**UNIVERSITÉ DENIS DIDEROT - PARIS 7**

## THÈSE DE DOCTORAT

Spécialité : **MATHÉMATIQUES**

présentée par **Claire AMIOT**

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# Sur les petites catégories triangulées

Soutenue le 11 juillet 2008, devant le jury composé de :

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# Introduction

La notion de catégorie triangulée a été introduite par Verdier et Grothendieck dans les années 60 (cf. [Ver96]). Leur but était d'axiomatiser certaines propriétés des catégories dérivées. A partir des années 80, ces catégories ont beaucoup été étudiées en théorie des représentations, notamment par D. Happel, C. Riedtmann, M. Broué et J. Rickard. Voici une liste des principales catégories utilisées en théorie des représentations (nous travaillons sur un corps $k$ algébriquement clos) :

– la catégorie stable $\underline{\mathsf{Mod}}\,A$ des $A$-modules à droite sur une $k$-algèbre auto-injective ;
– la catégorie dérivée bornée $\mathcal{D}^b(\mathsf{mod}\,A)$ des modules de type fini sur une $k$-algèbre de dimension finie ;
– la catégorie stable $\underline{CM}(R)$ des modules de Cohen-Macaulay sur une singularité isolée $R$ ;
– certaines catégories d'orbites de catégories triangulées.

Décrivons plus en détail cette dernière classe. Si $A$ est une $k$-algèbre de dimension finie, et $\Phi$ une auto-équivalence (algébrique) de la catégorie dérivée bornée $\mathcal{D}^b(\mathsf{mod}\,A)$ des $A$-modules (à droite) de dimension finie, alors on peut construire la *catégorie d'orbites* $\mathcal{D}^b(\mathsf{mod}\,A)/\Phi$ : les objets sont les mêmes que ceux de la catégorie $\mathcal{D}^b(\mathsf{mod}\,A)$ tandis que l'espace des morphismes entre $X$ et $Y$ est de la forme

$$\bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^bA}(X, \Phi^n Y).$$

B. Keller a montré le théorème essentiel suivant dans [Kel05] :

**Théorème 0.1** (Keller). *Soit $A$ une $k$-algèbre de dimension finie et $\Phi = ? \overset{L}{\otimes}_A X$ une auto-équivalence algébrique de $\mathcal{D}^b(\mathsf{mod}\,A)$. Supposons que*

– *la catégorie $\mathcal{D}^b(\mathsf{mod}\,A)$ est équivalente à une catégorie de la forme $\mathcal{D}^b(\mathcal{H})$ où $\mathcal{H}$ est une catégorie héréditaire,*
– *pour tout indécomposable $X$ de $\mathcal{H}$, il n'y a qu'un nombre fini d'entiers $n$ tels que $\Phi^n(X)$ appartient à $\mathcal{H}$.*
– *il existe un entier $N \geq 0$ tel que l'orbite sous $\Phi$ de tout objet indécomposable de $\mathcal{D}^b(\mathsf{mod}\,A)$ contient un objet $U[n]$, où $0 \leq n \leq N$ et $U$ appartient à $\mathcal{H}$.*

*Alors la catégorie d'orbites est triangulée.*

Appelons *acyclique* un carquois sans cycle orienté. Alors le théorème implique en particulier que la catégorie amassée $\mathcal{C}_Q$ associée à un carquois acyclique $Q$ est triangulée.

Ce travail de thèse se compose de deux parties indépendantes. Le but de la première (chapitre 2) est la classification des catégories triangulées $k$-linéaires n'ayant qu'un nombre fini d'indécomposables (et vérifiant certaines conditions supplémentaires). Dans la deuxième (chapitres 4, 5, 6 et 7), le but est de généraliser la construction de catégories amassées en partant non plus d'un carquois acyclique mais d'une algèbre de dimension finie et de dimension globale 2. Le chapitre 1 est dédié à des préliminaires sur les notions de base utilisées dans cette thèse.

## Première partie

Dans cette partie (correspondant au chapitre 2 de la thèse) on classifie, dans une large mesure, les petites catégories triangulées $k$-linéaires $\mathcal{T}$ (où $k$ est un corps algébriquement clos) vérifiant la propriété de Krull-Schmidt et les conditions de finitude :

*a)* $\mathcal{T}$ est $\mathsf{Hom}$-*finie*, *i.e.* les espaces de morphismes sont de dimension finie sur $k$.

*b)* $\mathcal{T}$ est *localement finie*, *i.e.* pour tout objet $X$ indécomposable de $\mathcal{T}$, il existe un nombre fini de classes d'isomorphisme d'objets indécomposables $Y$ tels que l'espace $\mathsf{Hom}_{\mathcal{T}}(X,Y)$ est non nul.

Il est montré dans [XZ05] que la condition *b*) implique son dual. On dira que $\mathcal{T}$ est *additivement finie* si le nombre de classes d'isomorphisme d'indécomposables est fini. Notons que si la catégorie $\mathcal{T}$ est additivement finie, elle est localement finie.

Ces conditions peuvent paraître très restrictives, mais beaucoup de catégories construites de manières très différentes rentrent dans cette description. En particulier, on retrouve des catégories de tous les types décrits ci-dessus :

- Si $A$ est une $k$-algèbre auto-injective de dimension finie et de représentation finie, la catégorie $\underline{\mathsf{mod}}\,A$ vérifie toutes les hypothèses. Dans [Rie80a], [Rie80b], [Rie83b] et [Rie83a], C. Riedtmann a classifié ces algèbres et décrit la structure $k$-linéaire de leur catégorie de modules stables.
- Dans [Hap87], D. Happel a montré que la catégorie dérivée bornée $\mathcal{D}^b(kQ)$ (où $Q$ est un carquois connexe et acyclique) est localement finie si et seulement si $Q$ est un carquois de Dynkin, c'est-à-dire que le graphe sous-jacent à $Q$ est un diagramme de Dynkin simplement lacé.
- Grâce à ce résultat, il est facile de voir que la catégorie amassée $\mathcal{C}_Q$ associée à un carquois connexe acyclique $Q$ est additivement finie si et seulement si $Q$ est un carquois de Dynkin.
- Si $R$ est une singularité isolée de dimension 1, M. Auslander et I. Reiten ont montré dans [AR86] que la catégorie stable des modules de Cohen-Macaulay est additivement finie, et ont calculé son carquois d'Auslander-Reiten.

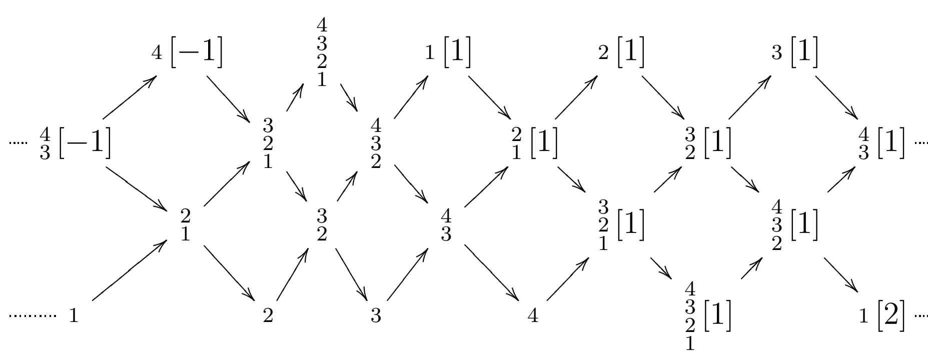
## Carquois d'Auslander-Reiten

La stratégie pour classifier les catégories $\mathcal{T}$ vérifiant les hypothèses de finitude comporte comme première étape le calcul d'un invariant important : *le carquois d'Auslander-Reiten* de $\mathcal{T}$. Rappelons d'abord quelques résultats sur les exemples précédents :

Le théorème suivant de D. Happel (cf. [Hap87]) donne une description explicite du carquois d'Auslander-Reiten de la catégorie dérivée $\mathcal{D}^b(kQ)$, où $Q$ est un carquois de Dynkin.

**Théorème 0.2** (Happel)**.** *Soit $Q$ un carquois de Dynkin. Alors le carquois d'Auslander-Reiten de la catégorie dérivée $\mathcal{D}^b(kQ)$ est le carquois de répétition $\mathbb{Z}Q$, muni de la translation canonique.*

*Exemple.* Soit $Q$ le carquois suivant $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$. Le carquois d'Auslander-Reiten de la catégorie $\mathcal{D}^b(kQ)$ a alors la forme suivante :



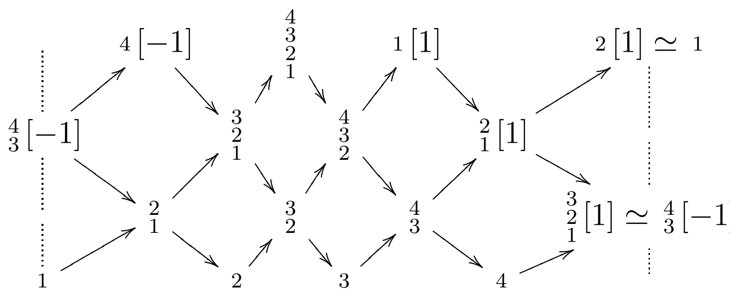
Chaque module est ici représenté par sa filtration radicale. Le module $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$ a une tête isomorphe au module simple $S_3$ associé au sommet 3, la tête de son radical est $S_2$, et son socle est $S_1$.

Utilisant ce dernier théorème, A. Buan, R. Marsh, M. Reineke, I. Reiten et G. Todorov ont determiné dans [BMR⁺06] la structure du carquois d'Auslander-Reiten d'une catégorie d'orbites de la forme $\mathcal{D}^b(kQ)/\Phi$, où $Q$ est un carquois de Dynkin.

**Théorème 0.3** (Buan-Marsh-Reineke-Reiten-Todorov)**.** *Soit $Q$ un carquois de Dynkin, et $\Phi$ une auto-équivalence d'ordre infini de la catégorie $\mathcal{D}^b(kQ)$. Alors le carquois d'Auslander-Reiten de la catégorie $\mathcal{D}^b(kQ)/\Phi$ est de la forme $\mathbb{Z}Q/\varphi$, où $\varphi$ est l'automorphisme de carquois à translation de $\mathbb{Z}Q$ correspondant à l'auto-équivalence $\Phi$.*

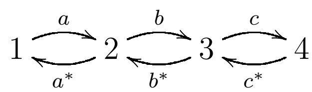
*Exemple.* Soit $Q$ le carquois précédent $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$. Alors la catégorie amassée

$\mathcal{C}_Q$ a pour carquois d'Auslander-Reiten :




Les bords droit et gauche de ce carquois sont identifiés. On obtient ainsi un "ruban de Möbius".

*Exemple.* Soit $A$ l'algèbre préprojective de type $A_4$, c'est-à-dire l'algèbre définie par le carquois :




et par les relations

$$a^*a = 0, \quad aa^* + b^*b = 0, \quad bb^* + c^*c = 0 \quad \text{et} \quad cc^* = 0.$$

La figure 1 présente le carquois de la catégorie $\mathsf{mod}\, A$. Les modules projectifs-injectifs sont entourés en rouge. On voit alors que la catégorie stable $\underline{\mathsf{mod}}\, A$ a le même carquois que la catégorie amassée associée au carquois $D_6$.

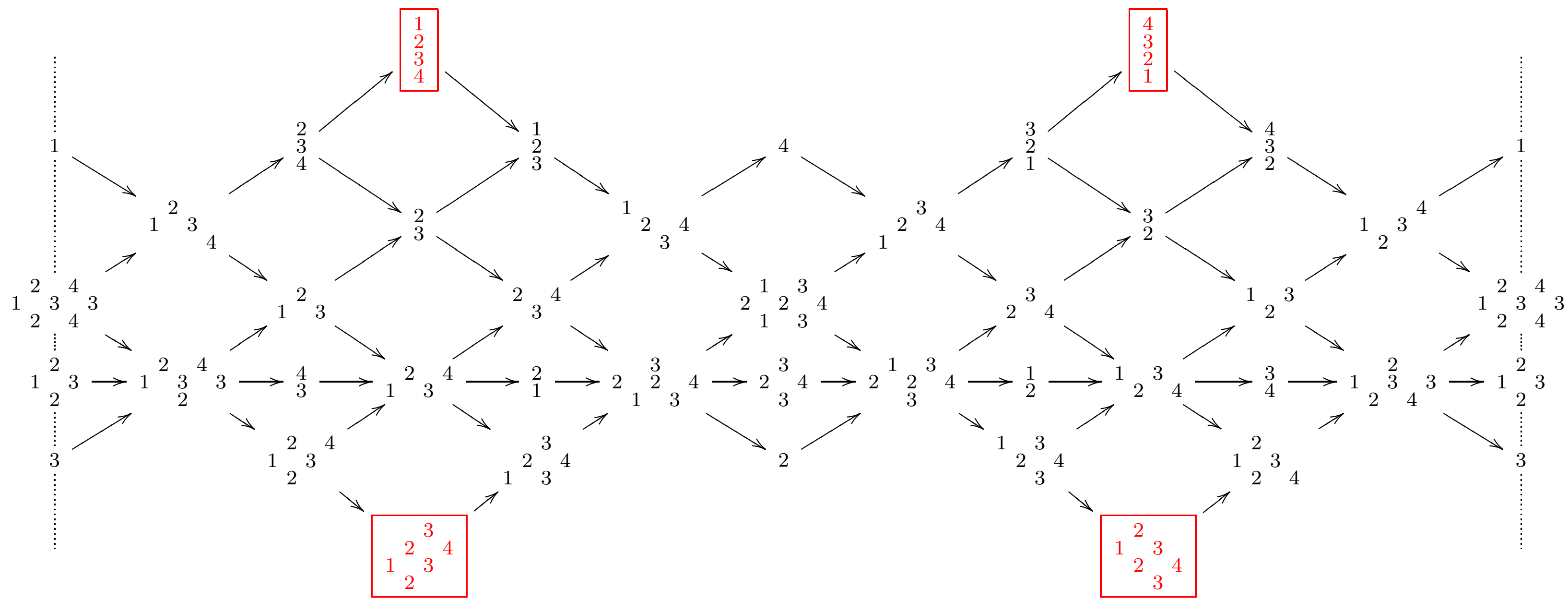
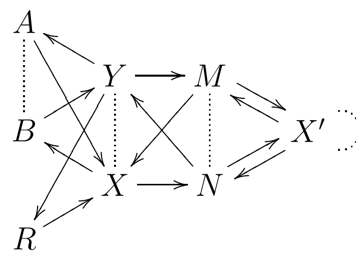
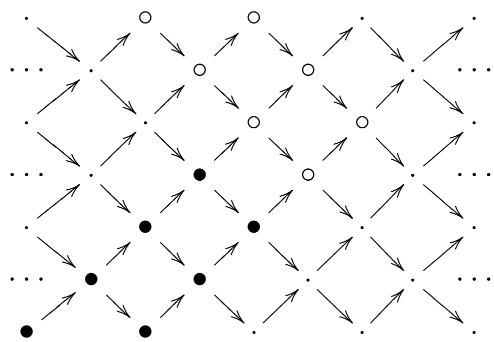


FIG. 1 – Carquois d'Auslander-Reiten de la catégorie $\mathsf{mod}\,\Lambda(A_4)$

*Exemple.* Soit $R$ la singularité isolée $k\{x,y\}/(x^2y+y^4)$ de dimension 1, où $k\{x,y\}$ désigne l'anneau des séries formelles. Alors le carquois d'Auslander-Reiten de $CM(R)$ a la forme suivante (cf. [Yos90], Exemple 9.11) :




Les pointillés représentent la translation d'Auslander-Reiten qui, dans ce cas, est d'ordre 2. L'unique module de Cohen-Macaulay projectif est le module $R$. Il est alors facile de voir que ce carquois muni de la translation est isomorphe au quotient de $\mathbb{Z}A_7$ par l'automorphisme envoyant le domaine fondamental $\bullet$ sur le domaine fondamental $\circ$.



Dans le deuxième chapitre de cette thèse, section 2.4, on donne une autre preuve du résultat suivant dû à J. Xiao et B. Zhu [XZ05], où contrairement à eux, on ne traite pas à part le cas où le carquois d'Auslander-Reiten comporte une boucle.

**Théorème** (Xiao-Zhu)**.** *(Théorème 2.9) Soit $\mathcal{T}$ une catégorie triangulée de Krull-Schmidt localement finie. Soit $\Gamma$ une composante connexe de son carquois d'Auslander-Reiten. Alors il existe un arbre de Dynkin $Q$ de type $A$, $D$ ou $E$, et un automorphisme $\varphi$ d'ordre infini (ou trivial) de $\mathbb{Z}Q$, tel qu'on ait un isomorphisme de carquois à translation* $\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}Q/\varphi$ .

Dans cette thèse, la preuve consiste d'abord à démontrer l'existence d'un foncteur de Serre dans la catégorie $\mathcal{T}$ (section 2.1), c'est-à-dire d'une auto-équivalence $\nu$ et d'un isomorphisme bifonctoriel :

$$\eta_{X,Y} : \mathsf{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\sim} D\mathsf{Hom}_{\mathcal{T}}(Y,\nu X)$$

où $D = \mathsf{Hom}_k(?,k)$ est la dualité par rapport au corps de base. Ceci revient à démontrer l'existence de triangles d'Auslander-Reiten. Puis il est possible de construire une fonction sous-additive sur le carquois d'Auslander-Reiten et d'utiliser des résultats de combinatoire de D. Happel, U. Preiser et C. M. Ringel (cf. [HPR80a], [HPR80b]) pour conclure sur la structure de $\Gamma_{\mathcal{T}}$ (sections 2.2, 2.3 et 2.4).

Etant donné un carquois à translation de la forme $\mathbb{Z}\Delta/\varphi$, il existe une catégorie triangulée admettant ce carquois comme carquois d'Auslander-Reiten : la catégorie d'orbites $\mathcal{D}^b(k\Delta)/\Phi$. La question naturelle qui se pose donc est la suivante :

*Si $\mathcal{T}$ est une catégorie triangulée localement finie dont le carquois d'Auslander-Reiten de la forme $\mathbb{Z}Q/\varphi$, peut-on construire une équivalence entre $\mathcal{T}$ et $\mathcal{D}^b(\mathsf{mod}\, kQ)/\Phi$ ?*

Si on distingue le cas d'équivalence seulement $k$-linéaire (on dira dans ce cas que la catégorie $\mathcal{T}$ est *standard*), du cas plus fort d'une équivalence triangulée, cette question se divise en réalité en deux.

## Equivalence k-linéaire

Pour construire une équivalence $k$-linéaire entre $\mathcal{T}$ et la catégorie d'orbites $\mathcal{D}^b(kQ)/\Phi$, le plus simple est d'utiliser la propriété universelle de la catégorie $k$-linéaire sous-jacente à la catégorie d'orbites. Il faut donc construire un foncteur de revêtement :

$$F : \mathcal{D}^b(\mathsf{mod}\, kQ) \twoheadrightarrow \mathcal{T} \quad (\circlearrowleft \Phi)$$

et un isomorphisme de foncteurs entre $F$ et $F \circ \Phi$. Suivant la méthode de C. Riedtmann dans [Rie80a], il est facile de construire un foncteur de revêtement (section 2.5), avec donc pour corollaire immédiat (Corollaire 2.11) que si le carquois $\Gamma$ est isomorphe à $\mathbb{Z}Q$, alors la catégorie $\mathcal{T}$ est standard et donc $k$-linéairement équivalente à $\mathcal{D}^b(kQ)$.

Ensuite, on montre qu'il est possible de construire un isomorphisme de foncteurs entre $F$ et $F \circ \Phi$ si la catégorie est "suffisamment grande" et on obtient le théorème suivant :

**Théorème.** *(cf. Théorèmes 2.12 et 2.13) Soit $\mathcal{T}$ une catégorie triangulée de Krull-Schmidt localement finie de carquois d'Auslander-Reiten $\mathbb{Z}Q/\varphi$. Alors $\mathcal{T}$ est standard, i.e. $k$-linéairement équivalente à $\mathcal{D}^b(\mathsf{mod}\, kQ)/\Phi$ où $\Phi$ est l'équivalence de $\mathcal{D}^b(\mathsf{mod}\, kQ)$ induite par $\varphi$, si on est dans un des deux cas suivants :*
- *le carquois $Q$ est de type $A_n$ et $\varphi$ est une puissance de la translation d'Auslander-Reiten ;*
- *le nombre de classes d'isomorphisme d'indécomposables de la catégorie $\mathcal{T}$ est supérieur au nombre d'indécomposables de la catégorie $\mathsf{mod}\, kQ$.*

En particulier, si $\mathcal{T}$ est maximale $d$-Calabi-Yau avec $d \geq 2$, alors $\mathcal{T}$ est $k$-linéairement équivalente à la catégorie $d$-amassée $\mathcal{C}_Q^d$ (Corollaire 2.14).

Néanmoins en utilisant les travaux de J. Białkowski, K. Erdmann, et A. Skowroński [BES07], il est possible de trouver des catégories 1-Calabi-Yau non standard. Ceci est traité à la fin du deuxième chapitre (section 2.8). On montre le théorème suivant :

**Théorème.** *(Théorème 2.21) Soit $\mathcal{P}$ une catégorie $k$-linéaire munie d'une auto-équivalence $S$, telle que $\mathsf{mod}\,\mathcal{P}$ soit une catégorie de Frobenius. Supposons qu'il existe une suite exacte de foncteurs exacts de $\mathsf{mod}\,\mathcal{P}$ dans $\mathsf{mod}\,\mathcal{P}$ :*

$$0 \longrightarrow Id \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0$$

*où les $X^i$ sont à valeurs dans $\mathsf{proj}\,\mathcal{P}$, et où $S$ est le foncteur de $\mathsf{mod}\,\mathcal{P}$ induit par $S$. Alors la catégorie $\mathcal{P}$ a une structure naturelle de catégorie triangulée avec foncteur suspension $S$.*

Ce théorème nous permet de montrer que les catégories des modules projectifs de dimension finie sur des algèbres préprojectives déformées de type Dynkin généralisé ont une structure de catégorie triangulée (Corollaire 2.24). Puis en utilisant les résultats de J. Białkowski, K. Erdmann, et A. Skowroński (cf.[BES07]), on montre l'existence de catégories triangulées non standard en caractéristique 2.

Le chapitre 3 de cette thèse est un appendice dans lequel il est montré (Théorème 3.1) que la structure naturelle triangulée des complexes parfaits $\mathsf{per}\,A$ sur une dg-algèbre $A$ provient d'une suite exacte comme dans le théorème 2.21.

## Equivalence triangulée

Montrer qu'une catégorie triangulée est triangle-équivalente à une catégorie d'orbites est beaucoup plus complexe. On ne peut parvenir à une réponse partielle qu'en rajoutant une hypothèse supplémentaire sur la structure triangulée. Nous supposons que la structure triangulée de $\mathcal{T}$ est algébrique, c'est-à-dire que $\mathcal{T}$ est la catégorie stable d'une catégorie de Frobenius, ou de manière équivalente qu'elle admet un renforcement en une catégorie différentielle graduée (dg) (cf. [Kel06]). Notons que toutes les catégories citées en exemple comme étant des catégories triangulées utilisées en théorie des représentations sont algébriques.

Supposons donc que $\mathcal{T}$ est une catégorie algébrique de la forme $\mathsf{per}\,\mathcal{B}$, où $\mathcal{B}$ est une dg-catégorie. Alors pour construire une équivalence algébrique entre $\mathcal{T}$ et $\mathcal{D}^b(kQ)/\Phi$, où $\Phi$ est un foncteur algébrique de la forme $? \overset{L}{\otimes}_{kQ} Y$, on doit construire un foncteur de revêtement algébrique $F = ? \otimes_{kQ} X$ et un isomorphisme dans $\mathcal{D}(kQ^{op} \otimes \mathcal{B})$ entre $Y \overset{L}{\otimes}_{kQ} X$ et $X$. Nous parvenons alors au résultat suivant :

**Théorème.** *(Théorème 2.16) Toutes les catégories triangulées localement finies standard connexes et algébriques sont des catégories d'orbites de la forme $\mathcal{D}^b(kQ)/\Phi$, où $Q$ est un carquois de Dynkin, et où $\Phi$ est une auto-équivalence d'ordre infini de $\mathcal{D}^b(kQ)$.*

Ces résultats s'appliquent en particulier à de nombreuses catégories stables $\underline{\mathsf{mod}}\,A$ d'algèbres autoinjectives de représentation finie. Ces algèbres ont été classifiées à équivalence stable près par C. Riedtmann dans [Rie80b] et [Rie83b] et par H. Asashiba dans [Asa99]. Dans [BS06], J. Białkowski et A. Skowroński donnent une une condition nécessaire et suffisante sur ces algèbres pour que leur catégorie stable $\underline{\mathsf{mod}}\,A$ soit Calabi-Yau. Dans [HJ06a] et [HJ06b] T. Holm et P. Jørgensen prouvent que certaines catégories stables $\underline{\mathsf{mod}}\,A$ sont en fait des catégories $d$-amassées en utilisant le théorème 2.16.

# Deuxième partie

Dans la deuxième partie de cette thèse, correspondant aux chapitres 3, 4, 5 et 6, nous nous intéressons à généraliser la construction des catégories amassées.

**Définition 0.4.** [BMR+06] Soit $k$ un corps algébriquement clos. Soit $Q$ un carquois fini connexe acyclique. Notons $\nu$ le foncteur de Serre de la catégorie dérivée bornée $\mathcal{D}^b(\mathsf{mod}\,kQ)$ des $kQ$-modules à droite de dimension finie et [1] son foncteur suspension. Alors la *catégorie amassée* est la catégorie d'orbites $\mathcal{D}^b(\mathsf{mod}\,kQ)/\nu[-2]$.

Notons qu'une autre définition a été donnée indépendamment par [CCS06] pour le cas où le carquois $Q$ est de type $A_n$.

Ces catégories ont été introduites dans le but de "catégorifier" les algèbres amassées, inventées par S. Fomin et A. Zelevinski en 2000 (cf. [FZ02], [FZ03],[FZ07]) l'objectif étant de mieux en comprendre la combinatoire. De très nombreux articles ([MRZ03], [BMR+06], [CK08],[CC06],[BMR07], [BMR08], [BMRT07],[CK06]) traitent du problème de catégorification d'algèbres amassées par des catégories amassées associée à un carquois acyclique.

Un autre point de vue de la théorie consiste à catégorifier des algèbres amassées par des sous-catégories de modules d'une algèbre préprojective associée à un carquois acyclique (cf. [GLS07a], [GLS06a], [GLS06b], [GLS07b], [BIRS07]).

Dans ces deux cadres, les catégories $\mathcal{T}$ étudiées vérifient les propriétés-clés suivantes :

– la propriété *2-Calabi-Yau*, *i.e.* il existe un isomorphisme

$$\mathsf{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\sim} D\mathsf{Hom}_{\mathcal{T}}(Y,X[2])$$

bifonctoriel en les objets $X$ et $Y$ de $\mathcal{T}$ ;

– l'existence d'objets *amas-basculants*, c'est-à-dire d'objets basiques tels que pour tout objet $X$, l'espace $\mathsf{Ext}^1_{\mathcal{T}}(T,X)$ s'annule si et seulement si $X$ appartient à $add(T)$, la plus petite sous-catégorie de $\mathcal{T}$ contenant $T$ et stable par facteurs directs.

Des catégories vérifiant ces propriétés ont donc été étudiées de manière plus générale (cf. [IY06],[KR06], [KR07], [Kel08a], [Pal], [Tab07]). Dans [IY06] O. Iyama et Y. Yoshino ont montré en particulier que dans une catégorie triangulée ayant ces deux propriétés, il existe une "mutation", concept essentiel à la catégorification.

Le but de cette partie est de construire une “catégorie amassée” vérifiant ces deux propriétés en partant non plus d’une algèbre héréditaire $kQ$ mais d’une algèbre de dimension globale 2, de dimension finie. Le candidat naturel pour cette catégorie est donc la catégorie d’orbites

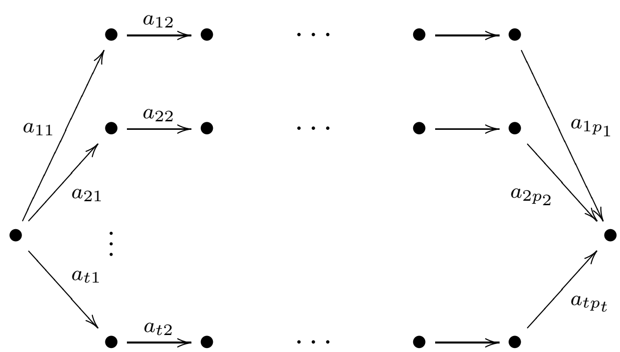
$$\mathcal{D}^b(\mathsf{mod}\,A)/\nu[-2]$$

où $\nu$ désigne le foncteur de Serre de $\mathcal{D}^b(\mathsf{mod}\,A)$. Mais on ne peut conclure sur la structure triangulée de cette catégorie d’orbites que dans le cas où on a une équivalence dérivée entre $\mathsf{mod}\,A$ et une catégorie héréditaire $\mathcal{H}$ en utilisant le théorème 0.1 de B. Keller. D’après la classification de D. Happel et I. Reiten (cf.[HR02] [Hap01]), ceci se produit si et seulement si $A$ est équivalente par dérivation à une algèbre héréditaire ou à une algèbre canonique. On se trouve donc dans un des deux cas suivants :

- Soit $A$ l’algèbre des endomorphismes d’un module basculant $T$ d’une algèbre héréditaire $kQ$. Alors $A$ est de dimension globale $\leq 2$. De plus, d’après D. Happel [Hap88], on a une équivalence de catégories

  $$\mathcal{D}^b(\mathsf{mod}\,kQ) \xrightarrow{R\mathsf{Hom}_{kQ}(?,T)} \mathcal{D}^b(\mathsf{mod}\,A)\ .$$

  Donc d’après le théorème 0.1, la catégorie d’orbites $\mathcal{D}^b(\mathsf{mod}\,A)/\nu_A[-2]$ est triangulée et équivalente à la catégorie amassée $\mathcal{C}_Q$.
- Dans [Rin84], C. M. Ringel a introduit les algèbres canoniques $A(p,\lambda)$, dépendant d’une suite de poids $p=(p_1,\ldots,p_t)$ d’entiers positifs et d’une suite de paramètres $\lambda=(\lambda_3,\ldots,\lambda_t)$ dans $k$ deux à deux distincts. Plus précisément l’algèbre $A(p,\lambda)$ est définie par le carquois :

  et les $t-3$ relations suivantes :

  $$\text{pour tout } i=3,\ldots,t \qquad a_{ip_i}\cdots a_{i2}a_{i1}=a_{2p_2}\cdots a_{22}a_{21}-\lambda_i a_{1p_1}\cdots a_{12}a_{11}.$$

  Cette algèbre est de dimension globale 2. W. Geigle et H. Lenzing ont montré dans [GL] l’équivalence entre les catégories $\mathcal{D}^b(\mathsf{mod}\,A(p,\lambda))$ et $\mathcal{D}^b(coh\mathbb{X})$ où $\mathbb{X}$ est la droite projective pondérée $\mathbb{X}(p,\lambda)$. La catégorie des faisceaux cohérents $coh(\mathbb{X})$ est une catégorie héréditaire, donc là encore le théorème 0.1 s’applique, et la catégorie d’orbites $\mathcal{D}^b(\mathsf{mod}\,A(p,\lambda))/\nu_A[-2]$ est triangulée.

## Construction de la catégorie amassée et propriété 2-CY

Mais pour une algèbre $A$ de dimension globale $\leq 2$ quelconque, la catégorie d'orbites $\mathcal{D}^b(\mathsf{mod}\,A)/\nu_A[-2]$ n'est pas triangulée. Il semble donc judicieux de prendre son enveloppe triangulée calculée par B. Keller dans [Kel05] (Théorème 2). Ceci nous amène à poser la définition suivante :

**Définition.** Soit $A$ une $k$-algèbre de dimension globale 2. Posons $B = A \oplus DA[-3]$ l'algèbre différentielle graduée, où $DA$ est le $A$-$A$-bimodule $\mathsf{Hom}_k(A,k)$. Alors on définit la *catégorie amassée* $\mathcal{C}_A$ associée à $A$ comme la sous-catégorie épaisse engendrée par $A$ dans le quotient :

$$\mathcal{C}_A^+ = \mathcal{D}^b(B)/\mathsf{per}\,B$$

où $\mathcal{D}^b(B)$ est la catégorie dérivée des dg-B-modules dont l'homologie est de dimension totale finie, et où $\mathsf{per}\,B$ est la sous-catégorie épaisse engendrée par $B$ dans $\mathcal{D}^b(B)$.

La catégorie quotient $\mathcal{C}_A^+$ n'est pas $\mathsf{Hom}$-finie en général, ce qui pose donc un problème si on veut montrer que $\mathcal{C}_A$ est 2-Calabi-Yau. En revanche, il existe une forme bilinéaire bifonctorielle non-dégénérée $\beta$ :

$$\beta_{NX} : \mathsf{Hom}_{\mathcal{D}}(N,X) \times \mathsf{Hom}_{\mathcal{D}}(X,N[3]) \to k$$

pour tout $N$ dans $\mathsf{per}\,B$ et pour tout $X$ dans $\mathcal{D}^b(B)$. Ceci nous permet (chapitre 4, section 1) de construire une forme bilinéaire bifonctorielle $\beta'$

$$\beta'_{XY} : \mathsf{Hom}_{\mathcal{C}}(X,Y) \times \mathsf{Hom}_{\mathcal{C}}(Y,X[2]) \to k$$

pour tous $X$ et $Y$ objets de $\mathcal{C}_A^+$. Cette forme $\beta'$ sera non dégénérée (chapitre 4, sections 2 et 3) si les objets de $\mathcal{D}^b B$ sont "limites" d'objets de $\mathsf{per}\,B$. En particulier on obtient le corollaire suivant :

**Corollaire.** *(Corollaire 4.4) Soit $A$ une $k$-algèbre de dimension finie et de dimension globale $\leq 2$. Supposons que le foncteur $\mathsf{Tor}_2^A(?,DA)$ est nilpotent. Alors la catégorie amassée $\mathcal{C}_A$ est $\mathsf{Hom}$-finie et 2-Calabi-Yau.*

Je remercie R. Rouquier de m'avoir informée que ces résultats ont été obtenus de façon indépendante, sous une forme beaucoup plus forte, dans la prépublication en préparation [CR].

## Objet amas-basculant

Le but est maintenant de trouver un objet amas-basculant dans cette catégorie 2-Calabi-Yau. Il y a un candidat canonique qui est l'algèbre $A$ elle-même, de même que l'algèbre $kQ$ vue comme objet dans la catégorie amassée $\mathcal{C}_Q$ est un objet amas-basculant. Cet objet $A$ est en effet rigide (Proposition 5.4.1). Et de plus, si le foncteur $\mathsf{Tor}_2^A(?,DA)$

est nilpotent, l'objet $A$ est amas-basculant dans la catégorie d'orbites $\mathcal{D}^b(\mathsf{mod}\, A)/\nu_A[-2]$ (Proposition 5.4.2). De manière plus précise, si $X$ est un objet de $\mathcal{D}^b(\mathsf{mod}\, A)/\nu_A[-2]$ tel que $\mathsf{Ext}^1_{\mathcal{C}}(T, X)$ est nul, alors $X$ est dans $add(A)$ la sous-catégorie de $\mathcal{C}_A$ des facteurs directs de somme directes de $A$. Au chapitre 7, nous montrerons par d'autres méthodes un résultat plus fort : l'objet $A$ est amas-basculant dans la catégorie amassée toute entière.

On peut calculer l'algèbre des endomorphismes de cet objet $A$ dans la catégorie amassée, et on obtient l'algèbre tensorielle $T_A \mathsf{Ext}^2_A(DA, A)$ (Proposition 5.2.1). Cette algèbre est de dimension finie si et seulement si le foncteur $\mathsf{Tor}^A_2(?, DA)$ est nilpotent (Théorème 5.1).

*Remarque.* Si $A$ est l'algèbre des endomorphismes d'un objet basculant $T$ de $\mathsf{mod}\, kQ$ où $Q$ est un carquois acyclique, alors les catégories amassées $\mathcal{C}_A$ et $\mathcal{C}_Q$ sont équivalentes. L'objet $T$ vu dans $\mathcal{C}_Q$ est amas-basculant, et il est montré dans [ABS06] que son algèbre des endomorphismes $\mathcal{C}_Q$ est

$$\mathsf{End}_{\mathcal{C}_A}(A) \simeq \mathsf{End}_{\mathcal{C}_Q}(T) \simeq A \oplus \mathsf{Ext}^2_A(DA, A).$$

En effet dans ce cas le produit tensoriel $\mathsf{Ext}^2_A(DA, A) \otimes_A \mathsf{Ext}^2_A(DA, A)$ est nul.

## Cas d'une algèbre d'endomorphismes d'un morceau "postprojectif"

On s'intéresse dans le chapitre 6 au cas particulier où $A$ est l'algèbre des endomorphismes d'un morceau "postprojectif" $\mathcal{M}$ stable par prédecesseurs d'une algèbre dérobée (=concealed) $B$ (cf. [Rin84]). Plus précisément, soit $T$ un module basculant préinjectif d'une algèbre héréditaire $kQ$ et $B$ l'algèbre d'endomorphismes $\mathsf{End}_{kQ}(T)$. On pose

$$\mathcal{M} = \{X \in \mathsf{mod}\, B \mid \mathsf{Ext}^1_B(X, H) = 0\}$$

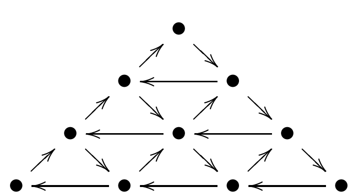
où $H$ est une tranche post-projective de $\mathsf{mod}\, B$.

Dans ce cas, d'après le résultat précédent et les résultats de I. Assem, T. Brüstle et R. Schiffler, le carquois de l'algèbre des endomorphismes de $A$ dans la catégorie amassée $\mathcal{C}_A$ est le carquois d'Auslander-Reiten de $\mathcal{M}$ auquel on rajoute, pour chaque module $x$ de $\mathcal{M}$ non-projectif, une flèche $x \to \tau_B x$ où $\tau_B$ est la translation d'Auslander-Reiten de la catégorie $\mathsf{mod}\, B$ (Proposition 5.2.2).

*Exemple.* Soit $A$ l'algèbre d'Auslander du carquois de Dynkin :

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4.$$

Alors le carquois de l'algèbre $\mathsf{End}_{\mathcal{C}_A}(A)$ est le suivant :



qui est le carquois d'un objet amas-basculant de la catégorie $\underline{\mathsf{mod}}\,\Lambda(A_5)$ d'après [GLS06a]. On peut donc se demander si les catégories $\underline{\mathsf{mod}}\,\Lambda(A_5)$ et $\mathcal{C}_A$ sont équivalentes.

Plus généralement, C. Geiss, B. Leclerc et J. Schröer ont construit [GLS07b] des sous-catégories $\mathcal{C}_M$ de $\mathsf{mod}\,\Lambda$ (où $\Lambda = \Lambda_Q$ est une algèbre préprojective) associées à certains $kQ$-modules terminaux. On montre dans cette thèse (Théorème 6.5) que la catégorie stable de cette catégorie de Frobenius est triangle équivalente à une catégorie amassée $\mathcal{C}_A$ où $A$ est une l'algèbre des endomorphismes d'un module postprojectif d'une algèbre héréditaire.

Suivant un autre point de vue, A. Buan, O. Iyama, I. Reiten et J. Scott ont construit dans [BIRS07] des catégories triangulées 2-Calabi-Yau de la forme $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ où $\mathcal{I}_w$ est un idéal d'une algèbre préprojective $\Lambda = \Lambda_Q$ associé à un élément $w$ du groupe de Weyl du graphe de $Q$. Pour certains éléments $w$ du groupe de Weyl, qui sont associés à des $kQ$-modules basculants préinjectifs, on construit une équivalence triangulée entre $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ et une catégorie amassée $\mathcal{C}_A$ (Théorème 6.8) où $A$ est l'algèbre des endomorphismes d'un module post-projectif d'une algèbre dérobée.

## Algèbre préprojective dérivée

En utilisant la théorie du basculement généralisée aux dg-algèbres élaborée par B. Keller dans [Kel94], on trouve dans le chapitre 7 une autre construction de la catégorie amassée $\mathcal{C}_A$. La catégorie amassée est définie comme le quotient de catégories triangulées

$$\mathcal{C}_A = \langle A\rangle_B/\mathsf{per}\,B,$$

où $B$ est la dg-algèbre $A\oplus DA[-3]$ et $\langle A\rangle_B$ est la sous-catégorie épaisse de $\mathcal{D}^bB$ engendrée par $A$. Notons $\Pi_3A$ la dg-algèbre $R\mathcal{H}om_B(A_B, A_B)$. D'après [Kel94], le foncteur

$$R\mathcal{H}om_B(A_B, ?) : \mathcal{D}^bB \to \mathsf{per}\,\Pi_3A$$

induit une équivalence triangulée entre les catégories

$$\begin{array}{rccc} R\mathcal{H}om_B(A_B,?): & \langle A\rangle_B & \xrightarrow{\sim} & \mathsf{per}\,\Pi_3A \\ & \uparrow & & \uparrow \\ & \mathsf{per}\,B & \xrightarrow{\sim} & \mathcal{D}^b\Pi_3(A) \end{array}.$$

On obtient donc une autre définition de la catégorie amassée comme le quotient

$$\mathcal{C}_A = \mathsf{per}\,\Pi_3A/\mathcal{D}^b\Pi_3A.$$

L'image de $A$ par ce foncteur est le dg-module libre $\Pi_3A$.

Cette dg-algèbre est isomorphe dans la catégorie homotopique des dg-algèbres à la dg-algèbre $T_A\Theta[2]$ où $\Theta$ est le dg-$A$-bimodule $R\mathcal{H}om^{\bullet}_A(DA, A)$. Cette dg-algèbre (la 3-algèbre préprojective dérivée) a été introduite très récemment par B. Keller et vérifie les propriétés (cf.[Kel08a], [Kel08b]) :

– $\Pi_3 A$ est homologiquement lisse (au sens de M. Kontsevich et Y. Soibelman [KS06]) ;
– $\Pi_3 A$ a son homologie nulle en degrés strictement positifs ;
– $\Pi_3 A$ est 3-Calabi-Yau en tant que bimodule (au sens de V. Ginzburg [Gin06]).

La condition de nilpotence de $\mathsf{Tor}_2^A(?, DA)$ est équivalente au fait que $H^0(\Pi_3 A)$ est de dimension finie. On obtient dans cette thèse le résultat suivant :

**Théorème.** *(Théorème 7.1) Soit $\Gamma$ une dg-algèbre homologiquement lisse, 3-Calabi-Yau en tant que bimodule, d'homologie nulle en degré strictement positif et telle que $H^0(\Gamma)$ est de dimension finie. Alors la catégorie $\mathsf{per}\,\Gamma/\mathcal{D}^b(\Gamma)$ est $\mathsf{Hom}$-finie, 2-Calabi-Yau et l'objet libre $\Gamma$ est un objet amas-basculant.*

Ce théorème implique donc que l'objet $A$ est un objet amas-basculant de la catégorie amassée $\mathcal{C}_A$ (Corollaire 7.2).

