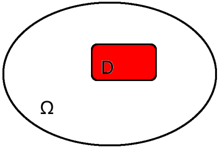
Superlensing using hyperbolic metamaterials

Eric Bonnetier, Université de Grenoble-Alpes
Hoai Minh Nguyen, EPF Lausanne

Outline:

1. Introduction



Consider Ω smooth bounded domain in $\mathbb{R}^d$ and $D \subset\subset \Omega$

Let $k \geq 0$. Metamaterials generally concern PDE's of the type

$$\begin{cases} \operatorname{div}(s_\delta a \nabla u_\delta) + k^2 s_\delta \sigma u_\delta = f & \text{in } \Omega \\ a\nabla u_\delta \cdot \nu - iku_\delta = 0 & \text{on } \partial\Omega \end{cases}$$

where the coefficients $a(x), \sigma(x)$ satisfy for some $a_0, \sigma_0 > 0$

$$\begin{array}{ccccccc} 1/a_0|\xi|^2 & \leq & a(x)\xi \cdot \xi & \leq & a_0|\xi|^2, & & \xi \in \mathbb{R}^d \\ 1/\sigma_0 & \leq & \sigma(x) & \leq & \sigma_0 & & \end{array}$$

and where for $\delta > 0$

$$s_\delta(x) = \begin{cases} 1 & x \in \Omega \setminus \overline{D} \\ -1 - i\delta & x \in D \end{cases}$$

The parameter δ models absorption of EM energy in the medium

When $\delta > 0$, the PDE (with proper BC's) has a unique solution

- Under what conditions do the u_δ's remain bounded or do they converge to some limiting u_0 ?

- If this is the case, in what sense does u_0 solve the limiting equation

$$\begin{cases} \operatorname{div}(s_0(x)a(x)\nabla u_0(x)) + k^2 s_0(x)\sigma(x)u_0(x) = f & \text{in } \Omega \\ a\nabla u_0 \cdot \nu - iku_0 = 0 & \text{on } \partial\Omega \end{cases} \quad ?$$

- What are the particular properties (localization, blow up,...) of the u_δ's or of u_0 ?

Metamaterials have become an active research area :

- Construction of metamaterials : Bouchitté-Bourel-Felbacq, Pendry, Shelby et al, Milton-McPhedran,. . .
- Well-posedness of transmission problems with sign-changing coefficients : Costabel-Stephan, Bonnet BenDiah-Ciarlet-Chesnel, Hoai-Minh Nguyen, . . .
- Extraordinary properties : cloaking, fields enhancement, superlensing : Veselago, Nicorovici-Milton-McPhedran, Ammari-Ciraolo-Kang-Lee-Milton, Kohn-Lu-Schweizer-Weinstein, Hoai-Minh Nguyen, . . .

We are particularly interested in hyperbolic metamaterials, where for $\delta = 0$ the coefficient matrix in the inclusion has positive and negative eigenvalues

Some natural material exhibit hyperbolic properties in certain frequency ranges

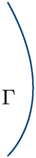
Interesting effects have been observed experimentally, particularly superlensing, focusing, and enhancement of nonlinear response
[Poddubny et al, Nature Photonics, Vol 7. Dec. 2013]

2. Superlensing using NIM's

2.1. Complementary media [Hoai-Minh Nguyen]

Assume that u solves a conduction equation in a medium that contains 2 phases separated by an interface Γ

$$\mathrm{div}(A_1 \nabla u) = f \qquad\qquad \mathrm{div}(A_2 \nabla u) = f$$



Γ $\qquad$ Ω_2

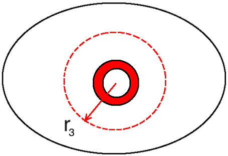
Let F be a reflection through Γ and $\quad v(x) = u \circ F^{-1}(x), \quad x \in \Omega_2$

