ and $g \in \mathcal{O}_{P_{i,n+1}}$
$$\mathcal{F}(\pi_{P_{i,n+1}})[s, dg]_{\tilde{\mathcal{P}}_{i,n+1}} = [s, d\pi^*_{P_{i,n+1}} g]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}.$$
But we have seen that $f_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}(\overline{h}) = \phi_{i,j}(dg)$ and hence in $\mathcal{F}((\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1})$ we have that

$$[-\phi_{i,j}(dg), d(\pi^*_{P_{i,n+1}} g_i - \pi^*_{P_{j,n+1}} g_j) \otimes \mathcal{O}_U]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}} = 0$$

and hence

$$[0, d\pi^*_{P_{i,n+1}} g_i]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}} = [\phi_{i,j}(dg), d\pi^*_{P_{j,n+1}} g_j]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}.$$

We therefore have that

$$\mathcal{F}(\pi_{P_{i,n+1}})[s, dg_i]_{\tilde{\mathcal{P}}_{i,n+1}} = [s + \phi_{i,j}(dg), d\pi^*_{P_{j,n+1}} g_j]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}} = [s + \phi_{i,j}(dg), d\pi^*_{P_{j,n+1}} g_j]_{(\tilde{\mathcal{P}}_i \times \tilde{\mathcal{P}}_j)_{n+1}}$$

and hence

$$\mathcal{F}(\pi_{P_{i,n+1}})[s, dg_i]_{\tilde{\mathcal{P}}_{i,n+1}} = \mathcal{F}(\pi_{P_{j,n+1}})[s + \phi_{i,j}(dg), dg_j]_{\tilde{\mathcal{P}}_{j,n+1}}.$$

We note that for any $s \in S_U$ and any $g_i \in \mathcal{O}_{P_{i,n+1}}$ $R_i[s, dg_i \otimes \mathcal{O}_U]_{\tilde{\mathcal{P}}_{i,n+1}} = (s, dg)$ and

$$R_j[s + \phi_{i,j}(dg), dg_j \otimes \mathcal{O}_U] = (s + \phi_{i,j}(dg), dg)$$

and so it follows that for any $s \in S_U$ and $\omega \in \Omega^1_{P_{i,n+1}} \otimes \mathcal{O}_U$ we have that

$$R_j \circ \mathcal{F}(\pi_{P_{j,n+1}})^{-1} \circ \mathcal{F}(\pi_{P_{i,n+1}}) \circ R_i^{-1}(s, w) = (s + \phi_{i,j}(w), w) = \tilde{\phi}_{i,j}(s, w).$$

or in other words $R_j \circ J^{\tilde{\mathcal{P}}_{j,n+1}}_{\tilde{\mathcal{P}}_{i,n+1}} \circ R_i^{-1} = \tilde{\phi}_{i,j}(s, w)$ which completes the proof of equation 2. But Proposition 4.29 now follows because this implies that the maps

$$R'_i : \mathcal{D}(\mathcal{X}^1_{n+1}, \mathcal{X}^2_{n+1})|_{U_i} \to E_i$$

given by $R'_i = (r_i)^{-1} \circ R_i \circ s_{\tilde{\mathcal{P}}_{i,n+1}}$ satisfy the equation

$$(R'_j) \circ R_i'^{-1} = r_j^{-1} \circ R_j \circ J^{\tilde{\mathcal{P}}_{j,n+1}}_{\tilde{\mathcal{P}}_{i,n+1}} \circ R_i^{-1} \circ r_i = r_j^{-1} \circ \tilde{\phi}_{ij} \circ r_i = \mathrm{Id}_E$$

for any pair (i, j). These maps therefore glue together to give a global isomorphism of extensions. □

The following lemma establishes the existence of $\mathcal{X}^2_{n+1}$ locally.

LEMMA 4.31. *Let $\mathcal{X}_n$, $\mathcal{X}^1_{n+1}$ be as above. Let U be an open affine subset of X such that V_U is a trivial bundle. There is then an extension of $\mathcal{U}_n$, $\mathcal{U}^2_{n+1}$ such that $\mathcal{D}(\mathcal{U}^1_{n+1}, \mathcal{U}^2_{n+1})$ is isomorphic to $E|_U$.*

Proof of Lemma 4.31.

By Lemma 3.25 there is a smooth affine variety P, an $n+1$-th order infinitesimal neighbourhood of P, $\mathcal{P}_{n+1}$ and a map $f^1_{n+1} : U^1_{n+1} \to P_{n+1}$ such that $(P, \mathcal{P}_{n+1}, f^1_{n+1})$ is an embedding. We consider the exact sequence

$$0 \to N^*_{\mathcal{P}_n} \to \Omega^1_{P_n} \otimes \mathcal{O}_U \to \Omega_U \to 0. \tag{3}$$

Since P is smooth and affine and $\Omega^1_{P_n} \otimes \mathcal{O}_U$ is a locally free sheaf, the map $\mathrm{Hom}(N^*_{\mathcal{P}_n}, S_U) \to \mathrm{Ext}^1(\Omega_U, S_U)$ is a surjection. We can therefore find a $\phi \in \mathrm{Hom}(N^*_{\mathcal{P}_n}, S)$ such that the pull back of (3) along ϕ is $\cong$ E_i. We define a subscheme $U^2_{n+1} \in P_{n+1}$ by

$$I_{U^2_{i,n+1}} = \{g' - \epsilon | g' \in I_{U^1_{n+1}}, \epsilon \in I_{P_n|P_{n+1}}, \mathrm{Sym}^{n+1}(f^*_n)(\mathrm{Sym}^{n+1}(\alpha_{\mathcal{P}_{n+1}})^{-1}\epsilon) = \phi(g)\}.$$

Let $\tilde{\mathcal{P}}_{i,n+1}$ be the element of $\mathcal{C}(U_i)$ defined by taking $P_{i,n+1}$, U^1_{n+1} and U^2_{n+1}, with f^1_{n+1} and f^2_{n+1} given by the inclusion maps. It is immediate from the definition of $f_{\tilde{\mathcal{P}}_{i,n+1}}$ that $f_{\tilde{\mathcal{P}}_{i,n+1}} = \phi$, and hence $\mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{i,n+1})$, which is simply the push-forward of (3) along $f_{\tilde{\mathcal{P}}_{i,n+1}}$, satisfies $\mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{i,n+1}) = E_i$. This completes the proof of Lemma 4.31. □

LEMMA 4.32. *Let $\mathcal{X}_n$ and $\mathcal{X}^1_{n+1}$ be as above. Let U_i be a covering of X by open affines such that V_{U_i} is trivial for every U_i and for each i let $\mathcal{U}^2_{i,n+1}$ be an extension of $\mathcal{U}_{i,n}$ such that $\mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{i,n+1}) \cong E$. Then there is an extension of $\mathcal{X}_n$, $\mathcal{X}^{2'}_{n+1}$ such that $\mathcal{X}^{2'}_{n+1}|_{U_i}$ is isomorphic to $\mathcal{U}^2_{n+1}$ for every i.*

Proof of Lemma 4.32.

For each i we let $T_i : \mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{i,n+1}) \to E_i$ be an isomorphism.

It will be enough to construct for each pair (i,j) an isomorphism $B_{i,j} : \mathcal{U}^2_{i,n+1} \to \mathcal{U}^2_{j,n+1}$ which are compatible on the triple intersections. Over the open set $U_i \cap U_j = U_{i,j}$ we have chains $C_{i,j} = (\mathcal{U}^2_{i,n+1}, \mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{j,n+1})$ and $C'_{i,j} = (\mathcal{U}^2_{i,n+1}, \mathcal{U}^2_{j,n+1})$. For each pair (i,j) we have a series of maps

(4)
$$J_{i,j} : \mathcal{D}(\mathcal{U}^2_{i,n+1}, \mathcal{U}^2_{j,n+1}) \overset{\phi^{C_{i,j}}_{C'_{i,j}}}{\to} \mathcal{D}(\mathcal{U}^2_{i,n+1}, \mathcal{U}^1_{i,n+1}) + \mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{j,n+1}) \overset{\tau'_{\mathcal{U}^2_{i,n+1},\mathcal{U}^1_{i,n+1}}}{\to}$$
$$-\mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{i,n+1}) + \mathcal{D}(\mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{j,n+1}) \overset{-T_i+T_j}{\to} -E_i|_{U_{i,j}} + E_j|_{U_{i,j}}$$
(5)

where $C_{i,j}$ is the chain $(\mathcal{U}^2_{i,n+1}, \mathcal{U}^1_{i,n+1}, \mathcal{U}^2_{j,n+1})$ and $C'_{i,j}$ is the chain $(\mathcal{U}^2_{i,n+1}, \mathcal{U}^2_{j,n+1})$ We have a canonical map $c_{-E_{i,j}} : -E_i|_{U_{i,j}} + E_j|_{U_{i,j}} \to S_{U_{i,j}} \oplus \Omega_{U_{i,j}}$ and hence for every pair i,j there is a map

$$S_{i,j} = c_{-E_{i,j}} \circ J_{i,j} : \mathcal{D}(\mathcal{U}^2_{i,n+1}, \mathcal{U}^2_{j,n+1}) \to S_{U_{i,j}} \oplus \Omega_{U_{i,j}}.$$

By Proposition 4.2, this splitting corresponds to a gluing $b(S_{ij}) = B_{i,j} : \mathcal{U}^2_{i,n+1} \to \mathcal{U}^2_{j,n+1}$. It will be enough to show that these gluings are compatible on the triple intersections. We start with the following lemma.