De plus, le théorème 7.1 peut s'appliquer aux dg-algèbres de Ginzburg (cf. [Gin06]) $\Gamma(Q, W)$ où $(Q, W)$ est un carquois à potentiel (cf. [DWZ07]) dans le cas où l'algèbre de Jacobi $\mathcal{J}(Q, W)$ est de dimension finie. On construit alors une catégorie amassée $\mathsf{Hom}$-finie 2-Calabi-Yau qu'on notera $\mathcal{C}_{(Q,W)}$. Le résultat suivant est alors un corollaire immédiat :

**Théorème.** *(Théorème 7.10) Soit $(Q, W)$ un carquois à potentiel. Si l'algèbre Jacobienne $\mathcal{J}(Q, W)$ est de dimension finie, alors $\mathcal{J}(Q, W)$ est Calabi-Yau-amassée (Calabi-Yau-tilted au sens de Reiten [Rei07]).*

En combinant ce dernier résultat avec des résultats récents de B. Keller ([Kel08b]), de B. Keller et D. Yang [KY08] et de A. Buan, O. Iyama, I. Reiten et D. Smith [BIRS08], on obtient en particulier le résultat suivant :

**Proposition.** *(Corollaire 7.13) Soit $Q$ un carquois acyclique et $T$ un objet amas-basculant de la catégorie amassée $\mathcal{C} = \mathcal{C}_Q$. Soit $(Q', W)$ le carquois à potentiel associé à l'algèbre des endomorphismes $\mathsf{End}_{\mathcal{C}_Q}(T)$ (cf. [BIRS08] et [Kel08b]). Alors la catégorie $\mathcal{C}_{(Q',W)}$ est triangle équivalente à la catégorie $\mathcal{C}_Q$.*

## Perspectives

Les résultats de cette deuxième partie amènent de nombreuses nouvelles questions.

– Soit $\mathcal{C}$ une catégorie triangulée 2-Calabi-Yau et $T = T_1 \oplus \cdots \oplus T_n$ un objet amas-basculant. Alors d'après [IY06], on peut "muter" l'objet $T$ pour chaque $i = 1, \ldots, n$. Si le carquois $Q_T$ de $T$ et le carquois de son "muté" $\mu_i(T)$ n'ont ni boucle ni 2-cycle, alors le carquois du "muté" $\mu_i(T)$ est le "muté" du carquois $\mu_i(Q_T)$. Il s'agirait donc de trouver des conditions sur le carquois à potentiel $(Q, W)$ associé à une algèbre $A$ de dimension globale 2 pour qu'il soit rigide au sens de [DWZ07]. Ainsi, on pourrait muter $(Q, W)$ indéfiniment sans qu'il n'apparaisse ni boucle ni 2-cycle.
– B. Keller et I. Reiten ont montré dans [KR06] que si une catégorie triangulée algébrique 2-Calabi-Yau $\mathcal{C}$ a un objet amas-basculant $T$ dont le carquois $Q_T$ est

acyclique, alors il existe une équivalence triangulée entre $\mathcal{C}$ et la catégorie amassée $\mathcal{C}_Q$. Ceci serait faux dans le cas où le carquois de $T$ comporte des cycles orientés (cf. [Tep]), mais il semblerait que le carquois à potentiel $(Q, W)$ soit un meilleur invariant de la catégorie amassée $\mathcal{C}_{(Q,W)}$. Formulons donc la question suivante :

*Soit $\mathcal{C}$ une catégorie 2-Calabi-Yau algébrique admettant un objet amas-basculant. Existe-t-il un carquois à potentiel $(Q, W)$ et une équivalence triangulée entre $\mathcal{C}$ et $\mathcal{C}_{(Q,W)}$ ?*

– La motivation initiale des catégories amassées est de 'catégorifier' des algèbres amassées. Il serait donc intéressant d'explorer la classe des algèbres pouvant être catégorifiées par les catégories $\mathcal{C}_{(Q,W)}$. Nous montrons une première propriété de clôture à la section 7.4.3 : la classe des carquois à potentiels $(Q, W)$ Jacobi-finis est stable par extension triangulaire (Proposition 7.4.1). Soient deux algèbres amassées $\mathcal{A}_1$ et $\mathcal{A}_2$ catégorifiées par des catégories $\mathcal{C}_{(Q_1,W_1)}$ et $\mathcal{C}_{(Q_2,W_2)}$ Jacobi-finies. Soit $\mathcal{A}$ un "recollement" (=gluing) des algèbres $\mathcal{A}_1$ et $\mathcal{A}_2$ comme le décrivent C. Fu et B. Keller (cf. [FK07] section 5). Alors l'algèbre $\mathcal{A}$ pourra être catégorifiée par une catégorie $\mathcal{C}_{(\bar{Q},\bar{W})}$ où le carquois à potentiel $(\bar{Q}, \bar{W})$ est une extension triangulaire de $(Q_1, W_1)$ par $(Q_2, W_2)$.

# Summary of results

This thesis is divided into two independant parts. The first one, corresponding to chapters 2 and 3, is devoted to the problem of classifying triangulated categories with finitely many indecomposables. In the second one (chapters 4, 5, 6, and 7), we are concerned in generalizing the construction of cluster categories. The first chapter is devoted to basic definitions and properties used in this thesis.

## Part 1

The results of the first part are communicated in the article [Ami07]. The aim is to classify the small triangulated $k$-categories $\mathcal{T}$ (where $k$ is an algebraically closed field) with the Krull-Schmidt property and satisfying the following finiteness properties:

- $\mathcal{T}$ is $\mathsf{Hom}$-finite *i.e.* the morphism spaces $\mathsf{Hom}_{\mathcal{T}}(X, Y)$ are finite dimensional for all objects $X$ and $Y$ in $\mathcal{T}$;
- $\mathcal{T}$ is locally finite, *i.e.* for each indecomposable $X$ of $\mathcal{T}$, there are at most finitely many isoclasses of indecomposables $Y$ such that $\mathsf{Hom}_{\mathcal{T}}(X, Y) \neq 0$ (this condition implies its dual by [XZ02]).

### Auslander-Reiten quiver

The strategy to classify such categories consists first in computing an invariant of the category $\mathcal{T}$: the Auslander-Reiten quiver. Here we give another proof of a theorem by B. Xiao and J. Zhu [XZ02].

**Theorem** (Xiao-Zhu)**.** *(theorem 2.9) Let $\mathcal{T}$ be a triangulated Krull-Schmidt $k$-category which is $\mathsf{Hom}$-finite and locally finite. Let $\Gamma$ be a connected component of its Auslander-Reiten quiver. There exists a simply laced Dynkin quiver $Q$ and an automorphism $\varphi$ of the repetition quiver $\mathbb{Z}Q$ of infinite order (or trivial) and an isomorphism of translation quivers $\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}Q/\varphi$ .*

The first step of the proof consists in showing the existence of Auslander-Reiten triangles in $\mathcal{T}$ (section 2.1). Next we construct a subadditive function on the Auslander-Reiten quiver. Finally we conclude the proof using combinatorial results of D. Happel, U. Preiser and C. M. Ringel [HPR80a], [HPR80b](sections 2.2, 2.3 and 2.4).

In the other hand, by [Kel05] and [BMR+06], for each quiver $\mathbb{Z}Q/\varphi$ where $Q$ is Dynkin, there exists a triangulated category whose quiver is $\mathbb{Z}Q/\varphi$: the orbit category $\mathcal{D}^b(kQ)/\Phi$. Thus we formulate the following question:

*If $\mathcal{T}$ is a triangulated locally finite category with Auslander-Reiten quiver $\mathbb{Z}Q/\varphi$, is it possible to construct an equivalence between the categories $\mathcal{T}$ and $\mathcal{D}^b(kQ)/\Phi$?*

In order to answer to this question, we have to make precise what we mean by 'equivalence'. We consider two possibilities: $k$-linear equivalence and triangle equivalence.

### $k$-linear equivalence

In order to give a $k$-linear equivalence between $\mathcal{T}$ and $\mathcal{D}^b(kQ)/\Phi$, we use the universal property of the orbit category. First we construct a covering functor (section 2.5)

$$F : \underset{\overset{\circlearrowright}{\Phi}}{\mathcal{D}^b(kQ)} \twoheadrightarrow \mathcal{T}$$

following the method of C. Riedtmann [Rie80a]. We provide then an isomorphism of functors between $F$ and $F \circ \Phi$ when the category $\mathcal{T}$ has 'enough' indecomposables. More explicitly, we obtain the following result:

**Theorem.** *(cf. Theorems 2.12 and 2.13) Let $\mathcal{T}$ be a Krull-Schmidt locally finite triangulated category with Auslander-Reiten quiver $\mathbb{Z}Q/\varphi$. The category $\mathcal{T}$ is standard, i.e. $k$-linearly equivalent to $\mathcal{D}^b(kQ)/\Phi$ where $\Phi$ is the equivalence of $\mathcal{D}^b(kQ)$ induced by $\varphi$, if we are in one of the following two cases:*

- *the quiver of $Q$ is of type $A_n$ and $\varphi$ is a power of the Auslander-Reiten translate;*
- *the number of isoclasses of indecomposables of the category $\mathcal{T}$ is at least equal to the number of indecomposables of the category $\mathsf{mod}\, kQ$.*

In particular, if $\mathcal{T}$ is maximal $d$-Calabi-Yau, with $d \geq 2$, then $\mathcal{T}$ is $k$-linearly equivalent to the $d$-cluster category (corollary 2.14).

By using results of J. Białkowski, K. Erdmann, and A. Skowroński [BES07], we succeed in constructing non-standard 1-Calabi-Yau categories in caracteristic 2 (Theorem 2.25). We show the existence of a triangulated structure on the category of projective modules of finite dimension over deformed preprojective algebras of generalized Dynkin type (Corollary 2.24).

### Triangle equivalence

We make some additional assumptions on the triangulated category $\mathcal{T}$. We assume that $\mathcal{T}$ is algebraic, *i.e.* triangle equivalent to the stable category of a Frobenius category. We prove the following result:

**Theorem.** *(Theorem 2.16) If $\mathcal{T}$ is a triangulated category which is locally finite, connected, algebraic and standard, then $\mathcal{T}$ is triangle equivalent to some category $\mathcal{D}^b(kQ)/\Phi$, where $Q$ is a Dynkin quiver and $\Phi$ is an autoequivalence of infinite order of $\mathcal{D}^b(kQ)$.*

## Part 2

In the second part of the thesis (chapters 4, 5, 6, and 7), we generalize the construction of cluster categories.

Let $k$ be an algebraically closed field, and $Q$ an acyclic quiver. The cluster category $\mathcal{C}_Q$ is the orbit category

$$\mathcal{D}^b(kQ)/\nu[-2]$$

where $\nu$ is the Serre functor of the derived category $\mathcal{D}^b(kQ)$ and where [1] denotes its suspension functor.

This category has been introduced by [BMR$^+$06] (and by [CCS06] in the $A_n$ case) in order to 'categorify' cluster algebras. In the 'categorification' process, all categories $\mathcal{T}$ satisfy the following fundamental properties:

- $\mathcal{T}$ is a triangulated category;
- $\mathcal{T}$ satisfies the 2-Calabi-Yau property, *i.e.* there exists an isomorphism

  $$\mathsf{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\sim} D\mathsf{Hom}_{\mathcal{T}}(Y,X[2])$$

  which is bifunctorial in the objects $X$ and $Y$ of $\mathcal{T}$;
- there exist cluster-tilting objects, *i.e.* basic objects $T$ with the property that the space $\mathsf{Ext}^1_{\mathcal{T}}(T,X)$ vanishes if and only if $X$ is in $add(T)$ (= the smallest subcategory of $\mathcal{T}$ which contains $T$ and which is stable under direct summands).

If $\mathcal{T}$ is a category with such properties, then by [IY06], it is possible to mutate the cluster-tilting objects. This is an essential property of the 'categorification' process.

### Cluster category and 2-CY property

We want to generalize the construction of $\mathcal{C}_Q$ by replacing the hereditary algebra $kQ$ with a finite dimensional algebra $A$ of global dimension $\leq 2$. A candidate might be the orbit category $\mathcal{D}^b(A)/\nu[-2]$, where $\nu$ is the Serre functor of the derived category $\mathcal{D}^b(A)$. By

[Kel05], such a category is triangulated if $A$ is derived equivalent to an hereditary category $\mathcal{H}$. By [HR02] and [Hap01], this is true if and only if $A$ is a canonical algebra, or if $A$ is the endomorphism algebra of a tilting module over an hereditary algebra. However in general, it is not triangulated.

Thus a more appropiate candidate is the triangulated hull of the orbit category $\mathcal{D}^b(A)/\nu[-2]$. It is defined in [Kel05] as follows: Let $B$ be the dg algebra $A \oplus DA[-3]$ where $DA$ is the dual $\mathsf{Hom}_k(A,k)$ of $A$ over $k$. The category $\mathcal{D}^b(B)$ is the derived category of dg $B$-modules whose homology is of finite total dimension. Denote by $\mathsf{per}\, B$ its thick subcategory generated by $B$. The cluster category of $A$ is then defined as the thick subcategory $\mathcal{C}_A$ of the quotient

$$\mathcal{C}_A^+ = \mathcal{D}^b(B)/\mathsf{per}\, B$$

generated by $A$.

The category $\mathcal{C}_A^+$ is not $\mathsf{Hom}$-finite in general. This can be a problem since we want to show that $\mathcal{C}_A$ is 2-Calabi-Yau. Nevertheless there exists a non degenerate bilinear bifunctorial form $\beta$:

$$\beta_{NX} : \mathsf{Hom}_{\mathcal{D}}(N,X) \times \mathsf{Hom}_{\mathcal{D}}(X,N[3]) \to k$$

for each $N$ in $\mathsf{per}\, B$ and each $X$ in $\mathcal{D}^b(B)$. This allows us to construct a bilinear bifunctorial form (chapter 4, section 1)

$$\beta'_{XY} : \mathsf{Hom}_{\mathcal{C}}(X,Y) \times \mathsf{Hom}_{\mathcal{C}}(Y,X[2]) \to k$$

for all $X$ and $Y$ objects of $\mathcal{C}_A^+$. The form $\beta'$ will be non degenerate (chapter 4, sections 2 and 3) if each object of $\mathcal{D}^b(B)$ is a 'limit' of objects of $\mathsf{per}\, B$. In particular we obtain the following result:

**Corollary.** *(Corollary 4.4) Let $A$ be a finite dimensional $k$-algebra of global dimension $\leq 2$. If the functor $\mathsf{Tor}_2^A(?,DA)$ is nilpotent then the cluster category $\mathcal{C}_A$ is $\mathsf{Hom}$-finite and 2-Calabi-Yau.*

I thank R. Rouquier for informing me that these results have been independently obtained, in a much stronger form, in the forthcoming preprint [CR].

### Cluster-tilting object

The next step is to find a cluster-tilting object in this 2-Calabi-Yau category. Since $kQ$ is a cluster-tilting object of $\mathcal{C}_Q$, the canonical candidate would be the object $A$ itself. This object $A$ is rigid (Proposition 5.4.1). Moreover, if the functor $\mathsf{Tor}_2^A(?,DA)$ is nilpotent, the object $A$ is orbit-cluster-tilting (Proposition 5.4.2). More precisely, if $X$ is an object of the orbit category $\mathcal{D}^b(A)/\nu[-2]$ such that $\mathsf{Ext}^1_{\mathcal{C}_A}(T,X)$ vanishes, then $X$ is in $add(A)$. In chapter 7 we will show that $A$ is in fact cluster-tilting using completely different methods.

Its endomorphism algebra is the tensor algebra $T_A\mathsf{Ext}^2_A(DA, A)$ (Proposition 5.2.1). It is finite dimensional if and only if the functor $\mathsf{Tor}_2^A(?, DA)$ is nilpotent (Theorem 5.1). Using same techniques as in the paper [ABS06] we can compute its quiver (Proposition 5.2.2).

**Endomorphism algebra of a postprojective module**

In [GLS07b], C. Geiss, B. Leclerc and J. Schröer constructed subcategories $\mathcal{C}_M$ of $\mathsf{mod}\,\Lambda$ (where $\Lambda = \Lambda_Q$ is a preprojective algebra of an acyclic quiver) associated to certain terminal $kQ$-modules $M$. We show in chapter 6, that the stable category in such a Frobenius category $\mathcal{C}_M$ is triangle equivalent to a cluster category $\mathcal{C}_A$ where $A$ is an endomorphism algebra of a postprojective module over an hereditary algebra (Theorem 6.5).

Another approach is given by A. Buan, O. Iyama, I. Reiten and J. Scott in [BIRS07]. They construct 2-Calabi-Yau triangulated categories $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ where $\mathcal{I}_w$ is a two-sided ideal of the preprojective algebra $\Lambda = \Lambda_Q$ associated with an element $w$ of the Weyl group of $Q$. For certain elements $w$ of the Weyl group (namely those coming from preinjective tilting modules), we construct a triangle equivalence between $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ and a cluster category $\mathcal{C}_A$ where $A$ is the endomorphism algebra of a postprojective module over a concealed algebra (Theorem 6.8).

**Derived preprojective algebra**

Using generalized tilting theory (cf. [Kel94]), we give another construction of the cluster category in chapter 7. Let $A$ be a finite dimensional algebra of global dimension $\leq 2$ and let $B$ be the dg algebra $A \oplus DA[-3]$. Let $\Pi_3(A)$ be the dg algebra $R\mathcal{H}om_B(A_B, A_B)$. Using the results of [Kel94] we show that the functor $R\mathcal{H}om_B(A_B, ?) : \mathcal{D}^bB \to \mathsf{per}\,\Pi_3A$ induce the following triangle equivalences:

$$\begin{array}{rccc} R\mathcal{H}om_B(A_B,?): & \langle A\rangle_B & \xrightarrow{\ \sim\ } & \mathsf{per}\,\Pi_3A \ , \\ & \uparrow & & \uparrow \\ & \mathsf{per}\,B & \xrightarrow{\ \sim\ } & \mathcal{D}^b\Pi_3(A) \end{array}$$

where $\langle A\rangle_B$ is the thick subcategory of $\mathcal{D}^b(B)$ generated by $A_B$. We then obtain another definition of the cluster category as the quotient:

$$\mathcal{C}_A = \mathsf{per}\,\Pi_3A/\mathcal{D}^b\Pi_3A.$$

The image of the rigid object $A$ is the free dg module $\Pi_3(A)$. This dg algebra is in fact isomorphic, in the homotopy category of dg algebras, to the derived 3-preprojective algebra defined by B. Keller. As a consequence of results in [Kel08a] and [Kel08b], it satisfies the following properties:

1. it is homologically smooth in the sense of M. Kontsevich and Y. Soibelman (cf. [KS06]);

2. it has its homology concentrated in negative degrees;

3. it is bimodule 3-Calabi-Yau (*i.e.* 3-Calabi-Yau in the sense of V. Ginzburg [Gin06]).

The nilpotence of the functor $\mathsf{Tor}_2^A(?, DA)$ is equivalent to the fact that $H^0(\Pi_3(A))$ is finite-dimensional. Therefore we study in chapter 7 dg algebras with such properties and we prove the theorem

**Theorem.** *(Theorem 7.1) Let $\Gamma$ be a dg $k$-algebra with properties (1), (2) and (3). Suppose that $H^0(\Gamma)$ is finite dimensional. The category $\mathsf{per}\,\Gamma/\mathcal{D}^b(\Gamma)$ is $\mathsf{Hom}$-finite, 2-Calabi-Yau and the free dg module $\Gamma$ is a cluster-tilting object.*

As a direct consequence of this theorem, $A$ becomes a cluster-tilting object of the cluster category $\mathcal{C}_A$ (Corollary 7.2).

Furthermore it is possible to apply Theorem 7.1 to Ginzburg's dg algebras $\Gamma(Q, W)$ (cf. [Gin06]) where $(Q, W)$ is a quiver with potential (cf. [DWZ07]) when the Jacobi algebra $\mathcal{J}(Q, W)$ is finite dimensional. We construct a cluster category $\mathcal{C}_{(Q,W)}$ which is $\mathsf{Hom}$-finite, 2-Calabi-Yau. This category admits a cluster-tilting object whose endomorphism algebra is isomorphic to $\mathcal{J}(Q, W)$.

Combining this last result with some results of [Kel08b], [KY08] and [BIRS08], we obtain the corollary:

**Corollary.** *(Corollary 7.13) Let $Q$ be an acyclic quiver, and $T$ a cluster-tilting object of the cluster category $\mathcal{C}_Q$. Let $(Q', W')$ be the quiver with potential associated to the endomorphism algebra $\mathsf{End}_{\mathcal{C}_Q}(T)$. The cluster category $\mathcal{C}_{(Q,W')}$ is triangle equivalent to the category $\mathcal{C}_Q$.*

### Perspectives

The results of the second part of this thesis lead to many questions:

- Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category and $T = T_1 \oplus \cdots \oplus T_n$ a cluster-tilting object. By [IY06], for each $i = 1, \ldots, n$ it is possible to mutate the object $T$ into another cluster-tilting object $\mu_i(T)$. If the quivers of $T$ and of $\mu_i(T)$ have no loops nor 2-cylces, then the quiver of the mutated object $\mu_i(T)$ is the mutation $\mu_i(Q_T)$ of the quiver $Q_T$ of the endomorphism algebra of $T$. Therefore, it would be very useful to find conditions on the quiver with potential $(Q, W)$ associated to an algebra $A$ of global dimension 2 to be rigid in the sense of [DWZ07]. We could then mutate indefinitely the quiver with potential $(Q, W)$.

- B. Keller and I. Reiten showed in [KR06] that if a 2-Calabi-Yau algebraic triangulated category $\mathcal{C}$ has a cluster-tilting object whose quiver $Q$ is acyclic, then $\mathcal{C}$ and $\mathcal{C}_Q$ are triangle equivalent. This is not true when the quiver $Q$ has oriented cycles, but the quiver with potential $(Q, W)$ might be a better invariant. Therefore we formulate the following question:

*Let $\mathcal{C}$ be a 2-Calabi-Yau algebraic triangulated category with a cluster-tilting object. Does there exist a quiver with potential $(Q, W)$ such that $\mathcal{C}$ and $\mathcal{C}_{(Q,W)}$ are triangle equivalent?*

- Cluster categories have been constructed in order to categorify cluster algebras. Hence it might be interesting to explore the class of cluster algebras that may be categorified by categories of the form $\mathcal{C}_{(Q,W)}$. We show a first closure property of this class in section 7.4.3 which can be related to the 'gluing process' of cluster algebras described in section 5 of [FK07].

# Chapitre 1

# Préliminaires

## 1.1 Catégories triangulées

### 1.1.1 Définitions et propriétés de base

Soit $\mathcal{T}$ une (petite) catégorie additive munie d'une auto-équivalence $\Sigma$. Soit $\mathcal{S}$ l'ensemble des *sextuplets* $(X, Y, Z, u, v, w)$ où $X$, $Y$ et $Z$ sont des objets de $\mathcal{T}$ et $u : X \to Y$, $v : Y \to Z$ et $w : Z \to \Sigma X$ des morphismes. On notera un tel sextuplet :

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

Un *morphisme de sextuplets* est la donnée d'un diagramme commutatif

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array} .$$

Si $f$, $g$ et $h$ sont des isomorphismes, on dira que $(f, g, h)$ est un isomorphisme de sextuplets.

**Définition 1.1.** Une *catégorie triangulée* $\mathcal{T}$ est une catégorie additive munie d'une auto-équivalence $\Sigma$ et d'un sous-ensemble $\Delta$ de $\mathcal{S}$ appelé l'ensemble des *triangles* qui vérifient les axiomes suivants :

**TR0** : L'ensemble des triangles est stable par isomorphisme. Pour tout objet $X$ de $\mathcal{T}$, le sextuplet $X = X \longrightarrow 0 \longrightarrow \Sigma X$ est un triangle.

**TR1** : Pour tout morphisme $u : X \to Y$ dans $\mathcal{T}$, il existe un triangle :

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma X$$

**TR2** : Si $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ est un triangle, alors le sextuplet

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

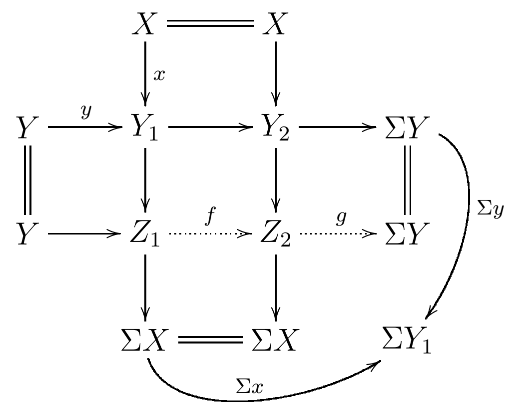
est un triangle.

**TR3** : Pour tout diagramme commutatif de la forme

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

où les deux lignes sont des triangles, il existe un morphisme $h : Z \to Z'$ (non unique) tel que $(f, g, h)$ est un morphisme de triangles.

**TR4 (Axiome de l'octahèdre)** : Etant donné un diagramme commutatif de la forme




où les deux colonnes et la première ligne sont des triangles, il existe des morphismes $f : Z_1 \to Z_2$ et $g : Z_2 \to \Sigma Y$, tels que la deuxième ligne soit un triangle et que tous les carré commutent, y compris le carré :

$$\begin{array}{ccc} Z_2 & \xrightarrow{g} & \Sigma Y \\ \downarrow & & \downarrow{\scriptstyle \Sigma y} \\ \Sigma X & \xrightarrow{\Sigma u} & \Sigma Y_1 \end{array}$$

Si les axiomes **TR1-TR3** sont vérifiés, cet axiome est équivalent à l'axiome suivant [Nee01] :

**TR4'** : Etant donné un diagramme

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

où les lignes sont des triangles, il existe un morphisme $h : Z \to Z'$ rendant le diagramme commutatif et tel que le cône est un triangle :

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

Pour les propriété de bases des catégories triangulées, nous renvoyons au premier chapitre de [Hap88]. Rappelons juste la propriété bihomologique du bifoncteur Hom.

**Proposition 1.1.1.** *Soit $\mathcal{T}$ une catégorie triangulée. Alors, tout triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

*induit des suites exactes longues :*

$$\cdots \to \mathsf{Hom}_{\mathcal{T}}(?, \Sigma^{-1}Z) \to \mathsf{Hom}_{\mathcal{T}}(?, X) \to \mathsf{Hom}_{\mathcal{T}}(?, Y) \to \mathsf{Hom}_{\mathcal{T}}(?, Z) \to \mathsf{Hom}_{\mathcal{T}}(?, \Sigma X) \to \cdots$$

$$\cdots \to \mathsf{Hom}_{\mathcal{T}}(\Sigma X, ?) \to \mathsf{Hom}_{\mathcal{T}}(Z, ?) \to \mathsf{Hom}_{\mathcal{T}}(Y, ?) \to \mathsf{Hom}_{\mathcal{T}}(X, ?) \to \mathsf{Hom}_{\mathcal{T}}(\Sigma^{-1}Z, ?) \to \cdots$$

**Définition 1.2.** Soit $(\mathcal{T}, \Sigma)$ et $(\mathcal{T}', \Sigma')$ deux catégories triangulées. Une foncteur triangulé $(F, \Phi) : \mathcal{T} \to \mathcal{T}'$ est la donnée d'un foncteur $F : \mathcal{T} \to \mathcal{T}'$ de catégories additives et d'un isomorphisme de foncteurs $\Phi : F \circ \Sigma \to \Sigma' \circ F$ tel que pour tout triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ de $\mathcal{T}$, le sextuplet

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\Phi_X \circ Fw} \Sigma' FX$$

est un triangle de $\mathcal{T}'$.

### 1.1.2 Dualité de Serre et catégories de Calabi-Yau

Soit $k$ un corps commutatif.

**Définition 1.3.** Une catégorie triangulée $k$-linéaire $\mathcal{T}$ vérifie la propriété de *Krull-Remak-Schmidt* si tout objet est isomorphe à une unique (à permutation près) somme directe de d'objets indécomposables et si l'anneau des endomorphismes d'un objet indécomposable est local. Celà est équivalent au fait que les idempotents se scindent, *i.e.* si $e$ est un idempotent de $X$, alors $e$ s'écrit $\sigma\rho$ où $\sigma$ est une section et $\rho$ une rétraction [Hap88](I.3.2).

**Dans toute cette thèse, les catégories triangulées étudiées sont $k$-linéaires et vérifient la propriété de Krull-Remak-Schmidt.**

La catégorie est dite Hom-finie si pour tous objets $X$ et $Y$ dans $\mathcal{T}$, le $k$-espace vectoriel $\mathsf{Hom}_{\mathcal{T}}(X, Y)$ est de dimension finie.

**Définition 1.4.** Soit $\mathcal{T}$ une catégorie triangulée $k$-linéaire et Hom-finie. Un *foncteur de Serre* est la donnée d'une auto-équivalence $k$-linéaire $\nu : \mathcal{T} \to \mathcal{T}$ et d'un isomorphisme de foncteur

$$D\mathsf{Hom}_{\mathcal{T}}(X, ?) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(?, \nu X)$$

pour tout objet $X$ de $\mathcal{T}$, où $D$ est le foncteur dual $\mathsf{Hom}_k(?, k)$.

Si $\mathcal{T}$ admet un foncteur de Serre, alors celui-ci est unique à isomorphisme près.

**Définition 1.5.** Soit $\mathcal{T}$ une catégorie triangulée $k$-linéaire et Hom-finie, de foncteur suspension $\Sigma$. La catégorie $\mathcal{T}$ est dite *$d$-Calabi-Yau*, si le foncteur $\Sigma^d$ est un foncteur de Serre.

### 1.1.3 Exemples

**Algèbre préprojective**

Un carquois $Q = (Q_0, Q_1, s, t)$ est la donnée d'un ensemble de sommets $Q_0$, d'un ensemble de flèches $Q_1$ et de deux applications $s : Q_1 \to Q_0$ (l'application source) et $t : Q_1 \to Q_0$ (l'application but).

Soit $Q$ un carquois fini et sans cycle orienté. On définit $\bar{Q}$ le carquois double à partir de $Q$ en ajoutant à chaque flèche $a : i \to j$ une flèche $\overline{a} : j \to i$. *L'algèbre préprojective* $\Lambda_Q = k\bar{Q}/\mathcal{I}_Q$ associée à $Q$ est définie comme le quotient de l'algèbre des chemins $k\bar{Q}$ du carquois double quotienté par l'idéal $\mathcal{I}_Q$ engendré par les relations :

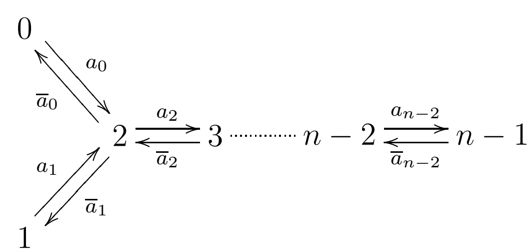
$$\sum_{a \in Q_1} (\overline{a}a + a\overline{a}).$$

Les résultats suivants sont classiques :

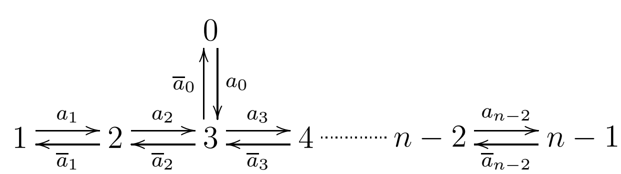
**Théorème 1.6.** *Si le graphe sous-jacent à $Q$ est Dynkin de type $A$, $D$ ou $E$, alors l'algèbre $\Lambda_Q$ est de dimension finie, auto-injective, et la catégorie stable $\underline{\mathsf{mod}}\,\Lambda_Q$ est triangulée 2-Calabi-Yau.*

$Q = \mathbb{A}_n$ $(n \geq 1)$ :

$Q = \mathbb{D}_n$ $(n \geq 4)$ :



$Q = \mathbb{E}_n$ $(n = 6, 7, 8)$ :



**Théorème 1.7.** *([BBK02], [Boc07]) Si $Q$ n'est pas de type Dynkin, alors l'algèbre $\Lambda_Q$ est de dimension infinie. Notons $\bar{\Lambda}_Q$ la complétion de l'algèbre $\Lambda$, et $\mathsf{f.l.}\,\bar{\Lambda}_Q$ la catégorie des $\bar{\Lambda}_Q$-modules de longueur finie. Alors la catégorie dérivée $\mathcal{D}^b(\mathsf{f.l.}\,\bar{\Lambda}_Q)$ est triangulée 2-Calabi-Yau.*

### Catégorie amassée

Soit $Q$ un carquois fini sans cycle orienté. La *catégorie amassée* est définie comme la catégorie d'orbites :

$$\mathcal{C}_Q = \mathcal{D}^b(kQ)/\nu[-2]$$

où $\nu$ est le foncteur de Serre de la catégorie $\mathcal{D}^b(kQ)$ et $[1]$ le foncteur décalage, qui est la suspension de la catégorie triangulée $\mathcal{D}^b(kQ)$. Les objets de cette catégorie sont les mêmes que ceux de $\mathcal{D}^b(kQ)$, et étant donnés deux objets $X$ et $Y$ de $\mathcal{D}^b(kQ)$, l'espace des morphismes dans $\mathcal{C}_Q$ entre $X$ et $Y$ est donné par :

$$\mathsf{Hom}_{\mathcal{C}_Q}(X, Y) = \bigoplus_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b(kQ)}(X, \nu^p Y[-2p]).$$

Un corollaire du théorème 0.1 est le suivant :

**Corollaire 1.8.** *Soit $Q$ un carquois fini sans cycle orienté. La catégorie amassée $\mathcal{C}_Q$ est* $\mathsf{Hom}$*-finie, triangulée, 2-Calabi-Yau. La projection canonique $\pi : \mathcal{D}^b(kQ) \longrightarrow \mathcal{C}_Q$ est un foncteur triangulé.*

La structure $k$-linéaire sous-jacente à la catégorie $\mathcal{C}_Q$ vérifie la proposition universelle suivante : soit $\mathcal{T}$ une catégorie triangulée et $F$ un foncteur $k$-linéaire $F : \mathcal{D}^b(\mathsf{mod}\, kQ) \to \mathcal{T}$. Si il existe un isomorphisme de foncteur entre $F$ et $F \circ \nu[-2]$, alors $F$ se factorise à travers $\pi$. En particulier, si $\mathcal{T}$ est 2-Calabi-Yau et que $F$ est triangulé, alors $F$ induit un foncteur $k$-linéaire $\mathcal{C}_Q \to \mathcal{T}$.

## 1.2 Généralités sur les dg-catégories

Cette section reprend les notations et définitions de [Kel06].
Soit $k$ un corps commutatif.

### 1.2.1 Cas général

#### $k$-modules différentiels gradués

Un *$k$-module gradué* est un $k$-espace vectoriel $V$ muni d'une décomposition :

$$V = \bigoplus_{p \in \mathbb{Z}} V^p.$$

On note $V[1]$ le $k$-module gradué tel que pour tout $p$ and $\mathbb{Z}$, on a $V[1]^p = V^{p+1}$.

Un *morphisme de $k$-module gradué homogène de degré $n$* est une application linéaire $f : V \to V'$ telle que pour tout $p$ dans $\mathbb{Z}$, on a $f(V^p) \subset V'^{p+n}$.

Le produit tensoriel de deux $k$-modules gradués est un $k$-module gradué par

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q.$$

Si $f : V \to V'$ est un morphisme gradué et $g : W \to W'$ est un morphisme homogène de degré $p$, alors l'application $f \otimes g$ est définie par

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(w)$$

si $v \in V$ est homogène de degré $q$.

On note $\mathcal{G}(k)$ la $k$-catégorie tensorielle des $k$-modules gradués où les morphismes sont les morphismes homogènes de degré 0.

Un *$k$-module différentiel gradué* (dg-$k$-module) est un $k$-module gradué $V$ muni d'un endomorphisme de degré 1 $d_V : V \to V$ appelé *différentielle* tel que $d_V^2 = 0$. Alors on définit le décalage et le produit tensoriel de dg-k-modules par :

$$(V, d)[1] = (V[1], -d) \quad \text{et} \quad (V, d_V) \otimes (W, d_W) = (V \otimes W, d_V \otimes 1_W + 1_V \otimes d_W).$$

Un morphisme de dg-$k$-module est un morphisme homogène de degré 0 qui commute à la différentielle.

### dg-catégories

**Définition 1.9.** Une *dg-catégorie* $\mathcal{A}$ est une $k$-catégorie dont les espaces de morphismes sont des dg-$k$-modules et dont les compositions

$$\mathcal{A}(x, y) \otimes \mathcal{A}(y, z) \to \mathcal{A}(x, z)$$

sont des morphismes de dg-$k$-modules.

*Exemple.* Soit $\mathcal{C}_{dg}(k)$ la catégorie définie comme suit :
- les objets de $\mathcal{C}_{dg}(k)$ sont les dg-$k$-modules,
- si $V$ et $W$ sont des dg-$k$-modules, l'espace $\mathcal{C}_{dg}(k)(V, W) = \mathcal{H}om_k^\bullet(V, W)$ est le complexe suivant :

$$\cdots \longrightarrow \mathcal{G}(k)(V, W[p]) \xrightarrow{d} \mathcal{G}(k)(V, W[p+1]) \longrightarrow \cdots,$$

  où $d$ est définie comme $df = d_W \circ f - (-1)^p f \circ d_V$, si $f$ est dans $\mathcal{G}(k)(V, W[p])$, c'est-à-dire homogène de degré $p$.

Cette catégorie est une dg-catégorie.

**Définition 1.10.** Soient $\mathcal{A}$ et $\mathcal{A}'$ deux dg-catégories. Un *dg-foncteur* est la donnée d'une application $F : obj(\mathcal{A}) \to obj(\mathcal{A}')$ et de morphismes de dg-$k$-modules

$$F_{xy} : \mathcal{A}(x, y) \to \mathcal{A}'(Fx, Fy)$$

pour tous objets $x$ et $y$ de $\mathcal{A}$.

### dg-$\mathcal{A}$-modules

**Définition 1.11.** Un *dg-$\mathcal{A}$-module* $M$ est un dg-foncteur de $\mathcal{A}^{op} \to \mathcal{C}_{dg}(k)$. En particulier, $M$ est la donnée :

- pour tout objet $x$ de $\mathcal{A}$, d'un complexe de $k$-espaces vectoriels $Mx$,
- pour tous objets $x$, $y$ de $\mathcal{A}$ d'un morphisme de complexes :

$$M(x,y) : \mathcal{A}(y,x) \longrightarrow \mathcal{C}_{dg}(k)(Mx, My)$$

qui doit être compatible avec la composition.

Comme $M(x,y) = M$ est un morphisme de dg-$k$-module, il commute à la différentielle et alors si $f$ est dans $\mathcal{A}(y,x)$, on a $M \circ d(f^p) = d_{Mx} \circ Mf^p - (-1)^p Mf^p \circ d_{My}$.

Notons $\mathcal{C}_{dg}\mathcal{A}$ la classe des dg-$\mathcal{A}$-modules. Un morphisme $f$ entre deux dg-$\mathcal{A}$-modules $M$ et $N$, est la donnée pour $x$ dans $\mathcal{A}$ d'un élément $f_x$ de $\mathcal{G}(k)(Mx, My)$ tel que si $g$ est dans $\mathcal{A}^p(y,x)$, alors pour tout $q$ on a le diagramme commutatif suivant :

$$\begin{array}{ccc} Mx & \xrightarrow{f_x^q} & Nx[q] \\ \downarrow{\scriptstyle Mg} & & \downarrow{\scriptstyle Ng} \\ My[p] & \xrightarrow{f_y^q} & Ny[p+q] \end{array} .$$

On munit cet espace gradué de morphismes de la différentielle induite par celle de $\mathcal{C}_{dg}(k)$. C'est-à-dire que si $f$ est homogène de degré $q$ entre $M$ et $N$, on pose $(df)_x = d(f_x)$. Munie de ces espaces de morphismes, la catégorie des dg-$\mathcal{A}$-modules $\mathcal{C}_{dg}(\mathcal{A})$ forme alors une dg-catégorie.

Pour tout objet $x$ de $\mathcal{A}$, le foncteur $x^\wedge = \mathcal{A}(?, x)$ est un dg-$\mathcal{A}$-module. Le foncteur de Yoneda :

$$\begin{array}{rcl} \mathsf{Yon} : \mathcal{A} & \longrightarrow & \mathcal{C}_{dg}(\mathcal{A}) \\ x & \longmapsto & x^\wedge \end{array}$$

est un dg-foncteur pleinement fidèle.

### Catégorie dérivée d'une dg-catégorie

On définit ensuite les catégories $\mathcal{CA}$ et $\mathcal{HA}$. Les objets sont les mêmes que ceux de $\mathcal{C}_{dg}(\mathcal{A})$ et les espaces de morphismes sont donnés par :

$$\mathcal{CA}(M,N) = Z^0(\mathcal{C}_{dg}(\mathcal{A})(M,N)),$$

$$\text{et } \mathcal{HA}(M,N) = H^0(\mathcal{C}_{dg}(\mathcal{A})(M,N)).$$

Donc si $f$ est dans $\mathcal{CA}(M,N)$, et $x$ dans $\mathcal{A}$, alors $f_x$ est un morphisme de complexes entre $Mx$ et $Nx$. Si $f$ est dans $\mathcal{HA}(M,N)$ alors $f_x$ est un morphisme de complexes modulo homotopie entre $Mx$ et $Nx$.

Un quasi-isomorphisme $h$ est un morphisme de $\mathcal{HA}$ tel que pour tout $x$ de $\mathcal{A}$ et pour tout $p$, $H^p(f_x)$ est inversible. On définit la catégorie dérivée $\mathcal{DA}$ comme la localisation de la catégorie $\mathcal{HA}$ par les quasi-isomorphismes.

La catégorie $\mathcal{CA}$ est une catégorie exacte pour les conflations

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

qui sont les suites exactes scindées de $\mathcal{CA}$. Les objets contractiles, c'est-à-dire homotopes à 0 sont les projectifs-injectifs de cette catégorie. La catégorie $\mathcal{CA}$ est une catégorie de Frobenius. Sa catégorie stable est la catégorie $\mathcal{HA}$ qui est donc une catégorie triangulée. La catégorie $\mathcal{DA}$ est alors triangulée, comme localisation d'une catégorie triangulée.

On a la suite de foncteurs suivante :

$$\mathcal{A} \xhookrightarrow{Yon} \mathcal{C}_{dg}(\mathcal{A}) \xrightarrow{Z^0} \mathcal{CA} \longrightarrow \mathcal{HA} \longrightarrow \mathcal{DA}$$

La catégorie $\mathsf{per}\,\mathcal{A}$ est définie comme la plus petite sous-catégorie triangulée de $\mathcal{DA}$ contenant les $x^\wedge$ et stable par facteurs directs (=sous-catégorie épaisse). La catégorie $\mathcal{D}^b\mathcal{A}$ est la sous-catégorie de $\mathcal{DA}$ formée des objets $M$ tels que pour tout $x$ de $\mathcal{A}$, l'espace vectoriel

$$\bigoplus_{p\in\mathbb{Z}} H^p(Mx)$$

est de dimension finie. C'est une sous-catégorie triangulée de $\mathcal{DA}$.

### 1.2.2 Cas où la dg-catégorie provient d'une dg-algèbre

#### Cas général

Une dg-algèbre $A$ est une algèbre $\mathbb{Z}$-graduée munie d'une différentielle vérifiant la règle de Leibniz, c'est-à-dire que $d$ est un morphisme $k$-linéaire homogène de degré 1 et si $a$ et $b$ sont dans $A$ et $a$ est homogène de degré $p$, alors $d(ab) = (da)b + (-1)^p a d(b)$.

On peut voir $A$ comme une dg-catégorie : la dg-catégorie n'ayant qu'un seul objet $*$ et où $\mathsf{End}(*)$ est la dg-algèbre $A$.