Then u and v solve
$$\begin{cases} \mathrm{div}(A_2 \nabla u) = f & \text{in } \Omega_2 \\ \mathrm{div}(F_* A_1 \nabla v) = f & \text{in } \Omega_2 \\ v = u & \text{on } \Gamma \\ F_* A_1 \nabla v \cdot \nu = A_2 \nabla u \cdot \nu & \text{on } \Gamma \end{cases}$$

where $\quad F_* A_1(y) = \dfrac{DF(x) A_1(x) DF^T(x)}{|\det DF(x)|}$

If $F_* A_1 = A_2$ then unique continuation implies that $\quad u \equiv v$

2.2. Cloaking by complementary media



Consider the annulus $\{r_2 < r < r_3\}$ filled with NIM

in a domain Ω, such that $B_{r_3} \subset\subset \Omega, \quad r_3 = r_2^2/r_1$

Let $f \in L^2(\Omega)$ with $\quad Sptf \subset \Omega \setminus B_{r_3} \quad$ and let u_δ solve

$$\mathrm{div}(\varepsilon_\delta(x)\nabla u_\delta(x)) \;=\; f \qquad \text{in } \Omega$$

where $\quad \varepsilon_\delta \;=\; \begin{cases} -(1+i\delta) & \text{for } \; r_1 < |x| < r_2 \\ \quad 1 & \text{otherwise} \end{cases}$

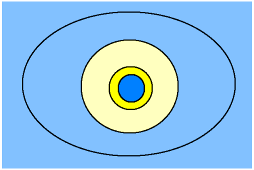
Assume that u_δ is uniformly bounded wrt δ and that $u_\delta \to u_0$ in $H^1(\Omega)$. Then

$$\mathrm{div}(\varepsilon_0(x)\nabla u_0(x)) \quad = \quad f \qquad \text{in } \Omega$$

Let $F(x) = r_2^2 x/|x|^2$ be the reflection through ∂B_{r_2} and

$$v(x) \quad = \quad u_0 \circ F^{-1}(x), \quad x \in \mathbb{R}^2 \setminus B_{r_2}$$

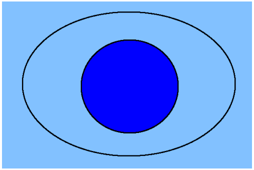
Then $v(x) \; = \; u_0(x), \quad x \in B_{r_3} \setminus B_{r_2}$



Next, let $G(x) = r_3^2 x/|x|^2$ be the reflection through ∂B_{r_3} and

$$w(x) \quad = \quad v \circ G^{-1}(x), \quad x \in B_{r_3}$$

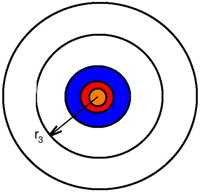
Then $\left\{ \begin{array}{lllll} \Delta w & = & 0 & & \text{in } B_{r_3} \\ w & = & v \; = \; u_0 & & \text{on } \partial B_{r_3} \\ \partial_\nu w|_- & = & \partial_\nu v|_- \; = \; \partial_\nu u_0 & & \text{on } \partial B_{r_3} \end{array} \right.$



It follows that $\mathcal{U}(x) \; = \; \left\{ \begin{array}{ll} u_0(x) & \text{in } \Omega \setminus B_{r_3} \\ w(x) & \text{in } B_{r_3} \end{array} \right.$ satisfies $\Delta \mathcal{U} = f$ in Ω

$\rightarrow$ An observer located outside of B_{r_3} cannot see the annulus

2.3. Superlensing via complementary media



Consider $0 < r_0 < r_1 < r_2 < r_3 < R \qquad m, \alpha > 1$

$$mr_0 = r_2 \qquad mr_1 = r_3 \qquad r_3 = r_2^{\alpha}/r_1^{\alpha-1}$$

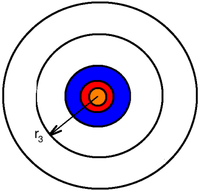
Let $F \; : B_{r_2} \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \overline{B_{r_2}}$ defined by $F(x) = r_2^{\alpha} x/|x|^{\alpha}$

Consider a conductivity map of the form

$$A_\delta(x) = \begin{cases} a(x) & \text{in } B_{r_0} \quad \text{(to be imaged)} \\ m^{d-2} I & \text{in } B_{r_1} \setminus B_{r_0} \\ (-1 - i\delta)\, F_*^{-1} I & \text{in } B_{r_2} \setminus B_{r_1} \\ I & \text{otherwise} \end{cases}$$