LEMMA 4.33. *Let $\mathcal{U}_n$ be an n-th order infinitesimal neihgbourhood of U and let $\mathcal{U}^1_{n+1}$, $\mathcal{U}^2_{n+1}$ and $\mathcal{U}^3_{n+1}$ be three extensions of $\mathcal{U}_n$. Suppose given three isomorphisms of extensions*

$$\phi^{1,2} : \mathcal{U}^1_{n+1} \to \mathcal{U}^2_{n+1}, \phi^{2,3} : \mathcal{U}^2_{n+1} \to \mathcal{U}^3_{n+1} \text{ and } \phi^{1,3} : \mathcal{U}^1_{n+1} \to \mathcal{U}^3_{n+1}.$$

Let $S_{i,j} : \mathcal{D}(\mathcal{X}^i_{n+1}, \mathcal{X}^j_{n+1}) \to S \oplus \Omega$ be the isomorphism associated to $\phi^{i,j}$. We then have that $\phi^{1,3} = \phi^{2,3} \circ \phi^{1,2}$ if and only if

$$c_{S\oplus\Omega} \circ (S_{1,2} + S_{2,3}) = S_{1,3} \circ \phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3}$$

where $c_{S\oplus\Omega} : (S \oplus \Omega) + (S \oplus \Omega) \to S \oplus \Omega$ is the canonical contraction map.

Proof of Lemma 4.33.

The statement of the lemma may be checked locally, so we may assume that there is a triple over U of the form $(P, \mathcal{P}_{n+1}, f^3_{n+1} \circ \phi^{1,2} \circ \phi^{2,3}, f^3_{n+1} \circ \phi^{2,3}, f^3_{n+1})$, where $f^3_{n+1} : U^3_{n+1} \to P_{n+1}$ is an embedding. We then have that the isomorphism associated to $\phi^{2,3} \circ \phi^{1,2}$ is given by

$$[s,\omega]_{\tilde{\mathcal{P}}^{13}_{n+1}} \to [s, \omega|_{X_n}]$$

for any $s \in S$ and $\omega \in \Omega^1_{P_{n+1}} \otimes \mathcal{O}_U$. In particular, we have that $\phi^{13} = \phi^{23} \circ \phi^{12}$ if and only if $S_{1,3}[s,\omega]_{\tilde{\mathcal{P}}^{13}_{n+1}} = [s, \omega|_{X_n}]$ for any $s \in S$ and $\omega \in \Omega^1_{P_{n+1}} \otimes \mathcal{O}_U$. We also have that $\phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3}[[s_1,\omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}[s_2,\omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}] = [s_1 + s_2, \omega]_{\tilde{\mathcal{P}}^{1,3}_{n+1}}$. It is therefore the case that $\phi^{1,3} = \phi^{2,3} \circ \phi^{1,2}$ if and only if $S_{1,3} \circ \phi_{\mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3}[[s_1,\omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}[s_2,\omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}] = [s_1 + s_2, \omega|_{\mathcal{X}_n}$ for any $s_1, s_2 \in S$ and $\omega \in \Omega^1_{P_{n+1}} \otimes \mathcal{O}_U$. On the other hand we have that

$$S_{2,3}[s_2,\omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}} = [s_2, \omega|_{X_n}]$$

and

$$S_{1,2}[s_1,\omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}} = [s_1, \omega|_{X_n}].$$

In particular,

$$c_{S\oplus\Omega} \circ (S_{1,2} + S_{2,3})[[s_1,\omega]_{\tilde{\mathcal{P}}^{1,2}_{n+1}}[s_2,\omega]_{\tilde{\mathcal{P}}^{2,3}_{n+1}}] = [s_1 + s_2, \omega|_{X_n}].$$

This completes the proof of Lemma 4.33. □

To prove the lemma 4.32 it will therefore be enough to show that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{j,n+1})+\mathcal{D}(\mathcal{U}^2_{j,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{c_{E_{i,j,k}}\circ J_{i,j}+c_{E_{i,j,k}}\circ J_{j,k}} & S\oplus\Omega+S\oplus\Omega \\ \downarrow{\scriptstyle \phi^{C_2}_{C_2'}} & & \downarrow{\scriptstyle c_{S\oplus\Omega}} \\ \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{c_{E_{i,j,k}}\circ J_{i,k}} & S\oplus\Omega \end{array}$$

where here C_2' is the chain $(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{j,n+1},\mathcal{U}^2_{k,n+1})$ and C_2 is the chain $(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{k,n+1})$. Since the diagram

$$\begin{array}{ccc} (E_i-E_j)+(E_j-E_k) & \xrightarrow{c_{E_{i,j}}+c_{E_{j,k}}} & S\oplus\Omega+S\oplus\Omega \\ \downarrow{\scriptstyle c_{E_j}} & & \downarrow{\scriptstyle c_{S\oplus\Omega}} \\ -E_i+E_k & \xrightarrow{c_{E_{i,k}}} & S\oplus\Omega \end{array}$$

is commutative it will be enough to show that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{j,n+1})+\mathcal{D}(\mathcal{U}^2_{j,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{J_{i,j}+J_{j,k}} & -E_i+E_j-E_j+E_k \\ \downarrow{\scriptstyle \phi^{C_2}_{C_2'}} & & \downarrow{\scriptstyle c_{E_j}} \\ \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{J_{i,k}} & -E_i+E_k \end{array}$$

The following diagram commutes since it simply says that the contraction maps commute with isomorphisms:

$$\begin{array}{ccc} \begin{array}{c}-\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{i,n+1})+\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{j,n+1})\\ -\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{j,n+1})+\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{k,n+1})\end{array} & \xrightarrow{-T_i+T_j-T_j+T_k} & -E_i+E_j-E_j+E_k \\ \downarrow{\scriptstyle c_{\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{j,n+1})}} & & \downarrow{\scriptstyle \mathrm{Id}_{-E_i}+c_{E_j}+\mathrm{Id}_{E_k}} \\ \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{-T_i+T_k} & -E_i+E_k \end{array}$$

so it will be enough to show that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{j,n+1})+\mathcal{D}(\mathcal{U}^2_{j,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow[+\tau_j'\circ\phi^{C_{j,k}}_{C'_{j,k}}]{\tau_i'\circ\phi^{C_{i,j}}_{C'_{i,j}}} & \begin{array}{c}-\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{i,n+1})+\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{j,n+1})\\ -\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{j,n+1})+\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{k,n+1})\end{array} \\ \downarrow{\scriptstyle \phi^{C_2}_{C_2'}} & & \downarrow{\scriptstyle c_{\mathcal{D}(\mathcal{U}^1_{j,n+1},\mathcal{U}^2_{j,n+1})}} \\ \mathcal{D}(\mathcal{U}^2_{i,n+1},\mathcal{U}^2_{k,n+1}) & \xrightarrow{\tau_i'\circ\phi^{C_{i,k}}_{C'_{i,k}}} & -\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{i,n+1})+\mathcal{D}(\mathcal{U}^1_{i,n+1},\mathcal{U}^2_{k,n+1}) \end{array}$$

where here for any indice α the map τ_α' denotes $\tau'_{\mathcal{U}^2_{\alpha,n+1},\mathcal{U}^1_{\alpha,n+1}}$. But this diagram commutes by Remark 4.23 and Lemma 4.27. This completes the proof of Lemma 4.32. □

End of the proof of Proposition 4.28. We consider the extension $\mathcal{X}'^2_{n+1}$ constructed in the above lemma. We have that $\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}'^2_{n+1})|_{U_i}=E_i$. It follows that the extension $E-\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}'^2_{n+1})|_{U_i}$ is trivial and hence, by Proposition 4.29, there exists a global extension $\mathcal{X}^2_{n+1}$ such that

$$\mathcal{D}(\mathcal{X}'^2_{n+1},\mathcal{X}^2_{n+1})\cong E-\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}'^2_{n+1})$$

or in other words

$$\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}'^2_{n+1})+\mathcal{D}(\mathcal{X}'^2_{n+1},\mathcal{X}^2_{n+1})\cong E.$$

By Proposition 4.11 we have that $\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^2_{n+1})=E$. This completes the proof of the torsor character of $\mathcal{D}(\mathcal{X}^1_{n+1},\mathcal{X}^2_{n+1})$. □

This completes the proof of Theorem 1.10. □

5. OBSTRUCTIONS.

This section will be devoted to a proof of the following theorem.