Un dg-$A$-module est alors la donnée :

– d'un complexe de $k$-espaces vectoriels $M*$, que l'on notera $M$,

– et d'un morphisme de dg-algèbres $M(*, *) : A \to \mathcal{C}_{dg}(k)(M, M)$.

$M(*, *)$ est un morphisme gradué, donc pour tout $p$, $A^p$ s'envoie dans $\mathcal{C}_{dg}(k)(M, M[p])$, c'est-à-dire que pour tout $n$, on a une application

$$\begin{array}{rcl} M^n \times A^p & \longrightarrow & M^{n+p} \\ (m, a) & \longmapsto & m.a. \end{array}$$

De plus, $M(*, *)$ est un morphisme de complexes, il commute donc à la différentielle. Ceci signifie que si $a$ est dans $A^p$ et $m$ dans $M^n$, alors

$$m.d_A a = d_M(m.a) - (-1)^p (d_M m).a.$$

Enfin $M(*,*)$ est un morphisme de $k$-algèbre, donc pour $m$ dans $M^n$, $a$ dans $A^p$ et $b$ dans $A^q$, on a $(m.a).b = m.(ab)$ dans $M^{n+p+q}$.

Soient $M$ et $N$ deux dg-$A$-modules. Un morphisme $f$ de dg-$A$-modules entre $M$ et $N$ est un élément de $\mathcal{C}_{dg}(k)(M,N)$ tel que pour tout $a$ dans $A^p$, et pour tous $n,q$, le carré suivant commute :

$$\begin{array}{ccc} M^n & \xrightarrow{f^q} & N^{n+q} \\ \downarrow{\scriptstyle .a} & & \downarrow{\scriptstyle .a} \\ M^{n+p} & \xrightarrow{f^q} & N^{n+p+q}. \end{array}$$

Dans la catégorie $\mathcal{C}A$ les morphismes sont les morphismes de complexes de $k$-espaces vectoriels gradués entre $M$ et $N$ tels que si $a$ est dans $A_p$, on a le carré commutatif :

$$\begin{array}{ccc} M^n & \xrightarrow{f} & N^n \\ \downarrow{\scriptstyle .a} & & \downarrow{\scriptstyle .a} \\ M^{n+p} & \xrightarrow{f} & N^{n+p}. \end{array}$$

Le dg-$\mathcal{A}$-module représentable $*^\wedge$ est le complexe $A$ muni de la multiplication de l'algèbre.

**Cas d'une algèbre**

Soit $A$ une $k$-algèbre, on peut la voir comme une dg-algèbre concentrée en degré 0. Les objets de $\mathcal{C}_{dg}(A)$ sont alors les complexes de $A$-modules à droite. La composante homogène de degré $p$ de l'espace des morphismes de $\mathcal{C}_{dg}(A)$ de $M$ dans $N$ est l'ensemble des morphismes gradués de $A$-modules de $M$ dans $N[p]$ ($f$ commute avec l'action de $A$ mais pas avec la différentielle de $M$).

La catégorie $\mathcal{C}A$ est simplement la catégorie des complexes de $A$-modules.

La catégorie $\mathcal{D}A$ est la catégorie $\mathcal{D}(\mathsf{Mod}\,A)$. Si $A$ est de dimension finie, on a toujours :

$$\mathsf{per}\,A \subset \mathcal{D}^b(\mathsf{mod}\,A) \subset \mathcal{D}^bA \subset \mathcal{D}A = \mathcal{D}(\mathsf{Mod}\,A),$$

où $\mathsf{mod}\,A$ désigne la catégorie des modules de présentation finie sur $A$ (=modules de type fini).

Si $A$ est de dimension globale finie, on a de plus l'égalité $\mathsf{per}\,A = \mathcal{D}^b(\mathsf{mod}\,A) = \mathcal{D}^bA$.

### 1.2.3 Foncteur de Serre

Soit $A$ une dg-algèbre de dimension finie. Notons $\mathcal{D}$ la catégorie $\mathcal{D}^bA$, et pour $X$ et $Y$ des dg-$A$-modules, notons $\mathcal{H}om_A^\bullet(X,Y)$ l'espace des morphismes entre $X$ et $Y$ dans la dg-catégorie $\mathcal{C}_{dg}(A)$. Il a donc une structure de complexe de $k$-espaces vectoriels.

**Lemme 1.2.1.** *Soit $X$ un objet de $\mathsf{per}\,A$ et $Y$ un objet de $\mathcal{D}^b A$, alors on a un isomorphisme bifonctoriel*

$$D\mathsf{Hom}_{\mathcal{D}}(X,Y) \simeq \mathsf{Hom}_{\mathcal{D}}(Y, X \overset{L}{\otimes}_A DA).$$

*Démonstration.* Pour $X$ dans $\mathsf{per}\,A$, notons $\nu X = D\mathcal{H}om^{\bullet}_A(X,A)$. Nous allons construire un morphisme bifonctoriel :

$$F_{XY} : D\mathcal{H}om^{\bullet}_A(X,Y) \longrightarrow \mathcal{H}om^{\bullet}_A(Y,\nu X)$$

Soit $\varphi$ un élément du dual de $\mathcal{H}om^{\bullet}_A(X,Y)$. On utilise l'isomorphisme canonique

$$\mathcal{H}om^{\bullet}_A(Y,\nu X) \simeq \mathcal{H}om^{\bullet}_A(\mathcal{H}om^{\bullet}_A(A,Y), D\mathcal{H}om^{\bullet}_A(X,A)).$$

A un élément $f$ de $\mathcal{H}om^{\bullet}_A(A,Y)$ on associe la forme $F_{XY}(f)$, qui à un $g$ de $\mathcal{H}om^{\bullet}_A(X,A)$ associe $\varphi(gf)$.

En utilisant les formules classiques d'adjonction, on vérifie facilement que $F_{AY}$ est un isomorphisme. Donc pour tout $X$ appartenant à la sous-catégorie épaisse $\mathcal{H}A$ contenant $A$, $F_{XY}$ est un quasi-isomorphisme. Donc, $H^0(F_{XY})$ est un isomorphisme, *i.e.* on a un isomorphisme fonctoriel :

$$D\mathsf{Hom}_{\mathcal{H}A}(X,Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{H}A}(Y,\nu X)$$

Pour tout objet cofibrant $X$ de $\mathsf{per}\,A$, $\nu X$ est fibrant et on a donc un isomorphisme :

$$D\mathsf{Hom}_{\mathcal{D}A}(X,Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}A}(Y,\nu X)$$

Il reste à montrer que pour tout $X$ de $\mathsf{per}\,A$, on a un quasi-isomorphisme entre $\nu X = D\mathcal{H}om^{\bullet}_A(X,A)$ et $X \overset{L}{\otimes}_A DA$. Si $X$ est un dg-module, on a un morphisme fonctoriel :

$$\begin{aligned} X \overset{L}{\otimes}_A DA &\longrightarrow D\mathcal{H}om^{\bullet}_A(X,A) \\ x \otimes \varphi &\longmapsto \big(g \mapsto \varphi(g(x))\big) \end{aligned}$$

qui s'étend à un morphisme pour tout $X$ dans $\mathsf{per}\,A$. Comme c'est un isomorphisme pour $X = A$, c'est un quasi-isomorphisme pour $X$ dans $\mathsf{per}\,A$.

□

*Remarque.* En utilisant les formules d'adjonction, on obtient immédiatement que si $X$ est dans $\mathsf{per}\,A$ et $Y$ dans $\mathcal{D}^b(A)$ alors on a un isomorphisme

$$D\mathsf{Hom}_{\mathcal{D}}(X,Y) \simeq \mathsf{Hom}_{\mathcal{D}}(Y \overset{L}{\otimes}_A R\mathcal{H}om_A(A,DA), X).$$

Ce lemme nous donne immédiatement le corollaire suivant :

**Corollaire 1.12.** *Soit $A$ une dg-algèbre de dimension finie. Si $DA$ est isomorphe à $A[d]$ en tant que $A$-$A$-bimodule, alors pour tout $X$ dans* $\mathsf{per}\,A$ *et pour tout $Y$ dans $\mathcal{D}^bA$, on a un isomorphisme fonctoriel :*

$$D\mathsf{Hom}_{\mathcal{D}}(X,Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(Y, X[d])\ .$$

*En particulier la catégorie* $\mathsf{per}\,A$ *est $d$-Calabi-Yau.*

*Remarque.* Plus généralement, Keller a montré dans [Kel08a] que si $A$ est un dg-algèbre de dimension quelconque, alors pour tout $X$ dans $\mathsf{per}\,A$ et $Y$ dans $\mathcal{D}^b(A)$ on a un isomorphisme

$$D\mathsf{Hom}_{\mathcal{D}}(X,Y) \simeq \mathsf{Hom}_{\mathcal{D}}(Y \overset{L}{\otimes}_A R\mathcal{H}om_{A^e}(A, A^e), X).$$

Si $A$ est une algèbre de dimension finie et de dimension globale finie alors le foncteur $\nu = ?\overset{L}{\otimes}_A DA$ est une équivalence dont l'inverse est $?\overset{L}{\otimes}_A R\mathcal{H}om_A(A, DA)$. Donc la catégorie $\mathcal{D}^b(A) = \mathsf{per}\,A$ admet un foncteur de Serre.

### 1.2.4 Catégories triangulées algébriques

Soit $\mathcal{T}$ une catégorie triangulée $k$-linéaire. On dit que $\mathcal{T}$ est *algébrique* s'il existe une équivalence triangulée entre $\mathcal{T}$ et $\underline{\mathcal{E}}$ où $\mathcal{E}$ est une catégorie de Frobenius $k$-linéaire. En fait d'après Keller [Kel06], une catégorie triangulée est algébrique si et seulement si elle admet un renforcement en catégorie différentielle-graduée, *i.e.* il existe une équivalence triangulée entre $\mathcal{T}$ et $\mathcal{D}\mathcal{A}$ où $\mathcal{A}$ est une dg-catégorie. Cette notion est stable par passage à une sous-catégorie triangulée et par localisation. Ainsi, toutes les catégories triangulées apparaissant en algèbre et en géométrie sont algébriques.

Si $\mathcal{D}\mathcal{A}$ et $\mathcal{D}\mathcal{B}$ sont des catégories triangulées algébriques, alors un foncteur algébrique $F$ est un foncteur 'provenant' des renforcements $\mathcal{A}$ et $\mathcal{B}$, *i.e.* $F = ?\overset{L}{\otimes}_{\mathcal{A}} X$ est le produit tensoriel dérivé par un objet $X$ de $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$.

Cette définition nous permet de donner la propriété universelle de la catégorie amassée. Soit $\mathcal{T} = \mathcal{D}\mathcal{A}$ une catégorie algébrique, et soit $X$ un objet de $\mathcal{D}(kQ^{op} \otimes \mathcal{A})$.

$$\mathcal{D}^b(\mathsf{mod}\,kQ) \xrightarrow{?\overset{L}{\otimes}_{kQ} X} \mathcal{D}\mathcal{A}$$
$$\circlearrowleft\ ?\overset{L}{\otimes}_{kQ} DkQ[-2]$$

Si il existe un isomorphisme dans $\mathcal{D}(kQ^{op} \otimes \mathcal{A})$ entre $DkQ \otimes_{kQ} X[-2]$ et $X$, alors le foncteur $?\overset{L}{\otimes}_{kQ} X$ se factorise en un foncteur algébrique de $\mathcal{C}_Q$ vers $\mathcal{D}\mathcal{A}$.

## 1.3 Une autre construction de la catégorie amassée

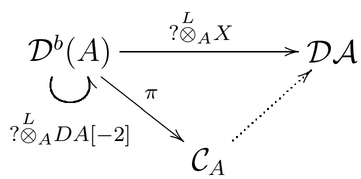
### 1.3.1 Enveloppe triangulée

Soit $A$ une $k$-algèbre de dimension finie et de dimension globale finie. Alors la catégorie $\mathcal{D}^b(A)$ est triangulée et admet un foncteur de Serre $\nu_A = ? \overset{L}{\otimes}_A DA$ où $DA = \mathsf{Hom}_k(A,k)$ est le dual de $A$. La catégorie d'orbites

$$\mathcal{D}^b(A)/? \overset{L}{\otimes}_A DA[-2]$$

n'est pas triangulée en général si la dimension globale de $A$ est $\geq 2$. Son *enveloppe triangulée* est la catégorie $\mathcal{C}_A$ vérifiant la propriété universelle suivante :

- il existe un foncteur triangulé algébrique $\pi : \mathcal{D}^b(A) \to \mathcal{C}_A$ (pas essentiellement surjectif en général) ;
- Soit $? \overset{L}{\otimes}_A X : \mathcal{D}^b(A) \to \mathcal{D}\mathcal{B}$ un foncteur algébrique où $\mathcal{B}$ est une dg-catégorie. Si on a un isomorphisme dans $\mathcal{D}(A^{op} \otimes \mathcal{B})$ entre $DA \overset{L}{\otimes}_A X[-2]$ et $X$, alors le foncteur $? \overset{L}{\otimes}_A X$ se factorise par $\pi$.




### 1.3.2 Construction du foncteur $\pi$

Soit $B$ la dg-algèbre suivante :

$$A \oplus DA[-3] = \cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow DA \longrightarrow 0 \longrightarrow \cdots .$$

Un objet $M$ de $\mathcal{C}B$ est un complexe de $A$-modules muni d'une action de $DA$ de degré 3 qui anticommute à la différentielle. Ce qui signifie que pour tout $n$ et pour tout $a^*$ dans $DA$ le diagramme suivant anticommute :

$$\begin{array}{ccc} M^n \times DA & \xrightarrow{a^*} & M^{n+3} \\ \downarrow{\scriptstyle (d_M, id)} & & \downarrow{\scriptstyle d_M} \\ M^{n+1} \times DA & \xrightarrow{a^*} & M^{n+4} \end{array}$$

et que pour tous $a^*, b^*$ dans $DA$, $m.a^*.b^* = 0$.

La projection $p : B \to A$ induit un foncteur restriction pleinement fidèle $p_* : \mathcal{C}A \to \mathcal{C}B$. Ce foncteur envoie un complexe de $A$-modules sur lui même muni de l'action de $DA$ nulle. L'objet $A$ peut être vu comme un $A$-$B$-bimodule, et alors le foncteur $p_*$ est égal au foncteur $? \otimes_A A_B$. Deux complexes homotopes dans $\mathcal{C}A$ vont être homotopes dans $\mathcal{C}B$ donc ce foncteur induit un foncteur $\mathcal{H}A \to \mathcal{H}B$. De même les quasi-isomorphismes de $\mathcal{C}A$ sont des quasi-isomorphismes de $\mathcal{C}B$, donc la projection induit un foncteur $\mathcal{D}A \to \mathcal{D}B$. De plus si l'homologie de $M$ est de dimension totale finie alors l'homologie de son image sera la même et donc de dimension totale finie. Finalement, on a un foncteur :

$$p_* : \mathcal{D}^b A \longrightarrow \mathcal{D}^b B.$$

Maintenant, comme $A$ (et donc $DA$) est de dimension finie, $\mathsf{per}\, B$ est inclus dans $\mathcal{D}^b B$. On obtient donc un foncteur :

$$F : \mathcal{D}^b A \longrightarrow \mathcal{D}^b B/\mathsf{per}\, B =: \mathcal{C}_A^+.$$

Notons $i : A \to B$ l'injection canonique, et $i_* : \mathcal{D}^b B \to \mathcal{D}^b A$, le foncteur 'oubli' associé. Alors on a les adjonctions suivantes :

$$\begin{array}{ccc} \mathcal{D}^b B & \text{et} & \mathcal{D}^b A \\ {\scriptstyle -\overset{L}{\otimes}_A B}\uparrow \downarrow {\scriptstyle i_*} & & {\scriptstyle -\overset{L}{\otimes}_B A}\uparrow \downarrow {\scriptstyle p_*} \\ \mathcal{D}^b A & & \mathcal{D}^b B \end{array} .$$

La suite exacte de $A$-$B$-bimodules :

$$0 \longrightarrow DA[-3] \longrightarrow B \longrightarrow A \longrightarrow 0$$

nous donne un triangle dans $\mathcal{D}(A^{op} \otimes B)$

$${}_A DA_B[-3] \longrightarrow {}_A B_B \longrightarrow {}_A A_B \longrightarrow {}_A DA_B[-2] .$$

L'objet $B_B$ est parfait donc le morphisme $A_B \to DA_B[-2]$ est un isomorphisme dans $\mathcal{C}_A^+ = \mathcal{D}^b B/\mathsf{per}\, B$. D'après la propriété universelle de l'enveloppe triangulée de la catégorie d'orbites, $p_*$ induit un foncteur triangulé algébrique :

$$p_* : \mathcal{C}_A \to \mathcal{C}_A^+ = \mathcal{D}^b B/\mathsf{per}\, B.$$

Ce foncteur est pleinement fidèle et Keller a montré le théorème suivant [Kel05] :

**Théorème 1.13** (Keller)**.** *Soit $A$ une algèbre de dimension finie et de dimension globale finie. Soit $B$ la dg-algèbre $A \oplus DA[-3]$ et $p_* : \mathcal{D}^b A \to \mathcal{D}^b B$ le foncteur restriction de la projection $p : B \to A$. Notons $\langle A \rangle_B$ la sous-catégorie épaisse (=sous-catégorie triangulée stable par facteurs directs) de $\mathcal{D}^b B$ engendrée par l'image de $A$ par $p_*$. Alors l'enveloppe triangulée de la catégorie d'orbites*

$$\mathcal{D}^b(A)/? \overset{L}{\otimes}_A DA[-2]$$

*est algébriquement équivalente à la catégorie $\langle A \rangle_B/\mathsf{per}\, B$.*

Notons que dans le cas où la dimension globale de $A$ est 1, ou que $A$ est dérivée équivalente à une catégorie héréditaire, la catégorie d'orbites est triangulée et donc on a une équivalence triangulée :

$$\mathcal{D}^b(A)/? \overset{L}{\otimes}_A DA[-2] \simeq \langle A \rangle_B / \mathsf{per}\, B$$

# Chapter 2

# On the structure of triangulated categories with finitely many indecomposables

Ce chapitre correspond à l'article [Ami07].

## Notation and terminology

We work over an algebraically closed field $k$. By a *triangulated category*, we mean a $k$-linear triangulated category $\mathcal{T}$. We write $S$ for the suspension functor of $\mathcal{T}$ and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that $\mathcal{T}$ is $\mathsf{Hom}$-*finite* if for each pair $X$, $Y$ of objects in $\mathcal{T}$, the space $\mathsf{Hom}_{\mathcal{T}}(X,Y)$ is finite-dimensional over $k$. The category $\mathcal{T}$ will be called a *Krull-Remak-Schmidt* category if each object is isomorphic to a finite direct sum of indecomposable objects with unicity (up to reordering) of this decomposition, and if the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of $\mathcal{T}$ split, *i.e.* if $e$ is an idempotent of $X$, then $e = \sigma\rho$ where $\sigma$ is a section and $\rho$ is a retraction [Hap88, I 3.2]. The category $\mathcal{T}$ will be called *locally finite* if for each indecomposable $X$ of $\mathcal{T}$, there are only finitely many isoclasses of indecomposables $Y$ such that $\mathsf{Hom}_{\mathcal{T}}(X,Y) \neq 0$. This property is selfdual by [XZ02, prop 1.1].

The *Serre functor* will be denoted by $\nu$ (see definition in section 2.1). The *Auslander-Reiten translation* will always be denoted by $\tau$ (section 2.1).

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories. An *$S$-functor* $(F,\phi)$ is given by a $k$-linear functor $F : \mathcal{T} \to \mathcal{T}'$ and a functor isomorphism $\phi$ between the functors $F \circ S$ and $S' \circ F$, where $S$ is the suspension of $\mathcal{T}$ and $S'$ the suspension of $\mathcal{T}'$. The notion of $\nu$-functor, or $\tau$-functor is then clear. A *triangle functor* is an $S$-functor $(F,\phi)$ such that for each triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ of $\mathcal{T}$, the sequence $FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{\phi_U \circ Fw} S'FU$

is a triangle of $\mathcal{T}'$.

The category $\mathcal{T}$ is *Calabi-Yau* if there exists an integer $d > 0$ such that we have a triangle functor isomorphism between $S^d$ and $\nu$. We say that $\mathcal{T}$ is *maximal d-Calabi-Yau* if $\mathcal{T}$ is $d$-Calabi-Yau and if for each covering functor $\mathcal{T}' \to \mathcal{T}$ with $\mathcal{T}'$ $d$-Calabi-Yau, we have a $k$-linear equivalence between $\mathcal{T}$ and $\mathcal{T}'$.

For an additive $k$-category $\mathcal{E}$, we write $\mathsf{mod}\,\mathcal{E}$ for the category of contravariant finitely presented functors from $\mathcal{E}$ to $\mathsf{mod}\,k$ (section 2.8), and if the projectives of $\mathsf{mod}\,\mathcal{E}$ coincide with the injectives, $\underline{\mathsf{mod}}\,\mathcal{E}$ will be the *stable category*.

## 2.1 Serre duality and Auslander-Reiten triangles

### 2.1.1 Serre duality

Recall from [RVdB02] that a *Serre functor* for $\mathcal{T}$ is an autoequivalence $\nu : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\mathsf{Hom}_{\mathcal{T}}(X, ?) \simeq \mathsf{Hom}_{\mathcal{T}}(?, \nu X)$ for each $X \in \mathcal{T}$, where $D$ is the duality $\mathsf{Hom}_k(?, k)$.

**Theorem 2.1.** *Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Then $\mathcal{T}$ has a Serre functor $\nu$.*

*Proof.* Let $X$ be an object of $\mathcal{T}$. We write $X^\wedge$ for the functor $\mathsf{Hom}_{\mathcal{T}}(?, X)$ and $F$ for the functor $D\mathsf{Hom}_{\mathcal{T}}(X, ?)$. Using the lemma [RVdB02, I.1.6] we just have to show that $F$ is representable. Indeed, the category $\mathcal{T}^{op}$ is locally finite as well. The proof is in two steps.

*Step 1: The functor $F$ is finitely presented.*

Let $Y_1, \ldots, Y_r$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$ such that $FY_i$ is not zero. The space $\mathsf{Hom}(Y_i^\wedge, F)$ is finite-dimensional over $k$. Indeed it is isomorphic to $FY_i$ by the Yoneda lemma. Therefore, the functor $\mathsf{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge$ is representable. We get an epimorphism from a representable functor to $F$:

$$\bigoplus_{i=1}^{r} \mathsf{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \longrightarrow F.$$

By applying the same argument to its kernel we get a projective presentation of $F$ of the form $U^\wedge \longrightarrow V^\wedge \longrightarrow F \longrightarrow 0$, with $U$ and $V$ in $\mathcal{T}$.

*Step 2: A cohomological functor $H : \mathcal{T}^{op} \to \mathsf{mod}\,k$ is representable if and only if it is finitely presented.*

Let $U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{\phi} H \longrightarrow 0$ be a presentation of $H$. We form a triangle

$$U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU.$$

We get an exact sequence

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge\ .$$

Since the composition of $\phi$ with $u^\wedge$ is zero and $H$ is cohomological, the morphism $\phi$ factors through $v^\wedge$. But $H$ is the cokernel of $u^\wedge$, so $v^\wedge$ factors through $\phi$. We obtain a commutative diagram:

$$\begin{array}{ccccccc} U^\wedge & \xrightarrow{u^\wedge} & V^\wedge & \xrightarrow{v^\wedge} & W^\wedge & \xrightarrow{w^\wedge} & SU. \\ & & {\scriptstyle \phi}\downarrow & \overset{i}{\nearrow} \swarrow_{\phi'} & & & \\ & & H & & & & \end{array}$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of $H$ because $\phi$ is an epimorphism. We deduce that $H$ is a direct factor of $W^\wedge$. The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \mathsf{End}(W)$ splits and we get $H = W'^\wedge$ for a direct factor $W'$ of $W$.

□

### 2.1.2 Auslander-Reiten triangles

**Definition 2.2.** [Hap87] A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $\mathcal{T}$ is called an *Auslander-Reiten triangle* or *AR-triangle* if the following conditions are satisfied:

(AR1) $X$ and $Z$ are indecomposable objects;
(AR2) $w \neq 0$;
(AR3) if $f : W \to Z$ is not a retraction, there exists $f' : W \to Y$ such that $vf' = f$;
(AR3') if $g : X \to V$ is not a section, there exists $g' : Y \to V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3') are equivalent. We say that a triangulated category $\mathcal{T}$ *has Auslander-Reiten triangles* if, for any indecomposable object $Z$ of $\mathcal{T}$, there exists an AR-triangle ending at $Z$:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX.$$

In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of $Z$.

The following proposition is proved in [RVdB02, Proposition I.2.3]

**Proposition 2.1.1.** *Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Then the category $\mathcal{T}$ has Auslander-Reiten triangles.*

The composition $\tau = S^{-1}\nu$ is called the Auslander-Reiten translation. An AR-triangle of $\mathcal{T}$ ending at $Z$ has the form:

$$\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.$$

## 2.2 Valued translation quivers and automorphism groups

### 2.2.1 Translation quivers

In this section, we recall some definitions and notations concerning quivers [Die87]. A quiver $Q = (Q_0, Q_1, s, t)$ is given by the set $Q_0$ of its vertices, the set $Q_1$ of its arrows, a source map $s$ and a tail map $t$. If $x \in Q_0$ is a vertex, we denote by $x^+$ the set of direct successors of $x$, and by $x^-$ the set of its direct predecessors. We say that $Q$ is *locally finite* if for each vertex $x \in Q_0$, there are finitely many arrows ending at $x$ and starting at $x$ (in this case, $x^+$ and $x^-$ are finite sets). The quiver $Q$ is said to be *without double arrows*, if two different arrows cannot have the same tail and source.

**Definition 2.3.** A *stable translation quiver* $(Q, \tau)$ is a locally finite quiver without double arrows with a bijection $\tau : Q_0 \to Q_0$ such that $(\tau x)^+ = x^-$ for each vertex $x$. For each arrow $\alpha : x \to y$, let $\sigma\alpha$ be the unique arrow $\tau y \to x$.

Note that a stable translation quiver can have loops.

**Definition 2.4.** A *valued translation quiver* $(Q, \tau, a)$ is a stable translation quiver $(Q, \tau)$ with a map $a : Q_1 \to \mathbb{N}$ such that $a(\alpha) = a(\sigma\alpha)$ for each arrow $\alpha$. If $\alpha$ is an arrow from $x$ to $y$, we write $a_{xy}$ instead of $a(\alpha)$.

**Definition 2.5.** Let $\Delta$ be an oriented tree. The *repetition of* $\Delta$ is the quiver $\mathbb{Z}\Delta$ defined as follows:

- $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$
- $(\mathbb{Z}\Delta)_1 = \mathbb{Z} \times \Delta_1 \cup \sigma(\mathbb{Z} \times \Delta_1)$ with arrows $(n, \alpha) : (n, x) \to (n, y)$ and $\sigma(n, \alpha) : (n-1, y) \to (n, x)$ for each arrow $\alpha : x \to y$ of $\Delta$.

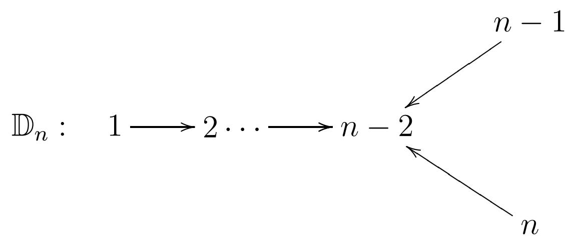
The quiver $\mathbb{Z}\Delta$ with the translation $\tau(n, x) = (n-1, x)$ is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of $\Delta$ (see [Rie80a]).

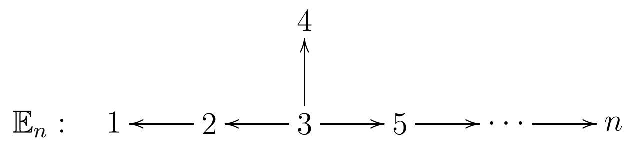
### 2.2.2 Groups of weakly admissible automorphisms

**Definition 2.6.** An automorphism group $G$ of a quiver is said to be *admissible [Rie80a]* if no orbit of $G$ intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It said to be *weakly admissible [Die87]* if, for each $g \in G - \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \emptyset$.

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

$$\mathbb{A}_n : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$




Let $\Delta$ be a Dynkin tree. We define an automorphism $S$ of $\mathbb{Z}\Delta$ as follows:

- if $\Delta = \mathbb{A}_n$, then $S(p, q) = (p + q, n + 1 - q)$;
- if $\Delta = \mathbb{D}_n$ with $n$ even, then $S = \tau^{-n+1}$;
- if $\Delta = \mathbb{D}_n$ with $n$ odd, then $S = \tau^{-n+1}\phi$ where $\phi$ is the automorphism of $\mathbb{D}_n$ which exchanges $n$ and $n-1$;
- if $\Delta = \mathbb{E}_6$, then $S = \phi\tau^{-6}$ where $\phi$ is the automorphism of $\mathbb{E}_6$ which exchanges 2 and 5, and 1 and 6;
- if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
- and if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [Rie80a, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result in which we describe all weakly admissible automorphism groups of Dynkin diagrams:

**Theorem 2.7.** *Let $\Delta$ be a Dynkin tree and $G$ a non trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. Then $G$ is isomorphic to $\mathbb{Z}$, and here is a list of its possible generators:*

- *if $\Delta = \mathbb{A}_n$ with $n$ odd, possible generators are $\tau^r$ and $\phi\tau^r$ with $r \geq 1$, where $\phi = \tau^{\frac{n+1}{2}}S$ is an automorphism of $\mathbb{Z}\Delta$ of order 2;*
- *if $\Delta = \mathbb{A}_n$ with $n$ even, then possible generators are $\rho^r$, where $r \geq 1$ and where $\rho = \tau^{\frac{n}{2}}S$. (Since $\rho^2 = \tau^{-1}$, $\tau^r$ is a possible generator.)*
- *if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are $\tau^r$ and $\tau^r\phi$, where $r \geq 1$ and where $\phi = (n-1, n)$ is the automorphism of $\mathbb{D}_n$ exchanging $n$ and $n-1$.*
- *if $\Delta = \mathbb{D}_4$, then possible generators are $\phi\tau^r$, where $r \geq 1$ and where $\phi$ belongs to $\mathfrak{S}_3$ the permutation group on 3 elements seen as subgroup of automorphisms of $\mathbb{D}_4$.*

- *if $\Delta = \mathbb{E}_6$, then possible generators are $\tau^r$ and $\phi\tau^r$, where $r \geq 1$ and where $\phi$ is the automorphism of $\mathbb{E}_6$ exchanging 2 and 5, and 1 and 6.*
- *if $\Delta = \mathbb{E}_n$ with $n = 7, 8$, possible generators are $\tau^r$, where $r \geq 1$.*

*The unique weakly admissible automorphism group which is not admissible exists for $\mathbb{A}_n$, $n$ even, and is generated by $\rho$.*

## 2.3 Property of the Auslander-Reiten translation

We define the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of the category $\mathcal{T}$ as a valued quiver $(\Gamma, a)$. The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects $X$ and $Y$ of $\mathcal{T}$, we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ is not zero, where $\mathcal{R}(?,?)$ is the radical of the bifunctor $\mathsf{Hom}_{\mathcal{T}}(?,?)$. A morphism of $\mathcal{R}(X,Y)$ which does not vanish in the quotient $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ will be called *irreducible*. Then we put

$$a_{xy} = \dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y).$$

Remark that the fact that $\mathcal{T}$ is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_{\mathcal{T}}$ with the translation $\tau$ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

**Proposition 2.3.1.** *If $X$ and $Y$ are indecomposable objects of $\mathcal{T}$, we have the equality*

$$\dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y) = \dim_k \mathcal{R}(\tau Y,X)/\mathcal{R}^2(\tau Y,X).$$

Let us recall some definitions [Hap88].

**Definition 2.8.** A morphism $g : Y \to Z$ is called *sink morphism* if the following hold
(1) $g$ is not a retraction;
(2) if $h : M \to Z$ is not a retraction, then $h$ factors through $g$;
(3) if $u$ is an endomorphism of $Y$ which satisfies $gu = g$, then $u$ is an automorphism.
Dually, a morphism $f : X \to Y$ is called *source morphism* if the following hold:
(1) $f$ is not a section;
(2) if $h : X \to M$ is not a section, then $h$ factors through $f$;
(3) if $u$ is an endomorphism of $Y$ which satisfies $uf = f$, then $u$ is an automorphism.

These conditions imply that $X$ and $Z$ are indecomposable. Obviously, if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ is an AR-triangle, then $u$ is a source morphism and $v$ is a sink morphism. Conversely, if $v \in \mathsf{Hom}_{\mathcal{T}}(Y,Z)$ is a sink morphism (or if $u \in \mathsf{Hom}_{\mathcal{T}}(X,Y)$ is a source morphism), then there exists an AR-triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ (see [Hap88, I 4.5]).

The following lemma (and the dual statement) is proved in [Rin84, 2.2.5].

**Lemma 2.3.1.** *Let $g$ be a morphism from $Y$ to $Z$, where $Z$ is indecomposable and $Y = \bigoplus_{i=1}^{r} Y_i^{n_i}$ is the decomposition of $Y$ into indecomposables. Then the morphism $g$ is a sink morphism if and only if the following hold:*

*(1) For each $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$, the restriction $g_{i,j}$ of $g$ to the $j^{th}$ component of the $i^{th}$ isotopic part of $Y$ belongs to the radical $\mathcal{R}(Y_i, Z)$.*

*(2) For each $i = 1, \ldots, r$, the family $(\overline{g}_{i,j})_{j=1,\ldots,n_i}$ forms a $k$-basis of the space $\mathcal{R}(Y_i, Z)/\mathcal{R}^2(Y_i, Z)$.*

*(3) If $h \in \mathsf{Hom}_{\mathcal{T}}(Y', Z)$ is irreducible and $Y'$ indecomposable, then $h$ factors through $g$ and $Y'$ is isomorphic to $Y_i$ for some $i$.*

Using this lemma, it is easy to see that proposition 2.3.1 holds. Thus, the Auslander-Reiten quiver $\Gamma_{\mathcal{T}} = (\Gamma, \tau, a)$ of the category $\mathcal{T}$ is a valued translation quiver.

## 2.4 Structure of the Auslander-Reiten quiver

This section is dedicated to another proof of a theorem due to J. Xiao and B. Zhu ([XZ05]):

**Theorem 2.9.** *[XZ05] Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Let $\Gamma$ be a connected component of the AR-quiver of $\mathcal{T}$. Then there exists a Dynkin tree $\Delta$ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, a weakly admissible automorphism group $G$ of $\mathbb{Z}\Delta$ and an isomorphism of valued translation quivers*

$$\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}\Delta/G.$$

*The underlying graph of the tree $\Delta$ is unique up to isomorphism (it is called the* type of $\Gamma$*), and the group $G$ is unique up to conjugacy in $\mathsf{Aut}(\mathbb{Z}\Delta)$.*

*In particular, if $\mathcal{T}$ has an infinite number of isoclasses of indecomposable objects, then $G$ is trivial, and $\Gamma$ is the repetition quiver $\mathbb{Z}\Delta$.*

### 2.4.1 Auslander-Reiten quivers with a loop

In this section, we suppose that the Auslander-Reiten quiver of $\mathcal{T}$ contains a loop, *i.e.* there exists an arrow with same tail and source. Thus, we suppose that there exists an indecomposable $X$ of $\mathcal{T}$ such that

$$\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1.$$

**Proposition 2.4.1.** *Let $X$ be an indecomposable object of $\mathcal{T}$. Suppose that we have $\dim_k \mathcal{R}(X, X)/\mathcal{R}^2(X, X) \geq 1$. Then $\tau X$ is isomorphic to $X$.*

To prove this, we need a lemma.

**Lemma 2.4.1.** *Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1}$ be a sequence of irreducible morphisms between indecomposable objects with $n \geq 2$. If the composition $f_n \circ f_{n-1} \cdots f_1$ is zero, then there exists an $i$ such that $\tau^{-1} X_i$ is isomorphic to $X_{i+2}$.*

*Proof.* The proof proceeds by induction on $n$. Let us show the assertion for $n = 2$. Suppose $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence such that $f_2 \circ f_1 = 0$. We can then construct an AR-triangle:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{(f_1,f)^T} & X_2 \oplus X & \xrightarrow{(g_1,g_2)} & \tau^{-1}X_1 & \longrightarrow & SX_1 \\ & & \downarrow{\scriptstyle (f_2,0)} & \swarrow{\scriptstyle \beta} & & & \\ & & X_3 & & & & \end{array}$$

The composition $f_2 \circ f_1$ is zero, thus the morphism $f_2$ factors through $g_1$. As the morphisms $g_1$ and $f_2$ are irreducible, we conclude that $\beta$ is a retraction, and $X_3$ a direct summand of $\tau^{-1}X_1$. But $X_1$ is indecomposable, so $\beta$ is an isomorphism between $X_3$ and $\tau^{-1}X_1$.

Now suppose that the property holds for an integer $n-1$ and that we have $f_n f_{n-1} \cdots f_1 = 0$. If the composition $f_{n-1} \cdots f_1$ is zero, the proposition holds by induction. So we can suppose that for $i \leq n-2$, the objects $\tau^{-1}X_i$ and $X_{i+2}$ are not isomorphic. We show now by induction on $i$ that for each $i \leq n-1$, there exists a map $\beta_i : \tau^{-1}X_i \to X_{n+1}$ such that $f_n \cdots f_{i+1} = \beta_i g_i$ where $g_i : X_{i+1} \to \tau^{-1}X_i$ is an irreducible morphism. For $i = 1$, we construct an AR-triangle:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{(f_1,f'_1)^T} & X_2 \oplus X'_1 & \xrightarrow{(g_1,g'_1)} & \tau^{-1}X_1 & \longrightarrow & SX_1 \\ & & \downarrow{\scriptstyle (f_n \cdots f_2,0)} & \swarrow{\scriptstyle \beta_1} & & & \\ & & X_{n+1} & & & & \end{array}$$

As the composition $f_n \cdots f_1$ is zero, we have the factorization $f_n \cdots f_2 = \beta_1 g_1$.

Now for $i$, as $\tau^{-1}X_{i-1}$ is not isomorphic to $X_{i+1}$, there exists an AR-triangle of the form:

$$\begin{array}{ccccccc} X_i & \xrightarrow{(g_{i-1},f_i,f'_i)^T} & \tau^{-1}X_{i-1} \oplus X_{i+1} \oplus X'_i & \xrightarrow{(g''_i,g_i,g'_i)} & \tau^{-1}X_i & \longrightarrow & SX_i \\ & & \downarrow{\scriptstyle (-\beta_{i-1},f_n \cdots f_{i+1},0)} & \swarrow{\scriptstyle \beta_i} & & & \\ & & X_{n+1} & & & & \end{array}$$

By induction, $-\beta_{i-1}g_{i-1} + f_n \cdots f_{i+1}f_i$ is zero, thus $f_n \cdots f_{i+1}$ factors through $g_i$. This property is true for $i = n-1$, so we have a map $\beta_{n-1} : \tau^{-1}X_{n-1} \to X_{n+1}$ such that $\beta_{n-1}g_{n-1} = f_n$. As $g_{n-1}$ and $f_n$ are irreducible, we conclude that $\beta_{n-1}$ is an isomorphism between $X_{n+1}$ and $\tau^{-1}X_{n-1}$. □

Now we are able to prove proposition 2.4.1. There exists an irreducible map $f : X \to X$. Suppose that $X$ and $\tau X$ are not isomorphic. Then from the previous lemma, the endomorphism $f^n$ is non zero for each $n$. But since $\mathcal{T}$ is a Krull-Remak-Schmidt, locally finite category, a power of the radical $\mathcal{R}(X, X)$ vanishes. This is a contradiction.

### 2.4.2 Proof of theorem 2.9

Let $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ be the valued quiver obtained from $\Gamma$ by removing the loops, *i.e.* we have $\tilde{\Gamma}_0 = \Gamma_0$, $\tilde{\Gamma}_1 = \{\alpha \in \Gamma_1$ such that $s(\alpha) \neq t(\alpha)\}$, and $\tilde{a} = a_{|_{\tilde{\Gamma}_1}}$.

**Lemma 2.4.2.** *The quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{a})$ with the translation $\tau$ is a valued translation quiver without loop.*

*Proof.* We have to check that the map $\sigma$ is well-defined. But from proposition 2.4.1, if $\alpha$ is a loop on a vertex $x$, $\sigma(\alpha)$ is the unique arrow from $\tau x = x$ to $x$, *i.e.* $\sigma(\alpha) = \alpha$. Thus $\tilde{\Gamma}$ is obtained from $\Gamma$ by removing some $\sigma$-orbits and it keeps the structure of stable valued translation quiver. □

Now, we can apply Riedtmann's Struktursatz [Rie80a] and the result of Happel-Preiser-Ringel [HPR80b]. There exist a tree $\Delta$ and an admissible automorphism group $G$ (which may be trivial) of $\mathbb{Z}\Delta$ such that $\tilde{\Gamma}$ is isomorphic to $\mathbb{Z}\Delta/G$ as a valued translation quiver. The underlying graph of the tree $\Delta$ is then unique up to isomorphism and the group $G$ is unique up to conjugacy in $\mathsf{Aut}(\mathbb{Z}\Delta)$. Let $x$ be a vertex of $\Delta$. We write $\overline{x}$ for the image of $x$ by the map:

$$\Delta \longrightarrow \mathbb{Z}\Delta \xrightarrow{\pi} \mathbb{Z}\Delta/G \simeq \tilde{\Gamma} \hookrightarrow \Gamma.$$

Let $C : \Delta_0 \times \Delta_0 \to \mathbb{Z}$ be the matrix defined as follows:

- $C(x, y) = -a_{\overline{x}\,\overline{y}}$ (resp. $-a_{\overline{y}\,\overline{x}}$) if there exists an arrow from $x$ to $y$ (resp. from $y$ to $x$) in $\Delta$,
- $C(x, x) = 2 - a_{\overline{x}\,\overline{x}}$,
- $C(x, y) = 0$ otherwise.

The matrix $C$ is symmetric; it is a 'generalized Cartan matrix' in the sense of [HPR80a]. If we remove the loops from the 'underlying graph of $C$' (in the sense of [HPR80a]), we get the underlying graph of $\Delta$.

In order to apply the result of Happel-Preiser-Ringel [HPR80a, section 2], we have to show:

**Lemma 2.4.3.** *The set $\Delta_0$ of vertices of $\Delta$ is finite.*

*Proof.* Riedtmann's construction of $\Delta$ is the following. We fix a vertex $x_0$ in $\tilde{\Gamma}_0$. Then the vertices of $\Delta$ are the paths of $\tilde{\Gamma}$ beginning on $x_0$ and which do not contain subpaths of the form $\alpha\sigma(\alpha)$, where $\alpha$ is in $\tilde{\Gamma}_1$. Now suppose that $\Delta_0$ is an infinite set. Then for each $n$, there exists a sequence:

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1} \xrightarrow{\alpha_n} x_n$$

such that $\tau x_{i+2} \neq x_i$. Then there exist some indecomposables $X_0, \ldots, X_n$ such that the vector space $\mathcal{R}(X_{i-1}, X_i)/\mathcal{R}^2(X_{i-1}, X_i)$ is not zero. Thus from the lemma 2.4.1, there exists irreducible morphisms $f_i : X_{i-1} \to X_i$ such that the composition $f_n f_{n-1} \cdots f_1$ does not vanish. But the functor $\mathsf{Hom}_{\mathcal{T}}(X_0, ?)$ has finite support. Thus there is an indecomposable $Y$ which appears an infinite number of times in the sequence $(X_i)_i$. But since $\mathcal{R}^N(Y,Y)$ vanishes for an $N$, we have a contradiction. □

Let $\mathcal{S}$ a system of representatives of isoclasses of indecomposables of $\mathcal{T}$. For an indecomposable $Y$ of $\mathcal{T}$, we put

$$l(Y) = \sum_{M \in \mathcal{S}} \dim_k \mathsf{Hom}_{\mathcal{T}}(M, Y).$$

This sum is finite since $\mathcal{T}$ is locally finite.