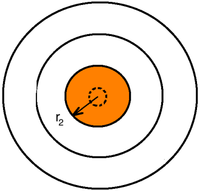
Let u_δ be the solution in $H^1_0(B_R)$ to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega = B_R$$



and let $\mathcal{U} \in H^1_0(B_R)$ be the solution to $\operatorname{div}(\mathcal{A}\nabla \mathcal{U}) = f$

with $\mathcal{A} = \begin{cases} a(x/m) & \text{in } B_{r_2} \\ I & \text{otherwise} \end{cases}$



Thm [HM Nguyen 15] Assume that $Spt f \cap B_{r_3} = \emptyset$, then

$u_\delta \rightharpoonup \mathcal{U}$ in $H^1(\Omega \setminus B_{r_3})$ as $\delta \to 0$

- Superlensing is achieved: for an observer outside B_{r_3}, the medium $a(x)$ that occupies B_{r_0} is perceived as

$$m^{2-d}a(x/m) \ \text{in} \ B_{r_2} = B_{mr_0}$$

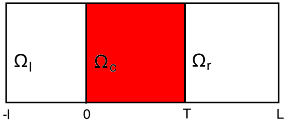
- A similar construction is valid in 3D and for the Helmholtz equation

- It uses reflecting complementary media, i.e. such that

$$F_*(x)A(x) \quad = \quad A(x), \ \ x \in B_{r_3} \setminus \overline{B_{r_2}}$$

so that u_δ and $u_\delta \circ F^{-1}$ satisfy the same equation and the same Cauchy data across ∂B_{r_2}

3. A toy problem for an elliptic/hyperbolic equation



Consider

$$\begin{aligned}\Omega &= \Omega_l \cup \Omega_c \cup \Omega_r \\ &= (-l,0)\times(0,2\pi) \cup (0,T)\times(0,2\pi) \cup (T,L)\times(0,2\pi)\end{aligned}$$

and $\Gamma = \partial\Omega, \quad \Gamma_l = \{0\}\times(0,2\pi), \quad \Gamma_r = \{T\}\times(0,2\pi)$

Let a be a uniformly elliptic matrix-valued conductivity and

$$a_\delta = \begin{pmatrix} 1-i\delta & 0 \\ 0 & -1-i\delta \end{pmatrix}$$

Consider the conductivity

$$A_\delta(x) = \begin{cases} a(x) & \text{in } \Omega_l \cup \Omega_r \\ a_\delta(x) & \text{in } \Omega_c \end{cases}$$

Let $f \in L^2(\Omega)$, with $\quad Spt(f) \cap \Omega_c = \emptyset$

and consider the equation

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) = f & \text{in } \Omega \\ u_\delta \in H^1_0(\Omega) \end{cases}$$

Assume that u_δ is uniformly bounded in $H^1(\Omega)$ and that $u_\delta \rightharpoonup u_0$ weakly in $H^1(\Omega)$

Then $u_0 \in H^1_0(\Omega)$ is a solution to

$$\begin{cases} \operatorname{div}(a\nabla u_0) & = & f & \text{in } \Omega \setminus \Omega_c \\ (\partial^2_{11} - \partial^2_{22})u_0 & = & 0 & \text{in } \Omega_c \end{cases}$$

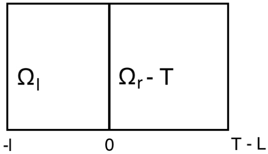
with the transmission conditions

$$\begin{cases} u_0|_{\Omega_c} & = & u_0|_{\Omega_l} \\ \partial_1 u_0|_{\Omega_c} & = & a\partial_1 u_0|_{\Omega_l} \end{cases} \text{on } \Gamma_l \qquad \begin{cases} u_0|_{\Omega_c} & = & u_0|_{\Omega_r} \\ \partial_1 u_0|_{\Omega_c} & = & a\partial_1 u_0|_{\Omega_r} \end{cases} \text{on } \Gamma_r$$

This is an ill-posed probem, except for special choices of T

Consider the effective domain

$$\hat{\Omega} \quad = \quad \Omega_l \cup \left(\begin{pmatrix} -T \\ 0 \end{pmatrix} + \Omega_r \right)$$



and define

$$\hat{a}(x_1, x_2), \hat{f}(x_1, x_2) \; = \begin{cases} a(x_1, x_2), f(x_1, x_2) & \text{in } \Omega_l \\ a(x_1 + T, x_2), f(x_1 + T, x_2) & \text{in } \hat{\Omega} \setminus \Omega_l \end{cases}$$