THEOREM 5.1. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of a reduced l.c.i. k-variety X of finite type with normal bundle V. We can associate an element* $\mathrm{ob}_{\mathcal{X}_n} \in \mathrm{Ext}^2(\Omega, S)$ *to $\mathcal{X}_n$ in such a way that there exists an extension of $\mathcal{X}_n$ if and only if* $\mathrm{ob}_{\mathcal{X}_n} = 0$.

Proof of Theorem 5.1.

We will use the notion of an *extension cocycle*. Section 5.1 below is slightly adapted from Vistoli [12]: we include it for completeness's sake.

5.1 EXTENSION COCYCLES AND CLASSES OF EXTENSION COCYCLES.

We fix an open affine covering U_i of X which has the following property: for any i there exists an extension $\mathcal{U}_{i,n+1}$ of $\mathcal{U}_{i,n}$. Throughout this section, U_i will refer to this choice of open affine covering and if $0 \to F \xrightarrow{i_E} E \xrightarrow{\pi_E} G \to 0$ is an extension of sheaves then we will denote $i_E(s)$ by s_E for any $s \in F$. We will also denote the intersection $U_{i_1} \cap U_{i_2} \ldots \cap U_{i_k}$ by $U_{i_1,\ldots,i_k}$. Given a sheaf $\mathcal{F}$ or a map of sheaves ϕ which is defined on a set containng $U_{i_1,\ldots,i_k}$ we denote the restriction of $\mathcal{F}$ (resp. ϕ) to $U_{i_1,\ldots,i_k}$ by $\mathcal{F}_{i_1,\ldots,i_k}$ (resp. $\phi_{i_1,\ldots,i_k}$).

DEFINITION 5.2. An extension cocycle of Ω by S with respect to U_i is a collection $\{E_{i,j}, F_{i,j,k}\}$ such that for every pair (i,j) $E_{i,j}$ is an extension of $\Omega_{U_{i,j}}$ by $S_{U_{i,j}}$ in the category of $\mathcal{O}_{U_{i,j}}$-modules and for every triple (i,j,k) $F_{i,j,k}$ is an isomorphism defined over $U_{i,j,k}$, $F_{i,j,k} : E_{i,j} + E_{j,k} \to E_{i,k}$, such that for any i,j,k,l we have the following associativity relation

$$F_{i,j,l} \circ (F_{i,j,k} + \mathrm{id}_{E_{k,l}}) = F_{i,k,l} \circ (id_{E_{i,j}} + F_{j,k,l}).$$

DEFINITION 5.3. Let $\{E_{i,j}, F_{i,j,k}\}$ and $\{E'_{i,j}, F'_{i,j,k}\}$ be two extension cocycles of Ω by S with respect to U_i. An isomorphism $\Phi : \{E_{i,j}, F_{i,j,k}\} \to \{E'_{i,j}, F'_{i,j,k}\}$ is a collection of morphisms of extensions $\phi_{i,j} : E_{i,j} \to E'_{i,j}$ indexed by pairs (i,j) such that for all triples i,j,k

$$\phi_{i,k} \circ F_{i,j,k} = F'_{i,j,k} \circ (\phi_{i,j} + \phi_{j,k}).$$

From now on, we will deal not with the set of extension cocycles but with the set of isomorphism classes of extension cocycles.

REMARK 5.4. Let $G_{1,\{U_i\}}(\Omega, S)$ be the set of isomorphism classes of extension cocycles of Ω by S with respect to U_i. $G_{1,\{U_i\}}(\Omega, S)$ is then an abelian group with group law given by

$$[\{E_{i,j}, F_{i,j,k}\}] + [\{E'_{i,j}, F'_{i,j,k}\}] = [\{E_{i,j} + E'_{i,j}, F_{i,j,k} + F'_{i,j,k}\}].$$

The zero element is the element $[(S_{i,j} \oplus \Omega_{i,j}), c_{S_{i,j,k} \oplus \Omega_{i,j,k}}]$.

We will now define coboundaries of collections of extensions.

DEFINITION 5.5. Let U_i be an open affine covering of X and for every i let E_i be an extension of S_i and Ω_i over U_i. We define the coboundary of the collection $\{E_i\}$, denoted by $\delta(\{E_i\})$, by

$$\delta(\{E_i\}) = \{E_i - E_j, \mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k}\}.$$

REMARK 5.6. The set $G_{2,\{U_i\}}(\Omega, S) = \oplus_i \mathrm{Ext}^1_{U_i}(\Omega_{U_i}, S_{U_i})$ is an abelian group with group law given by addition of extensions. The map $\delta : G_{2,\{U_i\}}(\Omega, S) \to G_{1,\{U_i\}}(\Omega, S)$ is a group morphism.

We are now in a position to define the set of cocycle classes of S and Ω.

DEFINITION 5.7. Let U_i be an open affine cover of X as above. We define $E_{\{U_i\}}(\Omega, S)$, the set of cocycle classes of S and Ω with respect to U_i, by

$$E_{\{U_i\}}(\Omega, S) = G_{1,\{U_i\}}(\Omega, S)/\delta(G_{2,\{U_i\}}(\Omega, S)).$$

PROPOSITION 5.8. *Under the above hypotheses, there is a group isomorphism* $\gamma_{\Omega,S,\{U_i\}} : E_{\{U_i\}}(\Omega, S) \to \mathrm{Ext}^2(\Omega, S)$.

Proof of Proposition 5.8.

Let $[\{E_{i,j}, F_{i,j,k}\}]$ be an element of $G_{1,\{U_i\}}(S, \Omega)$. We now construct $\gamma_{\Omega,S,\{U_i\}}[\{E_{i,j}, F_{i,j,k}\}]$ Choose an exact sequence of $\mathcal{O}_X$-modules $0 \to S \stackrel{i_K}{\to} K \stackrel{\pi_K}{\to} Q \to 0$ such that K is an injective sheaf on X and hence $\mathrm{Ext}^2(\Omega, S) = \mathrm{Ext}^1(\Omega, Q)$. As $\mathcal{E}xt^2(\Omega, S) = \mathcal{E}xt^1(\Omega, Q)$ and $\mathcal{E}xt^2(\Omega, S) = 0$ since X is a locally complete intersection, it follows that $\mathrm{Ext}^1(\Omega, Q) = H^1(\mathcal{H}om(\Omega, Q))$.

Over each $U_{i,j}$, we have the following digram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_{i,j} & \xrightarrow{i_{E_{i,j}}} & E_{i,j} & \xrightarrow{\pi_{E_{i,j}}} & \Omega_{i,j} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_{K_{i,j}}} & & & & & & \\ & & K_{i,j} & & & & & & \end{array}$$

Since K is injective, there exist maps $f_{i,j} : E_{i,j} \to K_{i,j}$ such that $f_{i,j} \circ i_{E_{i,j}} = i_{K_{i,j}}$. We say that the maps $f_{i,j}$ are compatible with the $F_{i,j,k}s$ if for any triple i, j, k we have that $f_{i,j} \circ F_{i,j,k} = (f_{i,j} + f_{j,k})$.

LEMMA 5.9. *There exist maps* $f_{i,j} : E_{i,j} \to K_{i,j}$ *such that for all pairs* i, j *we have that* $f_{i,j} \circ i_{E_{i,j}} = i_{K_{i,j}}$ *and for all triples* i, j, k *we have that* $f_{i,j} \circ F_{i,j,k} = (f_{i,j} + f_{j,k})$.

Proof of Lemma 5.9.