**Lemma 2.4.4.** *For $x$ in $\Delta_0$, we write $d_x = l(\overline{x})$. Then for each $x \in \Delta_0$, we have:*

$$\sum_{y \in \Delta_0} d_y C_{xy} = 2.$$

*Proof.* Let $X$ and $U$ be indecomposables of $\mathcal{T}$. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

be an AR-triangle. We write $(U, ?)$ for the cohomological functor $\mathsf{Hom}_{\mathcal{T}}(U, ?)$. Thus, we have a long exact sequence:

$$(U, S^{-1}Z) \xrightarrow{S^{-1}w_*} (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} (U, SX).$$

Let $S_Z(U)$ be the image of the map $w_*$. We have the exact sequence:

$$0 \longrightarrow S_{S^{-1}Z}(U) \longrightarrow (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} S_Z(U) \longrightarrow 0.$$

Thus we have the following equality:

$$\dim_k S_Z(U) + \dim_k S_{S^{-1}Z}(U) + \dim_k(U, Y) = \dim_k(U, X) + \dim_k(U, Z).$$

If $U$ is not isomorphic to $Z$, each map from $U$ to $Z$ is radical, thus $S_Z(U)$ is zero. If $U$ is isomorphic to $Z$, the map $w_*$ factors through the radical of $\mathsf{End}(Z)$, so $S_Z(Z)$ is isomorphic to $k$. Then summing the previous equality when $U$ runs over $\mathcal{S}$, we get:

$$l(X) + l(Z) = l(Y) + 2.$$

Clearly $l$ is $\tau$-invariant, thus $l(Z)$ equals $l(X)$. If the decomposition of $Y$ is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get:

$$l(Y) = \sum_i n_i l(Y_i) = \sum_{i, X \to Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i) + a_{XX} l(X).$$

We deduce the formula:

$$2 = (2 - a_{XX})l(X) - \sum_{i, X \to Y_i \in \tilde{\Gamma}} a_{XY_i} l(Y_i).$$

Let $x$ be a vertex of the tree $\Delta$ and $\overline{x}$ its image in $\tilde{\Gamma}$. Then an arrow $\overline{x} \to Y$ in $\tilde{\Gamma}$ comes from an arrow $(x, 0) \to (y, 0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x, 0) \to (y, -1)$ in $\mathbb{Z}\Delta$, *i.e.* from an arrow $(y, 0) \to (x, 0)$. Indeed the projection $\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$ is a covering. From this we deduce the following equality:

$$2 = (2 - a_{\overline{x}\,\overline{x}})d_x - \sum_{y, x \to y \in \Delta} a_{\overline{x}\,\overline{y}} d_y - \sum_{y, y \to x \in \Delta} a_{\overline{y}\,\overline{x}} d_y = \sum_{y \in \Delta_0} d_y C_{xy}.$$

□

Now we can prove theorem 2.9. The matrix $C$ is a 'generalized Cartan matrix'. The previous lemma gives us a subadditive function which is not additive. Thus by [HPR80a], the underlying graph of $C$ is of 'generalized Dynkin type'. As $C$ is symmetric, the graph is necessarily of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, or $\mathbb{L}$. But this graph is the graph $\Delta$ with the valuation $a$. We are done in the cases $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.

The case $\mathbb{L}_n$ occurs when the AR-quiver contains at least one loop. We can see $\mathbb{L}_n$ as $\mathbb{A}_n$ with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}\mathbb{L}_n$ are generated by $\tau^r$ for an $r \geq 1$. But proposition 2.4.1 tell us that a vertex $x$ with a loop satisfies $\tau x = x$. Thus $G$ is generated by $\tau$ and the AR-quiver has the following form:

$$1 \rightleftarrows 2 \rightleftarrows 3 \cdots\cdots \rightleftarrows n \circlearrowleft$$

This quiver is isomorphic to the quiver $\mathbb{Z}\mathbb{A}_{2n}/G$ where $G$ is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor $S$ sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism $S$ defined in section 2.2.2.

As shown in [XZ05], it follows from the results of [Kel05] that for each Dynkin tree $\Delta$ and for each weakly admissible group of automorphisms $G$ of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category $\mathcal{T}$ such that $\Gamma_{\mathcal{T}} \simeq \mathbb{Z}\Delta/G$. This category is of the form $\mathcal{T} = \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\varphi$ where $\varphi$ is an auto-equivalence of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$.

## 2.5 Construction of a covering functor

From now, we suppose that the AR-quiver $\Gamma$ of $\mathcal{T}$ is connected. We know its structure. It is natural to ask: Is the category $\mathcal{T}$ *standard*, *i.e.* equivalent as a $k$-linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$.

### 2.5.1 Construction

We write $\pi : \mathbb{Z}\Delta \to \Gamma$ for the canonical projection. As $G$ is a weakly admissible group, this projection verifies the following property: if $x$ is a vertex of $\mathbb{Z}\Delta$, the number of arrows of $\mathbb{Z}\Delta$ with source $x$ is equal to the number of arrows of $\mathbb{Z}\Delta/G$ with source $\pi x$. Let $\mathcal{S}$ be a system of representatives of the isoclasses of indecomposables of $\mathcal{T}$. We write $\mathsf{ind}\,\mathcal{T}$ for the full subcategory of $\mathcal{T}$ whose set of objects is $\mathcal{S}$. For a tree $\Delta$, we write $k(\mathbb{Z}\Delta)$ for the mesh category (see [Rie80a]). Using the same proof as Riedtmann [Rie80a], one shows the following theorem:

**Theorem 2.10.** *There exists a $k$-linear functor $F : k(\mathbb{Z}\Delta) \to \mathsf{ind}\,\mathcal{T}$ which is surjective and induces bijections:*

$$\bigoplus_{Fz=Fy} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \to \mathsf{Hom}_{\mathcal{T}}(Fx, Fy),$$

*for all vertices $x$ and $y$ of $\mathbb{Z}\Delta$.*

### 2.5.2 Infinite case

If the category $\mathcal{T}$ is locally finite not finite *i.e.* if there is infinitely many indecomposables, the constructed functor $F$ is immediately fully faithful. Thus we get the corollary.

**Corollary 2.11.** *If $\mathsf{ind}\,\mathcal{T}$ is not finite, then we have a $k$-linear equivalence between $\mathcal{T}$ and the mesh category $k(\mathbb{Z}\Delta)$.*

### 2.5.3 Uniqueness criterion

The covering functor $F$ can be see as a $k$-linear functor from the derived category $\mathcal{D}^b(\mathsf{mod}\,k\Delta)$ to the category $\mathcal{T}$. By construction, it satisfies the following property called the *AR-property*:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$ of $\mathcal{D}^b(\mathsf{mod}\,k\Delta)$, there exists a triangle of $\mathcal{T}$ of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\epsilon} SFX$ .

In fact, thanks to this property, $F$ is determined by its restriction to the subcategory $\mathsf{proj}\,k\Delta = k(\Delta)$, *i.e.* we have the following lemma:

**Lemma 2.5.1.** *Let $F$ and $G$ be $k$-linear functors from $\mathcal{D}^b(\mathsf{mod}\,k\Delta)$ to $\mathcal{T}$. Suppose that $F$ and $G$ satisfy the AR-property and that the restrictions $F_{|_{k(\Delta)}}$ and $G_{|_{k(\Delta)}}$ are isomorphic. Then the functors $F$ and $G$ are isomorphic as $k$-linear functors.*

*Proof.* It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [Nee01]). □

## 2.6 Particular cases of $k$-linear equivalence

From now we suppose that the category $\mathcal{T}$ is finite, *i.e.* $\mathcal{T}$ has finitely many isoclasses of indecomposable objects.

### 2.6.1 Equivalence criterion

Let $\Gamma$ be the AR-quiver of $\mathcal{T}$ and suppose that it is isomorphic to $\mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$. It induces an automorphism in the mesh category $k(\mathbb{Z}\Delta)$ that we still denote by $\varphi$. Then we have the following equivalence criterion:

**Proposition 2.6.1.** *The categories $k(\Gamma)$ and $\mathsf{ind}\,\mathcal{T}$ are equivalent as $k$-categories if and only if there exists a covering functor $F : k(\mathbb{Z}\Delta) \to \mathsf{ind}\,\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \to F$.*

The proof consists in constructing a $k$-linear equivalence between $\mathsf{ind}\,\mathcal{T}$ and the orbit category $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ using the universal property of the orbit category (see [Kel05]), and then constructing an equivalence between $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ and $k(\Gamma)$.

### 2.6.2 Cylindric case for $\mathbb{A}_n$

**Theorem 2.12.** *If $\Delta = \mathbb{A}_n$ and $\varphi = \tau^r$ for some $r \geq 1$, then there exists a functor isomorphism $\Phi : F\circ\varphi \to F$, i.e. for each object $x$ of $k(\mathbb{Z}\Delta)$ there exists an automorphism $\Phi_x$ of $Fx$ such that for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, the following diagram commutes:*

$$\begin{array}{ccc} Fx & \xrightarrow{\Phi_x} & Fx \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle F\varphi\alpha} \\ Fy & \xrightarrow{\Phi_y} & Fy. \end{array}$$

To prove this, we need the following lemma:

**Lemma 2.6.1.** *Let $\alpha : x \to y$ be an arrow of $\mathbb{Z}\mathbb{A}_n$ and let $c$ be a path from $x$ to $\tau^r y$, $r \in \mathbb{Z}$, which is not zero in the mesh category $k(\mathbb{Z}\mathbb{A}_n)$. Then $c$ can be written $c'\alpha$ where $c'$ is a path from $y$ to $\tau^r y$ (up to sign).*

*Proof.* There is a path from $x$ to $\tau^r y$, thus, we have $\mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \tau^r y) \simeq k$, and $x$ and $\tau^r y$ are opposite vertices of a 'rectangle' in $\mathbb{Z}\mathbb{A}_n$. This implies that there exists a path from $x$ to $\tau^r y$ beginning by $\alpha$. □

*Proof. (of theorem 2.12)* Combining proposition 2.6.1 and lemma 2.5.1, we have just to construct an isomorphism between the restriction of $F$ and $F \circ \varphi$ to a subquiver $\mathbb{A}_n$.

Let us fix a full subquiver of $\mathbb{Z}\mathbb{A}_n$ of the following form:

$$x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_n$$

such that $x_1 \dots, x_n$ are representatives of the $\tau$-orbits in $\mathbb{Z}\mathbb{A}_n$. We define the $(\Phi_{x_i})_{i=1\dots n}$ by induction. We fix $\Phi_{x_1} = Id_{Fx_1}$. Now suppose we have constructed some automorphisms $\Phi_{x_1}, \dots, \Phi_{x_i}$ such that for each $j \leq i$ the following diagram is commutative:

$$\begin{array}{ccc} Fx_{j-1} & \xrightarrow{\Phi_{x_{j-1}}} & Fx_{j-1} \\ {\scriptstyle F\alpha_{j-1}}\downarrow & & \downarrow{\scriptstyle F\varphi\alpha_{j-1}} \\ Fx_j & \xrightarrow{\Phi_{x_j}} & Fx_j. \end{array}$$

The composition $(F\varphi\alpha_i) \circ \Phi_{x_i}$ is in the morphism space $\mathsf{Hom}_{\mathcal{T}}(Fx_i, Fx_{i+1})$, which is isomorphic, by theorem 2.10, to the space

$$\bigoplus_{Fz=Fx_{i+1}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z).$$

Thus we can write

$$(F\varphi\alpha_i)\Phi_{x_i} = \lambda F\alpha_i + \sum_{z \neq x_{i+1}} F\beta_z$$

where $\beta_z$ belongs to $\mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z)$ and $Fz = Fx_{i+1}$. But $Fz$ is equal to $Fx_{i+1}$ if and only if $z$ is of the form $\tau^{rl}x_{i+1}$ for an $l$ in $\mathbb{Z}$. By the lemma, we can write $\beta_z = \beta'_z\alpha_i$. Thus we have the equality:

$$(F\varphi\alpha_i)\Phi_{x_i} = F(\lambda Id_{x_{i+1}} + \sum_z \beta'_z)F\alpha_i.$$

The scalar $\lambda$ is not zero. Indeed, $\Phi_{x_i}$ is an automorphism, thus the image of $(F\varphi\alpha_i)\Phi_{x_i}$ is not zero in the quotient

$$\mathcal{R}(Fx_i, Fx_{i+1})/\mathcal{R}^2(Fx_i, Fx_{i+1}).$$

Thus $\Phi_{x_{i+1}} = F(\lambda Id_{x_{i+1}} + \sum_z \beta'_z)$ is an automorphism of $Fx_{i+1}$ which verifies the commutation relation

$$(F\varphi\alpha_i) \circ \Phi_{x_i} = \Phi_{x_{i+1}} \circ F\alpha_i.$$

□

### 2.6.3 Other standard cases

In the mesh category $k(\mathbb{Z}\Delta)$, where $\Delta$ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms $\varphi$ such that, for an arrow $\alpha : x \to y$ of $\Delta$, the paths from $x$ to $\varphi^r y$ vanish in the mesh category for all $r \neq 0$. In other words, for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, we have:

$$\mathsf{Hom}_{k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}}(x, y) = \bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k,$$

where $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ is the orbit category (see section 2.6.1).

**Lemma 2.6.2.** *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$ and suppose that $\varphi$ verifies for each arrow $x \to y$ of $\mathbb{Z}\Delta$*

$$\bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k.$$

*Let $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$ and $G : k(\mathbb{Z}\Delta) \to \mathcal{T}$ be covering functors satisfying the AR-property. Suppose that $F$ and $G$ agree up to isomorphism on the objects of $k(\mathbb{Z}\Delta)$. Then $F$ and $G$ are isomorphic as $k$-linear functors.*

*Proof.* Using lemma 2.5.1, we have just to construct an isomorphism between the functors restricted to $\Delta$. Let $\alpha : x \to y$ be an arrow of $\Delta$. Using theorem 2.10 and the hypothesis, we have the following isomorphisms:

$$\mathsf{Hom}_{\mathcal{T}}(Fx, Fy) \simeq \bigoplus_{Fz = Fy} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \simeq \bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) \simeq k$$

and then

$$\mathsf{Hom}_{\mathcal{T}}(Gx, Gy) \simeq \mathsf{Hom}_{\mathcal{T}}(Fx, Fy) \simeq k.$$

Thus there exists a scalar $\lambda$ such that $G\alpha = \lambda F\alpha$. This scalar does not vanish since $F$ and $G$ are covering functors. As $\Delta$ is a tree, we can find some $\lambda_x$ for $x \in \Delta$ by induction such that

$$G\alpha = \lambda_x \lambda_y^{-1} F\alpha.$$

Now it is easy to check that $\Phi_x = \lambda_x Id_{Fx}$ is the functor isomorphism. □

This lemma gives us an isomorphism between the functors $F$ and $F \circ \varphi$, Moreover, using the same argument, one can show that the covering functor $F$ is an $S$-functor and a $\tau$-functor.

For each Dynkin tree $\Delta$ we can determine the automorphisms $\varphi$ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

**Theorem 2.13.** *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$. If one of these cases holds,*

- $\Delta = \mathbb{A}_n$ *with* $n$ *odd and* $G$ *is generated by* $\tau^r$ *or* $\varphi = \tau^r\phi$ *with* $r \geq \frac{n-1}{2}$ *and* $\phi = \tau^{\frac{n+1}{2}}S$;
- $\Delta = \mathbb{A}_n$ *with* $n$ *even and* $G$ *is generated by* $\rho^r$ *with* $r \geq n-1$ *and* $\rho = \tau^{\frac{n}{2}}S$;
- $\Delta = \mathbb{D}_n$ *with* $n \geq 5$ *and* $G$ *is generated by* $\tau^r$ *or* $\tau^r\phi$ *with* $r \geq n-2$ *and* $\phi$ *as in theorem 2.7;*
- $\Delta = \mathbb{D}_4$ *and* $G$ *is generated by* $\phi\tau^r$, *where* $r \geq 2$ *and* $\phi$ *runs over* $\sigma_3$;
- $\Delta = \mathbb{E}_6$ *and* $G$ *is generated by* $\tau^r$ *or* $\tau^r\phi$ *where* $r \geq 5$ *and* $\phi$ *is as in theorem 2.7;*
- $\Delta = \mathbb{E}_7$ *and* $G$ *is generated by* $\tau^r$, $r \geq 8$;
- $\Delta = \mathbb{E}_8$ *and* $G$ *is generated by* $\tau^r$, $r \geq 14$.

*then* $\mathcal{T}$ *is standard, i.e. the categories* $\mathcal{T}$ *and* $k(\Gamma)$ *are equivalent as* $k$*-linear categories.*

**Corollary 2.14.** *A finite maximal* $d$*-Calabi-Yau (see [Kel05, 8]) triangulated category* $\mathcal{T}$, *with* $d \geq 2$, *is standard,* i.e. *there exists a* $k$*-linear equivalence between* $\mathcal{T}$ *and the orbit category* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\tau^{-1}S^{d-1}$ *where* $\Delta$ *is Dynkin of type* $\mathbb{A}$, $\mathbb{D}$ *or* $\mathbb{E}$

## 2.7 Algebraic case

For some automorphism groups $G$, we know the $k$-linear structure of $\mathcal{T}$. But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. In this section, we distinguish two cases:

If $\mathcal{T}$ is locally finite, not finite, we have the following theorem which is proved in section 2.7.2:

**Theorem 2.15.** *Let* $\mathcal{T}$ *be a connected locally finite triangulated category with infinitely many indecomposables. If* $\mathcal{T}$ *is the base of a tower of triangulated categories [Kel91], then* $\mathcal{T}$ *is triangle equivalent to* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ *for some Dynkin diagram* $\Delta$.

Now if $\mathcal{T}$ is a finite standard category which is algebraic, *i.e.* $\mathcal{T}$ is triangle equivalent to $\underline{\mathcal{E}}$ for some $k$-linear Frobenius category $\mathcal{E}$ ([Kel06, 3.6]), then we have the following result which is proved in section 2.7.3:

**Theorem 2.16.** *Let* $\mathcal{T}$ *be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram* $\Delta$ *of type* $\mathbb{A}$, $\mathbb{D}$ *or* $\mathbb{E}$ *and an auto-equivalence* $\Phi$ *of* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ *such that* $\mathcal{T}$ *is triangle equivalent to the orbit category* $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$.

This theorem combined with corollary 2.14 yields the following result (compare to [Kel05, Cor 8.4]):

**Corollary 2.17.** *If $\mathcal{T}$ is a finite algebraic maximal $d$-Calabi-Yau category with $d \geq 2$, then $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/S^d\nu^{-1}$ for some Dynkin diagram $\Delta$.*

### 2.7.1 $\partial$-functor

We recall the following definition from [Kel91] and [Ver77].

**Definition 2.18.** Let $\mathcal{H}$ be an exact category and $\mathcal{T}$ a triangulated category. A *$\partial$-functor* $(I, \partial) : \mathcal{H} \to \mathcal{T}$ is given by:

- an additive $k$-linear functor $I : \mathcal{H} \to \mathcal{T}$;
- for each conflation $\epsilon : X \rightarrowtail^{i} Y \twoheadrightarrow^{p} Z$ of $\mathcal{H}$, a morphism $\partial\epsilon : IZ \to SIX$ functorial in $\epsilon$ such that $IX \xrightarrow{Ii} IY \xrightarrow{Ip} IZ \xrightarrow{\partial\epsilon} SIX$ is a triangle of $\mathcal{T}$.

For each exact category $\mathcal{H}$, the inclusion $I : \mathcal{H} \to \mathcal{D}^b(\mathcal{H})$ can be completed to a $\partial$-functor $(I, \partial)$ in a unique way. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. If $(F, \varphi) : \mathcal{T} \to \mathcal{T}'$ is an $S$-functor and $(I, \partial) : \mathcal{H} \to \mathcal{T}$ is a $\partial$-functor, we say that $F$ *respects* $\partial$ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \to \mathcal{T}'$ is a $\partial$-functor. Obviously each triangle functor respects $\partial$.

**Proposition 2.7.1.** *Let $\mathcal{H}$ be a $k$-linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a $\partial$-functor. Then there exists a unique (up to isomorphism) $k$-linear $S$-functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects $\partial$.*

*Proof.* On $\mathcal{H}$ (which can be seen as a full subcategory of $\mathcal{D}^b(\mathcal{H})$), the functor $F$ is uniquely determined. We want $F$ to be an $S$-functor, so $F$ is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since $\mathcal{H}$ is hereditary, each object of $\mathcal{D}^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, *i.e.* complexes concentrated in a single degree. Thus, the functor $F$ is uniquely determined on the objects. Now, let $X$ and $Y$ be stalk complexes of $\mathcal{D}^b(\mathcal{H})$ and $f : X \to Y$ a non-zero morphism. We can suppose that $X$ is in $\mathcal{H}$ and $Y$ is in $S^n\mathcal{H}$. If $n \neq 0, 1$, $f$ is necessarily zero. If $n = 0$, then $f$ is a morphism in $\mathcal{H}$ and $Ff$ is uniquely determined. If $n = 1$, $f$ is an element of $\mathsf{Ext}^1_{\mathcal{H}}(X, S^{-1}Y)$, so gives us a conflation $\epsilon : S^{-1}Y \rightarrowtail^{i} E \twoheadrightarrow^{p} X$ in $\mathcal{H}$. The functor $F$ respects $\partial$, thus $Ff$ has to be equal to $\varphi \circ \partial\epsilon$ where $\varphi$ is the natural isomorphism between $SFS^{-1}Y$ and $FY$. Since $\partial$ is functorial, $F$ is a functor. The result follows. ☐

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [Kel91, cor 2.7].

**Theorem 2.19.** *Let $\mathcal{H}$ be a $k$-linear exact category, and $\mathcal{T}$ be the base of a tower of triangulated categories [Kel91]. Let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a $\partial$-functor such that for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\mathsf{Hom}_{\mathcal{T}}(IX, S^nIY)$ vanishes. Then there* *exists a triangle functor* $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ *such that the following diagram commutes up to isomorphism of* $\partial$*-functors:*

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{D}^b(\mathcal{H}) \\ & {\scriptstyle (I,\partial)}\searrow \quad \swarrow{\scriptstyle F} & \\ & \mathcal{T} & \end{array}$$

From theorem 2.19, and the proposition above we deduce the following corollary:

**Corollary 2.20.** *(compare to [Rin06]) Let* $\mathcal{T}$*,* $\mathcal{H}$ *and* $(I, \partial) : \mathcal{H} \to \mathcal{T}$ *be as in theorem 2.19. If* $\mathcal{H}$ *is hereditary, then the unique functor* $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ *which respects* $\partial$ *is a triangle functor.*

### 2.7.2 Proof of theorem 2.15

Let $F$ be the $k$-linear equivalence constructed in theorem 2.10 between an algebraic triangulated category $\mathcal{T}$ and $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H} = \mathsf{mod}\, k\Delta$ and $\Delta$ is a simply-laced Dynkin graph. As we saw in section 2.6, the covering functor is an $S$-functor.

The category $\mathcal{H}$ is the heart of the standard $t$-structure on $\mathcal{D}^b(\mathcal{H})$. The image of this $t$-structure through $F$ is a $t$-structure on $\mathcal{T}$. Indeed, $F$ is an $S$-equivalence, so the conditions $(i)$ and $(ii)$ from [BBD82, Def 1.3.1] hold obviously. And since $\mathcal{H}$ is hereditary, for an object $X$ of $\mathcal{D}^b(\mathcal{H})$, the morphism $\tau_{>0}X \to S\tau_{\leq 0}X$ of the triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{>0}X \longrightarrow S\tau_{\leq 0}X$$

vanishes. Thus the image of this triangle through $F$ is a triangle of $\mathcal{T}$ and condition $(iii)$ of [BBD82, Def 1.3.1] holds. Then we get a $t$-structure on $\mathcal{T}$ whose heart is $\mathcal{H}$.

It results from [BBD82, Prop 1.2.4] that the inclusion of the heart of a $t$-structure can be uniquely completed to a $\partial$-functor. Thus we obtain a $\partial$-functor $(F_0, \partial) : \mathcal{H} \to \mathcal{T}$ with $F_0 = F_{|\mathcal{H}}$.

The functor $F$ is an $S$-equivalence. Thus for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\mathsf{Hom}_{\mathcal{T}}(FX, S^n FY)$ vanishes. Now we can apply theorem 2.19 and we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{D}^b(\mathcal{H}), \\ & {\scriptstyle (F_0,\partial)}\searrow \quad \overset{G}{\swarrow}\!\!\swarrow{\scriptstyle F} & \\ & \mathcal{T} & \end{array}$$

where $F$ is the $S$-equivalence and $G$ is a triangle functor. Note that a priori $F$ is an $S$-functor which does not respect $\partial$. The functors $F_{|\mathcal{H}}$ and $G_{|\mathcal{H}}$ are isomorphic. The functor $F$ is an $S$-functor thus we have an isomorphism $F_{|S^n\mathcal{H}} \simeq G_{|S^n\mathcal{H}}$ for each $n \in \mathbb{Z}$. Thus the functor $G$ is essentially surjective. Since $\mathcal{H}$ is the category $\mathsf{mod}\, k\Delta$, to show that $G$ is fully faithful, we have just to show that for each $p \in \mathbb{Z}$, there is an isomorphism induced by $G$

$$\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})}(A, S^p A) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(GA, S^p GA)$$

where $A$ is the free module $k\Delta$. For $p = 0$, this is clear because $A$ is in $\mathcal{H}$. And for $p \neq 0$ both sides vanish.

Thus $G$ is a triangle equivalence between $\mathcal{D}^b(\mathcal{H})$ and $\mathcal{T}$.

### 2.7.3 Finite algebraic standard case

For a small dg category $\mathcal{A}$, we denote by $\mathcal{CA}$ the category of dg $\mathcal{A}$-modules, by $\mathcal{DA}$ the derived category of $\mathcal{A}$ and by $\mathsf{per}\,\mathcal{A}$ the *perfect derived category* of $\mathcal{A}$, *i.e.* the smallest triangulated subcategory of $\mathcal{DA}$ which is stable under passage to direct factors and contains the free $\mathcal{A}$-modules $\mathcal{A}(?, A)$, where $A$ runs through the objects of $\mathcal{A}$. Recall that a small triangulated category is *algebraic* if it is triangle equivalent to $\mathsf{per}\,\mathcal{A}$ for a dg category $\mathcal{A}$. For two small dg categories $\mathcal{A}$ and $\mathcal{B}$, a triangle functor $\mathsf{per}\,\mathcal{A} \to \mathsf{per}\,\mathcal{B}$ is *algebraic* if it is isomorphic to the functor

$$F_X = ? \overset{L}{\otimes}_{\mathcal{A}} X$$

associated with a dg bimodule $X$, *i.e.* an object of the derived category $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$.

Let $\Phi$ be an algebraic autoequivalence of $\mathcal{D}^b(\mathsf{mod}\,k\Delta)$ such that the orbit category $\mathcal{D}^b(\mathsf{mod}\,k\Delta)/\Phi$ is triangulated. Let $Y$ be a dg $k\Delta$-$k\Delta$-bimodule such that $\Phi = F_Y$. In section 9.3 of [Kel05], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [loc. cit.], the projection functor

$$\pi : \mathcal{D}^b(\mathsf{mod}\,k\Delta) \to \mathcal{D}^b(\mathsf{mod}\,k\Delta)/\Phi$$

is algebraic.

The proof of theorem 7.0.5 is based on the following universal property of the triangulated orbit category $\mathcal{D}^b(\mathsf{mod}\,k\Delta)/\Phi$. For the proof, we refer to section 9.3 of [Kel05].

**Proposition 2.7.2.** *Let $\mathcal{B}$ be a small dg category and*

$$F_X = ? \overset{L}{\otimes}_{k\Delta} X : \mathcal{D}^b(\mathsf{mod}\,k\Delta) \to \mathsf{per}\,\mathcal{B}$$

*an algebraic triangle functor given by a dg $k\Delta$-$\mathcal{A}$-bimodule $X$. Suppose that there is an isomorphism between $Y \overset{L}{\otimes}_{k\Delta} X$ and $X$ in the derived bimodule category $\mathcal{D}(k\Delta^{op} \otimes \mathcal{B})$. Then the functor $F_X$ factors, up to isomorphism of triangle functors, through the projection*

$$\pi : \mathcal{D}^b(\mathsf{mod}\,k\Delta) \to \mathcal{D}^b(\mathsf{mod}\,k\Delta)/\Phi.$$

*Moreover, the induced triangle functor is algebraic.*

Let us recall a lemma of Van den Bergh [KR06]:

**Lemma 2.7.1.** *Let $Q$ be a quiver without oriented cycles and $\mathcal{A}$ be a dg category. We denote by $k(Q)$ the category of paths of $Q$ and by $Can : \mathcal{CA} \to \mathcal{DA}$ the canonical functor. Then we have the following properties:*

*a) Each functor $F : k(Q) \to \mathcal{DA}$ lifts, up to isomorphism, to a functor $\tilde{F} : k(Q) \to \mathcal{CA}$ which verifies the following property: For each vertex $j$ of $Q$, the induced morphism*

$$\bigoplus_i \tilde{F}i \to \tilde{F}j,$$

*where $i$ runs through the immediate predecessors of $j$, is a monomorphism which splits as a morphism of graded $\mathcal{A}$-modules.*

*b) Let $F$ and $G$ be functors from $k(Q)$ to $\mathcal{CA}$, and suppose that $F$ satisfies the property of a). Then any morphism of functors $\varphi : Can \circ F \to Can \circ G$ lifts to a morphism $\tilde{\varphi} : F \to G$.*

*Proof.* $a$) For each vertex $i$ of $Q$, the object $Fi$ is isomorphic in $\mathcal{DA}$ to its cofibrant resolution $X_i$. Thus for each arrow $\alpha : i \to j$, $F$ induces a morphism $f_\alpha : X_i \to X_j$ which can be lifted to $\mathcal{CA}$ since the $X_i$ are cofibrant. Since $Q$ has no oriented cycle, it is easy to choose the $f_\alpha$ such that the property is satisfied.

$b$) For each vertex $i$ of $Q$, we may assume that $Fi$ is cofibrant. Then we can lift $\varphi_i : Can \circ Fi \to Can \circ Gi$ to $\psi_i : Fi \to Gi$. For each arrow $\alpha$ of $Q$, the square

$$\begin{array}{ccc} Fi & \xrightarrow{F_\alpha} & Fj \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_j} \\ Gi & \xrightarrow{G_\alpha} & Gj \end{array}$$

is commutative in $\mathcal{DA}$. Thus the square

$$\begin{array}{ccc} \bigoplus_i Fi & \xrightarrow{F_\alpha} & Fj \\ \downarrow{\scriptstyle (\psi_i)} & & \downarrow{\scriptstyle \psi_j} \\ \bigoplus_i Gi & \xrightarrow{G_\alpha} & Gj \end{array}$$

is commutative up to nullhomotopic morphism $h : \bigoplus_i Fi \to Gj$. Since the morphism $f : \bigoplus_i Fi \to Fj$ is split mono in the category of graded $\mathcal{A}$-modules, $h$ extends along $f$ and we can modify $\Psi_j$ so that the square becomes commutative in $\mathcal{CA}$. The quiver $Q$ does not have oriented cycles, so we can construct $\tilde{\varphi}$ by induction. $\square$

*Proof. (of theorem 2.16)* The category $\mathcal{T}$ is small and algebraic, thus we may assume that $\mathcal{T} = \mathsf{per}\,\mathcal{A}$ for some small dg category $\mathcal{A}$. Let $F : \mathcal{D}^b(\mathsf{mod}\, k\Delta) \to \mathcal{T}$ be the covering functor of theorem 2.10. Let $\Phi$ be an auto-equivalence of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ such that the AR-quiver of the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quiver of $\mathcal{T}$. We may assume that $\Phi = -\overset{L}{\otimes}_{k\Delta} Y$ for an object $Y$ of $\mathcal{D}(k\Delta^{op}\otimes k\Delta)$. The orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ is algebraic, thus it is $\mathsf{per}\,\mathcal{B}$ for some dg category $\mathcal{B}$.

The functor $F_{|_{k(\Delta)}}$ lifts by lemma 2.7.1 to a functor $\tilde{F}$ from $k(\Delta)$ to $\mathcal{CA}$. This means that the object $X = \tilde{F}(k\Delta)$ has a structure of dg $k\Delta^{op}\otimes\mathcal{A}$-module. We denote by $X$ the image of this object in $\mathcal{D}(k\Delta^{op}\otimes\mathcal{A})$.

The functors $F$ and $-\overset{L}{\otimes}_{k\Delta} X$ become isomorphic when restricted to $k(\Delta)$. Moreover $-\overset{L}{\otimes}_{k\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by lemma 2.5.1, they are isomorphic as $k$-linear functors. So we have the following diagram:

$$\xymatrix{\mathcal{D}^b(\mathsf{mod}\, k\Delta) \ar[r]^-{-\overset{L}{\otimes}_{k\Delta} X} \ar@(dl,dr)_{-\overset{L}{\otimes}_{k\Delta} Y} & \mathsf{per}\,\mathcal{A} = \mathcal{T}}$$

The category $\mathcal{T}$ is standard, thus there exists an isomorphism of $k$-linear functors:

$$c : -\overset{L}{\otimes}_{k\Delta} X \longrightarrow -\overset{L}{\otimes}_{k\Delta} Y \overset{L}{\otimes}_{k\Delta} X.$$

The functor $-\overset{L}{\otimes}_{k\Delta} X$ restricted to the category $k(\Delta)$ satisfies the property of $a)$ of lemma 2.7.1. Thus we can apply $b)$ and lift $c_{|_{k(\Delta)}}$ to an isomorphism $\tilde{c}$ between $X$ and $Y \overset{L}{\otimes}_{k\Delta} X$ as dg-$k\Delta^{op}\otimes\mathcal{A}$-modules.

By the universal property of the orbit category, the bimodule $X$ endowed with the isomorphism $\tilde{c}$ yields a triangle functor from $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ to $\mathcal{T}$ which comes from a bimodule $Z$ in $\mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})$.

$$\xymatrix{\mathcal{D}^b(\mathsf{mod}\, k\Delta) \ar@(ul,ur)^{-\overset{L}{\otimes}_{k\Delta} Y} \ar[r]^-{-\overset{L}{\otimes}_{k\Delta} X} \ar[d]_{\pi} & \mathsf{per}\,\mathcal{A} = \mathcal{T} \\ \mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi = \mathsf{per}\,\mathcal{B} \ar@{-->}[ur]_{-\overset{L}{\otimes}_{k\Delta} Z} & }$$

The functor $-\overset{L}{\otimes}_{k\Delta} Z$ is essentially surjective. Let us show that it is fully faithful. For $M$ and $N$ objects of $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$ we have the following commutative diagram:

$$\xymatrix{ & \bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(M, \Phi^n N) \ar[dl]_{\pi} \ar[dr]^{-\overset{L}{\otimes}_{k\Delta} X = F} & \\ \mathsf{Hom}_{\mathcal{D}/\Phi}(\pi M, \pi N) \ar[rr]^{-\overset{L}{\otimes}_{k\Delta} Z} & & \mathsf{Hom}_{\mathcal{T}}(FM, FN),}$$

where $\mathcal{D}$ means $\mathcal{D}^b(\mathsf{mod}\, k\Delta)$. The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that $- \overset{L}{\otimes}_{k\Delta} Z$ is a triangle equivalence between the orbit category $\mathcal{D}^b(\mathsf{mod}\, k\Delta)/\Phi$ and $\mathcal{T}$. □

## 2.8 Triangulated structure on the category of projectives

Let $k$ be an algebraically closed field and $\mathcal{P}$ a $k$-linear category with split idempotents. The category $\mathsf{mod}\,\mathcal{P}$ of contravariant finitely presented functors from $\mathcal{P}$ to $\mathsf{mod}\, k$ is exact. As the idempotents split, the projectives of $\mathsf{mod}\,\mathcal{P}$ coincide with the representables. Thus the Yoneda functor gives a natural equivalence between $\mathcal{P}$ and $\mathsf{proj}\,\mathcal{P}$. Assume besides that $\mathsf{mod}\,\mathcal{P}$ has a structure of Frobenius category. The stable category $\underline{\mathsf{mod}}\,\mathcal{P}$ is a triangulated category, we write $\Sigma$ for the suspension functor.

Let $S$ be an auto-equivalence of $\mathcal{P}$. It can be extended to an exact functor from $\mathsf{mod}\,\mathcal{P}$ to $\mathsf{mod}\,\mathcal{P}$ and thus to a triangle functor of $\underline{\mathsf{mod}}\,\mathcal{P}$. The aim of this part is to find a necessary condition on the functor $S$ such that the category $(\mathcal{P}, S)$ has a triangulated structure. Heller already showed [Hel68, thm 16.4] that if there exists an isomorphism of triangle functors between $S$ and $\Sigma^3$, then $\mathcal{P}$ has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor $S$ and prove the following theorem:

**Theorem 2.21.** *Assume there exists an exact sequence of exact functors from* $\mathsf{mod}\,\mathcal{P}$ *to* $\mathsf{mod}\,\mathcal{P}$*:*

$$0 \longrightarrow Id \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0\,,$$

*where the* $X^i$*,* $i = 0, 1, 2$*, take values in* $\mathsf{proj}\mathcal{P}$*. Then the category* $\mathcal{P}$ *has a structure of triangulated category with suspension functor* $S$*.*

For an $M$ in $\mathsf{mod}\,\mathcal{P}$, denote $T_M : X^0M \longrightarrow X^1M \longrightarrow X^2M \longrightarrow SX^0M$ a *standard triangle*. A triangle of $\mathcal{P}$ will be a sequence $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ which is isomorphic to a standard triangle $T_M$ for an $M$ in $\mathsf{mod}\,\mathcal{P}$.

### 2.8.1 S-complexes, Φ-S-complexes and standard triangles

Let $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$ be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, *i.e.* the complexes homotopic to zero. The functor $Z^0 : \mathcal{A}cp(\mathsf{mod}\,\mathcal{P}) \to \mathsf{mod}\,\mathcal{P}$ which sends a complex

$$\cdots \longrightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots$$

to the kernel of $x^0$ is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between $\underline{\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})}$ and $\underline{\mathsf{mod}}\,\mathcal{P}$.

**Definition 2.22.** An object of $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$ is called an *$S$-complex* if it is $S$-periodic, *i.e.* if it has the following form:

$$\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

The category $S$-$\mathsf{comp}$ of $S$-complexes with $S$-periodic morphisms is a non full subcategory of $\mathcal{A}cp(\mathsf{mod}\,\mathcal{P})$. It is a Frobenius category. The projective-injectives are the $S$-contractibles, *i.e.* the complexes homotopic to zero with an $S$-periodic homotopy. Using the functor $Z^0$, we get an exact functor from $S$-$\mathsf{comp}$ to $\mathsf{mod}\,\mathcal{P}$ which induces a triangle functor:

$$\underline{Z^0} : \underline{S\text{-}\mathsf{comp}} \longrightarrow \underline{\mathsf{mod}}\,\mathcal{P}.$$

Fix a sequence as in theorem 2.21. Clearly, it induces for each object $M$ of $\underline{\mathsf{mod}}\,\mathcal{P}$, a functorial isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$, $\Phi_M : \Sigma^3 M \longrightarrow SM$ .

Let $Y$ be an $S$-complex,

$$Y : \cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

Let $M$ be the kernel of $u$. Then $Y$ induces an isomorphism $\theta$ (in $\underline{\mathsf{mod}}\,\mathcal{P}$) between $\Sigma^3 M$ and $SM$. If $\theta$ is equal to $\Phi_M$, we will say that $X$ is a *$\Phi$-$S$-complex.*

Let $M$ be an object of $\mathsf{mod}\,\mathcal{P}$. The standard triangle $T_M$ can be seen as a $\Phi$-$S$-complex:

$$\cdots \longrightarrow X^0M \longrightarrow X^1M \longrightarrow X^2M \longrightarrow SX^0M \longrightarrow SX^1M \longrightarrow \cdots .$$

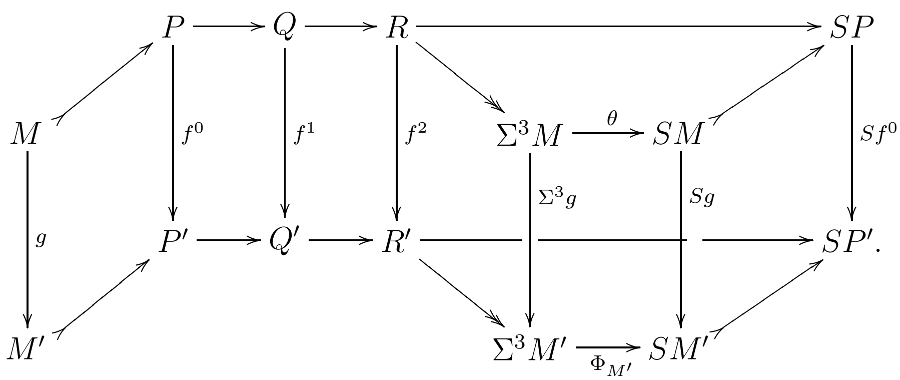
The functor $T$ which sends an object $M$ of $\mathsf{mod}\,\mathcal{P}$ to the $S$-complex $T_M$ is exact since the $X^i$ are exact. It satisfies the relation $Z^0 \circ T \simeq Id_{\mathsf{mod}\,\mathcal{P}}$. Moreover, as it preserves the projective-injectives, it induces a triangle functor:

$$T : \underline{\mathsf{mod}}\,\mathcal{P} \to \underline{S\text{-}\mathsf{comp}} .$$

## 2.8.2 Properties of the functors $Z^0$ and $T$

**Lemma 2.8.1.** *An $S$-complex which is homotopy-equivalent to a $\Phi$-$S$-complex is a $\Phi$-$S$-complex.*

*Proof.* Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be an $S$-complex homotopy-equivalent to the $\Phi$-$S$-complex $X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$. Let $M$ be the kernel of $u$ and $M'$ the kernel of $u'$. By assumption, there exists a $S$-periodic homotopy equivalence $f$ from $X$ to $X'$, which induces a morphism $g = Z^0 f : M \to M'$. Thus, we get the following commutative diagram:




The morphism $g$ is an isomorphism of $\underline{\mathsf{mod}}\,\mathcal{P}$ since $f$ is an isomorphism of $\underline{S\text{-}\mathsf{comp}}$ . Thus the morphisms $\Sigma^3 g$ and $Sg$ are isomorphisms of $\underline{\mathsf{mod}}\,\mathcal{P}$. The following equality in $\underline{\mathsf{mod}}\,\mathcal{P}$

$$\theta = (Sg)^{-1}\Phi_{M'}\Sigma^3 g = \Phi_M$$

shows that the complex $X$ is a $\Phi$-$S$-complex. ☐

**Lemma 2.8.2.** *Let*

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

*be two $\Phi$-$S$-complexes. Suppose that we have a commutative square:*

$$\begin{array}{ccc} P & \xrightarrow{u} & Q \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f^1} \\ P' & \xrightarrow{u'} & Q'. \end{array}$$

*Then, there exists a morphism $f^2 : R \to R'$ such that $(f^0, f^1, f^2)$ extends to an $S$-periodic morphism from $X$ to $X'$.*

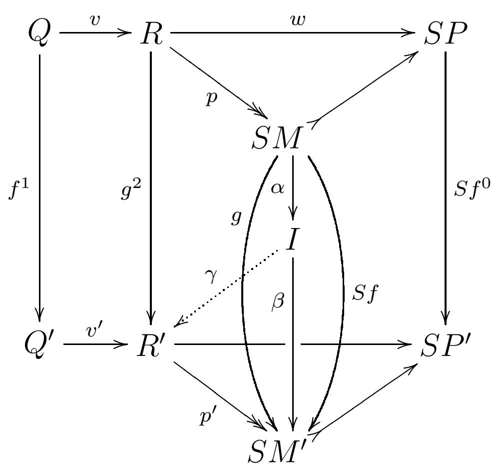
*Proof.* Let $M$ be the kernel of $u$, $M'$ be the kernel of $u'$ and $f : M \to M'$ be the morphism induced by the commutative square. As $R$ and $R'$ are projective-injective objects, we can find a morphism $g^2 : R \to R'$ such that the following square commutes:

$$\begin{array}{ccc} Q & \xrightarrow{v} & R \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle g^2} \\ Q' & \xrightarrow{v'} & R'. \end{array}$$

The morphism $g^2$ induces a morphism $g : SM \to SM'$ such that the following square is commutative in $\underline{\mathsf{mod}}\,\mathcal{P}$:

$$\begin{array}{ccc} \Sigma^3 M & \xrightarrow{\Phi_M} & SM \\ {\scriptstyle \Sigma^3 f}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma^3 M' & \xrightarrow{\Phi_{M'}} & SM'. \end{array}$$

Thus the morphisms $Sf$ and $g$ are equal in $\underline{\mathsf{mod}}\,\mathcal{P}$, *i.e.* there exists a projective-injective $I$ of $\mathsf{mod}\,\mathcal{P}$ and morphisms $\alpha : SM \to I$ and $\beta : I \to SM'$ such that $g - Sf = \beta\alpha$. Let $p$ (resp. $p'$) be the epimorphism from $R$ onto $SM$ (resp. from $R'$ onto $SM'$). Then, as $I$ is projective, $\beta$ factors through $p'$.