Assume that $\hat{a}$ is smooth, so that the solution to

$$\begin{cases} \operatorname{div}(\hat{a} \nabla \hat{u}) \; = \; \hat{f} & \text{in } \hat{\Omega} \\ \hat{u} \in H_0^1(\hat{\Omega}) & \end{cases}$$

is in $H^2(\hat{\Omega})$

Proposition : superlensing with tuned HHMs

Assume that T is a multiple of 2π and that $\quad Spt(f) \cap \Omega_c = \emptyset$

Then the solutions u_δ to

$$\left\{ \begin{array}{l} \mathrm{div}(A_\delta \nabla u_\delta) \;=\; f \qquad \text{in } \Omega \\ u_\delta \in H^1_0(\Omega) \end{array} \right.$$

are uniformly bounded and converge strongly in H^1 to u_0, the unique solution to

$$\left\{ \begin{array}{l} \mathrm{div}(A_0 \nabla u_0) \;=\; f \qquad \text{in } \Omega \\ u_0 \in H^1_0(\Omega) \end{array} \right.$$

Moreover, u_0 satisfies

$$u_0(x_1, x_2) \quad = \quad \left\{ \begin{array}{ll} \hat{u}(x_1, x_2) & \text{in } \Omega_l \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{array} \right.$$

In other words, u_0 can be computed in $\Omega_l \cup \Omega_r$ as if the part Ω_c had disappeared

Proof: Step 1. Construction of u_0

The smoothness assumption on $\hat{a}$ implies that $\hat{u} \in H^2(\hat{\Omega})$ and

$$||\hat{u}||_{H^2(\hat{\Omega})} \quad \leq \quad C\,||f||_{L^2(\hat{\Omega})}$$

Interpreting x_1 as a time variable in Ω_c, standard results for the wave equation show that there is a unique solution $v \in \mathcal{C}^0([0,T], H_0^1(0,2\pi)) \cap \mathcal{C}^1([0,T], L^2(0,2\pi))$ to

$$(\partial_{11}^2 - \partial_{22}^2)v \quad = \quad 0 \qquad \text{in } \Omega_c = (0,T) \times (0,2\pi)$$

with the boundary condition $v = 0$ on $\partial\Omega \cap \partial\Omega_c$, and with the initial conditions

$$\left\{ \begin{array}{lcl} v(0,x_2) & = & \hat{u}(0,x_2) \\ \partial_1 v(0,x_2) & = & \partial_1 \hat{u}(0,x_2) \end{array} \right.$$

Moreover, v satisfies

$$||\nabla v||_{L^2(\Omega_c)} \quad \leq \quad C \int_0^{2\pi} |\hat{u}(0,x_2)|^2 + |\partial_1 \hat{u}(0,x_2)|^2 \;\leq\; C||f||_{L^2(\Omega)}$$

As $\quad v(x) = \sum_{n\geq 1} \sin(nx_2)\,(a_n \cos(nx_1) + b_n \sin(nx_1))$

and since $T \in 2\pi\mathbf{N}$ we have

$$\left\{ \begin{array}{rcr} v(0,x_2) & = & v(T,x_2) \\ \partial_1 v(0,x_2) & = & \partial_1 v(T,x_2) \end{array} \right.$$

i.e., v satisfies the same Cauchy data at $x_1 = 0$ and at $x_1 = T$

One can then define the H_0^1 function

$$u_0(x_1,x_2) \quad = \quad \left\{ \begin{array}{ll} \hat{u}(x_1,x_2) & \text{in } \Omega_l \\ v(x_1,x_2) & \text{in } \Omega_c \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{array} \right.$$

which satisfies $\quad \mathrm{div}(A_0 \nabla u_0) \;=\; f \quad$ in Ω, and

$$||u_0||_{H^1(\Omega)} \quad \leq \quad C\,||f||_{L^2(\Omega)}$$

Step 2. Uniqueness of u_0

Assume that w_0 is another solution

In Ω_c, w_0 can be expanded as
$w_0(x) = \sum_{n\geq 1} \sin(nx_2)\,(\alpha_n \cos(nx_1) + \beta_n \sin(nx_1))$