We note that if s is an element of $S_{i,j,k}$ then

$$i_{K_{i,k}}(s) = f_{i,k} \circ F_{i,j,k}(s_{E_{i,j}+E_{j,k}}) = (f_{i,j} + f_{j,k})(s_{E_{i,j}+E_{j,k}}).$$

In particular, for each i, j, k there is a unique map $\tau_{i,j,k} : \Omega_{i,j,k} \to K_{i,j,k}$, such that

$$\tau_{i,j,k} \circ \pi_{E_{i,j}+E_{j,k}} = f_{i,k} \circ F_{i,j,k} - (f_{i,j} + f_{j,k}).$$

For any i, j, k, l we have that $\tau_{i,j,k} - \tau_{i,j,l} + \tau_{i,k,l} - \tau_{j,k,l} = 0$. In particular, the $\tau_{i,j,k}$ form a Čech cocycle and hence determine an element of $H^2(\mathcal{H}om(\Omega, K))$. Since K is injective, $H^2(\mathcal{H}om(\Omega, K)) = 0$, so there are elements $\tau_{i,j} \in \mathcal{H}om(\Omega_{i,j}, K_{i,j})$ such that for all i, j, k we have that $\tau_{i,j,k} = \tau_{i,j} - \tau_{i,k} + \tau_{j,k}$. We now define maps $f'_{i,j} : E_{i,j} \to K_{i,j}$ by setting $f'_{i,j} = f_{i,j} + \tau_{i,j} \circ \pi_{E_{i,j}}$. We note that the $f'_{i,j}$s form an alternative choice of liftings of the maps $i_{K_{i,j}} : S_{i,j} \to K_{i,j}$. It remains only to show that

$$f'_{i,k} \circ F_{i,j,k} = f'_{i,j} + f'_{j,k}.$$

We note that

$$f'_{i,k} \circ F_{i,j,k} - (f'_{i,j} + f'_{j,k}) = f_{i,k} \circ F_{i,j,k} - (f_{i,j} + f_{j,k}) + \tau_{i,k} \circ \pi_{E_{i,k}} \circ F_{i,j,k} - (\tau_{i,j} + \tau_{j,k}) \circ \pi_{E_{i,j}+E_{j,k}}$$
$$= \tau_{i,j,k} \circ \pi_{E_{i,j}+E_{j,k}} + (\tau_{i,k} - \tau_{i,j} - \tau_{j,k}) \circ (\pi_{E_{i,j}+E_{j,k}}) = 0.$$

This completes the proof of Lemma 5.9. □

Henceforth, we assume that the maps $f_{i,j}$ are compatible with the $F_{i,j,k}$s. Projecting onto $Q_{i,j}$, we obtain maps $\overline{f}_{i,j} = \pi_{K_{i,j}} \circ f_{i,j} : E_{i,j} \to Q_{i,j}$. Since $\overline{f}_{i,j}(s_{E_{i,j}}) = 0$ for any $s \in S_{i,j}$ there are unique maps $g_{i,j} : \Omega_{i,j} \to Q_{i,j}$ such that $\overline{f}_{i,j} = g_{i,j} \circ \pi_{E_{i,j}}$.

LEMMA 5.10. *The maps* $g_{i,j} : \Omega_{i,j} \to Q_{i,j}$ *defined above have the property that for any pair* (i,j) *we have that* $g_{i,k} = g_{i,j} + g_{j,k}$ *over* $U_{i,j,k}$.

Proof of Lemma 5.10.

We note that $f_{i,k} \circ F_{i,j,k} = (f_{i,j} + f_{j,k})$ whence $\overline{f}_{i,k} \circ F_{i,j,k} = (\overline{f}_{i,j} + \overline{f}_{j,k})$. Since $F_{i,j,k}$ is a morphism of extensions we have that $(\pi_{E_{i,j}+E_{j,k}}) = \pi_{E_{i,k}} \circ F_{i,j,k}$ and we deduce that

$$\overline{f}_{i,k} \circ F_{i,j,k} = g_{i,k} \circ \pi_{E_{i,k}} \circ F_{i,j,k} = g_{i,k} \circ \pi_{E_{i,j}+E_{j,k}}.$$

It follows that

$$g_{i,k} \circ (\pi_{E_{i,j}+E_{j,k}}) = (\overline{f}_{i,j} + \overline{f}_{j,k})$$

whence, for all $e_{i,j} \in E_{i,j}$, $e_{j,k} \in E_{j,k}$ such that $\pi_{E_{i,j}}(e_{i,j}) = \pi_{E_{j,k}}(e_{j,k})$ we have that

$$g_{i,k} \circ (\pi_{E_{i,j}+E_{j,k}})[e_{i,j}, e_{j,k}] = (\overline{f}_{i,j} + \overline{f}_{j,k})[e_{i,j}, e_{j,k}]$$

where $[e_{i,j}, e_{j,k}]$ is the equivalence class of $(e_{i,j}, e_{j,k})$ in $E_{i,j} + E_{j,k}$. It follows that

$$g_{i,k}(\pi_{E_{i,j}}(e_{i,j})) = f_{i,j}(e_{i,j}) + f_{j,k}(e_{j,k})$$

$$g_{i,k}(\pi_{E_{i,j}}(e_{i,j})) = g_{i,j}(\pi_{E_{i,j}}(e_{i,j})) + g_{j,k}(\pi_{E_{j,k}}(e_{j,k})).$$

But by definition of $E_{i,j} + E_{j,k}$ we have that $\pi_{E_{i,j}}(e_{i,j}) = \pi_{E_{j,k}}(e_{j,k})$ and it follows that

$$g_{i,k}(\pi_{E_{i,j}}(e_{i,j})) = g_{i,j}(\pi_{E_{i,j}}(e_{i,j}) + g_{j,k}(\pi_{E_{i,j}}(e_{i,j}))$$

for any $e_{i,j} \in E_{i,j}$ and hence $g_{i,k} = g_{i,j} + g_{j,k}$. This completes the proof of Lemma 5.10. □

DEFINITION 5.11. The element $\gamma_{\Omega,S,\{U_i\}}[\{E_{i,j}, F_{i,j,k}\}] \in E_{\{U_i\}}(\Omega, S) = H^1(\mathcal{H}om(\Omega, Q))$ is defined to be $[g_{i,j}]$, the class in $H^1(\mathcal{H}om(\Omega, Q))$ represented by the Čech cocycle $(g_{i,j})$.

It remains only to prove the three following results.

LEMMA 5.12. *Let* $\{E_{i,j}, F_{i,j,k}\}$ *be an extension cocycle of* S *and* Ω *with respect to the open covering* U_i. *Let* $f_{i,j} : E_{i,j} \to K_{i,j}$ *be a set of maps such that* $f_{i,j}(s_{E_{i,j}}) = s_{K_{i,j}}$ *for any* $s \in S$ *and for all* i,j,k *we have that* $f_{i,k} \circ F_{i,j,k} = (f_{i,j} + f_{j,k})$. *Let* $g_{i,j} : \Omega_{i,j} \to Q_{i,j}$ *be the unique maps such that* $g_{i,j} \circ \pi_{E_{i,j}} = \pi_{K_{i,j}} \circ f_{i,j}$. *The cohomology class* $[g_{i,j}] \in H^1(\mathcal{H}om(\Omega, Q))$ *is then independent of the choice of the maps* $f_{i,j}$.

PROPOSITION 5.13. *Let* $\{E_{i,j}, F_{i,j,k}\}$ *be an extension cocycle of* Ω *by* S. *We then have that* $\gamma_{\Omega,S,\{U_i\}}(\{E_{i,j}, F_{i,j,k}\}) = 0$ *if and only if there exists a collection of extensions* $\{E_i\}$ *such that* $\{E_{i,j}, F_{i,j,k}\}$ *is isomorphic to the boundary class* $\delta(\{E_i\})$.

LEMMA 5.14. *The map* $\gamma_{\Omega,S,\{U_i\}} : E_{\{U_i\}}(\Omega, S) \to \mathrm{Ext}^2(\Omega, S)$ *is surjective.*

Proof of Lemma 5.12.

Let $f'_{i,j}$ be an alternative choice of liftings and let $g'_{i,j}$ be the associated elements of $\mathcal{H}om(\Omega_{i,j}, Q_{i,j})$. We have that $f'_{i,j} - f_{i,j}|_{S_{i,j}} = 0$, so there is a unique map $h_{i,j} : \Omega_{i,j} \to K_{i,j}$ such that $f'_{i,j} - f_{i,j} = h_{i,j} \circ \pi_{E_{i,j}}$. In particular, $(g'_{i,j} - g_{i,j}) = \pi_{K_{i,j}} \circ h_{i,j}$. This implies that the cohomology class $[g'_{i,j} - g_{i,j}]$ has the property that

$$[g'_{i,j} - g_{i,j}] \in \pi_K(H^1(\mathcal{H}om(\Omega, K))) = \{0\}$$

where the last equality holds because K is injective. This completes the proof of Lemma 5.12. □

Proof of Proposition 5.13.

Assume that the class of $[g_{i,j}]$ is 0. Then there exist maps $g_i : \Omega_i \to Q_i$ such that for all (i,j) we have that $g_{i,j} = g_i - g_j$. We consider an extension E_i, which will be the pullback along g_i of the extension

$0 \to S_i \overset{i_{K_i}}{\to} K_i \overset{\pi_{K_i}}{\to} Q_i \to 0$. The extension E_i is then an extension of S_i by Ω_i equipped with a map $f_i : E_i \to K_i$, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_i & \xrightarrow{i_{E_i}} & E_i & \xrightarrow{\pi_{E_i}} & \Omega_i & \longrightarrow & 0 \\ & & & \searrow^{i_{K_i}} & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_i} & & \\ & & & & K_i & \xrightarrow{\pi_{K_i}} & Q_i & \longrightarrow & 0 \end{array}$$

and E_i has the following universality property.