We put $f^2 = g^2 - \gamma\alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = S f^0 w$. Thus the morphism $(f^0, f^1, f^2)$ extends to a morphism of $S$-$\mathsf{comp}$. □

**Proposition 2.8.1.** *The functor* $\underline{Z^0} : \underline{\Phi\text{-}S\text{-}\mathsf{comp}} \longrightarrow \underline{\mathsf{mod}}\,\mathcal{P}$ *is full and essentially surjective. Its kernel is an ideal whose square vanishes.*

*Proof.* The functor $\underline{Z^0}$ is essentially surjective since we have the relation $\underline{Z^0} \circ \underline{T} = Id_{\underline{\mathsf{mod}}\,\mathcal{P}}$.

Let us show that $\underline{Z^0}$ is full. Let

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two $\Phi$-$S$-complexes. Let $M$ (resp. $M'$) be the kernel of $u$ (resp. $u'$). As $P$, $Q$, $P'$ and $Q'$ are projective-injective, there exist morphisms $f^0 : P \to P'$ and $f^1 : Q \to Q'$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \rightarrowtail & P & \xrightarrow{u} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} \\ M' & \rightarrowtail & P' & \xrightarrow{u'} & Q'. \end{array}$$

Now the result follows from lemma 2.8.2.

Now let $\underline{f} : X \to X'$ be a morphism in the kernel of $\underline{Z}^0$. Up to homotopy, we can suppose that $\underline{f}$ has the following form:

$$\begin{array}{ccccccc}
P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle f^2}\downarrow & & \downarrow{\scriptstyle 0} \\
P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP'.
\end{array}$$

As the composition $w'f^2$ vanishes and as $Q'$ is projective-injective, $f^2$ factors through $v'$. For the same argument, $f^2$ factors through $w$. If $\underline{f}$ and $\underline{f'}$ are composable morphisms of the kernel of $\underline{Z}^0$, we get the following diagram:

$$\begin{array}{ccccccc}
P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & {\scriptstyle h^2}\swarrow & {\scriptstyle f^2}\downarrow & & \downarrow{\scriptstyle 0} \\
P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP' \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle f'^2}\downarrow & {\scriptstyle h'^3}\swarrow & \downarrow{\scriptstyle 0} \\
P'' & \xrightarrow{u''} & Q'' & \xrightarrow{v''} & R'' & \xrightarrow{w''} & SP''.
\end{array}$$

The composition $\underline{f'f}$ vanishes obviously.

□

**Corollary 2.23.** *A $\Phi$-$S$-complex morphism $f$ which induces an isomorphism $\underline{Z}^0(f)$ in $\underline{\mathsf{mod}}\,\mathcal{P}$ is an homotopy-equivalence.*

This corollary comes from the previous theorem and from the following lemma.

**Lemma 2.8.3.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a full functor between two additive categories. If the kernel of $F$ is an ideal whose square vanishes, then $F$ detects isomorphisms.*

*Proof.* Let $u \in \mathsf{Hom}_{\mathcal{C}}(A, B)$ be a morphism in $\mathcal{C}$ such that $Fu$ is an isomorphism. Since the functor $F$ is full, there exists $v$ in $\mathsf{Hom}_{\mathcal{C}}(B, A)$ such that $Fv = (Fu)^{-1}$. The morphism $w = uv - Id_B$ is in the kernel of $F$, thus $w^2$ vanishes. Then the morphism $v(Id_B - w)$ is a right inverse of $u$. In the same way we show that $u$ has a left inverse, so $u$ is an isomorphism. □

**Proposition 2.8.2.** *The category of $\Phi$-$S$-complexes is equivalent to the category of $S$-complexes which are homotopy-equivalent to standard triangles.*

*Proof.* Since standard triangles are $\phi$-$S$-complexes, each $S$-complex that is homotopy equivalent to a standard triangle is a $\Phi$-$S$-complex (lemma 2.8.1).

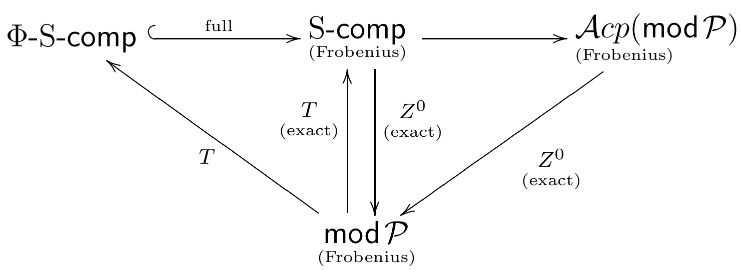
Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a $\Phi$-$S$-complex. Let $M$ be the kernel of $u$. Then there exist morphisms $f^1 : P \to X^0 M$ and $f^1 : Q \to X^1 M$ such that the following diagram is commutative:

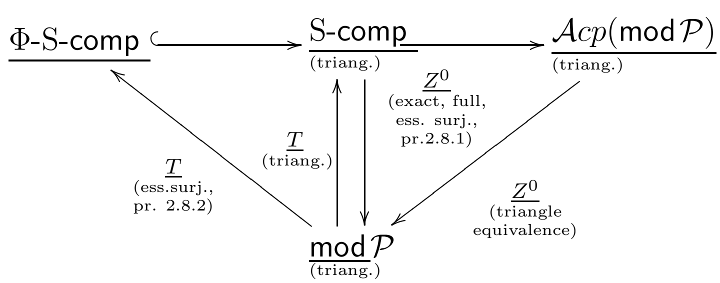
$$\begin{array}{ccccc} M & \rightarrowtail & P & \xrightarrow{u} & Q \\ \| & & \downarrow f^0 & & \downarrow f^1 \\ M & \rightarrowtail & X^0 M & \longrightarrow & X^1 M. \end{array}$$

We can complete (lemma 2.8.2) $f$ into an $S$-periodic morphism from $X$ in $T_M$. The morphism $f$ satisfies $Z^0 f = Id_M$, so $\underline{Z}^0(T_M)$ and $Z^0(X)$ are equal in $\underline{\mathsf{mod}}\mathcal{P}$. By the corollary, $T_M$ and $X$ are homotopy-equivalent. Thus the inclusion functor $T$ is essentially surjective.

□

These two diagrams summarize the results of this section:



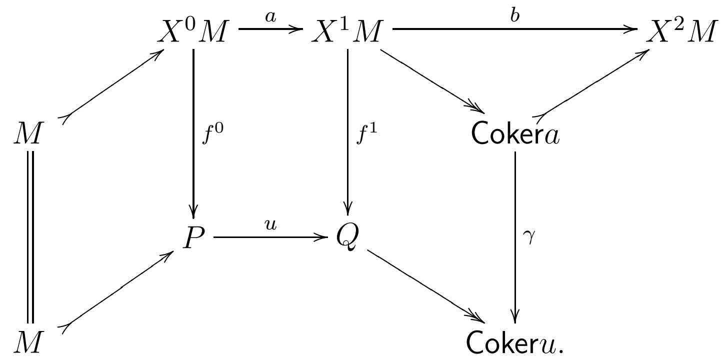




### 2.8.3 Proof of theorem 2.21

We are going to show that the $\Phi$-$S$-complexes form a system of triangles of the category $\mathcal{P}$. We use triangle axioms as in [Nee01].

**TR0**: For each object $M$ of $\mathcal{P}$, the $S$-complex $M = M \longrightarrow 0 \longrightarrow SM$ is homotopy-equivalent to the zero complex, so is a $\Phi$-$S$-complex.

**TR1**: Let $u : P \to Q$ be a morphism of $\mathcal{P}$, and let $M$ be its kernel. We can find morphisms $f^0$ and $f^1$ so as to obtain a commutative square:



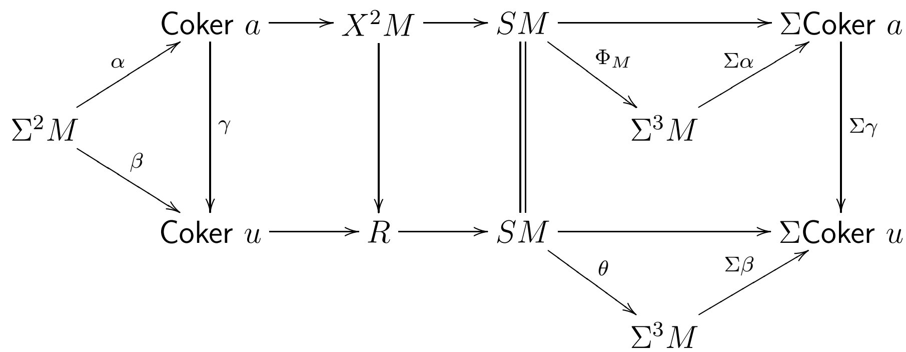

We form the following push-out:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{Coker}\,a & \longrightarrow & X^2M & \longrightarrow & SM & \longrightarrow & 0 \\
 & & \downarrow \gamma & \text{PO} & \downarrow & & \| & & \\
0 & \longrightarrow & \mathsf{Coker}\,u & \longrightarrow & R & \longrightarrow & SM & \longrightarrow & 0.
\end{array}$$

It induces a triangle morphism of the triangulated category $\underline{\mathsf{mod}}\,\mathcal{P}$:

$$\begin{array}{ccccccc}
\mathsf{Coker}\ a & \longrightarrow & X^2M & \longrightarrow & SM & \longrightarrow & \Sigma\mathsf{Coker}\ a \\
\downarrow \gamma & & \downarrow & & \| & & \downarrow \Sigma\gamma \\
\mathsf{Coker}\ u & \longrightarrow & R & \longrightarrow & SM & \longrightarrow & \Sigma\mathsf{Coker}\ u.
\end{array}$$

The morphism $\gamma$ is an isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$ since $\mathsf{Coker}\ a$ and $\mathsf{Coker}\ u$ are canonically isomorphic to $\Sigma^2M$ in $\underline{\mathsf{mod}}\,\mathcal{P}$. By the five lemma, $X^2M \to R$ is an isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$. Since $X^2M$ is projective-injective, so is $R$. Thus the complex $P \xrightarrow{u} Q \longrightarrow R \longrightarrow SP$ is an $S$-complex. Then we have to see that it is a $\Phi$-$S$-complex. Let $\theta$ be the isomorphism between $SM$ and $\Sigma^3M$ induced by this complex. We write $\alpha$ (resp. $\beta$) for the canonical isomorphism in $\underline{\mathsf{mod}}\,\mathcal{P}$ between $\Sigma^2M$ and $\mathsf{Coker}\ a$ (resp. $\mathsf{Coker}\ u$). From the commutative diagram:



we deduce the equality $\theta = (\Sigma\beta)^{-1}\Sigma\gamma\Sigma\alpha\Phi_M = \Phi_M$ in $\underline{\mathsf{mod}}\,\mathcal{P}$. The constructed $S$-complex is a $\Phi$-$S$-complex.

**TR2**: Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a $\Phi$-$S$-complex. It is homotopy-equivalent to a standard triangle $T_M$. Thus the $S$-complex

$$X' : Q \xrightarrow{-v} R \xrightarrow{-w} SP \xrightarrow{-Su} SQ$$

is homotopy-equivalent to $T_M[1]$. Since $\underline{T}$ is a triangle functor, the objects $T_{\Sigma M}$ and $T_M[1]$ are isomorphic in the stable category $\underline{S\text{-}\mathsf{comp}}$ , *i.e.* they are homotopy-equivalent. Thus, by lemma 2.8.1, $T_M[1]$ is a $\Phi$-$S$-complex and then so is $X'$.

**TR3**: This axiom is a direct consequence of lemma 2.8.2.

**TR4**: Let $X$ and $X'$ be two $\Phi$-$S$-complexes and suppose we have a commutative diagram:

$$\begin{array}{ccccccc} X : P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle Sf^0} \\ X' : P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP'. \end{array}$$

Let $M$ (resp. $M'$) be the kernel of $u$ (resp. $u'$), and $g : M \to M'$ the induced morphism. The morphism $Tg : T_M \to T_{M'}$ induces a $S$-complex morphism $\tilde{g} = (g^0, g^1, g^2)$ between $X$ and $X'$.

We are going to show that we can find a morphism $f^2 : R \to R'$ such that $(f^0, f^1, f^2)$ can be extended in an $S$-complex morphism that is homotopic to $\tilde{g}$. As $(g^0, g^1)$ and $(f^0, f^1)$ induce the same morphism $g$ in the kernels, we have some morphisms $h^1 : Q \to P'$ and $h^2 : R \to Q'$ such that $f^0 - g^0 = h^1 u$ and $f^1 - g^1 = u'h^1 + h^2 v$. We put $f^2 = g^2 + v'h^2$. We have the following equalities:

$$\begin{array}{rclcrcl} f^2 v & = & g^2 v + v'h^2 v & \text{and} & w'f^2 & = & w'g^2 \\ & = & v'(g^1 + h^2 v) & & & = & (Sg^0)w \\ & = & v'(f^1 - u'h^1) & & & = & (Sf^0 - Sh^1 Su)w \\ & = & v'f^1 & & & = & (Sf^0)w \end{array}$$

Thus $(f^0, f^1, f^2)$ can be extended to an $S$-periodic morphism $\tilde{f}$ which is $S$-homotopic to $\tilde{g}$. Their respective cones $C(\tilde{f})$ and $C(\tilde{g})$ are isomorphic as $S$-complexes. Moreover, since $\tilde{g}$ is a composition of $Tg : T_M \to T_{M'}$ with homotopy-equivalences, the cones $C(\tilde{g})$ and $C(Tg)$ are homotopy-equivalent.

In $\underline{\mathsf{mod}}\,\mathcal{P}$, we have a triangle

$$M \xrightarrow{g} M' \longrightarrow C(g) \longrightarrow \Sigma M.$$

Since $\underline{T}$ is a triangle functor, the sequence

$$T_M \xrightarrow{T_g} T_{M'} \longrightarrow T_{C(g)} \longrightarrow T_{\Sigma M}$$

is a triangle in $\underline{S\text{-comp}}$. But we know that

$$T_M \xrightarrow{Tg} T_{M'} \longrightarrow C(Tg) \longrightarrow T_M[1]$$

is a triangle in $\underline{S\text{-comp}}$. Thus the objects $C(Tg)$ and $T_{C(g)}$ are isomorphic in $\underline{S\text{-comp}}$, *i.e.* homotopy-equivalent. Thus, the cone $C(\tilde{f})$ of $\tilde{f}$ is a $\Phi$-$S$-complex by lemma 2.8.1.

## 2.9 Application to the deformed preprojective algebras

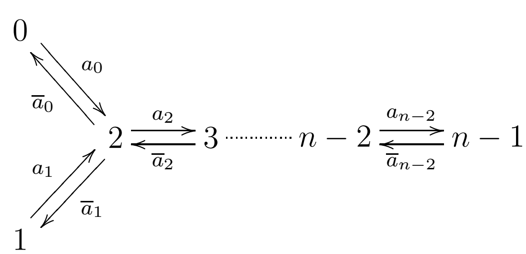
In this section, we apply the theorem 2.21 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [BES07]) is triangulated. This will give us some examples of non standard triangulated categories with finitely many indecomposables.

### 2.9.1 Preprojective algebra of generalized Dynkin type

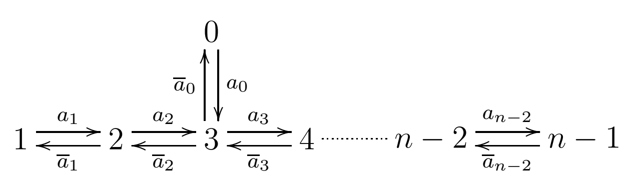
Recall the notations of [BES07]. Let $\Delta$ be a generalized Dynkin graph of type $\mathbb{A}_n$, $\mathbb{D}_n$ ($n \geq 4$), $\mathbb{E}_n$ ($n = 6, 7, 8$), or $\mathbb{L}_n$. Let $Q_\Delta$ be the following associated quiver:

$\Delta = \mathbb{A}_n$ ($n \geq 1$) :

$\Delta = \mathbb{D}_n$ ($n \geq 4$) :



$\Delta = \mathbb{E}_n$ ($n = 6, 7, 8$) :



$$\Delta = \mathbb{L}_n \ (n \geq 1): \qquad {\scriptstyle \epsilon=\overline{\epsilon}} \circlearrowleft 0 \underset{\overline{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\overline{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \cdots\cdots n-2 \underset{\overline{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1\,.$$

The *preprojective algebra* $P(\Delta)$ associated to the graph $\Delta$ is the quotient of the path algebra $kQ_\Delta$ by the relations:

$$\sum_{sa=i} a\overline{a}, \quad \text{for each vertex } i \text{ of } Q_\Delta.$$

The following proposition is classical [BES07, prop 2.1].

**Proposition 2.9.1.** *The preprojective algebra $P(\Delta)$ is finite dimensional and selfinjective. Its Nakayama permutation $\nu$ is the identity for $\Delta = \mathbb{A}_1$, $\mathbb{D}_{2n}$, $\mathbb{E}_7$, $\mathbb{E}_8$ and $\mathbb{L}_n$, and is of order 2 in all other cases.*

### 2.9.2 Deformed preprojective algebras of generalized Dynkin type

Let us recall the definition of deformed preprojective algebra introduced by [BES07]. Let $\Delta$ be a graph of generalized Dynkin type. We define an associated algebra $R(\Delta)$ as follows:

$$\begin{array}{rcl} R(\mathbb{A}_n) & = & k; \\ R(\mathbb{D}_n) & = & k\langle x,y\rangle/(x^2, y^2, (x+y)^{n-2}); \\ R(\mathbb{E}_n) & = & k\langle x,y\rangle/(x^2, y^3, (x+y)^{n-3}); \\ R(\mathbb{L}_n) & = & k[x]/(x^{2n}). \end{array}$$

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

$$\begin{array}{lll} 0 & \text{for} & \Delta = \mathbb{A}_n \text{ or } \mathbb{L}_n, \\ 2 & \text{for} & \Delta = \mathbb{D}_n, \\ 3 & \text{for} & \Delta = \mathbb{E}_n. \end{array}$$

Let $f$ be an element of the square $rad^2 R(\Delta)$ of the radical of $R(\Delta)$. The *deformed preprojective algebra* $P^f(\Delta)$ is the quotient of the path algebra $kQ_\Delta$ by the relations:

$$\sum_{sa=i} a\overline{a}, \quad \text{for each non exceptional vertex } i \text{ of } Q,$$

and

$$\begin{array}{ll} a_0\overline{a}_0 & \text{for} \quad \Delta = \mathbb{A}_n; \\ \overline{a}_0 a_0 + \overline{a}_1 a_1 + a_2\overline{a}_2 + f(\overline{a}_0 a_0, \overline{a}_1 a_1), \text{ and } (\overline{a}_0 a_0 + \overline{a}_1 a_1)^{n-2} & \text{for} \quad \Delta = \mathbb{D}_n; \\ \overline{a}_0 a_0 + \overline{a}_2 a_2 + a_3\overline{a}_3 + f(\overline{a}_0 a_0, \overline{a}_2 a_2), \text{ and } (\overline{a}_0 a_0 + \overline{a}_2 a_2)^{n-3} & \text{for} \quad \Delta = \mathbb{E}_n; \\ \epsilon^2 + a_0\overline{a}_0 + \epsilon f(\epsilon), \text{ and } \epsilon^{2n} & \text{for} \quad \Delta = \mathbb{L}_n. \end{array}$$

Note that if $f$ is zero, we get the preprojective algebra $P(\Delta)$.

### 2.9.3 Corollaries of [BES07]

The following proposition [BES07, prop 3.4] shows that the category $\mathsf{proj}\, P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of theorem 2.21.

**Proposition 2.9.2.** *Let $A = P^f(\Delta)$ be a deformed preprojective algebra. Then there exists an exact sequence of $A$-$A$-bimodules*

$$0 \longrightarrow {}_1A_{\Phi^{-1}} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

*where $\Phi$ is an automorphism of $A$ and where the $P_i$'s are projective as bimodules. Moreover, for each idempotent $e_i$ of $A$, we have $\Phi(e_i) = e_{\nu(i)}$.*

So we can easily deduce the corollary:

**Corollary 2.24.** *Let $P^f(\Delta)$ be a deformed preprojective algebra of generalized Dynkin type. Then the category $\mathsf{proj}\, P^f(\Delta)$ of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.*

Indeed, if $P_i = e_iA$ is a projective indecomposable, then $P_i \otimes_A A_\Phi$ is equal to $\Phi(e_i)A = e_{\nu(i)}A$ thus to $\nu(P_i)$.

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from the theorem [BES07, thm 1.3].

**Theorem 2.25.** *Let $k$ be an algebraically closed field of characteristic 2. Then there exist $k$-linear triangulated categories with finitely many indecomposables which are not standard.*

*Proof.* By theorem [BES07, thm 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P^f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\mathsf{proj}\, P^f(\Delta)$ and $\mathsf{proj}\, P(\Delta)$ can not be equivalent. But both are triangulated by corollary 2.24 and have the same AR-quiver $\mathbb{Z}\Delta/\tau = Q_\Delta$. □

Conversely, we have the following theorem:

**Theorem 2.26.** *Let $\mathcal{T}$ be a finite 1-Calabi-Yau triangulated category. Then $\mathcal{T}$ is equivalent to $\mathsf{proj}\, \Lambda$ as $k$-category, where $\Lambda$ is a deformed preprojective algebra of generalized Dynkin type.*

*Proof.* Let $M_1, \ldots, M_n$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$. The $k$-algebra $\Lambda = \mathsf{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since $\mathcal{T}$ has a Serre duality. It is easy to see that $\mathcal{T}$ and $\mathsf{proj}\, \Lambda$ are equivalent as $k$-categories.

Let $\mathsf{mod}\, \Lambda$ be the category of finitely presented $\Lambda$-modules. It is a Frobenius category. Denote by $\Sigma$ the suspension functor of the triangulated category $\underline{\mathsf{mod}}\, \Lambda$. The category $\mathcal{T}$

is 1-Calabi-Yau, that is to say that the suspension functor $S$ of the triangulated category $\mathcal{T}$ and the Serre functor $\nu$ are isomorphic. But in $\underline{\mathsf{mod}}\,\Lambda$, the functors $S$ and $\Sigma^3$ are isomorphic. Thus, for each non projective simple $\Lambda$-module $M$ we have an isomorphism $\Sigma^3 M \simeq \nu M$. By [BES07, thm 1.2], we get immediately the result. $\square$

# Chapitre 3

# Appendice

## Exemple des complexes parfaits

Les hypothèses du théorème 2.21 peuvent paraître très fortes. Dans cette partie nous allons voir l'exemple d'une catégorie triangulée dont la structure provient d'une suite exacte de foncteurs.

### Généralités sur les catégories de modules

Soient $\mathcal{C}$ et $\mathcal{D}$ deux catégories additives dans lesquelles les idempotents se scindent. Soit $i : \mathcal{C} \to \mathcal{D}$ un foncteur pleinement fidèle. Le foncteur de Yoneda

$$\begin{array}{rcl} \mathsf{Yon} : \mathcal{C} & \to & \mathsf{mod}\,\mathcal{C} \\ C & \mapsto & \mathsf{Hom}_{\mathcal{C}}(?, C) \end{array}$$

est un foncteur pleinement fidèle de la catégorie $\mathcal{C}$ dans la catégorie $\mathsf{mod}\,\mathcal{C}$. La catégorie $\mathcal{C}$ peut être vue comme la sous-catégorie pleine des projectifs de $\mathsf{mod}\,\mathcal{C}$.

Le foncteur $i$ induit immédiatement un foncteur *restriction* :

$$\begin{array}{rcl} R : \mathsf{mod}\,\mathcal{D} & \to & \mathsf{mod}\,\mathcal{C} \\ F & \mapsto & F \circ i. \end{array}$$

Remarquons que si $D$ est un objet de $\mathcal{D}$ vue comme sous-catégorie de $\mathsf{mod}\,\mathcal{D}$ alors $RD = \mathsf{Hom}_{\mathcal{D}}(i?, D)$ n'est pas forcément un foncteur représentable, donc pas forcément un objet de $\mathcal{C}$.

Un objet $F$ de $\mathsf{mod}\,\mathcal{C}$ est un foncteur de présentation projective finie, donc on peut écrire une suite exacte :

$$\mathsf{Hom}_{C}(?, C_1) \longrightarrow \mathsf{Hom}_{\mathcal{C}}(?, C_2) \longrightarrow F \longrightarrow 0.$$

Notons $LF$ le conoyau du morphisme de foncteur :

$$\mathsf{Hom}_{\mathcal{D}}(?, iC_1) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(?, iC_2).$$

Ceci nous donne un foncteur $L : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{D}$ tel que $L_{|\mathcal{C}} = i$. Ceci se résume en un diagramme :

$$\begin{array}{ccc} \mathcal{C} & \overset{\mathsf{Yon}}{\hookrightarrow} & \mathsf{mod}\,\mathcal{C} \\ {\scriptstyle i}\downarrow & & {\scriptstyle L}\downarrow \uparrow{\scriptstyle R} \\ \mathcal{D} & \overset{\mathsf{Yon}}{\hookrightarrow} & \mathsf{mod}\,\mathcal{D}. \end{array}$$

**Lemme 3.0.1.** *Les propriétés suivantes sont vérifiées :*

- *Le foncteur $L$ est pleinement fidèle.*
- *Le foncteur $R$ est exact.*
- *Le foncteur $L$ est adjoint à gauche de $R$.*
- $R \circ L \simeq Id_{\mathsf{mod}\,\mathcal{C}}$

*Démonstration.*

- Soient $F$ et $G$ deux objets de $\mathsf{mod}\,\mathcal{C}$. Notons $C_1$ et $C_2$ (resp. $C_1'$ et $C_2'$) des objets de $\mathcal{C}$ tels que $F$ (resp. $G$) soit le conoyau d'un morphisme $C_1 \to C_2$ (resp. $C_1' \to C_2'$). Alors se donner un morphisme de $F$ dans $G$ revient exactement à se donner un carré commutatif :

$$\begin{array}{ccc} C_1 & \longrightarrow & C_2 \\ \downarrow & & \downarrow \\ C_1' & \longrightarrow & C_2'. \end{array}$$

  Mais le foncteur $i$ est pleinement fidèle donc tout revient à se donner un carré commutatif :

$$\begin{array}{ccc} iC_1 & \longrightarrow & iC_2 \\ \downarrow & & \downarrow \\ iC_1' & \longrightarrow & iC_2', \end{array}$$

  ce qui est encore équivalent à se donner un morphisme de $LF$ dans $LG$.
- La seconde propriété est évidente.
- Soit $F$ un objet de $\mathsf{mod}\,\mathcal{C}$ et $G$ un objet de $\mathsf{mod}\,\mathcal{D}$. On veut montrer l'isomorphisme suivant :

$$\mathsf{Hom}_{\mathsf{mod}\,\mathcal{D}}(LF, G) \simeq \mathsf{Hom}_{\mathsf{mod}\,\mathcal{C}}(F, RG).$$

  Soient $C_1 \to C_2 \to F \to 0$ et $D_1 \to D_2 \to G \to 0$ des présentations projectives. Elles induisent les suites exactes $iC_1 \to iC_2 \to LF \to 0$ et $RD_1 \to RD_2 \to RG \to 0$ car $R$ est exact. Il suffit donc de prouver que pour tout objet $C$ de $\mathcal{C}$ et pour tout objet $D$ de $\mathcal{D}$, on a un isomorphisme :

$$\mathsf{Hom}_{\mathsf{mod}\,\mathcal{D}}(iC, D) \simeq \mathsf{Hom}_{\mathsf{mod}\,\mathcal{C}}(C, RD).$$

La catégorie $\mathcal{D}$ est une sous-catégorie pleine de $\mathsf{mod}\,\mathcal{D}$ donc le terme de gauche est égal à $\mathsf{Hom}_{\mathcal{D}}(iC, D)$. Par définition $RD$ est égal à $\mathsf{Hom}_{\mathcal{D}}(i?, D)$. Se donner un morphisme de $\mathsf{mod}\,\mathcal{C}$ de $C$ dans $RD$, c'est se donner un morphisme de $\mathsf{Hom}_{\mathcal{C}}(?, C)$ dans $\mathsf{Hom}_{\mathcal{D}}(i?, D)$. Mais comme $i$ est pleinement fidèle, on a un isomorphisme entre $\mathsf{Hom}_{\mathcal{C}}(?, C)$ et $\mathsf{Hom}_{\mathcal{D}}(i?, iC)$. En utilisant maintenant le fait que le foncteur $\mathsf{Yon}$ est pleinement fidèle, tout ceci revient à se donner un morphisme entre $iC$ et $D$ dans $\mathcal{D}$, ce qu'on voulait.

– La dernière propriété est une conséquence directe des autres.

□

**Lemme 3.0.2.** *Soit $F$ un objet de $\mathsf{mod}\,\mathcal{C}$ tel que $LF$ soit dans $\mathcal{D}$. Alors $F$ est un objet de $\mathcal{C}$.*

*Démonstration.* Soit $C_1 \to C_2 \to F \to 0$ une présentation projective de $F$. On a alors le suite exacte $iC_1 \to iC_2 \to LF \to 0$. Par hypothèse, l'objet $LF$ est projectif, donc il existe une section de $LF$ dans $iC_2$. En appliquant $R$ on trouve une section de $F$ dans $C_2$. Le foncteur $F$ est alors un facteur direct d'un projectif $C_2$ car les idempotents se scindent, donc $F$ est projectif. □

## Complexes parfaits

Soit $A$ une $k$-algèbre différentielle graduée. Dans toute la suite, sauf mention du contraire, les produits tensoriels seront sur $k$. La catégorie des *complexes parfaits* $\mathsf{per}\,A$ est la sous-catégorie triangulée de la catégorie dérivée $\mathcal{D}A$ stable par passage aux facteurs directs et engendrée par le module libre ${}_AA$. Nous supposerons que $\mathsf{per}\,A$ admet une dualité de Serre et nous noterons $\Sigma$ le foncteur suspension de la catégorie $\mathsf{per}\,A$ qui n'est autre que le foncteur décalage.

L'objet de ce paragraphe est de démontrer le théorème suivant :

**Théorème 3.1.** *Il existe une suite exacte de foncteurs de $\mathsf{mod}\,(\mathsf{per}\,A)$ dans $\mathsf{mod}\,(\mathsf{per}\,A)$ :*

$$0 \longrightarrow Id_{\mathsf{mod}\,(\mathsf{per}\,A)} \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \Sigma \longrightarrow 0,$$

*où les $X^i$ sont des foncteurs exacts à valeurs dans les projectifs.*

*La structure de catégorie triangulée qui en découle est la structure naturelle de $\mathsf{per}\,A$.*

**Lemme 3.0.3.** *Le foncteur suivant*

$$\begin{array}{rcl} (\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A & \to & \mathsf{per}\,(A^{op} \otimes A) \\ (P, Q) & \mapsto & P^{\check{}} \otimes Q, \end{array}$$

*où $P^{\check{}} = R\mathsf{Hom}_A(P, A)$, est pleinement fidèle.*

*Démonstration.* Soient $P$, $P'$ , $Q$ , $Q'$ des objets de $\mathsf{per}\, A$. Le foncteur suivant est une équivalence de catégorie :

$$\begin{array}{rcl} (\mathsf{per}\, A)^{op} & \to & \mathsf{per}\, (A^{op}) \\ P & \mapsto & P^{\check{}} \end{array}$$

De là, on déduit donc les isomorphismes suivants :

$$\begin{array}{rcl} \mathsf{Hom}_{(\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A}((P, Q), (P', Q')) & \simeq & \mathsf{Hom}_{(\mathsf{per}\, A)^{op}}(P, P') \otimes \mathsf{Hom}_{\mathsf{per}\, A}(Q, Q') \\ & \simeq & \mathsf{Hom}_{\mathsf{per}\, (A^{op})}(P^{\check{}}, P'^{\check{}}) \otimes \mathsf{Hom}_{\mathsf{per}\, A}(Q, Q'). \end{array}$$

Notons $F$ (resp. $G$) le multifoncteur $\mathsf{Hom}_{\mathsf{per}\, A}(?, ?) \otimes \mathsf{Hom}_{\mathsf{per}\, A}(?, ?)$ (resp. $\mathsf{Hom}_{\mathsf{per}\, (A^{op} \otimes A)}(? \otimes ?, ? \otimes ?)$) de $\mathsf{per}\, (A^{op}) \times \mathsf{per}\, A \times \mathsf{per}\, (A^{op}) \times \mathsf{per}\, A$ dans $\mathcal{D}k$. Les multifoncteurs $F$ et $G$ sont multitriangulés et additifs. De plus on a un isomorphisme entre $F(A^{op}, A, A^{op}, A)$ et $G(A^{op}, A, A^{op}, A)$. La fin de la preuve découle directement du lemme suivant. □

**Lemme 3.0.4.** *Soient $F$ et $G$ deux foncteurs additifs et triangulés de $\mathsf{per}\, A$ dans $\mathcal{D}k$. Supposons qu'on ait un morphisme de de foncteur $\Phi : F \to G$ tel que $\Phi_A : FA \to GA$ est un isomorphisme, alors $\Phi$ est un isomorphisme de foncteurs.*

*Démonstration.* Les foncteurs $F$ et $G$ sont triangulés, donc $\Phi$ est un isomorphisme pour tous les décalés de $A$.

Supposons que pour des objets $X$ et $Y$ de $\mathsf{per}\, A$, le morphisme $\Phi_{X \oplus Y}$ soit un isomorphisme. Alors, comme $F$ et $G$ sont additifs, on a le carré commutatif

$$\begin{array}{ccc} F(X \oplus Y) & \xrightarrow{\Phi_{X \oplus Y}} & G(X \oplus Y) \\ \downarrow & & \downarrow \\ FX \oplus FY & \xrightarrow[\begin{bmatrix} a & b \\ c & d \end{bmatrix}]{} & GX \oplus GY, \end{array}$$

où toutes les flèches sont des isomorphismes. Les carrés suivants sont commutatifs :

$$\begin{array}{ccccc} FX & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & FX \oplus FY & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & FY \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \begin{bmatrix} a & b \\ c & d \end{bmatrix}} & & \downarrow{\scriptstyle \Phi_Y} \\ GX & \xrightarrow[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]{} & GX \oplus GY & \xrightarrow[\begin{bmatrix} 0 & 1 \end{bmatrix}]{} & GY. \end{array}$$

De là, il vient que la matrice $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ est égale à la matrice $\begin{bmatrix} \Phi_X & 0 \\ 0 & \Phi_Y \end{bmatrix}$. Comme c'est un isomorphisme, $\Phi_X$ et $\Phi_Y$ sont des isomorphismes.

Enfin, si $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ est un triangle de $\mathsf{per}\, A$ et que $\Phi_X$ et $\Phi_Z$ sont des isomorphismes, le lemme des cinq nous dit que $\Phi_Y$ est aussi un isomorphisme. Comme la catégorie $\mathsf{per}\, A$ est la sous-catégorie triangulée de $\mathcal{D}A$ engendrée par $A$ et stable par passage aux facteurs directs, $\Phi$ est bien un isomorphisme de foncteurs.

□

Les résultats du paragraphe précédent nous donnent l'existence des foncteurs $L$ et $R$ adjoints. La catégorie $\mathsf{mod}\,(\mathsf{per}\,(A^{op} \otimes A))$ est de Frobenius comme catégorie de modules sur une petite catégorie triangulée. La catégorie $\mathsf{mod}\,((\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A)$ est aussi de Frobenius car nous avons supposé que $\mathsf{per}\, A$ admettait un foncteur de Serre. On a donc le diagramme suivant :

$$\begin{array}{ccc}
(\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A & \overset{\mathsf{Yon}}{\hookrightarrow} & \underset{\text{(Frobenius)}}{\mathsf{mod}\,((\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A)} \\
\Big\downarrow \text{(plein. fid.)} & & \text{L (exact, plein. fid.)} \Big\downarrow \text{(adj.)} \Big\uparrow \text{R (exact)} \\
\mathsf{per}\,(A^{op} \otimes A) & \overset{\mathsf{Yon}}{\hookrightarrow} & \underset{\text{(Frobenius)}}{\mathsf{mod}\,(\mathsf{per}\,(A^{op} \otimes A))}
\end{array}.$$

**Lemme 3.0.5.** *La catégorie des foncteurs exacts à droite de* $\mathsf{mod}\,(\mathsf{per}\, A)$ *dans* $\mathsf{mod}\,(\mathsf{per}\, A)$ *est équivalente à la catégorie* $\mathsf{mod}\,((\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A)$*.*

*Démonstration.* Soit $F$ un objet de $\mathsf{mod}\,((\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A)$, alors $F$ est un foncteur :

$$\begin{array}{rcl} F : (\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A & \to & \mathsf{mod}\, k \\ (P, Q) & \mapsto & F(P, Q) \end{array}.$$

Regardons le foncteur suivant :

$$\begin{array}{rcl} \tilde{F} : \mathsf{per}\, A & \to & \mathsf{mod}\,(\mathsf{per}\, A) \\ P & \mapsto & (Q \mapsto F(P, Q)) \end{array}$$

Ce foncteur se prolonge sur $\mathsf{mod}\,(\mathsf{per}\, A)$ en un foncteur exact à droite de manière unique.

Réciproquement, si $G$ est un foncteur exact à droite de $\mathsf{mod}\,(\mathsf{per}\, A)$ dans lui même, alors le foncteur $\tilde{G} : (P, Q) \mapsto G_P(Q)$ est un élément de $\mathsf{mod}\,((\mathsf{per}\, A)^{op} \otimes \mathsf{per}\, A)$. On a donc une équivalence de catégorie. □

Montrons maintenant le théorème 3.1.

*Démonstration.* Le foncteur $Id_{\mathsf{mod}\,(\mathsf{per}\,A)}$ correspond au bifoncteur $\mathsf{Hom}_{\mathsf{per}\,A}(?,-)$ par l'équivalence du lemme précédent. Prenons une coprésentation injective de ce foncteur dans $\mathsf{mod}\,((\mathsf{per}\,A)^{op}\otimes \mathsf{per}\,A)$. On a alors une suite exacte de la forme :

$$0 \longrightarrow Id_{\mathsf{mod}\,(\mathsf{per}\,A)} \longrightarrow \bigoplus_i (P_i, Q_i) \longrightarrow \bigoplus_j (P_j, Q_j)$$

Les objets $\bigoplus_i (P_i, Q_i)$ et $\bigoplus_j (P_j, Q_j)$ peuvent être vus comme des objets de $(\mathsf{per}\,A)^{op}\otimes \mathsf{per}\,A$ puisque ce sont des projectifs-injectifs de $\mathsf{mod}\,((\mathsf{per}\,A)^{op}\otimes \mathsf{per}\,A)$. Si on applique le foncteur exact $L$ à cette suite exacte, on obtient :

$$0 \longrightarrow LId_{\mathsf{mod}\,(\mathsf{per}\,A)} \longrightarrow \bigoplus_i (P_i^{\vee} \otimes Q_i) \longrightarrow \bigoplus_j (P_j^{\vee} \otimes Q_j)\ .$$

Les objets $\bigoplus_i (P_i^{\vee} \otimes Q_i)$ et $\bigoplus_j (P_j^{\vee} \otimes Q_j)$ sont dans $\mathsf{per}\,(A^{op} \otimes A)$ qui est triangulée. Formons donc un triangle dans $\mathsf{per}\,(A^{op} \otimes A)$ en utilisant l'axiome **TR1** :

$$\bigoplus_i (P_i^{\vee} \otimes Q_i) \longrightarrow \bigoplus_j (P_j^{\vee} \otimes Q_j) \longrightarrow X \longrightarrow \Sigma \bigoplus_i (P_i^{\vee} \otimes Q_i).$$

L'objet $X$ est un objet de $\mathsf{per}\,(A^{op} \otimes A)$. Appliquons le foncteur exact $R$ à ce triangle, on obtient :

$$\bigoplus_i (P_i, Q_i) \longrightarrow \bigoplus_j (P_j, Q_j) \longrightarrow RX \longrightarrow \Sigma \bigoplus_i (P_i, Q_i).$$

On obtient de cette manière une suite exacte :

$$0 \longrightarrow Id_{\mathsf{mod}\,(\mathsf{per}\,A)} \longrightarrow \bigoplus_i (P_i, Q_i) \longrightarrow \bigoplus_j (P_j, Q_j) \longrightarrow RX \longrightarrow \Sigma \longrightarrow 0.$$

L'objet $RX$ est dans $\mathsf{mod}\,((\mathsf{per}\,A)^{op}\otimes \mathsf{per}\,A)$. On voudrait montrer qu'il est dans $(\mathsf{per}\,A)^{op}\otimes \mathsf{per}\,A$, c'est à dire projectif en tant que bimodule. Ré-appliquons le foncteur $L$ à la suite précédente :

$$\begin{array}{ccccccc}
\bigoplus_i (P_i^{\vee} \otimes Q_i) & \longrightarrow & \bigoplus_j (P_j^{\vee} \otimes Q_j) & \longrightarrow & LRX & \longrightarrow & \Sigma \bigoplus_i (P_i^{\vee} \otimes Q_i) \\
\| & & \| & & \downarrow & & \| \\
\bigoplus_i (P_i^{\vee} \otimes Q_i) & \longrightarrow & \bigoplus_j (P_j^{\vee} \otimes Q_j) & \longrightarrow & X & \longrightarrow & \Sigma \bigoplus_i (P_i^{\vee} \otimes Q_i).
\end{array}$$

Le morphisme $LRX \to X$ est le morphisme d'adjonction. C'est un isomorphisme par le lemme des cinq. On peut alors utiliser le lemme 3.0.2 pour conclure que $RX$ est un objet de $(\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A$ donc projectif en tant que bimodule. Vu comme un foncteur de $\mathsf{mod}\,(\mathsf{per}\,A)$ dans $\mathsf{mod}\,(\mathsf{per}\,A)$, il envoie tout module dans les projectifs-injectifs.