$$\text{so that} \qquad \left\{ \begin{array}{rcr} w_0(0,x_2) & = & w_0(T,x_2) \\ \partial_1 w_0(0,x_2) & = & \partial_1 w_0(T,x_2) \end{array} \right.$$

one can then define

$$\hat{w}(x_1,x_2) \quad = \quad \left\{ \begin{array}{ll} w_0(x_1,x_2) & \text{in } \Omega_l \\ w_0(x_1+T,x_2) & \text{in } \hat{\Omega}\setminus\Omega_l \end{array} \right.$$

which is in $H^1_0(\hat{\Omega})$, and which solves $\quad \mathrm{div}(\hat{A}\nabla\hat{w}) \ = \ \hat{f} \quad$ in $\hat{\Omega}$

Uniqueness for this problem implies that $\hat{w} \equiv \hat{u}$, from which it follows that $w_0 \equiv u_0$

Step 3. Convergence

Set $\quad v_\delta = u_\delta - u_0 \quad$ in Ω. Then

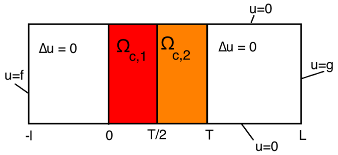
$$\begin{aligned} \mathrm{div}(A_\delta \nabla v_\delta) &= \mathrm{div}(A_\delta \nabla u_\delta) - \mathrm{div}(A_0 \nabla u_0) + \mathrm{div}(A_0 \nabla u_0) - \mathrm{div}(A_\delta \nabla u_0) \\ &= \mathrm{div}(i\delta 1_{\Omega_c} \nabla u_0) \end{aligned}$$

Multiplying by $\overline{v_\delta}$, integrating, taking the imaginary and real parts yields

$$\begin{aligned} ||\nabla v_\delta||^2_{L^2(\Omega_c)} &\leq \left| \int_\Omega \nabla u_0 \nabla v_\delta \right| \\ \int_\Omega |\nabla v_\delta|^2 &\leq C \left| \int_\Omega \nabla u_0 \nabla v_\delta \right| \end{aligned}$$

from which it follows that u_δ is uniformly bounded in H^1, and that v_δ converges strongly to 0

4. Another toy problem for a forward-backward device



$$\begin{aligned}\Omega &= (-l,0)\times(0,2\pi)\ \cup\ (0,T/2)\times(0,2\pi)\ \cup\ (T/2,T)\times(0,2\pi)\ \cup\ (T,L)\times(0,2\pi)\\ &= \Omega_l\ \cup\ \Omega_{c,1}\ \cup\ \Omega_{c,2}\ \cup\ \Omega_r\end{aligned}$$

We consider a conductivity of the form

$$A_\delta(x) = \begin{cases} a(x) & x\in\Omega_l\cup\Omega_r \\ \begin{pmatrix} 1+iO(\delta) & 0 \\ 0 & -1+iO(\delta)\end{pmatrix} & x\in\Omega_{c,1} \\ \begin{pmatrix} -1+iO(\delta) & 0 \\ 0 & 1+iO(\delta)\end{pmatrix} & x\in\Omega_{c,2}\end{cases}$$

Prop: superlensing via the complementary property

Assume that $Sptf \cap \Omega_c = \emptyset$

Then for some $C > 0$, independent of δ and f, we have

$$||u_\delta||_{H^1(\Omega)} \quad \leq \quad C||f||_{L^2(\Omega)}$$

Further, as $\delta \to 0$, $u_\delta \to u_0$ in $H^1(\Omega)$, where $u_0 \in H^1_0(\Omega)$ is the unique solution to

$$\mathrm{div}(A_0 \nabla u_0) \quad = \quad f \ \text{ in } \Omega$$

Additionnally, let $\hat{u}$ denote the $H^1_0(\hat{\Omega})$-solution to

$$\mathrm{div}(\hat{A} \nabla \hat{u}) \quad = \quad \hat{f} \ \text{ in } \hat{\Omega}$$

then $$u_0(x_1, x_2) = \begin{cases} \hat{u}(x_1, x_2) & \text{in } \Omega_l \\ \hat{u}(x_1 - T, x_2) & \text{in } \Omega_r \end{cases}$$