REMARK 5.15. Universality property of pullbacks. Let E_i be the pullback of

$$0 \to S_i \to K_i \to Q_i \to 0.$$

along the morphism $g_i : \Omega_i \to Q_i$. If any extension $0 \to S_i \to F_i \to \Omega_i \to 0$ is equipped with a morphism $\phi : F_i \to K_i$ such that $\phi(s_{F_i}) = s_{K_i}$ and for all $v \in F$ $\pi_{K_i} \circ \phi(v) = g_i \circ \pi_{F_i}(v)$ then there is a unique morphism of extensions $\Phi : F_i \to E_i$ such that $f_i \circ \Phi = \phi$.

In particular, two extension maps, $g_1, g_2 : E \to E_i$ are equal if and only if $f_i \circ g_1 = f_i \circ g_2$.

We now prove that $\delta\{E_i\} \cong \{E_{i,j}, F_{i,j,k}\}$.

LEMMA 5.16. *The extension* $(E_i - E_j)$ *is isomorphic to the pull back along* $g_{i,j}$ *of* $0 \to S_i \overset{i_{K_i}}{\to} K_i \overset{\pi_{K_i}}{\to} Q_i \to 0$

Proof of Lemma 5.16.

We recall that for any $e_i \in E_i$ and $e_j \in E_j$ such that $\pi_{E_i}(e_i) = \pi_{E_j}(e_j)$ we denote by $[e_i, e_j]'$ the equivalence class of (e_i, e_j) in $E_i - E_j$.

By remark 5.15 it will be enough to produce a map $\tilde{f}_{i,j} : E_i - E_j \to K_{i,j}$ such that $\pi_{K_{i,j}} \circ \tilde{f}_{i,j} = g_{i,j} \circ \pi_{E_i - E_j}$. and $\tilde{f}_{i,j}(s) = i_{K_i}(s)$. We define $\tilde{f}_{i,j}$ as follows:

$$\tilde{f}_{i,j}([e_i, e_j]') = f_i(e_i) - f_j(e_j).$$

for all $e_i \in E_i$ and $e_j \in E_j$ such that $\pi_{E_i}(e_i) = \pi_{E_j}(e_j)$. We note that for all $s \in S$

$$\tilde{f}_{i,j}(i_{E_i}(s), i_{E_j}(s)) = i_{K_{i,j}}(s) - i_{K_{i,j}}(s) = 0$$

and hence the above map is well defined on the equivalence class $[e_i, e_j]$. We have that $\tilde{f}_{i,j}(i_{E_i}(s), 0) = f_i(s_{E_i}) = i_{K_i}(s)$. Further, we have that

$$\pi_{K_{i,j}} \circ \tilde{f}_{i,j}([e_i, e_j]') = \pi_{K_{i,j}}(f_i(e_i) - f_j(e_j)) = g_i(\pi_{E_i}(e_i)) - g_j(\pi_{E_j}(e_j))$$

$$= (g_i - g_j) \circ \pi_{E_i - E_j}([e_i, e_j]') = g_{i,j} \circ \pi_{E_i - E_j}([e_i, e_j]').$$

where in the last equation we have used the fact that $\pi_{E_i - E_j}([e_i, e_j]') = \pi_{E_i}(e_i) = \pi_{E_j}(e_j)$. This completes the proof of Lemma 5.16. □

But now, applying Remark 5.15 to the maps $f_{i,j}$ (which is possible because $\pi_{K_{i,j}} \circ f_{i,j} = g_{i,j} \circ \pi_{E_{i,j}}$) we see that for all i, j there is a unique isomorphism of extensions $\Phi_{i,j} : (E_i - E_j) \to E_{i,j}$ such that $f_{i,j} \circ \Phi_{i,j} = \tilde{f}_{i,j}$. The following lemma says that the collection $\Phi_{i,j}$ is in fact an isomorphism of extension cocycles between $\delta\{E_i\}$ and $\{E_{i,j}, F_{i,j,k}\}$.

LEMMA 5.17. *We have that for all* i, j, k

$$\Phi_{i,k} \circ (\mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k}) = F_{i,j,k} \circ (\Phi_{i,j} + \Phi_{j,k}).$$

Proof of Lemma 5.17.

By Remark 5.15 it will be enough to show that

$$f_{i,k} \circ \Phi_{i,k} \circ (\mathrm{Id}_{E_i} + c_{E_j} + \mathrm{Id}_{-E_k}) = f_{i,k} \circ F_{i,j,k} \circ (\Phi_{i,j} + \Phi_{j,k}).$$

This is equivalent to

$$\tilde{f}_{i,k} \circ (\mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k}) = f_{i,k} \circ F_{i,j,k} \circ (\Phi_{i,j} + \Phi_{j,k}).$$

by definition of $f_{i,j}$ and this is equivalent to

$$\tilde{f}_{i,k} \circ (\mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k}) = (f_{i,j} + f_{j,k}) \circ (\Phi_{i,j} + \Phi_{j,k}).$$

We calculate that for all $e_{i,j} \in E_{i,j}$, $e_{j,k} \in E_{j,k}$ such that $\pi_{E_{i,j}}(e_{i,j}) = \pi_{E_{j,k}}(e_{j,k})$, we have that

$$(f_{i,j} + f_{j,k}) \circ (\Phi_{i,j} + \Phi_{j,k})[e_{i,j}, e_{j,k}] = (f_{i,j} + f_{j,k})([\Phi_{i,j}(e_{i,j}), \Phi_{j,k}(e_{j,k})])$$
$$= f_{i,j} \circ \Phi_{i,j}(e_{i,j}) + f_{j,k} \circ \Phi_{j,k}(e_{j,k}) = \tilde{f}_{i,j}(e_{i,j}) + \tilde{f}_{j,k}(e_{j,k}).$$

So it is enough to show that $\tilde{f}_{i,k} \circ (\mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k}) = \tilde{f}_{i,j}(e_{i,j}) + \tilde{f}_{j,k}(e_{j,k})$. We calculate that for all $e_i \in E_i$, $e_j \in E_j$, $e_k \in E_k$, we have that

$$\tilde{f}_{i,k} \circ (\mathrm{Id}_{E_i} + c_{-E_j} + \mathrm{Id}_{-E_k})([[e_i, e_j]', [e_j, e_k]'] = \tilde{f}_{i,k}([e_i, e_k]') = f_i(e_i) - f_k(e_k)$$

But on the other hand

$$(\tilde{f}_{i,j} + \tilde{f}_{j,k})([[e_i, e_j]', [e_j, e_k]']) = \tilde{f}_{i,j}([e_i, e_j]') + \tilde{f}_{j,k}([e_j, e_k]') = f_i(e_i) - f_k(e_k).$$

This completes the proof of Lemma 5.17. □

This proves that if $[g_{i,j} = 0]$ then $\gamma_{\Omega,S,\{U_i\}}(\{E_{i,j}, F_{i,j,k}\}) = 0$. It remains to prove the converse. Suppose that $\gamma_{\Omega,S,\{U_i\}}(\{E_{i,j}, F_{i,j,k}\}) = 0$, or in other words there exist extensions E_i over U_i such that $\{E_{i,j}, F_{i,j,k}\} \overset{\Phi}{\cong} \delta\{E_i\}$. We can choose maps $f_i : E_i \to K_i$ lifting $i_{K_i} : S_i \to K_i$ which give rise to liftings $f_i - f_j : E_i - E_j \to K_i$ given by $(f_i - f_j)([e_i, e_j]') = f_i(e_i) - f_j(e_j)$. Since there are isomorphisms $\Phi_{i,j} : E_{i,j} \to E_i - E_j$ we have a map $f_{i,j} = (f_i - f_j) \circ \Phi_{i,j} : E_{i,j} \to K_{i,j}$ lifting $i_{K_{i,j}} : S \to K_{i,j}$. We note that $f_{i,k} \circ F_{i,j,k} = (f_i - f_k) \circ \Phi_{i,k} \circ F_{i,j,k} = (f_i - f_k) \circ (\mathrm{Id} + c_{-E_j} + \mathrm{Id}) \circ \Phi_{i,j} + \Phi_{j,k}$. For any $e_i \in E_i$, $e_j \in E_j$ and $e_k \in E_k$ we have that $(f_i - f_k) \circ (\mathrm{Id} + c_{-E_j} + \mathrm{Id})([[e_i, e_j]'[e_j, e_k]']) = (f_i - f_k)([e_i, e_j]') = f_i(e_i) - f_k(e_k)$. On the other hand we have that $(f_i - f_j + f_j - f_k)([[e_i, e_j]', [e_j, e_k]']) = f_i(e_i) - f_j(e_j) + f_j(e_j) - f_k(e_k) = (f_i - f_k) \circ (\mathrm{Id} + c_{-E_j} + \mathrm{Id})([[e_i, e_j]'[e_j, e_k]'])$. Since any element of $(E_i - E_j) + (E_j - E_k)$ can be written in the form $[[e_i, e_j]'[e_j, e_k]']$ we then have that $f_{i,k} \circ F_{i,j,k} = f_{i,j} + f_{j,k}$ so the $f_{i,j}$s defined above are compatible with the $F_{i,j,k}$s. But the associated $g_{i,j}s$ are simply $g_{i,j} = g_i - g_j$ where g_i is the unique map $g_i : \Omega_i \to Q_i$ such that $\pi_{K_i} \circ f_i = g_i \circ \pi_{E_i}$. It follows that $[g_{i,j}] = 0$.