Le théorème 2.21 nous donne une structure de catégorie triangulée sur $\mathsf{per}\,A$. Vérifions que cette structure correspond à la structure classique. Commençons par énoncer un résultat immédiat :

**Lemme 3.0.6.** *Si* $P \longrightarrow Q \longrightarrow R \longrightarrow \Sigma P$ *est un triangle de* $\mathsf{per}\,(A^{op} \otimes A)$ *alors pour tout objet* $M$ *de* $\mathsf{per}\,A$*, la suite*

$$M \otimes_A P \longrightarrow M \otimes_A Q \longrightarrow M \otimes_A R \longrightarrow \Sigma(M \otimes_A P)$$

*est un triangle de* $\mathsf{per}\,A$*.*

La suite exacte d'objets de $\mathsf{mod}\,((\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A)$ que l'on vient de construire

$$0 \longrightarrow Id \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \Sigma \longrightarrow 0$$

est telle que

$$LX^0 \longrightarrow LX^1 \longrightarrow LX^2 \longrightarrow \Sigma LX^0$$

est un triangle de $\mathsf{per}\,(A^{op} \otimes A)$. Donc d'après le lemme ci-dessus,

$$M \otimes_A LX^0 \longrightarrow M \otimes_A LX^1 \longrightarrow M \otimes_A LX^2 \longrightarrow \Sigma(M \otimes_A LX^0)$$

est un triangle de $\mathsf{per}\,A$. Les "triangles standards" du théorème 2.21 sont les $\Sigma$-complexes de la forme

$$X^0M \longrightarrow X^1M \longrightarrow X^2M \longrightarrow \Sigma X^0M\ .$$

Il suffit donc de vérifier qu'on a un isomorphisme :

$$X^0M \simeq M \otimes_A LX^0.$$

L'objet $X^0$ est un élément de $(\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A$, supposons qu'il soit de la forme $(P, Q)$ avec $P$ et $Q$ dans $\mathsf{per}\,A$. Alors vu comme un foncteur de $(\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A$ dans $\mathsf{mod}\,k$ le foncteur $X^0$ s'écrit $\mathsf{Hom}_{(\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A}(?, (P, Q))$. On en tire les égalités suivantes :

$$\begin{array}{rcl} X^0M & = & \mathsf{Hom}_{(\mathsf{per}\,A)^{op} \otimes \mathsf{per}\,A}((M, ?), (P, Q)) \\ & = & \mathsf{Hom}_{\mathsf{per}\,A}(P, M) \otimes \mathsf{Hom}_{\mathsf{per}\,A}(?, Q) \end{array}.$$

De l'autre côté on a les égalités :

$$\begin{array}{rcl} M \otimes_A LX^0 & = & \mathsf{Hom}_{\mathsf{per}\,A}(?, M \otimes_A (P^{\check{}} \otimes Q)) \\ & = & \mathsf{Hom}_{\mathsf{per}\,A}(?, (M \otimes_A \mathsf{Hom}_{\mathsf{per}\,A}(P, A)) \otimes Q) \\ & = & \mathsf{Hom}_{\mathsf{per}\,A}(?, \mathsf{Hom}_{\mathsf{per}\,A}(P, M) \otimes Q) \\ & = & \mathsf{Hom}_{\mathsf{per}\,A}(P, M) \otimes \mathsf{Hom}_{\mathsf{per}\,A}(?, Q) \end{array}.$$

De là, on déduit le théorème 3.1.

$\square$

# Chapter 4

# Construction of a Serre functor in a quotient category

## 4.1 Bilinear form in a quotient category

Let $k$ be a field. Let $\mathcal{T}$ be a $k$-linear triangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{T}$ (*i.e.* a triangulated subcategory stable under taking direct summands). We denote by [1] the suspension functor of $\mathcal{T}$. We assume that there is an auto-equivalence $\nu$ in $\mathcal{T}$ such that $\nu(\mathcal{N}) \subset \mathcal{N}$. Moreover we assume that for each $N$ in $\mathcal{N}$ and each $X$ in $\mathcal{T}$ there is a bifunctorial non degenerate bilinear form:

$$\beta_{N,X} : \mathcal{T}(N, X) \times \mathcal{T}(X, \nu N) \longrightarrow k.$$

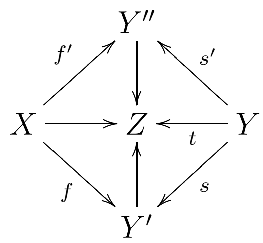
### 4.1.1 The quotient category $\mathcal{T}/\mathcal{N}$

The objects of the category $\mathcal{T}/\mathcal{N}$ are the objects of the category $\mathcal{T}$. Given two objects $X$ and $Y$ in $\mathcal{T}$, the morphisms from $X$ to $Y$ in $\mathcal{T}/\mathcal{N}$ are given by equivalence classes of diagrams:

$$s^{-1} \circ f : \quad X \xrightarrow{f} Y' \xleftarrow{s} Y$$

where $f$ and $s$ are morphisms in $\mathcal{T}$ and where the cone of $s$ is in $\mathcal{N}$.

Two diagrams $s^{-1} \circ f$ and $s'^{-1} \circ f'$ are equivalent if there exists a diagram



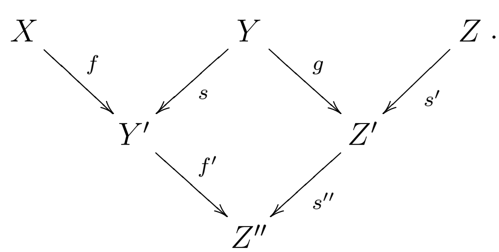

where all 'triangles' commute and where the cone of $t$ is in $\mathcal{N}$.

If $s^{-1} \circ f : X \to Y$ and $s'^{-1} \circ g : Y \to Z$ are morphisms in $\mathcal{T}/\mathcal{N}$, then the composition $(s'^{-1} \circ g) \circ (s^{-1} \circ f)$ is defined as follows:

Denote by $M$ the cone of $s$, and let $Z''$ be the object in $\mathcal{T}$ such that there is a triangle morphisms

$$\begin{array}{ccccccc} Y & \xrightarrow{s} & Y' & \longrightarrow & M & \longrightarrow & Y[1] \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \| & & \downarrow{\scriptstyle g[1]} \\ Z' & \xrightarrow{s''} & Z'' & \longrightarrow & M & \longrightarrow & Z'[1]. \end{array}$$

It yields the following diagram:




Then the composition $(s'^{-1} \circ g) \circ (s^{-1} \circ f)$ is defined by $(s\text{"}s')^{-1} \circ f'f$. Note that the cones of $s'$ and $s$ are in $\mathcal{N}$, so by the octahedral axiom, the cone of the composition $s''s'$ is also in $\mathcal{N}$.

It is now classical to check that this composition is well-defined on the equivalence classes of diagrams and that it is associative. Thus $\mathcal{T}/\mathcal{N}$ is a $k$-category.

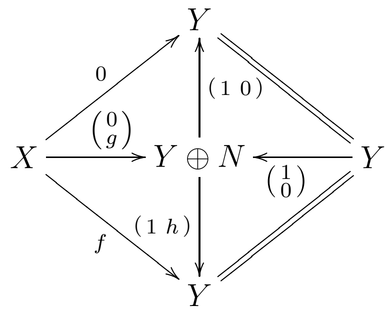
Moreover, the category $\mathcal{T}/\mathcal{N}$ is naturally triangulated and there is an exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{N} \xrightarrow{I} \mathcal{T} \xrightarrow{P} \mathcal{T}/\mathcal{N} \longrightarrow 0$$

where $I$ and $P$ are triangle functors.

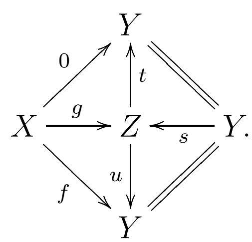
**Lemma 4.1.1.** *A morphism $f : X \to Y$ is in the kernel of $P$ if and only if it factorizes through an object of $\mathcal{N}$.*

*Proof.* Let $f : X \to Y$ be a morphism in $\mathcal{T}$ which factorizes through an object $N$ in $\mathcal{N}$. Then we have the following commutative diagram:

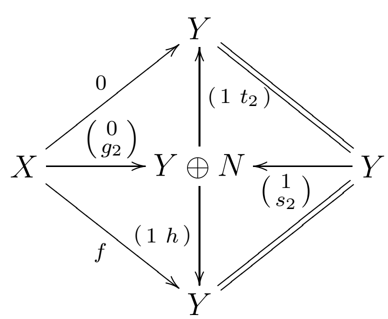



where $f = h \circ g$ is the factorization of $f$ through $N$. Thus the image of $f$ in $\mathcal{T}/\mathcal{N}$ vanishes.

Conversely, let $f : X \to Y$ be a morphism in $\mathcal{T}$ such that its image in $\mathcal{T}/\mathcal{N}$ vanishes. It means that there exists a commutative diagram:




The composition $t \circ s$ is the identity of $Y$, so $Z$ decomposes in $Y \oplus Cone(s)$. But by definition, the cone of $s$ is in $\mathcal{N}$. Finally, the previous diagram can be written:



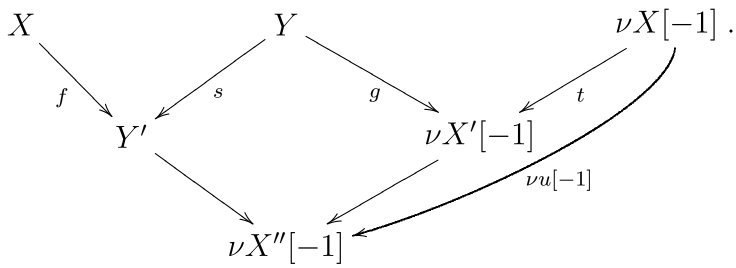

Thus $f$ is the composition $h \circ g_2$, and since $N = Cone(s)$ is in $\mathcal{N}$, the morphism $f$ factorizes through an object of $\mathcal{N}$. $\square$

### 4.1.2 Construction of a bilinear form in $\mathcal{T}/\mathcal{N}$

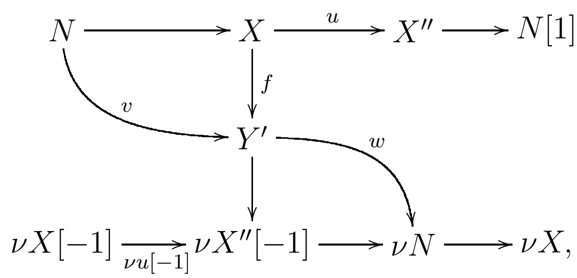
Let $X$ and $Y$ be objects in $\mathcal{T}$. The aim of this section is to construct a bifunctorial bilinear form:

$$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X, Y) \times \mathcal{T}/\mathcal{N}(Y, \nu X[-1]) \longrightarrow k.$$

Let $s^{-1} \circ f : X \to Y$ and $t^{-1} \circ g : Y \to \nu X[-1]$ be two morphisms in $\mathcal{T}/\mathcal{N}$. As we saw previously, we can construct a diagram




Denote by $N[1]$ the cone of $u$. It is in $\mathcal{N}$ since $\mathcal{N}$ is $\nu$-stable. Thus we get a diagram of the form:




where the two horizontal rows are triangles of $\mathcal{T}$. Then we define $\beta'_{X,Y}$ as follows:

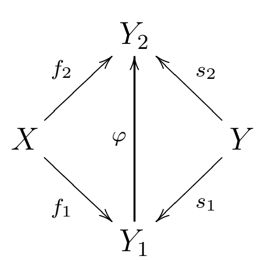
$$\beta'_{X,Y}(s^{-1} \circ f, t^{-1} \circ g) = \beta_{N,Y'}(v, w).$$

**Lemma 4.1.2.** *The form $\beta'$ is well-defined.*

*Proof.* Let $X$ and $Y$ two objects of $\mathcal{T}$. We have to show that if two fractions $s_1^{-1} \circ f_1$ and $s_2^{-1} \circ f_2$ are equal, then we have

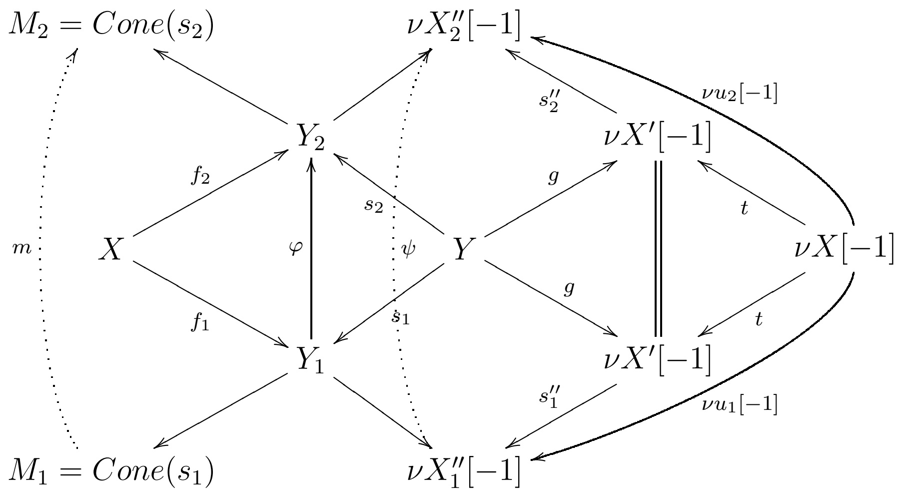
$$\beta'_{X,Y}(s_1^{-1} \circ f_1, t^{-1} \circ g) = \beta'_{X,Y}(s_2^{-1} \circ f_2, t^{-1} \circ g).$$

We can assume that there exists such a commutative diagram:




For $i = 1, 2$ let $M_i$ be the cone of $f_i : X \to Y_i$. Then $\varphi$ induces immediately a morphism

$m : M_1 \to M_2$. Then we get the following commutative diagram:




But $M_i$ is also the cone of $s''_i : \nu X'[-1] \to \nu X''_i[-1]$, so $m$ induces a morphism

$$\psi : \nu X''_1[-1] \to \nu X''_2[-1].$$

Now, if we denote by $N_i[1]$ the cone of $u_i : X \to X''_i$, the morphism $\psi$ induces a morphism $n : N_1 \to N_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} N_1 & \xrightarrow{v_1} & Y_1 & \xrightarrow{w_1} & \nu N_1 \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \nu n} \\ N_2 & \xrightarrow{v_2} & Y_2 & \xrightarrow{w_2} & \nu N_2 \end{array} .$$

Since the form $\beta$ is bifunctorial, we have immediately the equality $\beta_{N_1,Y_1}(v_1, w_1) = \beta_{N_2,Y_2}(v_2, w_2)$. By definition $\beta'_{X,Y}(s_i^{-1} \circ f_i, t^{-1} \circ g)$ is equal to $\beta_{N_i,Y_i}(v_i, w_i)$. Thus we get the equality:

$$\beta'_{X,Y}(s_1^{-1} \circ f_1, t^{-1} \circ g) = \beta'_{X,Y}(s_2^{-1} \circ f_2, t^{-1} \circ g).$$

We can show similary that $\beta'$ is well-defined on the second factor. □

**Lemma 4.1.3.** *The form $\beta'$ is bilinear and bifunctorial.*

*Proof.* These two properties come directly by the construction of $\beta'$ and by the bilinearity and bifunctoriality of $\beta$. □

### 4.1.3 Cluster category of an hereditary algebra

Let $A$ be a finite dimensional hereditary $k$-algebra. Denote by $B$ the dg-algebra $A \oplus DA[-3]$. The cluster category $\mathcal{C}_A$ is the orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\nu[-2]$ where $\nu =$

$? \overset{L}{\otimes}_A DA$ is the Serre functor of $\mathcal{D}^b(\mathsf{mod}\, A)$. By [Kel05], it is triangle equivalent to the quotient category $\mathcal{D}^b(B)/\mathsf{per}\, B$. The category $\mathsf{per}\, B$ is 3-Calabi-Yau, and there is even such a duality: for $X$ in $\mathcal{D}^b(B)$ and $N$ in $\mathsf{per}\, B$, there is an isomorphism (see chapter 1)

$$\mathsf{Hom}_{\mathcal{D}B}(N, X) \to D\mathsf{Hom}_{\mathcal{D}B}(X, N[3]).$$

Thus, by the last section, it is possible to construct a bifunctorial bilinear form

$$\beta'_{XY} : \mathcal{C}_A(X, Y) \times \mathcal{C}_A(Y, X[2]) \to k$$

on the cluster category $\mathcal{C}_A$. But in the other hand, the cluster category is 2-Calabi-Yau. Thus there is a bifunctorial duality

$$\eta_{XY} : \mathcal{C}_A(X, Y) \times \mathcal{C}_A(Y, X[2]) \to k$$

We show in this section that the bilinear form $\beta'$ constructed on the quotient $\mathcal{D}^b(B)/\mathsf{per}\, B$ coincide with $\eta$.

Recall by chapter 1, that the morphisms $A \xrightarrow{i} B \xrightarrow{p} A$ induce the following adjoint functors:

$$\begin{array}{ccc} \mathcal{D}^b B & & \mathcal{D}^b A \\ {\scriptstyle i^* = - \overset{L}{\otimes}_A B} \uparrow \ \downarrow {\scriptstyle i_*} & \text{and} & {\scriptstyle p^* - \overset{L}{\otimes}_B A} \uparrow \ \downarrow {\scriptstyle p_*} \\ \mathcal{D}^b A & & \mathcal{D}^b B. \end{array}$$

Let $X$ and $Y$ be two $A$-modules. The exact sequence of $A$-$B$-bimodules

$$0 \longrightarrow DA[-3] \xrightarrow{\iota} B \xrightarrow{p} A \longrightarrow 0$$

induces a canonical triangle in $\mathcal{D}^b B$

$$p_*(\nu X[-3]) \xrightarrow{u} i^* X \xrightarrow{v} p_* X \xrightarrow{w} p_*(\nu X[-2])$$

Let $f : X \to Y$ and $g : Y \to \nu X$ be morphisms in $\mathcal{D}^b A$. We have the following diagram

$$\begin{array}{ccccccc} X \overset{L}{\otimes}_A B & \xrightarrow{v} & p_* X & \xrightarrow{w} & p_*(\nu X[-2]) & \longrightarrow & X \overset{L}{\otimes}_A B[1] \\ & & \downarrow {\scriptstyle p_* f} & & & & \\ & & p_* Y & & & & \\ & & \downarrow {\scriptstyle p_* g} & & & & \\ p_* X[2] & \longrightarrow & p_*(\nu X) & \xrightarrow{u[3]} & X \overset{L}{\otimes}_A B[3] & \longrightarrow & p_*(X[3]), \end{array}$$

Since $i^*X = X \overset{L}{\otimes}_A B$ is an object of $\mathsf{per}\, B$, by definition of $\beta'$, we have $\beta'(p_*f, p_*g) = \beta(p_*f \circ v, u[3] \circ p_*g)$.

The morphism $v$ is induced by the morphism $p : B \to A$ of $A$-$B$-bimodules. The morphism $u[3]$ is induced by $\iota[3] : DA \to B[3]$ which is also equal to the composition $Dp : DA \to DB \simeq B[3]$. Then the equality $\beta(p_*f \circ p, Dp \circ p_*g) = \eta(f, g)$ comes directly by the following lemma.

**Lemma 4.1.4.** *Let $A$ and $B$ be $k$-algebras, and $p : B \to A$ a morphism of $B$-module, then the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{Hom}_B(A, DA) & \longrightarrow & k \\ \downarrow{\scriptstyle \pi} & \nearrow & \\ \mathsf{Hom}_B(B, DB) & & \end{array}$$

*where the application $\pi$ is given by $\pi(f) = Dp \circ f \circ p$ and where the other two arrows are standard duality.*

## 4.2 Non-degeneracy

In this section, we find conditions on $X$ and $Y$ such that the bilinear form $\beta'_{XY}$ is non-degenerate.

**Definition 4.1.** Let $X$ and $Y$ be objects in $\mathcal{T}$. A morphism $p : N \to X$ is called a *local $\mathcal{N}$-cover of $X$ relative to $Y$* if $N$ is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(X, Y) \xrightarrow{p^*} \mathcal{T}(N, Y).$$

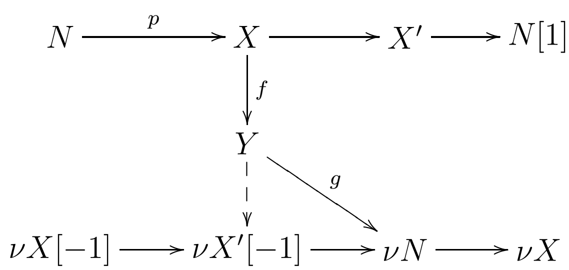
Let $Y$ and $Z$ be objects in $\mathcal{T}$. A morphism $i : Z \to N'$ is called a *local $\mathcal{N}$-envelope of $Z$ relative to $Y$* if $N'$ is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(Y, Z) \xrightarrow{i_*} \mathcal{T}(Y, N').$$

**Theorem 4.2.** *Let $X$ and $Y$ be objects of $\mathcal{T}$. If there exists a local $\mathcal{N}$-cover of $X$ relative to $Y$ and a local $\mathcal{N}$-envelope of $\nu X$ relative to $Y$, then the bilienar form $\beta'_{XY}$ constructed in the previous section is non-degenerate.*

*Proof.* Let $f : X \to Y$ be a morphism in $\mathcal{T}$ whose image in $\mathcal{T}/\mathcal{N}$ is in the kernel of $\beta'$. We have to show that it factorizes through an object of $\mathcal{N}$.

Let $p : N \to X$ be a local $\mathcal{N}$-cover of $X$ relative to $Y$, and let $X'$ be the cone of $p$. The morphism $f$ is in the kernel of $\beta'$. Thus for each morphism $g : Y \to \nu N$ which factorizes through $\nu X'[-1]$, $\beta(fp, g)$ vanishes.




This means that the linear form $\beta(fp, ?)$ vanishes on the image of the morphism

$$\mathcal{T}(Y, \nu X'[-1]) \longrightarrow \mathcal{T}(Y, \nu N).$$

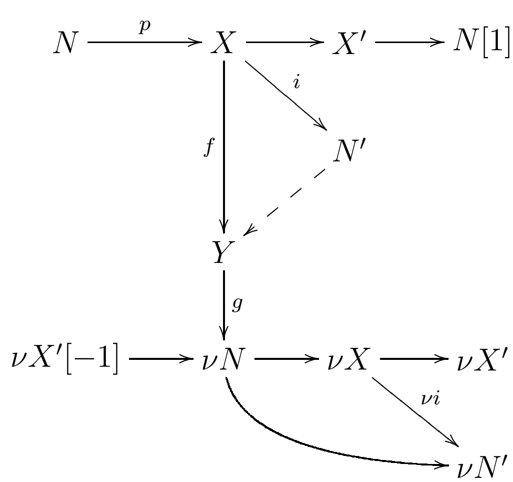
This image is canonically isomorphic to the kernel of the morphism

$$\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu X).$$

Let $\nu i : \nu X \to \nu N'$ be a local $\mathcal{N}$-envelope of $\nu X$ relative to $Y$. Then the sequence

$$0 \longrightarrow \mathcal{T}(Y, \nu X) \longrightarrow \mathcal{T}(Y, \nu N')$$

is exact. Therefore, the form $\beta(fp, ?)$ vanishes on $\mathsf{Ker}(\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu N'))$.




Now, $\beta$ is non-degenerate on

$$\mathsf{Coker}(\mathcal{T}(N', Y) \longrightarrow \mathcal{T}(N, Y)) \times \mathsf{Ker}(\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu N')).$$

Thus the morphism $fp$ lies on

$$\mathsf{Coker}(\mathcal{T}(N', Y) \longrightarrow \mathcal{T}(N, Y)),$$

that is to say that $fp$ factorizes through $ip$. Since $p : N \to X$ is a local $\mathcal{N}$-cover of $X$, $f$ factorizes through $N'$. $\square$

**Proposition 4.2.1.** *Let $X$ and $Y$ be objects in $\mathcal{T}$. If for each $N$ in $\mathcal{N}$ the vector spaces $\mathcal{T}(N,X)$ and $\mathcal{T}(Y,N)$ are finite-dimensional, then the existence of a local $\mathcal{N}$-cover of $X$ relative to $Y$ is equivalent to the existence of a local $\mathcal{N}$-envelope of $Y$ relative to $X$.*

*Proof.* Let $g : N \to X$ be a local $\mathcal{N}$-cover of $X$ relative to $Y$. It induces an injection

$$0 \longrightarrow \mathcal{T}(X,Y) \xrightarrow{g^*} \mathcal{T}(N,Y).$$

The space $\mathcal{T}(N,Y)$ is finite dimensional by hypothesis. Fix a basis $(f_1, f_2, \ldots, f_r)$ of this space. This space is in duality with the space $\mathcal{T}(Y,\nu N)$. Let $(f_1', f_2', \ldots, f_r')$ the dual basis of the basis $(f_1, f_2, \ldots, f_r)$. We show that the morphism

$$Y \xrightarrow{(f_1',\ldots,f_r')} \bigoplus_{i=1}^r \nu N$$

is a local $\mathcal{N}$-envelope of $Y$ relative to $X$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(X,Y) & \xrightarrow{(f_1',\ldots,f_r')_*} & \bigoplus \mathcal{T}(X,\nu N) \\ \downarrow g^* & & \downarrow g^* \\ \mathcal{T}(N,Y) & \xrightarrow{(f_1',\ldots,f_r')_*} & \bigoplus \mathcal{T}(N,\nu N). \end{array}$$

If $f$ is in the kernel of $(f_1', \ldots, f_r')_*$, then for all $i = 1, \ldots, r$, the morphism $f_i' \circ f \circ g$ is zero. Thus $f \circ g$ is orthogonal on the vectors of the basis $f_1', \ldots, f_r'$ and therefore vanishes. But since $g$ is a local $\mathcal{N}$-cover of $X$ relative to $Y$, $f$ is zero, and the morphism

$$\mathcal{T}(X,Y) \xrightarrow{(f_1',\ldots,f_r')_*} \bigoplus \mathcal{T}(X,\nu N)$$

is injective. Therefore, the morphism

$$Y \xrightarrow{(f_1',\ldots,f_r')} \bigoplus_{i=1}^r \nu N$$

is a local $\mathcal{N}$-envelope of $Y$ relative to $X$. The proof of the converse is dual. □

## 4.3 Examples

This section is dedicated to examples where the hypothesis of theorem 4.2 are satisfied.

### 4.3.1 Stable category

Let $A$ be a finite dimensional self-injective $k$-algebra. Denote by $\mathcal{T}$ the derived category $\mathcal{D}^b(\mathsf{mod}\, A)$ and by $\mathcal{N}$ the triangulated category $\mathsf{per}\, A$. Since $A$ is finite dimensional, there is an inclusion $\mathcal{N} \subset \mathcal{T}$. Moreover $A$ is self-injective so of infinite global dimension. Therefore the inclusion is strict. From [KV87], there is an exact sequence of triangulated categories:

$$0 \longrightarrow \mathsf{per}\, A \longrightarrow \mathcal{D}^b(\mathsf{mod}\, A) \longrightarrow \underline{\mathsf{mod}}\, A \longrightarrow 0.$$

The derived category $\mathcal{D}^b(\mathsf{mod}\, A)$ admits a Serre functor $\nu = ? \overset{L}{\otimes}_A DA$ which stabilizes $\mathsf{per}\, A$. Thus there is an induced functor in the quotient $\underline{\mathsf{mod}}\, A$ that we will also denote by $\nu$. Let $\Sigma$ be the suspension of the category $\underline{\mathsf{mod}}\, A$ which comes directly from the suspension $[1]$ of $\mathcal{D}^b(\mathsf{mod}\, A)$. Then we have the following proposition:

**Proposition 4.3.1.** *The functor $\Sigma^{-1} \circ \nu$ is a Serre functor for the stable category $\underline{\mathsf{mod}}\, A$.*

*Proof.* We have to check that $\underline{\mathsf{mod}}\, A$ satisfies the hypothesis of theorem 4.2. Since $\mathcal{D}^b(\mathsf{mod}\, A)$ is $\mathsf{Hom}$-finite, by proposition 4.2.1, it is sufficient to show the existence of local $\mathcal{N}$-covers. Let $X$ and $Y$ be in $\mathcal{D}^b(\mathsf{mod}\, A)$. We can assume that $X$ cofibrant, *i.e.*

$$X : \cdots \longrightarrow P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_m \longrightarrow 0 \longrightarrow \cdots$$

where the $P_i$ are projective. We can also assume that $Y$ is of the form:

$$\cdots 0 \longrightarrow 0 \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots Y_{n-1} \longrightarrow Y_n \longrightarrow 0 \longrightarrow \cdots$$

Now let $N$ be the stupid truncation of $X$ in degrees $\geq 0$.

$$\begin{array}{ccccccccccccc}
N : \cdots & \longrightarrow & 0 & \longrightarrow & P_0 & \longrightarrow & P_1 & \longrightarrow \cdots \longrightarrow & P_n & \longrightarrow & P_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \| & & \| & & \| & & \\
X : \cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \longrightarrow & P_1 & \longrightarrow \cdots \longrightarrow & P_n & \longrightarrow & P_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Y : \cdots & \longrightarrow & 0 & \longrightarrow & Y_0 & \longrightarrow & Y_1 & \longrightarrow \cdots \longrightarrow & Y_n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

It is then obvious that there is an exact sequence

$$0 \longrightarrow \mathcal{T}(X, Y) \longrightarrow \mathcal{T}(N, Y).$$

Since $N$ is left bounded with projective components, $N$ is perfect, and there exist local $\mathcal{N}$-covers.

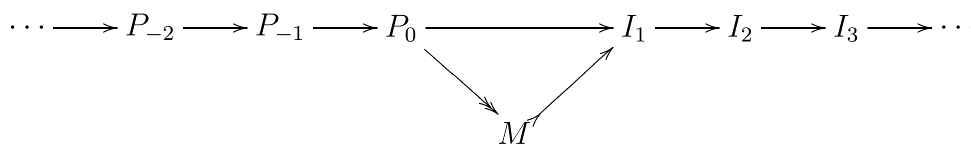
Thus by theorem 4.2, the stable category $\underline{\mathsf{mod}}\, A = \mathcal{D}^b(\mathsf{mod}\, A)/\mathsf{per}\, A$ admits a Serre functor induced by the functor $\nu \circ [-1]$ of the derived category $\mathcal{D}^b(\mathsf{mod}\, A)$.

□

Note that we could see directly, given $M$ and $N$ in $\mathsf{mod}\,A$, that there is a duality between $\underline{\mathsf{Hom}}_A(M,N)$ and $\underline{\mathsf{Hom}}_A(N,\Sigma^{-1}\nu M)$. Let

$$\cdots \longrightarrow P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \quad (\text{resp. } I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots)$$

be a projective (resp. injective) resolution of $M$. Since $A$ is self-injective, there is an acyclic complex with projective components



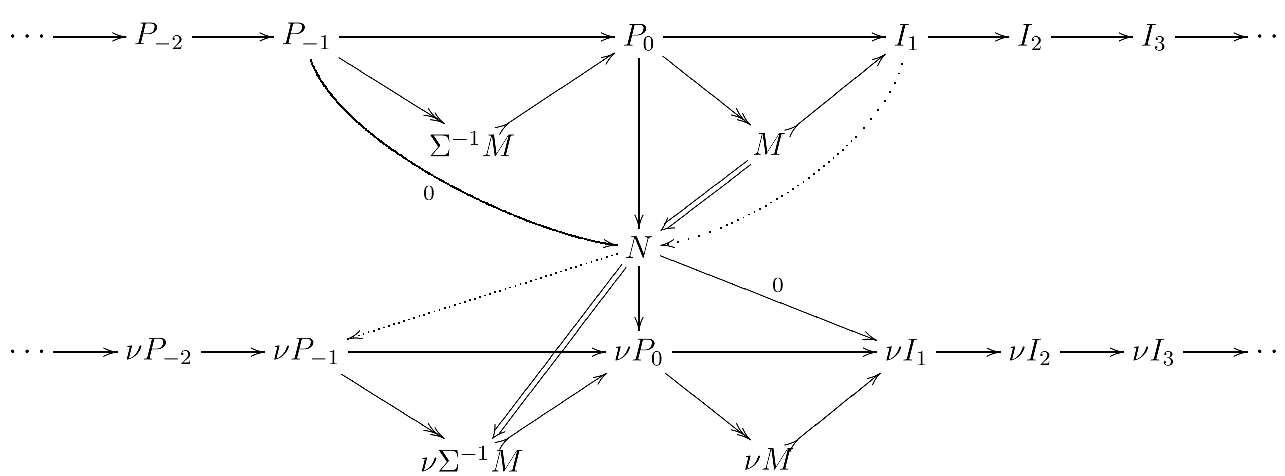

If we apply the functor $\mathsf{Hom}_A(-,N)$, we get the complex

$$\cdots \longleftarrow \mathsf{Hom}_A(P_{-1},N) \longleftarrow \mathsf{Hom}_A(P_0,N) \longleftarrow \mathsf{Hom}_A(I_1,N) \longleftarrow \cdots$$

By definition, the space $\underline{\mathsf{Hom}}_A(M,N)$ is exactly the zeroth cohomology of this complex. More precisely, it is the space of morphisms $P_0 \to N$ whose composition with $P_{-1} \to P_0$ is zero, modulo the morphisms which factorize through $I_1$. By definition of $\nu$, this complex is in duality with the complex:

$$\cdots \longrightarrow \mathsf{Hom}_A(N,\nu P_{-1}) \longrightarrow \mathsf{Hom}_A(N,\nu P_0) \longrightarrow \mathsf{Hom}_A(N,\nu I_1) \longrightarrow \cdots$$

Therefore $\underline{\mathsf{Hom}}_A(M,N)$ is in duality with the zeroth homology of this complex. More precisely, $\underline{\mathsf{Hom}}_A(M,N)$ is in duality with the space of morphisms $N \to \nu P_0$ whose composition with $\nu P_0 \to I_1$ vanishes modulo the morphims which factorize through $\nu P_1$. We have the following diagram:




Thus $\underline{\mathsf{Hom}}_A(M,N)$ is isomorphic to the dual of the space $\underline{\mathsf{Hom}}_A(N,\Sigma\nu M)$.

### 4.3.2 Cluster category: general case

Let $A$ be a finite dimensional $k$-algebra with finite global dimension. Let $B$ be the dg algebra $A \oplus DA[-3]$. The projection $p : B \to A$ yields a restriction functor $p_* : \mathcal{D}^b A \to \mathcal{D}^b B$. We denote by $\langle A \rangle_B$ the thick subcategory of $\mathcal{D}^b B$ containing the image of $\mathcal{D}^b A$. (This is not equal to $\mathcal{D}^b B$ in general.) We define the cluster category $\mathcal{C}_A$ as the quotient category $\langle A \rangle_B / \mathsf{per}\, B$. This is the triangulated hull of the orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\nu_A[-2]$ (cf [Kel05]) where $\nu_A$ is the auto-equivalence $? \overset{L}{\otimes}_A DA$ of $\mathcal{D}^b(\mathsf{mod}\, A)$.

The categories $\mathcal{T} = \langle A \rangle_B$ and $\mathcal{N} = \mathsf{per}\, B$ satisfy the hypothesis of the chapter: for all $N$ in $\mathsf{per}\, B$ and all $X$ in $\mathcal{D}^b B$, the spaces $\mathsf{Hom}_{\mathcal{D}}(N, X)$ and $\mathsf{Hom}_{\mathcal{D}}(X, N \overset{L}{\otimes}_B DB)$ are finite dimensional. Now by corollary 1.12, the functor $\nu_B = \overset{L}{\otimes}_B DB$ is isomorphic to [3] and therefore it stabilizes $\mathsf{per}\, B$. So we can construct a bifunctorial bilinear form:

$$\beta'_{XY} : \mathsf{Hom}_{\mathcal{C}_A}(X, Y) \times \mathsf{Hom}_{\mathcal{C}_A}(Y, X[2]) \to k.$$

**Theorem 4.3.** *Let $X$ and $Y$ be objects in $\mathcal{D}^b B$. If the spaces $\mathsf{Hom}_{\mathcal{D}}(X, Y)$ and $\mathsf{Hom}_{\mathcal{D}}(Y, \nu_B X) = \mathsf{Hom}_{\mathcal{D}}(Y, X[3])$ are finite dimensional, then the bilinear form*

$$\beta'_{XY} : \mathsf{Hom}_{\mathcal{C}_A}(X, Y) \times \mathsf{Hom}_{\mathcal{C}_A}(Y, X[2]) \to k$$

*is non-degenerate.*

Before proving this theorem, we recall some results about inverse limits of sequence of vector spaces that we will use in the proof. Let $\cdots \longrightarrow V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \longrightarrow V_1 \xrightarrow{\varphi} V_0$ be an inverse system of vector spaces (or vector space complexes). We then have the following exact sequence

$$0 \longrightarrow V_\infty = \varprojlim V_p \longrightarrow \prod_p V_p \xrightarrow{\Phi} \prod_q V_q \longrightarrow \varprojlim{}^1 V_p \longrightarrow 0$$

where $\Phi$ is defined by $\Phi(v_p) = v_p - \varphi(v_p) \in V_p \oplus V_{p-1}$ where $v_p$ is in $V_p$.

Recall two classical lemmas due to Mittag-Leffler:

**Lemma 4.3.1.** *If for all $p$, the sequence of vector spaces $W_i = \mathsf{Im}(V_{p+i} \to V_p)$ is stationary, then $\varprojlim{}^1 V_p$ vanishes.*

This happens in particular when all vector spaces $V_p$ are finite dimensional.

**Lemma 4.3.2.** *Let $\cdots \longrightarrow V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \longrightarrow V_1 \xrightarrow{\varphi} V_0$ be an inverse system of finite dimensional vector spaces such that $V_\infty = \varprojlim V_p$ is also finite dimensional. Let $V'_p$ be the image of $V_\infty$ in $V_p$. Then the sequence $V'_p$ is stationary and we have $V'_\infty = \varprojlim V'_p = V_\infty$.*

*Proof. (of theorem 4.3)* Let $X$ and $Y$ be objects of $\mathcal{D}^bB$ such that $\mathsf{Hom}_{\mathcal{D}^bB}(X,Y)$ is finite-dimensional. We will prove that there exists a local $\mathsf{per}\,B$-cover of $X$ relative to $Y$.

Let $P_\bullet : \ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0$ be a projective resolution of $X$. The complex $P_\bullet$ has components in $\mathsf{per}\,B$, and its homology vanishes in all degrees except in degree zero, where it is $X$. Let $P_{\leq n}$ and $P_{>n}$ be the natural truncations, and denote by $Tot(P)$ the total complex associated to $P_\bullet$. Then for all $n \in \mathbb{N}$, there is an exact sequence of dg $B$-modules:

$$0 \longrightarrow Tot(P_{\leq n}) \longrightarrow Tot(P) \longrightarrow Tot(P_{>n}) \longrightarrow 0$$

The complex $Tot(P)$ is quasi-isomorphic to $X$, and the complex $Tot(P_{\leq n})$ is in $\mathsf{per}\,B$. Moreover, $Tot(P)$ is the colimit of $Tot(P_{\leq n})$. Thus by definition, we have the following equalities

$$\begin{aligned} \mathcal{H}om^\bullet_B(Tot(P),Y) &= \mathcal{H}om^\bullet_B(\underrightarrow{\mathrm{colim}}\, Tot(P_{\leq n}),Y) \\ &= \varprojlim \mathcal{H}om^\bullet_B(Tot(P_{\leq n}),Y). \end{aligned}$$

Denote by $V_p$ the complex $\mathcal{H}om^\bullet_B(Tot(P_{\leq p}),Y)$. In the inverse system

$$\cdots \longrightarrow V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \longrightarrow V_1 \xrightarrow{\varphi} V_0 ,$$

all the maps are surjective. Thus by lemma 4.3.1, there is a short exact sequence

$$0 \longrightarrow V_\infty \longrightarrow \textstyle\prod_p V_p \xrightarrow{\Phi} \prod_q V_q \longrightarrow 0$$

which induces a long exact sequence in cohomology

$$\begin{array}{ccccccc} \cdots \prod_q H^{-1}V_q & \longrightarrow & H^0(V_\infty) & \longrightarrow & \prod H^0 V_p & \longrightarrow & \prod H^0 V_q \cdots . \\ \searrow & & \nearrow \quad \searrow & & \nearrow & & \\ & \varprojlim^1 H^{-1}V_p & & \varprojlim H^0 V_p & & & \end{array}$$

We have the equalities

$$\begin{aligned} H^0(V_\infty) &= H^0(\mathcal{H}om^\bullet_B(Tot(P),Y)) \\ &= \mathsf{Hom}_{\mathcal{H}}(Tot(P),Y) \\ &= \mathsf{Hom}_{\mathcal{D}}(X,Y). \end{aligned}$$

Denote by $W_p$ the complex $\mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}),Y)$ and by $U_p$ the complex $H^{-1}(V_p) = \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}),Y[-1])$. The vector spaces $U_p$ are finite dimensional. Thus by lemma 4.3.1, $\varprojlim^1 U_p$ vanishes and we have an isomorphism

$$H^0(\varprojlim V_p) = H^0(V_\infty) \simeq \varprojlim H^0(V_p).$$

The system $(W_p)_p$ satisfies the hypothesis of lemma 4.3.2. In fact, for each integer $p$, the space $\mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}), Y)$ is finite dimensional because $Tot(P_{\leq p})$ is in $\mathsf{per}\, B$. Moreover, by the last two equalities $W_\infty = \underleftarrow{\lim} W_p$ is isomorphic to $\mathsf{Hom}_{\mathcal{D}}(X, Y)$ which is finite dimensional by hypothesis. By lemma 4.3.2, the system $(W'_p)_p$ formed by the image of $W_\infty$ in $W_p$ is stationary. More precisely, there exists an integer $n$ such that $W'_n = \underleftarrow{\lim} W'_p$. But $W'_n$ is a subspace of $W_n = \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq n}), Y)$ and there is an injection

$$\mathsf{Hom}_{\mathcal{D}}(X, Y) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq n}), Y)\ .$$

This yields a local $\mathsf{per}\, B$-cover of $X$ relative to $Y$.

The spaces $\mathsf{Hom}_{\mathcal{D}}(N, X)$ and $\mathsf{Hom}_{\mathcal{D}}(X, N)$ are finite dimensional for $N$ in $\mathsf{per}\, B$ and $X$ in $\mathcal{D}^b B$. Thus by proposition 4.2.1, there exists local $\mathsf{per}\, B$ envelopes. Thus theorem 4.2 applies and $\beta'$ is non-degenerate.