Again, the part Ω_c has disappeared in the limit. There is no hypothesis on T here

- The proof consists in the same 3 steps as in the previous proof
- In the first step construct a solution to the wave equation

$$(\partial_{11}^2 - \partial_{22}^2)v \quad = \quad 0 \quad \text{in } \Omega_{c,1}$$

with Cauchy data

$$v(0, x_2) \;=\; \hat{u}(0, x_2) \qquad\qquad \partial_1 v(0, x_2) \;=\; \partial_1 \hat{u}(0, x_2)$$

The reflection of v across $x_1 = T/2$

$$v_r(x_1, x_2) \quad = \quad v(T - x_1, x_2)$$

is also a solution to the wave equation and satisfies the transmission conditions

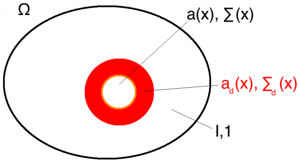
$$v(T/2, x_2) \;=\; v_r(T/2, x_2) \qquad\qquad \partial_1 v(T/2, x_2) \;=\; -\partial_1 v_r(T/2, x_2)$$

It follows that $\quad \hat{u}(0, x_2) = v_r(T, x_2) \qquad \partial_1 \hat{u}(0, x_2) \;=\; -\partial_1 v_r(T, x_2)$

so that one can define

$$u_0 \quad = \quad \begin{cases} \hat{u} & \text{in } \Omega_l \\ v & \text{in } \Omega_{c,1} \\ v_r & \text{in } \Omega_{c,2} \\ \hat{u} & \text{in } \Omega_r \end{cases}$$

5. More interesting devices :



The metamaterials lie in the red annulus

- Tuned superlensing in 2D, static (Laplace) and finite frequency (Helmholtz) case

- Tuned superlensing in 3D, finite frequency case

- Superlensing via the complementary property in 2D and 3D, static and finite frequency case

Example : Tuned superlensing in 3D, finite frequency case

Let Ω be a smooth bounded connected domain in $\mathbb{R}^3$

Assume that $0 < r_1 < r_2, \quad r_2 - r_1 \in 4\pi \mathbf{N}, \quad B_{r_2} \subset \Omega$

Assume that $k > 0$, $f \in L^2(\Omega \setminus B_{r_2})$

Medium description (using spherical coordinates r, θ, φ)

$$A_\delta, \Sigma_\delta \quad = \quad \begin{cases} a \quad \text{(uniformly elliptic, smooth)} & \sigma \quad \text{(bounded)} & \text{in } B_{r_1} \\ \dfrac{1}{r^2} e_r \otimes e_r - (e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) - i\delta & \dfrac{1}{4k^2 r^2} + i\delta & \text{in } B_{r_2} \setminus B_{r_1} \\ I & 1 & \text{in } \Omega \setminus B_{r_2} \end{cases}$$

Thm : The solutions u_δ to

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) + k^2 \Sigma_\delta u_\delta & = & f & \text{in } \Omega \\ \partial_\nu u_\delta - iku_\delta & = & 0 & \text{on } \partial\Omega \end{cases}$$

are uniformly bounded in $H^1(\Omega)$ and converge strongly to u_0 the unique solution to the above system with $\delta = 0$

Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$, where $\hat{u}$ is the unique solution to

$$\begin{cases} \operatorname{div}(\hat{A} \nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} & = & f & \text{in } \Omega \\ \\ \partial_\nu \hat{u} - ik\hat{u} & = & 0 & \text{on } \partial\Omega \end{cases}$$

where $\quad \hat{A}, \hat{\Sigma} \; = \begin{cases} I & 1 & \text{in } \Omega \setminus B_{r_2} \\ \dfrac{r_1}{r_2} a(\dfrac{r_1}{r_2} x) & \dfrac{r_1^3}{r_2^3} \sigma(\dfrac{r_1}{r_2} x) & \text{in } B_{r_2} \end{cases}$

Remarks :