This completes the proof of Proposition 5.13. □

Proof of Lemma 5.14.

Let $[g_{i,j}]$ be a cocycle class in $H^1(\mathcal{H}om(\Omega, Q))$. We define $E_{i,j}$ to be the pull-back of the extension $0 \to S_{i,j} \overset{i_{K_{i,j}}}{\to} K_{i,j} \overset{\pi_{K_{i,j}}}{\to} Q_{i,j} \to 0$ along $g_{i,j}$. There is therefore a unique map $f_{i,j} : E_{i,j} \to K_{i,j}$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_{i,j} & \longrightarrow & E_{i,j} & \xrightarrow{\pi_{E_{i,j}}} & \Omega_{i,j} & \longrightarrow & 0 \\ & & & \searrow & \downarrow{\scriptstyle f_{i,j}} & & \downarrow{\scriptstyle g_{i,j}} & & \\ & & & & K_{i,j} & \xrightarrow{\pi_{K_{i,j}}} & Q_{i,j} & \longrightarrow & 0 \end{array}$$

By remark 5.15, to construct a map $F_{i,j,k} : E_{i,j} + E_{j,k} \to E_{i,k}$ it will be enough to find a map $\tilde{f}_{i,j,k} : E_{i,j} + E_{j,k} \to K_{i,j,k}$ such that $\tilde{f}_{i,j,k}(s_{E_{i,j}+E_{j,k}}) = s_{K_{i,j,k}}$ and $\pi_{K_{i,j,k}} \circ \tilde{f}_{i,j,k} = g_{i,k} \circ \pi_{E_{i,j}+E_{j,k}}$.

We define $\tilde{f}_{i,j,k}$ as follows. For any $e_{i,j} \in E_{i,j}$ and $e_{j,k} \in E_{j,k}$ we set

$$\tilde{f}_{i,j,k}([e_{i,j},e_{j,k}]) = f_{i,j}(e_{i,j}) + f_{j,k}(e_{j,k}).$$

It is readily checked that $\tilde{f}_{i,j,k}$ satisfies the two given conditions. There is therefore a map of extensions $F_{i,j,k} : E_{i,j} + E_{j,k} \to E_{i,k}$ such that $f_{i,k} \circ F_{i,j,k} = \tilde{f}_{i,j,k}$. It will now be enough to check the compatibility condition

$$F_{i,j,l} \circ (F_{i,j,k} + \mathrm{id}_{E_{k,l}}) = F_{i,k,l} \circ (\mathrm{id}_{E_{i,j}} + F_{j,k,l}).$$

It will be enough to check that

$$f_{i,l} \circ F_{i,j,l} \circ (F_{i,j,k} + \mathrm{id}_{E_{k,l}}) = f_{i,l} \circ F_{i,k,l} \circ (\mathrm{id}_{E_{i,j}} + F_{j,k,l}).$$

This is equivalent to $\tilde{f}_{i,k,l} \circ (F_{i,j,k} + \mathrm{id}_{E_{k,l}}) = \tilde{f}_{i,k,l} \circ (\mathrm{id}_{E_{i,j}} + F_{j,k,l})$. We calculate that for $e_{i,j} \in E_{i,j}, e_{j,k} \in E_{j,k}$ and $e_{k,l} \in E_{k,l}$ such that $\pi_{E_{i,j}}(e_{i,j}) = \pi_{E_{j,k}}(e_{j,k}) = \pi_{E_{k,l}}(e_{k,l})$ we have that

$$\tilde{f}_{i,k,l} \circ (F_{i,j,k} + \mathrm{id}_{E_{k,l}})[e_{i,j}, e_{j,k}, e_{k,l}] = \tilde{f}_{i,k,l}[F_{i,j,k}([e_{i,j}, e_{j,k}]), e_{k,l}])$$

$$= f_{i,k} \circ F_{i,j,k}([e_{i,j}, e_{j,k}]) + f_{k,l}(e_{k,l}) = \tilde{f}_{i,j,k}([e_{i,j}, e_{j,k}]) + f_{k,l}(e_{k,l})$$

$$= f_{i,j}(e_{i,j}) + f_{j,k}(e_{j,k}) + f_{k,l}(e_{k,l}).$$

A similar calculation show that

$$\tilde{f}_{i,k,l} \circ (\mathrm{id}_{E_{i,j}} + F_{j,k,l})([e_{i,j}, e_{j,k}, e_{k,l}]) = f_{i,j}(e_{i,j}) + f_{j,k}(e_{j,k}) + f_{k,l}(e_{k,l}).$$

This completes the proof of Lemma 5.14. □

This completes the proof of Proposition 5.8. □

In the next section, we will use the isomorphism described above to prove Theorem 5.1.

5.2 Proof of Theorem 5.1.

We choose an open cover U_i such that $\mathcal{C}(U_i)$ is not empty for any i. Throughout this section, $\mathcal{D}(\mathcal{U}_{i,n+1}, \mathcal{U}_{j,n+1})$ will be denoted by $\mathcal{D}_{i,j}$, $\mathcal{D}(\mathcal{U}_{i,n+1}, \mathcal{U}'_{j,n+1})$ by $\mathcal{D}_{i,j'}$, $\mathcal{D}(\mathcal{U}'_{i,n+1}, \mathcal{U}_{j,n+1})$ by $\mathcal{D}_{i',j}$ and $\mathcal{D}(\mathcal{U}'_{i,n+1}, \mathcal{U}'_{j,n+1})$ by $\mathcal{D}_{i',j'}$. Likewise, $\tau(\mathcal{U}_{i,n+1}, \mathcal{U}_{j,n+1})$ (resp. $\tau(\mathcal{U}_{i',n+1}, \mathcal{U}_{j,n+1})$, $\tau(\mathcal{U}_{i,n+1}, \mathcal{U}_{j',n+1})$, $\tau(\mathcal{U}_{i',n+1}, \mathcal{U}_{j',n+1})$) will be denoted by $\tau_{i,j}$ (resp $\tau_{i',j}$ $\tau_{i,j'}$ $\tau_{i',j'}$) and $\tau'(\mathcal{U}_{i,n+1}, \mathcal{U}_{j,n+1})$ (resp. $\tau'(\mathcal{U}_{i',n+1}, \mathcal{U}_{j,n+1})$, $\tau'(\mathcal{U}_{i,n+1}, \mathcal{U}_{j',n+1})$, $\tau'(\mathcal{U}_{i',n+1}, \mathcal{U}_{j',n+1})$) will be denoted by $\tau'_{i,j}$ (resp $\tau'_{i',j}$ $\tau'_{i,j'}$ $\tau'_{i',j'}$).

Definition 5.18. For each i let $\mathcal{U}_{i,n+1}$ be an extension of $\mathcal{U}_{i,n}$. We set

$$\mathrm{ob}_{\mathcal{X}_n} = \gamma_{\Omega,S,\{U_i\}}([\{\mathcal{D}_{i,j}, \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}\}]).$$

Proposition 5.19. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V and let U_i be an open cover of X such that an extension $\mathcal{U}_{i,n+1}$ of $\mathcal{U}_{i,n}$ exists for all i. The element $[\{\mathcal{D}_{i,j}, \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}\}] \in E_{\{U_i\}}(\Omega, S)$ is then independent of the choice of extensions $\mathcal{U}_{i,n+1}$.*

Proof of Proposition 5.19.