□

By theorem 4.3, we can deduce the corollary:

**Corollary 4.4.** *Let $A$ be a finite dimensional $k$-algebra with finite global dimension. If the cluster category $\mathcal{C}_A$ is* $\mathsf{Hom}$*-finite, then it is 2-Calabi-Yau as a triangulated category.*

*Proof.* Denote by $p_* : \mathcal{D}^b A \to \mathcal{D}^b B$ the restriction of the projection $p : B \to A$. Let $X$ and $Y$ be in $\mathcal{D}^b(A)$. Then by hypothesis, the vector spaces

$$\bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b A}(X, \nu_A^p Y[-2p]) \quad \text{and} \quad \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b A}(Y, \nu_A^p X[-2p+3])$$

are finite dimensional. But by [Kel05], the space $\mathsf{Hom}_{\mathcal{D}^b B}(p_* X, p_* Y)$ is isomorphic to

$$\bigoplus_{p\geq 0} \mathsf{Hom}_{\mathcal{D}^b A}(X, \nu_A^p Y[-2p]),$$

so is finite dimensional. For the same reasons, the space $\mathsf{Hom}_{\mathcal{D}^b B}(Y, X[3])$ is also finite dimensional. Applying theorem 4.3, we get a non-degenerate bilinear form $\beta'_{p_*X, p_*Y}$. The non-degeneracy property is extension closed, so for each $M$ and $N$ in $\langle A \rangle_B$, the form $\beta'_{MN}$ is non-degenerate.

□

### 4.3.3 Relation with Tabuada's article

An article of G. Tabuada [Tab07] gives another example of such categories. Let $\mathcal{D}$ be an algebraic 2-Calabi-Yau category endowed with a cluster-tilting object. The author constructs a triangulated category $\mathcal{T}$ and a triangulated 3-Calabi-Yau subcategory $\mathcal{N}$ such that the quotient category $\mathcal{T}/\mathcal{N}$ is triangle equivalent to $\mathcal{D}$.

More precisely, let $\mathcal{C}$ be a cluster-tilting object of $\mathcal{D}$. The category $\mathcal{D}$ is algebraic, therefore it is the stable category $\underline{\mathcal{E}}$ of a certain Frobenius category $\mathcal{E}$. Denote by $\mathcal{M}$ the preimage of $\mathcal{C}$ under the projection $\mathcal{E} \to \underline{\mathcal{E}}$. The category $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of projective-injectives. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{C} & \hookrightarrow & \underline{\mathcal{E}} = \mathcal{D}. \end{array}$$

Denote by $\mathcal{HE}$ the homotopic category of $\mathcal{E}$, and let $\mathcal{T} = \mathcal{H}^{\mathcal{P}-\mathrm{b}}_{\mathcal{E}-\mathrm{ac}}$ be the full subcategory of $\mathcal{HE}$ of $\mathcal{E}$-acyclic complexes of the form

$$\cdots P_{i-2} \longrightarrow P_{i-1} \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where the $P_j$ are in $\mathcal{P}$ and where the $M_j$ are in $\mathcal{M}$. Denote by $\mathcal{N} = \mathcal{H}^b_{\mathcal{E}-\mathrm{ac}}$ the subcategory of $\mathcal{HE}$ of the $\mathcal{E}$-acyclic complexes which are bounded and which have components in $\mathcal{M}$. Then G. Tabuada shows the following result:

**Theorem 4.5.** *[Tab07] The category $\mathcal{H}^b_{\mathcal{E}-\mathrm{ac}}$ is 3-Calabi-Yau and there exists a triangle equivalence between $\mathcal{D}$ and the quotient $\mathcal{H}^{\mathcal{P}-\mathrm{b}}_{\mathcal{E}-\mathrm{ac}}/\mathcal{H}^b_{\mathcal{E}-\mathrm{ac}}$.*

The categories $\mathcal{T} = \mathcal{H}^{\mathcal{P}-\mathrm{b}}_{\mathcal{E}-\mathrm{ac}}$ and $\mathcal{N} = \mathcal{H}^b_{\mathcal{E}-\mathrm{ac}}$ satisfy the hypothesis of theorem 4.2. Indeed, let $X$ and $Y$ be objects in $\mathcal{H}^{\mathcal{P}-\mathrm{b}}_{\mathcal{E}-\mathrm{ac}}$. We can assume that for $n \geq 1$ the $X_n$ are projective-injective, and that for $n \leq 0$, the $Y_n$ are projective-injective.

$$Y = \cdots \longrightarrow Q_{-2} \longrightarrow Q_{-1} \longrightarrow Q_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \cdots, \qquad Q_0 \twoheadrightarrow K \rightarrowtail N_1$$

Denote by $K$ the kernel of the morphism $N_1 \to N_2$. Since $\mathcal{M}$ is a cluster-tilting subcategory of $\mathcal{E}$ there exists an admissible short exact sequence:

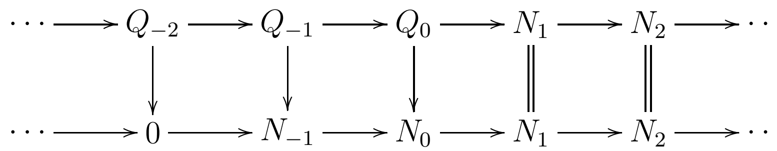
$$0 \longrightarrow N_{-1} \longrightarrow N_0 \longrightarrow K \longrightarrow 0$$

with $N_0$ and $N_{-1}$ in $\mathcal{M}$.

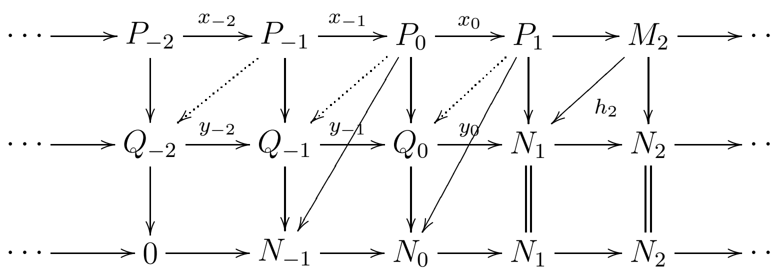
Let $N$ be the following complex:

$$N = \cdots \longrightarrow 0 \longrightarrow N_{-1} \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \cdots$$

This complex $N$ is acyclic and bounded, so it belongs to $\mathcal{N} = \mathcal{H}^b_{\mathcal{E}-\mathrm{ac}}$. Moreover since $Q_0$ and $Q_{-1}$ are projective, there exists a morphism of complexes:



Let us show that this morphism $Y \to N$ is a local $\mathcal{N}$-envelope of $Y$ relative to $X$. Let $f : X \to Y$ be a morphism of complexes such that the composition $X \to Y \to N$ is homotopic to zero.



Since $P_1$ is projective, it is possible to find a morphism $h_1 : P_1 \to Q_0$ such that $f_1 = y_0 h_1 + h_2 x_1$. Then since all $P_i$ and $Q_i$ are projective for $i \leq 0$, it is classical to construct an homotopy (see also [Pal08]). Thus there exist local $\mathcal{N}$-envelopes in $\mathcal{T}$.

Since the category $\mathcal{T} = \mathcal{H}_{\mathcal{E}-\mathrm{ac}}^{\mathcal{P}-\mathrm{b}}$ is $\mathsf{Hom}$-finite, by proposition 4.2.1 there exist local $\mathcal{N}$-covers.

# Chapter 5

# Cluster category of an algebra of global dimension 2

Let $A$ be a finite dimensional $k$-algebra with global dimension $\leq 2$. Since $A$ is of finite global dimension, the triangulated category $\mathcal{D}^b(\mathsf{mod}\, A)$ admits a Serre functor $\nu = ?\overset{L}{\otimes}_A DA$ (cf. chapter 1). In this chapter, we want to study the cluster category $\mathcal{C}_A$ associated to this algebra, namely the triangulated hull of the orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\nu[-2]$. We will still denote by $A$ the image of $A$ under the functor

$$\mathcal{D}^b(\mathsf{mod}\, A) \longrightarrow \mathcal{D}^b(\mathsf{mod}\, A)/\nu[-2] \hookrightarrow \mathcal{C}_A$$

## 5.1 Reminder on $t$-structures

The derived category $\mathcal{D}^b(\mathsf{mod}\, A) = \mathcal{D}^b A$ admits a $t$-structure. For each $n \in \mathbb{Z}$, let $\mathcal{D}_{\leq n}$ be the full subcategory of $\mathcal{D}^b A$ consisting of complexes whose cohomology vanishes in degrees $> n$, and $\mathcal{D}_{\geq n}$ the full subcategory of $\mathcal{D}^b A$ consisting of complexes whose cohomology vanishes in degrees $< n$. The categories $\mathcal{D}_{\leq n}$ and $\mathcal{D}_{\geq n}$ satisfy the following properties:

- $\mathcal{D}_{\leq 0}[n] = \mathcal{D}_{\leq -n}$ and $\mathcal{D}_{\geq 0}[n] = \mathcal{D}_{\geq -n}$;
- $\mathcal{D}_{\leq 0} \subset \mathcal{D}_{\leq 1}$, $\mathcal{D}_{\geq 1} \subset \mathcal{D}_{\geq 0}$;
- $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 1}) = 0$,
- for each object $Y$ in $\mathcal{D}^b A$, there exists a unique (up to unique triangle isomorphism) triangle

  $$\tau_{\leq 0} Y \longrightarrow Y \longrightarrow \tau_{\geq 1} Y \longrightarrow (\tau_{\leq 0} Y)[1]$$

  such that $\tau_{\leq 0} Y \in \mathcal{D}_{\leq 0}$ and $\tau_{\geq 1} Y \in \mathcal{D}_{\geq 1}$.

**Lemma 5.1.1.** *Let $A$ be a finite dimensional $k$-algebra with global dimension $\leq 2$. Denote by $\nu = ? \overset{L}{\otimes}_A DA$ the Serre functor of the derived category $\mathcal{D}^b A$. Then we have the following inclusions $\nu(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq -2}$ and $\nu^{-1}(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 2}$. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(U,V)$ vanishes for all $U$ in $\mathcal{D}_{\geq 0}$ and all $V$ in $\mathcal{D}_{\leq -3}$.*

# 5.2 Endomorphism algebra of the object $A$

## 5.2.1 Endomorphism algebra

In this section, we study the endomorphism algebra of the object $A$ in the cluster category $\mathcal{C}_A$.

**Proposition 5.2.1.** *Let $A$ be a finite dimensional $k$-algebra with global dimension $\leq 2$. Let $X$ be the $A$-$A$-bimodule $\mathsf{Ext}^2_A(DA, A)$. Then there is an isomorphism*

$$\widetilde{A} = \mathsf{End}_{\mathcal{C}_A}(A) \simeq T_A X$$

*where $T_A X$ is the tensor algebra of $X$ over $A$.*

*Proof.* By definition, we know that $\mathsf{End}_{\mathcal{C}_A}(A) = \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p])$. For $p \geq 1$, the object $\nu^p A[-2p]$ is in $\mathcal{D}_{\geq 2}$, so since $A$ is in $\mathcal{D}_{\leq 0}$, the space $\mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p])$ vanishes. Therefore, we can write $\mathsf{End}_{\mathcal{C}_A}(A) = \bigoplus_{p\geq 0} \mathsf{Hom}_{\mathcal{D}}(A, \nu^{-p} A[2p])$.

The functor $\nu = ? \overset{L}{\otimes}_A DA$ admits an inverse

$$\nu^{-1} = - \overset{L}{\otimes}_A \; R\mathcal{H}om_A(DA, A).$$

Since the global dimension of $A$ is $\leq 2$, the homology of the complex $R\mathcal{H}om_A(DA, A)$ is concentrated in degrees 0, 1 and 2 :

$$\begin{array}{rcl} H^0(R\mathcal{H}om_A(DA,A)) & = & \mathsf{Hom}_{\mathcal{D}}(DA,A) \\ H^1(R\mathcal{H}om_A(DA,A)) & = & \mathsf{Ext}^1_A(DA,A) \\ H^2(R\mathcal{H}om_A(DA,A)) & = & \mathsf{Ext}^2_A(DA,A) \end{array}$$

Let us denote by $Y$ the complex $R\mathcal{H}om_A(DA, A)[2]$. Then we have

$$\nu^{-p} A[2p] = A \overset{L}{\otimes}_A (Y^{\overset{L}{\otimes}_A p}) = Y^{\overset{L}{\otimes}_A p}.$$

Therefore we get the following equalities

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}A}(A, \nu^{-p} A[-2p]) & = & \mathsf{Hom}_{\mathcal{D}A}(A, Y^{\overset{L}{\otimes}_A p}) \\ & = & H^0(Y^{\overset{L}{\otimes}_A p}). \end{array}$$

Since $H^0(Y) = X$, we conclude using the following lemma. □

**Lemma 5.2.1.** *Let $M$ and $N$ two complexes of $A$-modules whose homology is concentrated in negative degrees. Then there is an isomorphism*

$$H^0(M \overset{L}{\otimes}_A N) \simeq H^0(M) \otimes_A H^0(N).$$

*Proof.* Without loss of generality, we can assume that $M$ and $N$ are cofibrant. Then, we have $M \overset{L}{\otimes}_A N = M \otimes_A N$ and so we have

$$\begin{aligned} H^0(M \otimes_A N) &= \mathsf{Coker}(M^{-1} \otimes_A N^0 \oplus M^0 \otimes_A N^{-1} \to M^0 \otimes_A N^0) \\ &\simeq (\mathsf{Coker} d_M) \otimes_A (\mathsf{Coker} d_N) \\ &\simeq H^0(M) \otimes_A H^0(N). \end{aligned}$$

□

### 5.2.2 Quiver of the endomorphism algebra

Let $A = kQ/I$ be a finite dimensional $k$-algebra of global dimension $\leq 2$. Suppose that $I$ is an admissible ideal generated by a finite set of minimal relations $r_i$, $i \in J$ where for each $i \in J$, the relation $r_i$ starts at the vertex $s(r_i)$ and ends at the vertex $t(r_i)$. Let $\widetilde{Q}$ the following quiver:

- the set of the vertices of $\widetilde{Q}$ equals that of $Q$;
- the set of arrows of $\widetilde{Q}$ is obtained from that of $Q$ by adding a new arrow $\rho_i$ with source $t(r_i)$ and target $s(r_i)$ for each $i$ in $J$.

Then we have the following proposition, which has essentially been proved by I. Assem, T. Brüstle and R. Schiffler [ABS06] (thm 2.6). The proposition is also proved in [Kel08b].

**Proposition 5.2.2.** *If the algebra $\mathsf{End}_{\mathcal{C}_A}(A) = \widetilde{A}$ is finite-dimensional, then its quiver is $\widetilde{Q}$.*

*Proof.* Let $B$ be a finite dimensional algebra. The vertices of its quiver are determined by the quotient $B/rad(B)$ and the arrows are determined by $rad(B)/rad^2(B)$. Denote by $X$ the $A$-$A$-bimodule $\mathsf{Ext}^2_A(DA, A)$. Since $X \otimes_A X$ is in $rad^2(B)$, the quiver of $\tilde{A} = T_A X$ is the same as the quiver of the algebra $A \ltimes X$. Then the proof is exactly the same as in [ABS06] (thm 2.6).

□

## 5.3 Condition of Hom-finiteness

### 5.3.1 Criterion

**Theorem 5.1.** *Let $A$ be a finite-dimensional $k$-algebra of global dimension $\leq 2$. Then the cluster category $\mathcal{C}_A$ is* $\mathsf{Hom}$*-finite if and only if the functor* $\mathsf{Tor}^2_A(?, DA)$ *is nilpotent.*

**Lemma 5.3.1.** *The functor* $\mathsf{Tor}^2_A(?, DA)$ *is nilpotent if and only if the functor* $? \otimes_A \mathsf{Ext}^2_A(DA, A)$ *is nilpotent. This is equivalent to the existence of an integer $N$ such that* $\Phi^N(\mathcal{D}_{\geq 0})$ *is included in* $\mathcal{D}_{\geq 1}$ *where $\Phi$ is the functor* $\nu[-2]$ *of* $\mathcal{D}^b(A)$.

*Proof.* The functor $\Phi$ is the following endofunctor $\nu(?)[-2] = ?\overset{L}{\otimes}_A DA[-2]$ of $\mathcal{D}^b(A)$. Thus it is obvious that $\mathsf{Tor}^2_A(?, DA)$ is the functor $H^0(\Phi(?))$. Since $A$ is of global dimension $\leq 2$, if $M$ is an $A$-module, the object $M \overset{L}{\otimes}_A DA[-2]$ lies in $\mathcal{D}_{\leq 0}$. Thus $H^0(\Phi H^0 \Phi(M)) = H^0\Phi^2(M)$ and we get for all $n \geq 0$

$$\mathsf{Tor}^2_A(?, DA)^n(M) \simeq H^0(\Phi^n M).$$

Using the formula $\nu^{-1} = - \overset{L}{\otimes}_A \; R\mathcal{H}om_A(DA, A)$, one can easily check that the functor $? \overset{L}{\otimes}_A \mathsf{Ext}^2_A(DA, A)$ is isomorphic to the functor $H^0(\Phi^{-1}(?))$. By lemma 5.2.1, we have for all $n \geq 0$,

$$M \overset{L}{\otimes}_A \mathsf{Ext}^2_A(DA, A)^{\overset{L}{\otimes}_A n} \simeq H^0(\Phi^{-n} M).$$

□

*Proof. (of theorem 5.1)* By proposition 5.2.1, the algebra $\widetilde{A} = \mathsf{End}_{\mathcal{C}_A}(A)$ is finite-dimensional if and only if the functor $? \otimes_A \mathsf{Ext}^2_A(DA, A)$ is nilpotent.

Now suppose that there exists some $N \geq 0$ such that $\Phi^N(\mathcal{D}_{\geq 0})$ is included in $\mathcal{D}_{\leq 1}$. For each object $X$ in $\mathcal{C}_A$, the class of the objects $Y$ such that the space $\mathsf{Hom}_{\mathcal{C}_A}(X, Y)$ (resp. $\mathsf{Hom}_{\mathcal{C}_A}(Y, X)$) is finite dimensional, is extension closed. Therefore, it is sufficient to show that for all simples $S$, $S'$, and each integer $n$, the space $\mathsf{Hom}_{\mathcal{C}_A}(S, S'[n])$ is finite dimensional.

There exists an integer $p_0$ such that for all $p \geq p_0$ $\Phi^p(S')$ is in $\mathcal{D}_{\geq n+1}$. Therefore, because of the defining properties of the $t$-structure, the space $\bigoplus_{p \geq p_0} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n])$ vanishes. Similary, there exists an integer $q_0$ such that for all $q \geq q_0$, we have $\Phi^q(S) \in \mathcal{D}_{\geq -n+3}$. Since the algebra $A$ is of global dimension $\leq 2$, the space $\bigoplus_{q \geq q_0} \mathsf{Hom}_{\mathcal{D}}(\Phi^q(S), S'[n])$ vanishes. Thus the space

$$\bigoplus_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n]) = \bigoplus_{p=-q_0}^{p_0} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n])$$

is finite dimensional. □

### 5.3.2 Examples

1. Let $Q$ be a Dynkin quiver. Let $A$ be its Auslander algebra. The algebra $A$ is of global dimension $\leq 2$. The category $\mathsf{mod}\, A$ is equivalent to the category $\mathsf{mod}\,(\mathsf{mod}\, kQ)$ of finitely presented functors $(\mathsf{mod}\, kQ)^{op} \to \mathsf{mod}\, k$. The projective indecomposables of $\mathsf{mod}\, A$ are the representable functors $U^\wedge = \mathsf{Hom}_{kQ}(?, U)$ where $U$ is an indecomposable $kQ$-module. Let $S$ be a simple $A$-module. Since $A$ is finite dimensional, this simple is associated to an indecomposable $U$ of $\mathsf{mod}\, kQ$. If $U$ is not projective, then it is easy to check that in $\mathcal{D}^b(A)$ the simple $S_U$ is isomorphic to the complex:

$$\dots \longrightarrow \underset{-3}{0} \longrightarrow \underset{-2}{(\tau U)^\wedge} \longrightarrow \underset{-1}{E^\wedge} \longrightarrow \underset{0}{U^\wedge} \longrightarrow \underset{1}{0} \longrightarrow \dots$$

where $0 \longrightarrow \tau U \longrightarrow E \longrightarrow U \longrightarrow 0$ is the Auslander-Reiten sequence associated to $U$. Thus $\Phi(S_U) = \nu S_U[-2]$ is the complex:

$$\dots \longrightarrow \underset{-1}{0} \longrightarrow \underset{0}{(\tau U)^\vee} \longrightarrow \underset{1}{E^\vee} \longrightarrow \underset{2}{U^\vee} \longrightarrow \underset{3}{0} \longrightarrow \dots$$

where $U^\vee$ is the injective $A$-module $D\mathsf{Hom}_{kQ}(U, ?)$. It follows from the Auslander-Reiten formula that this complex is quasi-isomorphic to the simple $S_{\tau U}$.

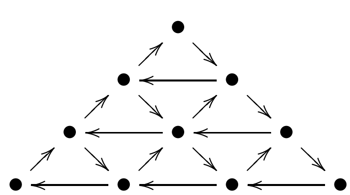
If $U$ is projective, then $S_U$ is isomorphic in $\mathcal{D}^b(A)$ to

$$\dots \longrightarrow \underset{-2}{0} \longrightarrow \underset{-1}{(radU)^\wedge} \longrightarrow \underset{0}{U^\wedge} \longrightarrow \underset{1}{0} \longrightarrow \dots,$$

and then $\Phi(S_U)$ is in $\mathcal{D}_{\geq 1}$. Since for each indecomposable $U$ there is some $N$ such that $\tau^N U$ is projective, there is some $M$ such that $\Phi^M(\mathcal{D}_{\geq 0})$ is included in $\mathcal{D}_{\geq 1}$. By theorem 6.1, the cluster category $\mathcal{C}_A$ is $\mathsf{Hom}$-finite, and 2-Calabi-Yau by corollary 4.4.

The quiver of $A$ is the Auslander-Reiten quiver of $\mathsf{mod}\, kQ$. The minimal relations of the algebra $A$ are given by the mesh relations. Thus the quiver of $\tilde{A}$ is the same as that of $A$ in which arrows $\tau x \to x$ are added for each non projective indecomposable $x$.

For instance, if $Q$ is $A_4$ with the orientation $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, then the quiver of the algebra $\tilde{A}$ is



2. Let $A$ be the Auslander algebra of the algebra of dual numbers $k[X]/X^2$. This algebra is defined by the quiver

$$P \underset{b}{\overset{a}{\rightleftarrows}} S \circlearrowright \quad \text{with the relations} \quad ab = 0.$$

   This a finite dimensional algebra of global dimension 2 since it is an Auslander algebra. The vertex $P$ corresponds to the unique projective-injective $k[X]/X^2$-module, and $S$ corresponds to the simple. Moreover we have $\tau S = S$, so $\tau$ is of infinite order. In the derived category $\mathcal{D}^b(\operatorname{mod} A)$, we have $\Phi(S_S) = S_S$. Therefore the space of morphism in the cluster category $\operatorname{Hom}_{\mathcal{C}}(S_S, S_S)$ is infinite dimensional.

3. Let $A$ the following algebra:

$$\begin{array}{ccc} & 2 & \\ {}^{b}\nearrow & & \searrow^{a} \\ 1 & \xrightarrow[c]{} & 3 \end{array} \quad \text{with the relations} \quad ab = 0.$$

   This is clearly a finite dimensional algebra of global dimension 2. The projectives and the injectives have the following composition series:

$$P_1 = {\scriptstyle 1}, P_2 = {\scriptstyle \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}, P_3 = {\scriptstyle \begin{smallmatrix} & 3 & \\ 1 & & 2 \end{smallmatrix}}, I_1 = {\scriptstyle \begin{smallmatrix} 2 & & 3 \\ & 1 & \end{smallmatrix}}, I_2 = {\scriptstyle \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}}, \text{ and } I_3 = {\scriptstyle 3}.$$

   The $A$-module ${\scriptstyle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}}$ is quasi-isomorphic to the complex:

$$\cdots \longrightarrow 0 \longrightarrow {\scriptstyle 1} \longrightarrow {\scriptstyle \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}} \longrightarrow {\scriptstyle \begin{smallmatrix} & 3 & \\ 1 & & 2 \end{smallmatrix}} \longrightarrow 0 \longrightarrow \cdots$$

   Thus the object $\Phi({\scriptstyle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}})$ is quasi-isomorphic to:

$$\cdots \longrightarrow 0 \longrightarrow {\scriptstyle \begin{smallmatrix} 2 & & 3 \\ & 1 & \end{smallmatrix}} \longrightarrow {\scriptstyle \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}} \longrightarrow {\scriptstyle 3} \longrightarrow 0 \longrightarrow \cdots$$

   and so to ${\scriptstyle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}}$. Therefore the space $\operatorname{Hom}_{\mathcal{C}}({\scriptstyle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}}, {\scriptstyle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}})$ is infinite-dimensional.

## 5.4 Cluster-tilting in the orbit category

### 5.4.1 Proof of the rigidity

**Proposition 5.4.1.** *Let $A$ be a finite dimensional $k$-algebra of global dimension $\leq 2$. Then the object $A$ is a rigid object in the category $\mathcal{C}_A$, in the sense that the space $\operatorname{Ext}^1_{\mathcal{C}_A}(A, A)$ vanishes.*

*Proof.* By definition, the space $\operatorname{Ext}^1_{\mathcal{C}_A}(A, A)$ is isomorphic to $\bigoplus_{p \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1])$. We show that for all $p$ in $\mathbb{Z}$, the space $\operatorname{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1])$ vanishes.

For $p = 0$, the space $\operatorname{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1]) = \operatorname{Hom}_{\mathcal{D}}(A, A[1])$ obviously vanishes.

Let $p$ be $\geq 1$. Then, since $\nu A = DA$ is in $\mathsf{mod}\, A$, $\nu A$ is in $\mathcal{D}_{\geq 0}$. Thus by lemma 5.1.1, the object $\nu^p A$ lies in $\mathcal{D}_{\geq -2(p-1)}$. Finally, the object $\nu^p A[-2p+1]$ is in $\mathcal{D}_{\geq 1}$. But, since $A$ is in $\mathcal{D}_{\leq 0}$, the space $\mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1])$ vanishes.

Now, let $p = -q$ be $\leq -1$. Then we have $\mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1]) = \mathsf{Hom}_{\mathcal{D}}(\nu^q A, A[2q+1])$. The object $\nu^q A$ is in $\mathcal{D}_{\geq -2q+2}$ and the object $A[2q+1]$ is in $\mathcal{D}_{\geq 2q+1}$. Since the space $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq -3})$ is zero, the space $\mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p+1])$ vanishes. □

### 5.4.2 Orbit-cluster-tilting

**Definition 5.2.** Let $A$ be a finite dimensional $k$-algebra of global dimension $\leq 2$. A rigid object $T$ of the orbit category will be called *orbit-cluster-tilting* if for all $X$ in the orbit category $\mathcal{D}^b A/\Phi$, the space $\mathsf{Ext}^1_{\mathcal{C}_A}(T, X)$ vanishes if and only if $X$ is in $add(T)$.

**Proposition 5.4.2.** *Let $A$ be a finite dimensional $k$-algebra with global dimension $\leq 2$. If the functor $\mathsf{Tor}_2^A(?, DA)$ is nilpotent, then the object $A$ is orbit-cluster-tilting.*

*Remark.* We will show in chapter 7, that under the hypotheses of the proposition, the object $A$ is in fact cluster-tilting in the triangulated hull of the orbit category.

*Proof.* Denote by $\Phi$ the auto-equivalence $\nu[-2]$ of $\mathcal{D} = \mathcal{D}^b(\mathsf{mod}\, A)$. Let $X$ be an object in $\mathcal{D}^b(\mathsf{mod}\, A)$ such that $\mathsf{Ext}^1_{\mathcal{C}_A}(A, X)$ vanishes. We have to show that $X$ lies in $add(A)$, *i.e.* that $X$ is a direct sum of objects of the form $\Phi^n(P)$, where $n$ is an integer and $P$ is a projective module.

The space $\mathsf{Ext}^1_{\mathcal{C}_A}(A, X)$ vanishes, so for each integer $n$ in $\mathbb{Z}$, the space $\mathsf{Hom}_{\mathcal{D}}(A, \Phi^n(X)[1])$ vanishes. In other words, for each $n$ we have $H^1(\Phi^n(X)) = 0$. By hypothesis, there exists an integer $N$ such that $\Phi^N(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 1}$ for the $t$-structure of $\mathcal{D}^b(\mathsf{mod}\, A)$. Therefore we can assume that $X$ lies in $\mathcal{D}_{\geq 0}$. Look at the canonical triangle:

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \xrightarrow{w} \tau_{\leq 0}X[1]\ .$$

Since $H^1(X)$ is zero, the complex $\tau_{\geq 1}X$ lies in $\mathcal{D}_{\geq 2}$. The global dimension of $A$ is $\leq 2$ and so $\tau_{\leq 0}X[1]$ lies in $\mathcal{D}_{\leq -1}$ Thus by lemma 5.1.1, the morphism $w$ is zero. Therefore we can write $X = \tau_{\leq 0}X \oplus \tau_{\geq 1}X$.

Since $X$ is in $\mathcal{D}_{\geq 0}$, $\tau_{\geq 0}X$ is the module $H^0(X)$. The space $\mathsf{Ext}^1_{\mathcal{C}_A}(A, H^0(X))$ is a direct summand of $\mathsf{Ext}^1_{\mathcal{C}_A}(A, X)$, so it vanishes. Therefore we have

$$\mathsf{Hom}_{\mathcal{D}}(A, \Phi(H^0(X))[1]) = 0.$$

But this space is $\mathsf{Hom}_{\mathcal{D}}(A, \nu H^0(X)[-1]) = \mathsf{Hom}_{\mathcal{D}}(A, \tau H^0(X))$. Thus $H^0(X)$ is a projective module, which may be zero.

Let $Y$ be the complex $\Phi^{-1}(\tau_{\geq 1}X)$. The complex $\tau_{\geq 1}X$ lies in $\mathcal{D}_{\geq 2}$, and so is $\nu^{-1}X$. Therefore $Y = \nu^{-1}\tau_{\geq 1}X[2]$ lies in $\mathcal{D}_{\geq 0}$. On the other hand, since $\tau_{\geq 1}X$ is a direct summand of $X$, for each integer $n$, the space $\mathsf{Hom}_{\mathcal{D}}(A, \Phi^n(\tau_{\geq 1}X)[1])$ vanishes. Thus for each $n$,

$\mathsf{Hom}_{\mathcal{D}}(A, \Phi^n(Y)[1])$ is zero. The complex $Y$ satisfies the same hypothesis as $X$ and we have the decomposition $Y = H^0(Y) \oplus \tau_{\geq 1} Y$ where $H^0(Y)$ is a projective module (possibly zero).

Finally, we have

$$\tau_{\geq 1} X = \Phi(H^0 \Phi^{-1}(\tau_{\geq 1} X)) \oplus \Phi \tau_{\geq 1} \Phi^{-1} \tau_{\geq 1} X.$$

We can continue and apply this process to $\Phi \tau_{\geq 1} \Phi^{-1} \tau_{\geq 1} X$, and finally we get, by an obvious induction, the decomposition:

$$X = H^0(X) \oplus \Phi(H^0 \Phi^{-1}(\tau_{\geq 1} X)) \oplus \Phi^2(H^0(\Phi^{-1} \tau_{\geq 1} \Phi^{-1} \tau_{\geq 1} X)) \oplus \cdots .$$

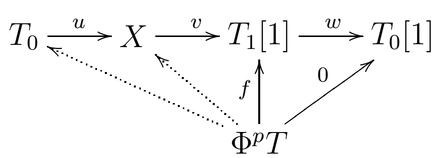
The decomposition ends since $X$ is the sum of finitely many indecomposables. □

**Proposition 5.4.3.** *Let $T$ be an orbit-cluster-tilting object of the cluster category $\mathcal{C}_A$. Then for each $X$ in the orbit category $\mathcal{D}^b(A)/\Phi$, there exists a triangle in $\mathcal{C}_A$*

$$T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow T_1[1]$$

*with $T_0$ and $T_1$ in $add(T)$.*

*Proof.* Let $X$ be an object in $\mathcal{D}^b A$. Let $t$ be the full subcategory of $\mathcal{D}^b A$ generated by the $\Phi^p T$ where $p$ is in $\mathbb{Z}$. The object $T$ is orbit-cluster-tilting, so the subcategory $add(t)$ of $\mathcal{D}^b(A)$ is a 2-cluster-tilting subcategory of $\mathcal{D}^b(A)$ in the sens of Iyama and Yoshino (see [IY06]). Let $T_0 \to X$ be a right $add(t)$-approximation in $\mathcal{D}^b A$. Denote by $T_1[1]$ the cone of this morphism. For $p$ in $\mathbb{Z}$, let $f : \Phi^p T \to T_1[1]$ be a morphism in $\mathcal{D}^b A$. We have the following diagram in $\mathcal{D}^b A$




The composition $w \circ f$ vanishes, because $T$ is a rigid object in $\mathcal{C}_A$. Thus $f$ factorizes through $v$. But since $u$ is an $add(t)$-approximation, this factorization factorizes through $u$. Thus $f$ is zero and $T_1$ is in $add(t)$. The image of this triangle through the triangle functor

$$\mathcal{D}^b(A) \to \mathcal{D}^b(A)/\Phi \to \mathcal{C}_A$$

gives the result. □

**Lemma 5.4.1.** *Let $\mathcal{C}$ be a 2-Calabi-Yau category endowed with a cluster-tilting object $R$. Let $A$ be a finite dimensional $k$-algebra of global dimension $\leq 2$ with an orbit-cluster-tilting object $T$. Assume that $F : \mathcal{C}_A \to \mathcal{C}$ is an algebraic triangle functor which induces an equivalence between $add(T)$ and $add(R)$. Then $F$ is triangle equivalence.*

*Proof.* Look at the following diagram:

$$\begin{array}{ccccc} \mathcal{D}^bA/\Phi & & \xrightarrow{F\circ i} & & \mathcal{C} \\ & \searrow^{i} & & \nearrow^{F} & \\ \uparrow & & \mathcal{C}_A & & \uparrow \\ add(T) & & \xrightarrow{\sim} & & add(R) \end{array}$$

First we show that the functor $F$ is essentially surjective. In fact, let $X$ be in $\mathcal{C}$, there exists a triangle in $\mathcal{C}$

$$R_1 \xrightarrow{r} R_0 \longrightarrow X \longrightarrow R_1[1] .$$

Since $F$ induces an equivalence between $add(T)$ and $add(R)$, the cone of $F^{-1}r$ yields an object which is sent on $X$.

Now let $X$ and $Y$ be objects in the orbit category $\mathcal{D}^bA/\Phi$. There exist triangles in $\mathcal{C}_A$

$$T_1 \xrightarrow{t} T_0 \longrightarrow X \longrightarrow T_1[1] \quad \text{and} \quad S_1 \xrightarrow{s} S_0 \longrightarrow Y \longrightarrow S_1[1]$$

where $S_i$ and $T_i$ are in $add(T)$. These triangles are just images through of the previous triangles in $\mathcal{D}^bA$. The space of morphisms $\mathsf{Hom}_{\mathcal{C}_A}(X,Y)$ is then isomorphic to the space of commutative squares

$$\begin{array}{ccc} T_1 & \xrightarrow{t} & T_0 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{s} & S_0 \end{array}$$

Thus it is isomorphic to the space of commutative squares of the form:

$$\begin{array}{ccc} FT_1 & \xrightarrow{Ft} & FT_0 \\ \downarrow & & \downarrow \\ FS_1 & \xrightarrow{Fs} & FS_0. \end{array}$$

which is clearly isomorphic to the space of morphisms $\mathsf{Hom}_{\mathcal{C}}(FX,FY)$ in $\mathcal{C}$. This proves that the functor $F\circ i$ is fully faithful. The functor $F$ is the 'triangulated hull' of the functor $F\circ i$. It stays fully faithful. Thus $F$ is an triangulated equivalence. $\square$

# Chapter 6

# Particular case coming from preprojective algebras

## 6.1 Definition and first properties

Let $Q$ be a finite quiver without oriented cycles with set of vertices $\{1, \cdots, n\}$. Let $T$ be a preinjective tilting $kQ$-module. We denote by $B$ the endomorphism algebra $\mathsf{End}_{kQ}(T)$. The algebra $B$ is a *concealed* algebra in the sense of Ringel [Rin84]. By Happel [Hap87], there is a triangle equivalence:

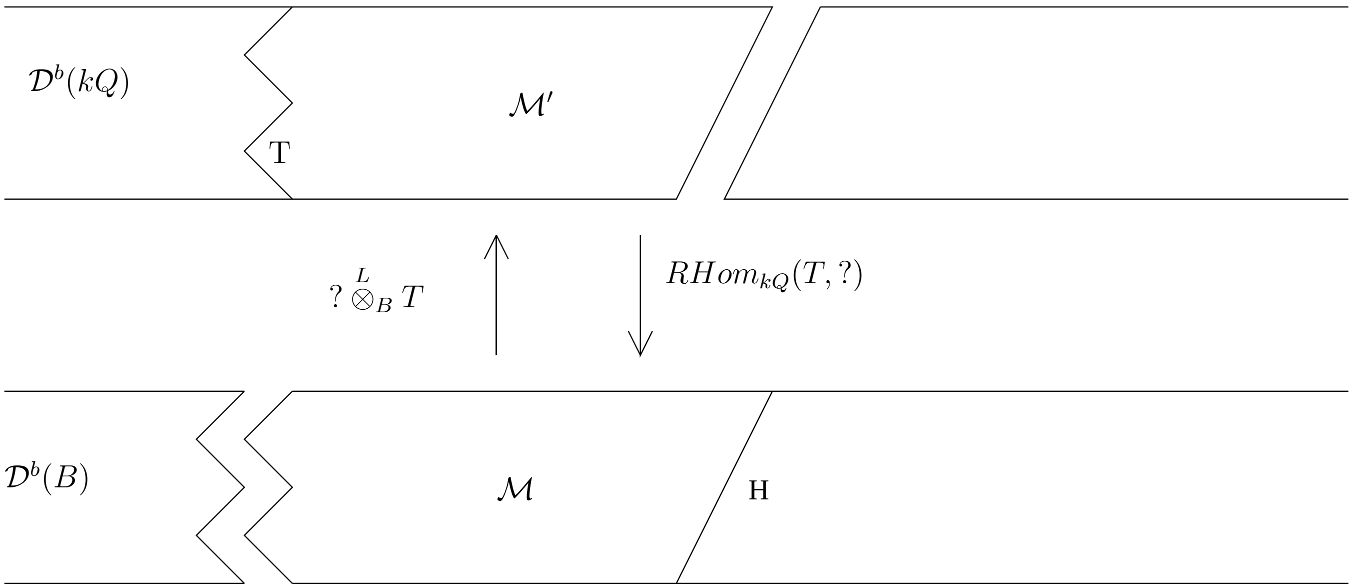
$$\mathcal{D}^b(\mathsf{mod}\, kQ) \underset{?\overset{L}{\otimes}_B T}{\overset{R\mathsf{Hom}_{kQ}(T,?)}{\rightleftarrows}} \mathcal{D}^b(\mathsf{mod}\, B).$$

We denote by $H$ the image of the injective module $DkQ$ under the functor $R\mathsf{Hom}_{kQ}(T, ?)$. The $B$-module $H$ is a 'slice' in the postprojective component of $\mathsf{mod}\, B$. Define $\mathcal{M}'$ and $\mathcal{M}$ as the following subcategories:

$$\mathcal{M}' = \{Y \in \mathsf{mod}\, kQ \mid \mathsf{Ext}^1_{kQ}(T, Y) = 0\} = \{Y \in \mathsf{mod}\, kQ \mid Y \text{ is generated by } T_{kQ}\}$$

and

$$\mathcal{M} = \{X \in \mathsf{mod}\, B \mid \mathsf{Ext}^1_B(X, H) = 0\} = \{X \in \mathsf{mod}\, B \mid X \text{ is cogenerated by } H_B\}$$




Let $\tau_B$ be the Auslander-Reiten translation of $\mathsf{mod}\,B$, and $\tau_{\mathcal{D}}$ the Auslander-Reiten translation of $\mathcal{D}^b(\mathsf{mod}\,kQ) \simeq \mathcal{D}^b(\mathsf{mod}\,B)$. Note that $\tau_{kQ}$ and $\tau_{\mathcal{D}}$ become isomorphic when we restrict them to the full subcategory of $kQ$-modules without non zero projective direct factors. But in general, $\tau_B$ and $\tau_{\mathcal{D}}$ are not isomorphic when restricted to the full subcategory of $B$-modules without non zero projective direct factors. If $X$ is an object of $\mathcal{M}$, then $X$ has projective dimension 1, so $X$ is isomorphic in $\mathcal{D}^b(\mathsf{mod}\,B)$ to a complex of the form:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

Thus $\tau_{\mathcal{D}}X$ is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \nu_B P_1 \longrightarrow \nu_B P_0 \longrightarrow 0 \longrightarrow \cdots$$

where $\nu_B$ is the Serre functor of the category $\mathsf{mod}\,B$. By definition, $\tau_B X$ is the kernel of $\nu_B P_1 \to \nu_B P_0$, so it is isomorphic in $\mathcal{D}^b(\mathsf{mod}\,B)$ to $H^0(\tau_{\mathcal{D}}X)$. Therefore there is a non trivial morphism $\tau_B X \to \tau_{\mathcal{D}}X$.

The following proposition is a classical result in tilting theory (see for example [Rin84]).

**Proposition 6.1.1.** *1. For each $X$ in $\mathcal{M}$ there exists a triangle*

$$X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X[1]$$

*in $\mathcal{D}^b(\mathsf{mod}\,B)$ functorial in $X$ with $H_0$ and $H_1$ in add$(H)$;*

2. *the functor $\mathsf{Hom}_{kQ}(T,?)$ induces an equivalence between the exact categories $\mathcal{M}'$ and $\mathcal{M}$;*

3. *$\mathcal{M}$ seen as a subcategory of $\mathsf{mod}\,B$, is closed under kernels so in particular, $\mathcal{M}$ is closed under $\tau_B$;*

4. *for each indecomposable $X$ in $\mathcal{M}$ there exists a unique $q \geq 0$ such that $\tau_B^{-q} X$ is in add$(H)$;*

5. *the category $\mathcal{M}$ has finitely many indecomposables.*

*Proof.* 1. Since $\mathcal{M}$ is the subcategory of $B$-modules cogenerated by $H$, we have just to see that the triangle $X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X[1]$ is functorial in $X$. This comes from the fact that $kQ$-modules admit functorial injective resolutions.

2. this is classical [Rin84][Theorem of Brenner-Butler, p. 170];

3. see [Rin84][(1), p. 179];

4. The indecomposable direct factors of $H$ form a slice in the postprojective component of the Auslander-Reiten quiver of $B$. Therefore, they contain a unique vertex of each $\tau_B^{-1}$-orbit of this component;

5. this is obvious since $H$ is a post-projective slice of $\mathsf{mod}\, B$.

□

### 6.1.1 Hom-finiteness

Let $\overline{\mathcal{M}}$ be the quotient $\mathcal{M}/add(H)$. Denote by $p : \mathcal{M} \to \overline{\mathcal{M}}$ the canonical projection. Since $H$ is a slice, we have the following properties.