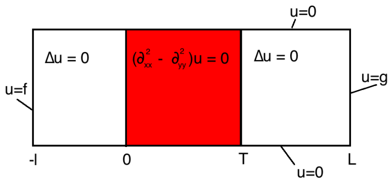
- The object in B_{r_1} is magnified in the limit $\delta \to 0$ by a factor $\dfrac{r_2}{r_1}$
- The equation for u_δ in $B_{r_2} \setminus B_{r_1}$ takes the form

$$\partial^2_{rr} u - \Delta_{\mathcal{S}^1} u + \frac{1}{4} u \quad = \quad 0$$

  Expansion in spherical harmonics shows that the Cauchy data are transported from ∂B_{r_2} to ∂B_{r_1}

6. Design stability

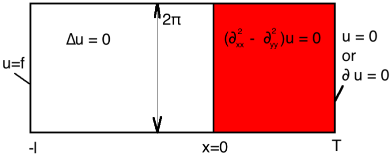
Consider again the toy problem



What happens when T is not a multiple of 2π ?

If we can define a limiting solution u_0 when $\delta = 0$, then the same proof as for the case $T = 2\pi$ shows that $u_\delta \to u_0$ (in particular the u_δ are uniformly bounded)

By linearity, we can restrict the study to



Seek

$$u_0(x,y) = \begin{cases} \sum_{n\geq 1} sin(ny)\left(a_n e^{nx} + b_n e^{-nx}\right) & -l < x < 0 \\ \sum_{n\geq 1} sin(ny)\left(\alpha_n e^{inx} + \beta_n e^{-inx}\right) & 0 < x < T \end{cases}$$

Expressing the transmission and boundary conditions fixes the values of the $a_n, b_n, \alpha_n, \beta_n$'s provided the determinant of the associated linear system does not vanish

The case of a homogeneous Dirichlet BC at $x = T$

$$\det \quad = \quad 2i \left[e^{nl} \left(\cos(nT) + \sin(nT)\right) - e^{-nl} \left(\cos(nT) - \sin(nT)\right) \right]$$

- If $l = T/\pi$ is irrational and diophantine of class r

$$\forall\, (p,q) \in \mathbf{Z} \times \mathbf{Z}^* \quad |l - \frac{p}{q}\pi| \quad > \quad \frac{\varepsilon}{q^r}$$

for some $\varepsilon > 0$, then

$$\begin{aligned}
&|e^{nl} \left(\cos(nT) + \sin(nT)\right)| \\
&\quad = \quad e^{nl} \left| \left(\cos(nT) + \sin(nT)\right) - \left(\cos(\frac{3\pi}{4} + 2\pi p) + \sin(\frac{3\pi}{4} + 2\pi p) \right) \right| \\
&\qquad\qquad\qquad \text{with} \quad 2\pi p < nT < 2\pi(p+1) \\
&\quad \geq \quad e^{nl} \frac{\sqrt{2}}{2} n \, |T - \frac{3+8p}{4n}\pi| \\
&\quad \geq \quad e^{nl} \frac{\sqrt{2}}{2} \frac{\varepsilon n}{(4n)^r} \ \geq \ c \ > \ 0
\end{aligned}$$

Thus, there is a unique solution u_0 to the elliptic/hyperbolic PDE

• If $T = \dfrac{4p+3}{4q}\pi,\ p, q \in \mathbf{Z}, q \neq 0$

then $\cos(nT) + \sin(nT)$ vanishes for an infinite number of n's

The determinant is $O(e^{-nl})$ and there is no solution u_0

Remarks :

See also [Bourgin-Duffin 1939, F. John 1941]

Diophantine numbers in $(0, 1)$ form a set of Lebesgue measure 1

The behavior of HMM's strongly depends on the geometry of the inclusions

7. Conclusion

- The superlensing properties of NIM's are related to the unique continuation principle; those of HMM's to the uniqueness of solutions to the Cauchy problem for the wave equation

- HMM's can be constructed by homogenization of 'metals', for instance by homogenization of laminates, at least concerning electric permittivity.

  Are there other constructions ?

- Concerning superlensing using HMM's, there are many open questions :

  Other geometries (in 2D, in 3D) ?

  Extension to the Maxwell system or to the system of elasticity ?