Let $\mathcal{U}'_{i,n+1}$ be another possible choice of extensions. We wish to show that $[\{\mathcal{D}_{i,j}, \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}\}]$ and $[\{\mathcal{D}_{i',j'}, \phi_{\mathcal{U}'_i,\mathcal{U}'_j,\mathcal{U}'_k}\}]$ are the same class in $E_{\{U_i\}}(\Omega, S)$. We consider the element of $G_{2,\{U_i\}}(\Omega, S)$ given by $\{\mathcal{D}_{i,i'}\}$. Over any set of the form $U_{i,j}$ we seek isomorphisms $\Phi_{i,j} : \delta\{\mathcal{D}_{i,i'}\} + \mathcal{D}_{i',j'} \to \mathcal{D}_{i,j}$ such that the following diagrams commute

$$\begin{array}{ccc}
\begin{array}{c}\mathcal{D}_{i,i'} - \mathcal{D}_{j,j'} + \mathcal{D}_{i',j'} \\ +\mathcal{D}_{j,j'} - \mathcal{D}_{k,k'} + \mathcal{D}_{j',k'}\end{array} & \xrightarrow{\Phi_{i,j}+\Phi_{j,k}} & \mathcal{D}_{i,j} + \mathcal{D}_{j,k} \\
\Big\downarrow {\scriptstyle c_{-\mathcal{D}_{j,j'}} + \phi_{\mathcal{U}_{i'},\mathcal{U}_{j'},\mathcal{U}_{k'}}} & & \Big\downarrow {\scriptstyle \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}} \\
\mathcal{D}_{i,i'} - \mathcal{D}_{k,k'} + \mathcal{D}_{i',k'} & \xrightarrow{\Phi_{i,k}} & \mathcal{D}_{i,k}
\end{array}$$

For all pairs (i,j) we consider the maps

$$\tau'_{j',j}{}^{-1} : \mathcal{D}_{i,i'} - \mathcal{D}_{j,j'} + \mathcal{D}_{i',j'} \to \mathcal{D}_{i,i'} + \mathcal{D}_{i',j'} + \mathcal{D}_{j',j}$$

and

$$\phi_{C_{i,j}}^{C'_{i,j}} : \mathcal{D}_{i,i'} + \mathcal{D}_{i',j'} + \mathcal{D}_{j',j} \to \mathcal{D}_{i,j}$$

where $C_{i,j}$ is the chain $(\mathcal{U}_i, \mathcal{U}'_i, \mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}_j)$ and $C'_{i,j}$ is the subchain $(\mathcal{U}_i, \mathcal{U}_j)$.

PROPOSITION 5.20. *The collection of maps* $\Phi_{i,j} = \phi_{C_{i,j}}^{C'_{i,j}} \circ \tau'_{j',j}{}^{-1} : \mathcal{D}_{i,i'} - \mathcal{D}_{j,j'} + \mathcal{D}_{i',j'} \to \mathcal{D}_{i,j}$ *is an isomorphism of extension cocycles.*

Proof of Proposition 5.20. We consider the following chains of extensions of U_n.

1. $C_1 = (\mathcal{U}_i, \mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}_j, \mathcal{U}'_j, \mathcal{U}'_k, \mathcal{U}_k)$
2. $C_2 = (\mathcal{U}_i, \mathcal{U}'_i, \mathcal{U}'_k, \mathcal{U}_k)$
3. $C_3 = (\mathcal{U}_i, \mathcal{U}_j, \mathcal{U}_k)$
4. $C_4 = (\mathcal{U}_i, \mathcal{U}_k)$

The following diagram of contraction maps commutes by Proposition 4.17.

$$\begin{array}{ccc}
\mathcal{D}(C_1) & \xrightarrow{\phi_{C_1}^{C_2}} & \mathcal{D}(C_2) \\
\Big\downarrow \phi_{C_1}^{C_3} & & \Big\downarrow \phi_{C_2}^{C_4} \\
\mathcal{D}(C_3) & \xrightarrow{\phi_{C_3}^{C_4}} & \mathcal{D}(C_4)
\end{array}$$

Expanding, we get a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}_{i,i'} + \mathcal{D}_{i',j'} + \mathcal{D}_{j',j} + \mathcal{D}_{j,j'} + \mathcal{D}_{j',k'} + \mathcal{D}_{k',k} & \xrightarrow{\phi_{C_1}^{C_2}} & \mathcal{D}_{i,i'} + \mathcal{D}_{i',k'} + \mathcal{D}_{k',k} \\
\Big\downarrow \phi_{C_1}^{C_3} = \phi_{C_{i,j}}^{C'_{i,j}} + \phi_{C_{j,k}}^{C'_{j,k}} & & \Big\downarrow \phi_{C_2}^{C_3} = \phi_{C_{i,k}}^{C'_{i,k}} \\
\mathcal{D}_{i,j} + \mathcal{D}_{j,k} & \xrightarrow{\phi_{C_3}^{C_4}} & \mathcal{D}_{i,k}
\end{array}$$

So, to establish that the choice $\Phi_{i,j} = \phi_{C_{i,j}}^{C'_{i,j}} \circ \tau'_{j',j}{}^{-1}$ is a cocycle isomorphism it will be enough to establish that the following diagram commutes.

$$\begin{array}{ccc}
\begin{array}{c}\mathcal{D}_{i,i'} - \mathcal{D}_{j,j'} + \mathcal{D}_{i',j'} \\ + \mathcal{D}_{j,j'} - \mathcal{D}_{k,k'} + \mathcal{D}_{j',k'}\end{array} & \xrightarrow{c_{-\mathcal{D}_{j,j'}} + \phi_{C_3}^{C_4}} & \mathcal{D}_{i,i'} - \mathcal{D}_{k,k'} + \mathcal{D}_{i',k'} \\
\Big\downarrow \tau'^{-1}_{j,j'} + \tau'^{-1}_{k,k'} & & \Big\downarrow \tau'_{k,k'}{}^{-1} \\
\mathcal{D}_{i,i'} + \mathcal{D}_{i',j'} + \mathcal{D}_{j',j} + \mathcal{D}_{j,j'} + \mathcal{D}_{j',k'} + \mathcal{D}_{k',k} & \xrightarrow{\phi_{C_1}^{C_2}} & \mathcal{D}_{i,i'} + \mathcal{D}_{i',k'} + \mathcal{D}_{k',k}
\end{array}$$

Eliminating $\mathcal{D}_{i,i'}$ and $\mathcal{D}_{k,k'}$ and permuting the terms we see that this is equivalent to proving that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{D}_{i',j'} - \mathcal{D}_{j,j'} + \mathcal{D}_{j,j'} + \mathcal{D}_{j',k'} & \xrightarrow{\phi_{C'_1}^{C'_2} \circ c_{(-\mathcal{D}_{j,j'})}} & \mathcal{D}_{i',k'} \\
\Big\downarrow \tau'^{-1}_{j,j'} & \nearrow_{\phi_{C'_3}^{C'_2}} & \\
\mathcal{D}_{i',j'} + \mathcal{D}_{j',j} + \mathcal{D}_{j,j'} + \mathcal{D}_{j',k'} & &
\end{array}$$

where here $C'_1 = (\mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}'_k)$, $C'_2 = (\mathcal{U}'_i, \mathcal{U}'_k)$ and $C'_3 = (\mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}_j, \mathcal{U}'_j, \mathcal{U}'_k)$. But this follows from Lemma 4.27 and Remark 4.23. This concludes the proof of Proposition 5.20. □

This completes the proof of Proposition 5.19. □

LEMMA 5.21. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X with normal bundle V. If there exists an extension of $\mathcal{X}_n$ then the cocycle class* $\mathrm{ob}_{\mathcal{X}_n} = 0 \in \mathrm{Ext}^2(\Omega, S)$.

Proof of Lemma 5.21.

Let $\mathcal{X}_{n+1}$ be the extension in question. We choose $\mathcal{U}_{i,n+1}$ to be the restriction to U_i of $\mathcal{X}_{n+1}$ and we consider the associated extension cocycle $\{\mathcal{D}(\mathcal{U}_{n+1}, \mathcal{U}_{n+1})|_{U_{i,j}}, \phi_{\mathcal{U}_{n+1},\mathcal{U}_{n+1},\mathcal{U}_{n+1}}|_{U_{i,j,k}}\}$. For each pair i, j we set

$$\Phi_{i,j} = r_{\mathrm{Id}}|_{U_{i,j}} : \mathcal{D}(\mathcal{X}_{n+1}, \mathcal{X}_{n+1})|_{U_{i,j}} \to S_{i,j} \oplus \Omega_{i,j}$$

To prove that the collection $\Phi_{i,j}$ is a isomorphism of extension cocycles it remains to show that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{D}(\mathcal{X}_{n+1}, \mathcal{X}_{n+1}) + \mathcal{D}(\mathcal{X}_{n+1}, \mathcal{X}_{n+1}) & \xrightarrow{\phi_C^{C'}} & \mathcal{D}(\mathcal{X}_{n+1}, \mathcal{X}_{n+1}) \\
\downarrow r_{\mathrm{Id}} & & \downarrow r_{\mathrm{Id}} \\
\mathcal{D}(\mathcal{X}_{n+1}, \mathcal{X}_{n+1}) & \xrightarrow{r_{\mathrm{Id}}} & S \oplus \Omega
\end{array}$$

where here $C = (\mathcal{X}_{n+1}, \mathcal{X}_{n+1}, \mathcal{X}_{n+1})$ and $C' = (\mathcal{X}_{n+1}, \mathcal{X}_{n+1})$. But this follows from Lemma 4.25. This completes the proof of Lemma 5.21. □

PROPOSITION 5.22. *Let $\mathcal{X}_n$ be an n-th order infinitesimal neighbourhood of X. If* $\mathrm{ob}_{\mathcal{X}_n}$ *is* 0 *then there exists an extension $\mathcal{X}_{n+1}$ of $\mathcal{X}_n$.*

Proof of Proposition 5.22.

We choose extensions $\mathcal{U}_{i,n+1}$ of $\mathcal{U}_{i,n}$ and we consider the associated extension cocycle $\{\mathcal{D}_{i,j}, \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}\}$. The fact that $[\{E_{i,j}, F_{i,j,k}\}] = 0$ in $E_{\{U_i\}}(\Omega, S)$ means we can choose extensions E_i over U_i and isomorphisms $I_{i,j} : \mathcal{D}(U_i, U_j) \to E_i - E_j$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{D}_{i,j} + \mathcal{D}_{j,k} & \xrightarrow{\phi_C^{C'}} & \mathcal{D}_{i,k} \\
\downarrow I_{i,j}+I_{j,k} & & \downarrow I_{i,k} \\
(E_i - E_j) + (E_j - E_k) & \xrightarrow{c_{-E_j}} & E_i - E_k
\end{array}$$

where here C is the chain $(\mathcal{U}_{i,n+1}, \mathcal{U}_{j,n+1}, \mathcal{U}_{k,n+1})$ and C' is the chain $(\mathcal{U}_{i,n+1}, \mathcal{U}_{k,n+1})$.

Alternatively, by Proposition 4.28 we can fix the following data.
1. Extensions $\mathcal{U}'_{i,n+1}$ of X over U_i,
2. Isomorphisms $I'_{i,j} : \mathcal{D}_{i,j} \to \mathcal{D}_{i,i'} - \mathcal{D}_{j,j'}$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{D}_{i,j} + \mathcal{D}_{j,k} & \xrightarrow{\phi_C^{C'}} & \mathcal{D}_{i,k} \\
\downarrow I'_{i,j}+I'_{j,k} & & \downarrow I'_{i,k} \\
(\mathcal{D}_{i,i'} - \mathcal{D}_{j,j'}) + (\mathcal{D}_{j,j'} - \mathcal{D}_{k,k'}) & \xrightarrow{c_{-\mathcal{D}_{j,j'}}} & \mathcal{D}_{i,i'} - \mathcal{D}_{k,k'}
\end{array}$$

There are maps

$$\phi_{C'_{i,j}}^{C_{i,j}} : \mathcal{D}_{i',j'} \to \mathcal{D}_{i',i} + \mathcal{D}_{i,j} + \mathcal{D}_{j,j'}$$

$$(\tau_{i,i'} + c_{-\mathcal{D}_{j,j'}}) : (\mathcal{D}_{i',i} + \mathcal{D}_{i,i'}) + (-\mathcal{D}_{j,j'} + \mathcal{D}_{j,j'}) \to S_{U_{i,j}} \oplus \Omega_{U_{i,j}}.$$

where $C_{i,j}$ is the chain $(\mathcal{U}'_{i,n+1}, \mathcal{U}_{i,n+1}, \mathcal{U}_{j,n+1}, \mathcal{U}'_{j,n+1})$ and $C'_{i,j}$ is the chain $(\mathcal{U}'_{i,n+1}, \mathcal{U}'_{j,n+1})$. Combining these maps, we get an isomorphism

$$T_{i,j} = (\tau_{i,j} + c_{\mathcal{D}(\mathcal{U}_j,\mathcal{U}'_j)}) \circ I'_{i,j} \circ \phi_{C'_{i,j}}^{C_{i,j}} : \mathcal{D}_{i',j'} \to S_{U_{i,j}} \oplus \Omega_{U_{i,j}}.$$

We can now consider the isomorphism $J_{i,j} = b(U_{i,j})^{-1}(T_{i,j}) : \mathcal{U}'_{i,n+1} \to \mathcal{U}'_{j,n+1}$ constructed in Section 4.1. To show that the $\mathcal{U}'_{i,n+1}$s actually glue together to get a global extension of $\mathcal{X}_n$, we will have to prove that $J_{j,k} \circ J_{i,j} = J_{i,k}$. By Lemma 4.33 this is the case if and only if $c_{S\oplus\Omega} \circ (T_{i,j} + T_{j,k}) = T_{i,k} \circ \phi_{\mathcal{U}_i,\mathcal{U}_j,\mathcal{U}_k}$.

LEMMA 5.23. *We have that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{D}_{i',j'}+\mathcal{D}_{j',k'} & \xrightarrow{\phi_C^{C'}} & \mathcal{D}_{i',k'} \\ \downarrow{\scriptstyle T_{i,j}+T_{j,k}} & & \downarrow{\scriptstyle T_{i,k}} \\ (S\oplus\Omega)+(S\oplus\Omega) & \xrightarrow{c_{S\oplus\Omega}} & S\oplus\Omega. \end{array}$$

where here C is the chain $(\mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}'_k)$ and C' is the chain $(\mathcal{U}'_i, \mathcal{U}'_k)$.

Proof of Lemma 5.23.

We consider the following diagram. Here we define C_1, C_2, C_3 and C_4 to be the following chains: $C_1 = (\mathcal{U}'_i, \mathcal{U}'_j, \mathcal{U}'_k)$, $C_2 = (\mathcal{U}'_i, \mathcal{U}'_k)$, $C_3 = (\mathcal{U}'_i, \mathcal{U}_i, \mathcal{U}_j, \mathcal{U}'_j, \mathcal{U}_j, \mathcal{U}_k, \mathcal{U}'_k)$ and $C_4 = (\mathcal{U}'_i, \mathcal{U}_i, \mathcal{U}_k, \mathcal{U}'_k)$.

$$\begin{array}{ccc} \mathcal{D}(C_1) & \xrightarrow{\phi_{C_1}^{C_2}} & \mathcal{D}(C_2) \\ \downarrow{\scriptstyle \phi_{C_1}^{C_3}} & & \downarrow{\scriptstyle \phi_{C_2}^{C_4}} \\ \mathcal{D}(C_3) & \xrightarrow{\phi_{C_3}^{C_4}} & \mathcal{D}(C_4) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ \begin{array}{c}\mathcal{D}_{i',i}+\mathcal{D}_{i,j}+\mathcal{D}_{j,k}+\\ \mathcal{D}_{k,k'}+\mathcal{D}_{j,j'}+\mathcal{D}_{j',j}\end{array} & \xrightarrow{\tau_{j,j'}} & \mathcal{D}(C_4) \\ \downarrow{\scriptstyle I'_{i,j}+I'_{j,k}} & & \downarrow{\scriptstyle I'_{i,k}} \\ \begin{array}{c}\mathcal{D}_{i,i'}+\mathcal{D}_{i,i'}-\mathcal{D}_{j,j'}+\mathcal{D}_{j,j'}\\ -\mathcal{D}_{k,k'}+\mathcal{D}_{k,k'}+\mathcal{D}_{j',j}+\mathcal{D}_{j,j'}\end{array} & \xrightarrow{c_{\mathcal{D}_{j,j'}}+\tau_{j,j'}} & \mathcal{D}_{i',i}+\mathcal{D}_{i,i'}-\mathcal{D}_{k,k'}+\mathcal{D}_{k,k'} \\ \downarrow{\scriptstyle \tau_{i',i}+c_{\mathcal{D}_{k,k'}}} & & \downarrow{\scriptstyle \tau_{i,i'}+c_{\mathcal{D}_{k,k'}}} \\ -\mathcal{D}_{j,j'}+\mathcal{D}_{j,j'}+\mathcal{D}_{j',j}+\mathcal{D}_{j,j'} & \xrightarrow{c_{S\oplus\Omega}\circ(c_{\mathcal{D}_{j,j'}}+\tau_{j,j'})} & S\oplus\Omega \end{array}$$

The first square of the diagram commutes by Proposition 4.23. The second square commutes by Lemma 4.27. The third square commutes by assumption and the last square commutes because if E_1 and E_2 are extensions and $\phi_1 : E_1 \to E_3$ and $\phi_2 : E_2 \to E_4$ are extension maps then $(\mathrm{Id} + \phi_2) \circ (\phi_1 + \mathrm{Id}) = (\phi_1 + \mathrm{Id}) \circ (\mathrm{Id} + \phi_2)$. But now, the right hand side map is $T_{i,k}$ and the the left hand side composed with the bottom map is $c_{S\oplus\Omega} \circ (T_{i,j} + T_{j,k})$. This completes the proof of Lemma 5.23. □

This completes the proof of Proposition 5.22. □

This completes the proof of Theorem 5.1. □

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