**Proposition 6.1.2.** *1. The category $\overline{\mathcal{M}}$ is equivalent to the full subcategory of $\mathcal{M}$ whose objects do not have non zero direct factors in add$(H)$. We denote by $i : \overline{\mathcal{M}} \to \mathcal{M}$ the associated inclusion.*

*2. The category $\overline{\mathcal{M}} \subset \mathsf{mod}\, B$ is closed under kernels, and hence under $\tau_B$.*

*3. The right exact functor $i : \mathsf{mod}\,\overline{\mathcal{M}} \to \mathsf{mod}\,\mathcal{M}$ induced by $i : \overline{\mathcal{M}} \to \mathcal{M}$ is isomorphic to the restriction along $p$.*

**Proposition 6.1.3.** *Let $A$ be the endomorphism algebra $\mathsf{End}_B(\bigoplus_{M \in \mathsf{ind}\,\overline{\mathcal{M}}} M)$. Then the global dimension of $A$ is at most 2.*

*Proof.* Let $F$ be the following functor

$$\begin{array}{ccc} \mathsf{mod}\,(\overline{\mathcal{M}}) & \longrightarrow & \mathsf{mod}\, A \\ F & \mapsto & F(\bigoplus_{M \in \mathsf{ind}\,\overline{\mathcal{M}}} M). \end{array}$$

It is right exact and induces an equivalence between the projectives. In fact, the indecomposable projectives of $\mathsf{mod}\,\overline{\mathcal{M}}$ are the representable functors $M^\wedge = \mathsf{Hom}_B(?, M)_{|\overline{\mathcal{M}}}$, where $M$ is an indecomposable of $\overline{\mathcal{M}}$. But their image through $F$ is exactly the direct summand of $A$ seen as right $A$-module. Thus $F$ is an equivalence of category.

Now let $F$ be in $\mathsf{mod}\,\overline{\mathcal{M}}$, it is of finite type. Thus there is an exact sequence:

$$U^\wedge \xrightarrow{\varphi} V^\wedge \longrightarrow F \longrightarrow 0,$$

where $U$ and $V$ are in $\overline{\mathcal{M}}$. By Yoneda's lemma, the morphism $\varphi$ comes from a morphism $f : U \to V$ in $\overline{\mathcal{M}}$. The category $\overline{\mathcal{M}}$ is stable under kernels, so the following sequence

$$0 \longrightarrow (\mathsf{Ker} f)^\wedge \longrightarrow U^\wedge \xrightarrow{f^\wedge} V^\wedge \longrightarrow F \longrightarrow 0$$

is exact and gives a projective resolution of length 2 of the module $F$. □

**Theorem 6.1.** *The cluster category $\mathcal{C}_A$ is a* $\mathsf{Hom}$*-finite, 2-Calabi-Yau category, and the object $A$ is an orbit-cluster-tilting object in $\mathcal{C}_A$.*

*Remark.* We will show in chapter 7 that the image of $A$ in $\mathcal{C}_A$ is in fact a cluster-tilting object.

*Proof.* Using corollary 4.4, theorem 5.1 and proposition 5.4.2, we have just to check that the functor $\mathsf{Tor}^2_A(?, DA)$ is nilpotent.

Denote by $\mathcal{D}$ the derived category $\mathcal{D}^b(\mathsf{mod}\,A)$ and $\Phi$ the endofunctor $\nu \circ [-2]$. We have to show the existence of a positive integer $N$ such that $\Phi^N(\mathcal{D}_{\geq 0})$ is included in $\mathcal{D}_{\geq 1}$.

Let $U$ be an indecomposable of $\overline{\mathcal{M}}$. Denote by $S_U$ the simple $A$-module associated to $U$. If $U$ is not projective, then look at the Auslander-Reiten sequence associated to $U$:

$$0 \longrightarrow \tau_B U \longrightarrow E \longrightarrow U \longrightarrow 0.$$

Then the simple $S_U$ is isomorphic in $\mathcal{D}$ to the complex

$$\ldots \longrightarrow \underset{-3}{0} \longrightarrow \underset{-2}{(\tau_B U)^\wedge} \longrightarrow \underset{-1}{E^\wedge} \longrightarrow \underset{0}{U^\wedge} \longrightarrow \underset{1}{0} \longrightarrow \ldots.$$

By proposition 6.1.2, $E$ and $\tau_B U$ are also in $\overline{\mathcal{M}}$. Thus this complex has projective components. Therefore the object $\nu S_U$ is isomorphic in $\mathcal{D}$ to the complex:

$$\ldots \longrightarrow \underset{-3}{0} \longrightarrow \underset{-2}{(\tau_B U)^\vee} \longrightarrow \underset{-1}{E^\vee} \longrightarrow \underset{0}{U^\vee} \longrightarrow \underset{1}{0} \longrightarrow \ldots$$

where $U^\vee$ is the injective $A$-module $D\mathsf{Hom}_B(U,-)_{|\bar{\mathcal{M}}}$. Since $0 \longrightarrow \tau_B U \longrightarrow E \longrightarrow U \longrightarrow 0$ is an Auslander-Reiten sequence, this complex is exact except in degree $-2$ where its homology is $S_{\tau_B U}$. Therefore we get an isomorphism between $\nu S_U[-2] = \Phi(S_U)$ and $S_{\tau_B U}$.

If $U$ is projective, then $S_U$ is isomorphic in $\mathcal{D}$ to the complex

$$\dots \longrightarrow \underset{-2}{0} \longrightarrow \underset{-1}{(radU)^{\wedge}} \longrightarrow \underset{0}{U^{\wedge}} \longrightarrow \underset{1}{0} \longrightarrow \dots$$

Thus the object $\nu S_U[-2]$ is quasi-isomorphic to

$$\dots \longrightarrow \underset{0}{0} \longrightarrow \underset{1}{(radU)^{\vee}} \longrightarrow \underset{2}{U^{\vee}} \longrightarrow \underset{3}{0} \longrightarrow \dots$$

which lies in $\mathcal{D}_{\geq 1}$.

For each indecomposable $U$ in $\overline{\mathcal{M}}$, there exists an integer $N_U \geq 0$ such that $\tau_B^{N_U} U$ is a projective of $\mathsf{mod}\, B$. The category $\overline{\mathcal{M}}$ has only finitely many indecomposables, so for $N$ the maximum of the $N_U$, we get the inclusion $\Phi^N(S_U) \in \mathcal{D}_{\geq 1}$.

The algebra $A$ is finite dimensional, so all simples of $\mathsf{mod}\, A$ are of the form $S_U$. Moreover, each object $X$ of $\mathcal{D}_{\geq 0}$ is an iterated extension of shifts $S_U[-i]$, $i \geq 0$, of simples $S_U$, where $U$ is indecomposable in $\mathcal{M}$. Thus, for each object $X$ of $\mathcal{D}_{\geq 0}$, the object $\Phi^N(X)$ belongs to $\mathcal{D}_{\geq 1}$.

□

### 6.1.2 Construction of the functor $F : \mathsf{mod}\,\mathcal{M} \to \mathbf{f.l.}\Lambda$

Denote by $\mathcal{I}(kQ)$ the subcategory of the preinjective modules of $\mathsf{mod}\, kQ$.

**Proposition 6.1.4.** *There exists a k-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ unique up to isomorphism such that*

- *$P$ restricted to subcategory of the injective kQ-modules is isomorphic to the restriction functor $\mathsf{Hom}_{kQ}(T, ?)$;*
- *for each indecomposable $X$ in $\mathcal{I}(kQ)$ such that $P(X)$ is not projective, the image*

$$0 \longrightarrow P(\tau_{\mathcal{D}} X) \xrightarrow{Pi} P(E) \xrightarrow{Pp} P(X) \longrightarrow 0$$

*of an Auslander-Reiten sequence in $\mathsf{mod}\, kQ$ ending at $X$*

$$0 \longrightarrow \tau_{\mathcal{D}} X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

*is an Auslander-Reiten sequence in $\mathsf{mod}\, B$ ending at $P(X)$.*

*Moreover, this functor $P$ is full, essentially surjective, and satisfies $P \circ \tau_{\mathcal{D}} \simeq \tau_B \circ P$.*

*Proof.* The Auslander-Reiten quivers $\Gamma_{\mathcal{I}}$ of $\mathcal{I}(kQ)$ and $\Gamma_{\mathcal{M}}$ of $\mathcal{M}$ are connected translation quivers. Each vertex of $\Gamma_{\mathcal{I}}$ is of the form $\tau_{\mathcal{D}}^q x$ with $q \geq 0$ and $x$ indecomposable injective. Each vertex of $\Gamma_{\mathcal{M}}$ is of the form $\tau_B^q x$ where $x$ is in $add(H)$ (cf. (4) of prop 6.1.1). Moreover, there is a canonical isomorphism of quivers $\bar{P} : \Gamma_{DkQ} \to \Gamma_{add(H)}$. Thus we can construct inductively a morphism of quivers (that we will still denote by $\bar{P}$) $\bar{P} : \Gamma_{\mathcal{I}} \to \Gamma_{\mathcal{M}}$ extending $\bar{P}$ such that:

- $\bar{P}(\tau_{\mathcal{D}}x) = \tau_B \bar{P}(x)$ for each vertex $x$ of $\Gamma_{\mathcal{I}}$ ;
- $\bar{P}(\sigma_{\mathcal{D}}\alpha) = \sigma_B \bar{P}(\alpha)$ for each arrow $\alpha : x \to y$ of $\Gamma_{\mathcal{I}}$, where $\sigma_{\mathcal{D}}\alpha$ (resp. $\sigma_B\beta$) denotes the arrow $\tau_{\mathcal{D}}y \to x$ (resp. $\tau_B y \to x$) such that the mesh relations in $\Gamma_{\mathcal{I}}$ (resp. in $\Gamma_{\mathcal{M}}$) are of the form $\sum_{t(\alpha)=x} \sigma_{\mathcal{D}}(\alpha)\alpha$ (resp. $\sum_{t(\beta)=x} \sigma_B(\beta)\beta$).

Clearly, this morphism of translation quivers induces surjections in the sets of vertices and the sets of arrows.

The categories $\mathcal{I}(kQ)$ and $\mathcal{M}$ are standard, *i.e.* $k$-linearly equivalent to the mesh categories of their Auslander-Reiten quivers. Up to isomorphism, an equivalence $k(\Gamma_{\mathcal{I}}) \to \mathcal{I}(kQ)$ is uniquely determined by its restriction to a slice. Thus there exists a $k$-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ up to isomorphism which is equal to $\mathsf{Hom}_{kQ}(T, ?)$ on the slice of the injectives and such that the square

$$\begin{array}{ccc} k(\Gamma_{\mathcal{I}}) & \xrightarrow{\sim} & \mathcal{I}(kQ) \\ \downarrow{\scriptstyle \bar{P}} & & \downarrow{\scriptstyle P} \\ k(\Gamma_{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M} \end{array}$$

is commutative. This functor $P$ sends Auslander-Reiten sequences

$$0 \longrightarrow \tau_{\mathcal{D}}X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

to Auslander-Reiten sequences

$$0 \longrightarrow \tau_B P(X) \xrightarrow{Pi} P(E) \xrightarrow{Pp} P(X) \longrightarrow 0$$

if $P(X)$ is not projective. Since $\bar{P}$ is surjective, $P$ is full and essentially surjective. □

**Lemma 6.1.1.** *Let $X$ and $Y$ be indecomposables in $\mathcal{I}(kQ)$. The kernel of $\mathsf{Hom}_{kQ}(X,Y) \to \mathsf{Hom}_B(PX, PY)$ is generated by compositions of the form $X \longrightarrow Z \longrightarrow Y$ where $Z$ is indecomposable and $P(Z)$ is zero.*

*Proof.* If $P(X)$ or $P(Y)$ is zero this is obviously true. Suppose they are not. The mesh relations are minimal relations of the $k$-linear category $\mathcal{M}$ and $P$ is full. Thus the kernel of the functor $P$ is the ideal generated by the morphisms of the form

$$U \xrightarrow{g} V \xrightarrow{h} W$$

where

$$0 \longrightarrow P(U) \xrightarrow{Pg} P(V) \xrightarrow{Ph} P(W) \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathcal{M}$. Since $P(U)$ is isomorphic to $\tau_B P(W)$, the indecomposable $U$ is isomorphic to $\tau_{\mathcal{D}}(W)$. By the construction of $P$, $V$ is a direct factor of the middle term of the Auslander-Reiten sequence ending at $W$, and we can 'complete' the composition $\tau_{\mathcal{D}}W \xrightarrow{g} V \xrightarrow{h} W$ in an Auslander-Reiten sequence

$$0 \longrightarrow \tau_{\mathcal{D}}W \xrightarrow{\binom{g}{g'}} V \oplus V' \xrightarrow{(h\ h')} W \longrightarrow 0$$

with $P(V') = 0$ and $P(g') = P(h') = 0$. Thus, the morpism $hg = -h'g'$ does indeed factor through an object in the kernel of $P$. □

Now let $\Lambda$ be the preprojective algebra associated to $Q$. We denote by $e_i$ the idempotents of $\Lambda$ associated with the vertex $i$. Then we have a natural functor

$$\begin{array}{rcl} \mathsf{proj}\Lambda & \longrightarrow & \mathcal{I}^{\Pi}(kQ) \\ \Lambda e_i & \mapsto & \prod_{p\geq 0} \tau_{\mathcal{D}}^{p} I_i \end{array}$$

where $\mathcal{I}^{\Pi}(kQ)$ is the closure of $\mathcal{I}(kQ)$ under countable products. Composing this functor with the natural extension of $P$ to $\mathcal{I}^{\Pi}(kQ)$, we get a functor:

$$\begin{array}{rcl} \mathsf{proj}\Lambda & \longrightarrow & \mathcal{M} \\ e_i\Lambda & \mapsto & \bigoplus_{p\geq 0} \tau_{B}^{p} H_i. \end{array}$$

Therefore the restriction along this functor yields a functor $F : \mathsf{mod}\,\mathcal{M} \to \mathsf{mod}\,\Lambda$. Moreover, since $\mathcal{M}$ has finitely many indecomposables, the functor $F$ takes its values in the full subcategory f.l.$\Lambda$ formed by the $\Lambda$-modules of finite length.

This is an exact functor since it is a restriction functor. If $M$ is a $\mathcal{M}$-module, that is to say a finitely presented functor $\mathcal{M} \to \mathsf{mod}\,k$, then the vector space $F(M)e_j$ is isomorphic to $\bigoplus_{p\geq 0} M(\tau_B^p H_j)$. For $X$ in $\overline{\mathcal{M}}$, there exists $i \in Q_0$ and $q \geq 0$ such that $\tau^q H_i = X$. It is then easy to check that the image $F(S_X)$ of the simple associated to $X$ is the simple $\Lambda$-module $S_i$.

### 6.1.3 Fundamental propositions

**Proposition 6.1.5.** *For each $X$ in $\overline{\mathcal{M}}$, there exists a functorial sequence in* $\mathsf{mod}\,\Lambda$ *of the form*

$$0 \longrightarrow F \circ i_*(X^{\wedge}) \longrightarrow F(H_0^{\wedge}) \longrightarrow F(H_1^{\wedge}) \longrightarrow F \circ i_*(X^{\vee}) \longrightarrow 0$$

*where $i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to \mathsf{mod}\,\mathcal{M}$ is the right exact functor induced by $i : \overline{\mathcal{M}} \to \mathcal{M}$, and where $H_0$ and $H_1$ are in add($H$).*

*Proof.* Let $X$ be in $\overline{\mathcal{M}}$, and $iX$ its image in $\mathcal{M}$. By property 1 of proposition 6.1.1, there exists a triangle, functorial in $X$:

$$iX \longrightarrow H_0 \longrightarrow H_1 \longrightarrow (iX)[1]$$

with $H_0$ and $H_1$ in $add(H)$. It yields a long exact sequence in $\mathsf{mod}\,\mathcal{M}$:

$$0 \longrightarrow (iX)^\wedge \longrightarrow H_0^\wedge \longrightarrow H_1^\wedge \longrightarrow \mathsf{Ext}^1_B(?, iX)_{|\mathcal{M}} \longrightarrow \mathsf{Ext}^1_B(?, H_0)_{|\mathcal{M}} \longrightarrow \cdots.$$

By definition, the functor $\mathsf{Ext}^1_B(?, H_0)_{|\mathcal{M}}$ is zero. The Auslander-Reiten formula gives us an isomorphism

$$\mathsf{Ext}^1_B(?, iX)_{|\mathcal{M}} \simeq D\mathsf{Hom}_B(\tau_B^{-1} iX, ?)_{|\mathcal{M}}/\mathsf{proj}\,B.$$

Since $F$ is an exact functor, we get the following exact sequence in $f.l.\Lambda$:

$$0 \longrightarrow F((iX)^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B) \longrightarrow 0$$

By definition, we have $F((iX)^\wedge) \simeq (F \circ i_*)(X^\wedge)$. For $j = 1, \cdots, n$, we have an isomorphism:

$$F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j \simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(\tau_B^{-1} iX, \tau_B^p H_j)/\mathsf{proj}\,B.$$

For $p \geq 0$, we have $\tau_B^p(H_j) = \tau_B^{-1}(\tau_B^{p+1} H_j)$ if and only if $\tau_B^p H_j$ is not projective. Thus we have a vector space isomorphism

$$F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j \simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(\tau_B^{-1} iX, \tau_B^{-1}\tau_B^{p+1} H_j)/\mathsf{proj}\,B.$$

A morphism $f : \tau^{-1}X \to \tau^{-1}Y$ factorizes through a projective object if and only if $\tau(f) : X \to Y$ is not zero. Thus we have:

$$\begin{aligned} F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j &\simeq \bigoplus_{p \geq 1} D\mathsf{Hom}_B(iX, \tau_B^p H_j) \\ &\simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(X, \tau_B^p H_j)/[add(H)] \\ &\simeq (F \circ p^*)(X^\vee)e_j \simeq (F \circ i_*)(X^\vee)e_j. \end{aligned}$$

Therefore we get this exact sequence in $f.l.\Lambda$, functorial in $X$:

$$0 \longrightarrow (F \circ i_*)(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow (F \circ i_*)(X^\vee) \longrightarrow 0$$

□

**Proposition 6.1.6.** *Let $U$ and $V$ be indecomposables in $\overline{\mathcal{M}}$. Then we have*

$$\mathsf{Hom}_{\mathcal{C}_A}(U^\wedge, V^\wedge) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p U, V)/[add\tau_B^p H]$$

*where $\mathcal{M}(\tau_B^p U, V)/[add\tau_B^p H]$ is the cokernel of the composition map*

$$\mathcal{M}(\tau_B^p U, \tau_B^p H) \otimes \mathcal{M}(\tau_B^p H, V) \longrightarrow \mathcal{M}(\tau_B^p U, V).$$

We first show the following lemma:

**Lemma 6.1.2.** *Let $e_U$ and $e_V$ be the idempotents of $A$ associated to the indecomposables $U$ and $V$. Then we have an isomorphism*

$$e_U \mathsf{Ext}^2_A(DA, A)e_V \simeq \mathcal{M}(\tau_B U, V)/[add\tau_B H]$$

*where $\mathcal{M}(\tau_B U, V)/[add\tau_B H]$ is the coker of the composition map*

$$\mathcal{M}(\tau_B U, \tau_B H) \otimes \mathcal{M}(\tau_B H, V) \longrightarrow \mathcal{M}(\tau_B U, V).$$

*Proof.* We have the following isomorphisms:

$$\begin{array}{rcl} e_U \mathsf{Ext}^2_A(DA, A)e_V &=& \mathsf{Ext}^2_A(D(e_U A), Ae_V) \\ &\simeq& \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]). \end{array}$$

Denote by $\underline{\mathcal{M}}$ the category $\mathcal{M}/\mathsf{proj}\,B$. Then $\tau_B$ induces an equivalence of $k$-linear categories $\tau_B : \underline{\mathcal{M}} \to \overline{\mathcal{M}}$. Thus we get the following isomorphisms

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]) &\simeq& \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\underline{\mathcal{M}}(\tau_B^{-1}U, \tau_B^{-1}?), \underline{\mathcal{M}}(\tau_B^{-1}?, \tau_B^{-1}V)[2]) \\ &\simeq& \mathsf{Hom}_{\mathcal{D}(\underline{\mathcal{M}})}(D\underline{\mathcal{M}}(\tau_B^{-1}U, ?), \underline{\mathcal{M}}(?, \tau_B^{-1}V[2])) \\ &\simeq& \mathsf{Hom}_{\mathcal{D}(\mathcal{M})}(D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\,B, \mathcal{M}(?, \tau_B^{-1}V)/\mathsf{proj}\,B[2]) \end{array}$$

But by the previous lemma, we know a projective resolution in $\mathsf{mod}\,\mathcal{M}$ of the module $D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\,B$. Namely, there exists an exact sequence in $\mathsf{mod}\,\mathcal{M}$ of the form:

$$0 \longrightarrow \mathcal{M}(?, U) \longrightarrow \mathcal{M}(?, H_0) \longrightarrow \mathcal{M}(?, H_1) \longrightarrow D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\,B \longrightarrow 0$$

where $H_0$ and $H_1$ are in $add(H)$. Thus we get (using Yoneda's lemma)

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]) &\simeq& \mathsf{Hom}_{\mathcal{D}(\mathcal{M})}(\mathcal{M}(?, U), \mathcal{M}(?, \tau_B^{-1}V)/\mathsf{proj}\,B)/[add\mathcal{M}(?, H)] \\ &\simeq& \underline{\mathcal{M}}(U, \tau_B^{-1}V)/[addH] \\ &\simeq& \overline{\mathcal{M}}(\tau_B U, V)/[add\tau_B H]. \end{array}$$

Since $V$ is in $\overline{\mathcal{M}}$, a non-zero morphism of $\mathcal{M}(\tau_B U, V)$ cannot factorize through $add(H)$. Thus we get $\overline{\mathcal{M}}(\tau_B U, V)/[add\tau_B H] \simeq \mathcal{M}(\tau_B U, V)/[add\tau_B H]$.

□

*Proof. (of proposition 6.1.6)* In this proof, for simplicity we denote $\tau_B$ by $\tau$. Let $\tilde{A}$ be the algebra $\mathsf{End}_{\mathcal{C}_A}(A)$. By proposition 5.2.1, we have a vector space isomorphism

$$e_U \tilde{A} e_V \simeq e_U A e_V \oplus e_U \mathsf{Ext}^2_A(DA, A)e_V \oplus e_U \mathsf{Ext}^2_A(DA, A)^{\otimes_A 2} e_V \oplus \ldots$$

We prove by induction that

$$e_U \mathsf{Ext}^2_A(DA, A)^{\otimes_A p} e_V \simeq \mathcal{M}(\tau^p U, V)/[add \tau^p H].$$

For $p = 0$, $e_U A e_V$ is isomorphic to $\overline{\mathcal{M}}(U, V)$ by Yoneda's lemma, and so to $\mathcal{M}(U, V)/[add(H)]$. Suppose the proposition holds for an integer $p - 1 \geq 0$. Then we have

$$e_u \mathsf{Ext}^2_A(DA, A)^{\otimes_A p} e_V \simeq \sum_{W \in \mathsf{ind}\,(\overline{\mathcal{M}})} e_u \mathsf{Ext}^2_A(DA, A)^{\otimes_A p-1} e_W \otimes e_W \mathsf{Ext}^2_A(DA, A) e_V.$$

The sum means here the direct sum modulo the mesh relations of the category $\overline{\mathcal{M}}$. Thus this vector space is the sum over the indecomposables $W$ of $\overline{\mathcal{M}}$ of

$$\mathcal{M}(\tau^{p-1} U, W)/[add(\tau^{p-1} H)] \otimes \mathcal{M}(\tau W, V)/[add(\tau H)]$$

modulo the mesh relations of $\overline{\mathcal{M}}$. This is isomorphic to the cokernel of the map $\varphi^{p-1}_{\tau^{p-1}U,W} \otimes 1_{\tau W,V} + 1_{\tau^{p-1}U,W} \otimes \varphi^1_{\tau W,V}$ where

$$\varphi^j_{X,Y} : \mathcal{M}(X, \tau^j H) \otimes \mathcal{M}(\tau^j H, Y) \longrightarrow \mathcal{M}(X, Y)$$

is the composition map and where

$$1_{X,Y} : \mathcal{M}(X, Y) \longrightarrow \mathcal{M}(X, Y)$$

is the identity. The cokernel of this map is isomorphic to the cokernel of the map $\varphi^p_{\tau^p U, \tau W} \otimes 1_{\tau W,V} + 1_{U,\tau W} \otimes \varphi^1_{\tau W,V}$. But we have an isomorphism

$$\sum_{W \in \mathsf{ind}\,\overline{\mathcal{M}}} \mathcal{M}(\tau^p U, \tau W) \otimes \mathcal{M}(\tau W, V) \simeq \mathcal{M}(\tau^p U, V).$$

Finally we get

$$\mathsf{Coker}\left(\sum_{W \in \mathsf{ind}\,\overline{\mathcal{M}}} \varphi^p_{\tau^p U, \tau W} \otimes 1_{\tau W,V} + 1_{U,\tau W} \otimes \varphi^1_{\tau W,V}\right) \simeq \mathsf{Coker}(\varphi^p_{\tau^p U,V} + \varphi^1_{\tau^p U,V}).$$

Furthermore, a morphism in $\mathcal{M}(\tau^p U, V)$ which factorizes through $\tau H$ factorizes through $\tau^p H$ since $H$ is a slice and $U$ is in $\overline{\mathcal{M}}$. Thus this cokernel is in fact isomorphic to the cokernel of $\varphi^p_{\tau^p U,V}$ that is to say to the space

$$\mathcal{M}(\tau^p U, V)/[add \tau^p H].$$

□

## 6.2 Case where $B = \mathsf{End}_{kQ}(T)$ is hereditary

### 6.2.1 Results of Geiss, Leclerc and Schröer

Let us first recall some definitions and results of Geiss, Leclerc and Schröer [GLS07b].

Let $Q$ be a finite connected quiver without oriented cycles with $n$ vertices. Denote by $\mathcal{P}$ the postprojective component of the Auslander-Reiten quiver of $\mathsf{mod}\,kQ$, and by $P_1, \ldots, P_n$ the indecomposable projectives.

**Definition 6.2** (Geiss-Leclerc-Schröer, [GLS07b])**.** A $kQ$-module $M = M_1 \oplus \cdots \oplus M_r$ with $M_i$ indecomposables and pairwise non isomorphic, is called *initial* if the following conditions hold:

- for all $i = 1, \ldots, r$, $M_i$ is postprojective;
- if $X$ is an indecomposable $kQ$-module with $\mathsf{Hom}_{kQ}(X, M) \neq 0$, then $X$ is in $add(M)$;
- and $P_i \in add(M)$ for each indecomposable projective $kQ$-module $P_i$.

We define the integers $t_i$ as

$$t_i = max\{j \geq 0 | \tau^{-j}(P_i) \in add(M) - \{0\}\}.$$

Denote by $\Lambda$ the preprojective algebra associated to $Q$. There is a canonical algebras embedding $kQ \hookrightarrow \Lambda$. Denote by $\pi_Q : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,kQ$ the corresponding restriction functor.

Geiss, Leclerc and Schröer showed the following theorem:

**Theorem 6.3** (Geiss-Leclerc-Schröer, [GLS07b])**.** *Let $M$ be an initial $kQ$-module, and let $\mathcal{C}_M = \pi_Q^{-1}(add(M))$ be the subcategory of all $\Lambda$-modules $X$ with $\pi_Q(X) \in add(M)$. Then the following holds:*

*(i) the category $\mathcal{C}_M$ is a Frobenius category with $n$ projective-injectives;*

*(ii) the stable category $\underline{\mathcal{C}}_M$ is a 2-Calabi-Yau triangulated category.*

Recall from Ringel [Rin98] that the category $\mathsf{mod}\,\Lambda$ can be seen as $\mathsf{mod}\,kQ(\tau^{-1}, 1)$. The objects are pairs $(X, f)$ where $X$ is in $\mathsf{mod}\,kQ$ and $f : \tau^{-1}X \to X$ is a morphism in $\mathsf{mod}\,kQ$.The morphisms $\varphi$ between $(X, f)$ and $(Y, g)$ are commutative squares:

$$\begin{array}{ccc} \tau^{-1}X & \xrightarrow{f} & X \\ {\scriptstyle \tau^{-1}\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \tau^{-1}Y & \xrightarrow{g} & Y \end{array}$$

The image of an object $(X, f)$ under $\pi_Q : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,kQ$ is then the module $X$.

Let $X = \tau^{-l}P_i$ be an indecomposable summand of an initial module $M$. Then let $R_X = (Y, f)$ be the following object in $\mathsf{mod}\, kQ(\tau^{-1}, 1) \simeq \mathsf{mod}\,\Lambda$:

$$Y = \bigoplus_{j=0}^{l} \tau^{-j}P_i \quad \text{and} \quad \mathrm{f} : \bigoplus_{\mathrm{j=1}}^{\mathrm{l+1}} \tau^{-\mathrm{j}}\mathrm{P_i} \longrightarrow \bigoplus_{\mathrm{j=0}}^{\mathrm{l}} \tau^{-\mathrm{j}}\mathrm{P_i}$$

is given by the matrix

$$f = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Geiss, Leclerc and Schröer proved the following:

**Proposition 6.2.1** (Geiss-Leclerc-Schröer,[GLS07b])**.** *The category $\mathcal{C}_M$ has a canonical maximal rigid object $R = \bigoplus_{X \in \mathsf{ind}\, add(M)} R_X$. The projective-injectives of $\mathcal{C}_M$ are the $R_{\tau^{-t_i}P_i}$, $i = 1, \ldots, n$. Therefore, $R$ is a cluster-tilting object in $\underline{\mathcal{C}}_M$.*

### 6.2.2 Endomorphism algebra of the cluster-tilting object

Let $Q$ be a connected quiver without oriented cycles and denote by $B$ the path algebra $kQ$. Let $M$ be an initial $B$-module. Let $H$ be the following $B$-module $H = \bigoplus_{i=1}^{n} \tau^{-t_i}P_i$. Thus $\mathsf{End}_B(H)$ is an hereditary algebra isomorphic to $kQ'$ where $Q'$ is a quiver with the same underlying graph as $Q$. We have the following triangle equivalence:

$$\mathcal{D}^b(\mathsf{mod}\, kQ') \underset{DR\mathsf{Hom}_{kQ}(?,H)}{\overset{?\overset{L}{\otimes}_{kQ'}\nu^{-1}H}{\rightleftarrows}} \mathcal{D}^b(\mathsf{mod}\, kQ')$$

which sends $H$ on $DkQ' \in \mathsf{mod}\, kQ'$. Let $T$ be the image in $\mathcal{D}^b(\mathsf{mod}\, kQ')$ of $kQ$ under this equivalence. Thus $T$ is a tilting object in the preinjective component of $\mathcal{D}^b(\mathsf{mod}\, kQ')$, and $B = \mathsf{End}_{kQ'}(T) \simeq kQ$. The previous equivalence can be also written

$$\mathcal{D}^b(\mathsf{mod}\, kQ') \underset{?\overset{L}{\otimes}_{kQ}T}{\overset{R\mathsf{Hom}_{kQ'}(T,?)}{\rightleftarrows}} \mathcal{D}^b(\mathsf{mod}\, kQ).$$

Let us define, as in the previous section, the subcategory $\mathcal{M}$ of $\mathcal{D}^b(\mathsf{mod}\, kQ)$ as

$$\mathcal{M} = \{X \in \mathsf{mod}\, kQ \ / \mathsf{Ext}^1_B(X, H) = 0\}.$$

Then it is obvious that $\mathcal{M} = add(M)$. As previously, we put $\Lambda = \Lambda_Q = \Lambda_{Q'}$ because $Q$ and $Q'$ have the same underlying graph. Recall that we have $\overline{\mathcal{M}} = \mathcal{M}/add(H)$, and that $A = \mathsf{End}_B(\overline{\mathcal{M}})$ is an algebra of global dimension 2. Note that in this case $\tau_B$ and $\tau_{\mathcal{D}}$ coincide on $\mathsf{mod}\, B$ since $B$ is hereditary.

**Lemma 6.2.1.** *Let $U$ and $V$ be indecomposables in $\overline{\mathcal{M}}$. Then we have*

$$\mathsf{Hom}_\Lambda(R_U, R_V) \simeq \bigoplus_{j\geq 0} \mathcal{M}(\tau^j U, V).$$

*Proof.* Let $P$ and $Q$ be projective indecomposables such that $U = \tau^{-q}Q$ and $V = \tau^{-p}P$. Then the morphisms bewteen $R_U$ and $R_V$ are given by commutative diagrams

$$\begin{array}{ccc} \bigoplus_{i=1}^{q+1}\tau^{-i}Q & \xrightarrow{\begin{pmatrix}0&&&\\1&\ddots&&\\&\ddots&\ddots&\\&&1&0\end{pmatrix}} & \bigoplus_{i=0}^{q}\tau^{-i}Q \\ \Big\downarrow{\scriptstyle \tau^{-1}f} & & \Big\downarrow{\scriptstyle f} \\ \bigoplus_{j=1}^{p+1}\tau^{-j}P & \xrightarrow{\begin{pmatrix}0&&&\\1&\ddots&&\\&\ddots&\ddots&\\&&1&0\end{pmatrix}} & \bigoplus_{j=0}^{p}\tau^{-j}P \end{array}$$

*Case 1: $p \leq q$*
An easy computation gives the following equalities

$$\begin{aligned} \mathsf{Hom}_\Lambda(R_U, R_V) &\simeq \bigoplus_{j=0}^{p} \mathcal{M}(Q, \tau^{-j}P) \\ &\simeq \bigoplus_{j=0}^{p} \mathcal{M}(\tau^{-p+j}Q, \tau^{-p}P) \\ &\simeq \bigoplus_{j=0}^{p} \mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q), \tau^{-p}P) \\ &\simeq \bigoplus_{j=q-p}^{q} \mathcal{M}(\tau^j U, V) \end{aligned}$$

Since $\mathcal{M}(\tau^k U, V)$ vanishes for $k \leq q-p+1$ and since $\tau^k U$ vanishes for $k \geq q+1$, we get an isomorphism

$$\mathsf{Hom}_\Lambda(R_U, R_V) \simeq \bigoplus_{j\geq 0} \mathcal{M}(\tau^j U, V).$$

*Case 2: $p > q$*
In this case, a morphism from $R_U$ to $R_V$ is given by morphisms $a_j \in \mathcal{M}(Q, \tau^{-j}P)$, with $j = 0, \ldots, p$ such that $\tau^{-q+1}a_j = 0$ for $j = 0, \ldots, p-q-1$. But since $\tau^{-q+1-j}P$ is not zero for $j = 0, \ldots, p-q-1$, the morphism $\tau^{-q+1}a_j : \tau^{-q+1}Q \to \tau^{-q+1-j}P$ vanishes if and

only if $a_j$ vanishes. Thus we get

$$\begin{aligned}\mathsf{Hom}_\Lambda(R_U,R_V) &\simeq \bigoplus_{j=p-q}^{p}\mathcal{M}(Q,\tau^{-j}P)\\ &\simeq \bigoplus_{j=p-q}^{p}\mathcal{M}(\tau^{-p+j}Q,\tau^{-p}P)\\ &\simeq \bigoplus_{j=p-q}^{p}\mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q),\tau^{-p}P)\\ &\simeq \bigoplus_{j=0}^{q}\mathcal{M}(\tau^{j}U,V)\end{aligned}$$

Since $\tau^j U$ vanishes for $j \geq q+1$ we get

$$\mathsf{Hom}_\Lambda(R_U,R_V) \simeq \bigoplus_{j\geq 0}\mathcal{M}(\tau^j U,V).$$

□

**Corollary 6.4.** *Let $U$ and $V$ be indecomposable objects in $\overline{\mathcal{M}}$. Then we have*

$$\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U,R_V) \simeq e_U\tilde{A}e_V$$

*and therefore the algebras $\tilde{A}$ and $\mathsf{End}_{\underline{\mathcal{C}}_M}(R)$ are isomorphic.*

*Proof.* The projective-injectives in the category $\mathcal{C}_M$ are the $R_{H_i}$ with $i=1,\ldots,n$. Denote by $R_H$ the sum $\bigoplus_{i=1}^n R_{H_i}$. Then $\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U,R_V)$ is the cokernel of the composition map

$$\mathsf{Hom}_{\mathcal{C}_M}(R_U,R_H)\otimes\mathsf{Hom}_{\mathcal{C}_M}(R_H,R_V)\longrightarrow\mathsf{Hom}_{\mathcal{C}_M}(R_U,R_V).$$

By the previous lemma this map is isomorphic to the following

$$\bigoplus_{i,j\geq 0}\mathcal{M}(\tau^iU,H)\otimes\mathcal{M}(\tau^jH,V)\xrightarrow{\quad\Phi\quad}\bigoplus_{p\geq 0}\mathcal{M}(\tau^pU,V)$$

Given two morphisms $f\in\mathcal{M}(\tau^iU,H)$ and $\mathcal{M}(\tau^jH,V)$, $\Phi(f\otimes g)$ is the composition $\tau^j f\circ g\in\mathcal{M}(\tau^{i+j}U,V)$. Thus the cokernel of this map is the cokernel of the map

$$\bigoplus_{p\geq 0}\bigoplus_{i=0}^{p}\mathcal{M}(\tau^pU,\tau^iH)\otimes\mathcal{M}(\tau^iH,V)\xrightarrow{\quad\Phi\quad}\bigoplus_{p\geq 0}\mathcal{M}(\tau^pU,V)\ .$$

Since $H$ is a slice and since $U$ is in $\overline{\mathcal{M}}$, a morphism in $\mathcal{M}(\tau^pU,V)$ which factorizes through $\tau^iH$ with $i\leq p$ factorizes through $\tau^pH$. Finally we get

$$\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U,R_V)\simeq\bigoplus_{p\geq 0}\mathcal{M}(\tau^pU,V)/[add\tau^pH],$$

and we conclude using proposition 6.1.6. □

### 6.2.3 Triangle equivalence

The aim of this section is to prove the following theorem:

**Theorem 6.5.** *The functor* $F \circ i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to f.l.\Lambda$ *yields a triangle equivalence between* $\mathcal{C}_{\overline{\mathcal{M}}}$ *and* $\underline{\mathcal{C}}_M$.

*Proof.* Let $X = \tau_B^{-l} P_i$ be an indecomposable of $\mathcal{M}$. The $\mathcal{M}$-module $X^\wedge = \mathsf{Hom}_B(?, X)_{|\mathcal{M}}$ is projective. The underlying vector space of $F(X^\wedge)$ is

$$\begin{aligned} F(X^\wedge) &\simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(\tau_B^q H, \tau_B^{-l} P_i) \\ &\simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(\tau_B^{-q} B, \tau_B^{-l} P_i) \\ &\simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(B, \tau_B^{q-l} P_i) \simeq \bigoplus_{q=0}^{l} \tau_B^{-q} P_i \end{aligned}$$

The action on the arrows is obviously given by the morphism

$$f : \bigoplus_{j=1}^{l+1} \tau^{-j} P_i \longrightarrow \bigoplus_{j=0}^{l} \tau^{-j} P_i$$

with

$$f = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Thus each projective $X^\wedge$ is sent on an object of $\mathcal{C}_M$. Moreover, $H$ is equal to $\bigoplus_{i=1}^n \tau^{-t_i} P_i$ so $F(H^\wedge)$ is equal to $\bigoplus_{i=1}^n R_{\tau^{-t_i} P_i}$ the sum of all the projective-injectives of $\mathcal{C}_M$. Therefore $F$ induces a functor $F : \mathcal{D}^b(\mathsf{mod}\,\mathcal{M}) \to \mathcal{D}^b(\mathcal{C}_M)$. We have the following composition:

$$\underset{\circlearrowright\ ?\overset{L}{\otimes}_A DA[-2]}{\mathcal{D}^b(\mathsf{mod}\,\overline{\mathcal{M}})} \simeq \mathcal{D}^b(\mathsf{mod}\,A) \xrightarrow{i_*} \mathcal{D}^b(\mathsf{mod}\,\mathcal{M}) \xrightarrow{F} \mathcal{D}^b(\mathcal{C}_M) \xrightarrow{\pi} \mathcal{D}^b(\mathcal{C}_M)/\mathsf{per}\,\mathcal{C}_M \simeq \underline{\mathcal{C}}_M$$

The functor $F \circ i_*$ is clearly isomorphic to the left derived tensor product with the $A$-$\Lambda$-bimodule $R = F \circ i_*(A)$. By proposition 6.1.5, for $X$ in $\overline{\mathcal{M}}$, we have the following exact sequence, functorial in $X$:

$$0 \longrightarrow F \circ i_*(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F \circ i_*(X^\vee) \longrightarrow 0$$

with $H_0$ and $H_1$ in $add(H)$. It yields a morphism

$$F \circ i_*(DA) \to F \circ i_*(A)[2]$$

in the derived category of $A$-$\Lambda$-bimodules. Since the objects $F(H_0^\wedge)$ and $F(H_1^\wedge)$ vanish in the stable category $\underline{\mathcal{C}}_M$, the image

$$F \circ i_*(DA) \to F \circ i_*(A)[2]$$

of this morphism in the category of $A$-$\mathcal{B}$-bimodules is invertible, where $\mathcal{B}$ is a dg category whose perfect derived category is algebraically equivalent to the stable category $\underline{\mathcal{C}}_M$. In other words, in the derived category $\mathcal{D}(A^{op} \otimes \mathcal{B})$, we have an isomorphism

$$DA \overset{L}{\otimes}_A \pi F i_*(A) \simeq \pi F i_*(A)[-2].$$

By the universal property of the orbit category, we have the factorization

$$\begin{array}{ccc} \mathcal{D}^b(\mathsf{mod}\,\overline{\mathcal{M}}) & \xrightarrow{\;?\overset{L}{\otimes}_A \underline{R}\;} & \underline{\mathcal{C}}_M . \\ & \searrow \qquad \nearrow & \\ & \mathcal{C}_{\overline{\mathcal{M}}} & \end{array}$$

This factorization is an algebraic functor between 2-Calabi-Yau categories which sends the orbit-cluster-tilting object $A$ on the cluster-tilting object $\underline{R}$. Moreover by corollary 6.4, it yields an equivalanece between the categories $add(A)$ and $add(\underline{R})$. Thus using the lemma 5.4.1, it is an algebraic triangle equivalence.

□

Note that if $M$ is the initial module $kQ \oplus \tau^{-1}kQ$, Geiss, Leclerc and Schröer proved, using a result of Keller and Reiten [KR06], that the 2-Calabi-Yau category $\underline{\mathcal{C}}_M$ is triangle equivalent to the cluster category $\mathcal{C}_Q$. Here, $H$ is $\tau^{-1}kQ$ and then $\overline{\mathcal{M}}$ is $kQ$, so we get another proof of this fact.

### 6.2.4 Example: Dynkin case

Let $Q$ be $A_4$ with the following orientation:

$$1 \longleftarrow 2 \longrightarrow 3 \longrightarrow 4.$$

Let $M$ be the following initial module:

$$M = {\scriptstyle 2} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 3 & & 4 \\ & 2 & & & \end{smallmatrix} \oplus {\scriptstyle 3} \oplus {\scriptstyle 1} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$$

Thus we have

$$H = \begin{smallmatrix} 1 & & 3 & & 4 \\ & 2 & & & \end{smallmatrix} \oplus {\scriptstyle 3} \oplus {\scriptstyle 1} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \quad \text{and} \quad \overline{\mathrm{M}} = {\scriptstyle 2} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix}.$$

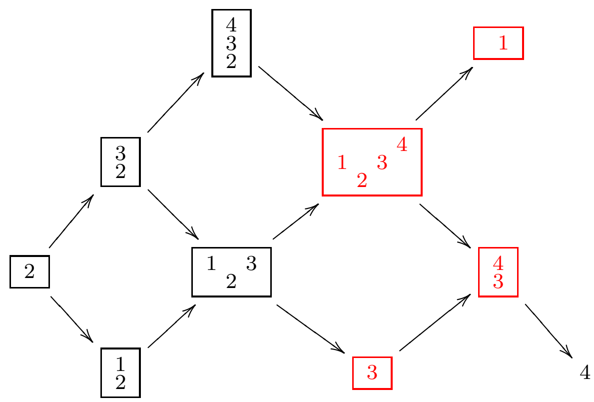


Figure 6.1: Auslander-Reiten quiver of the category $\mathsf{mod}\, kQ$

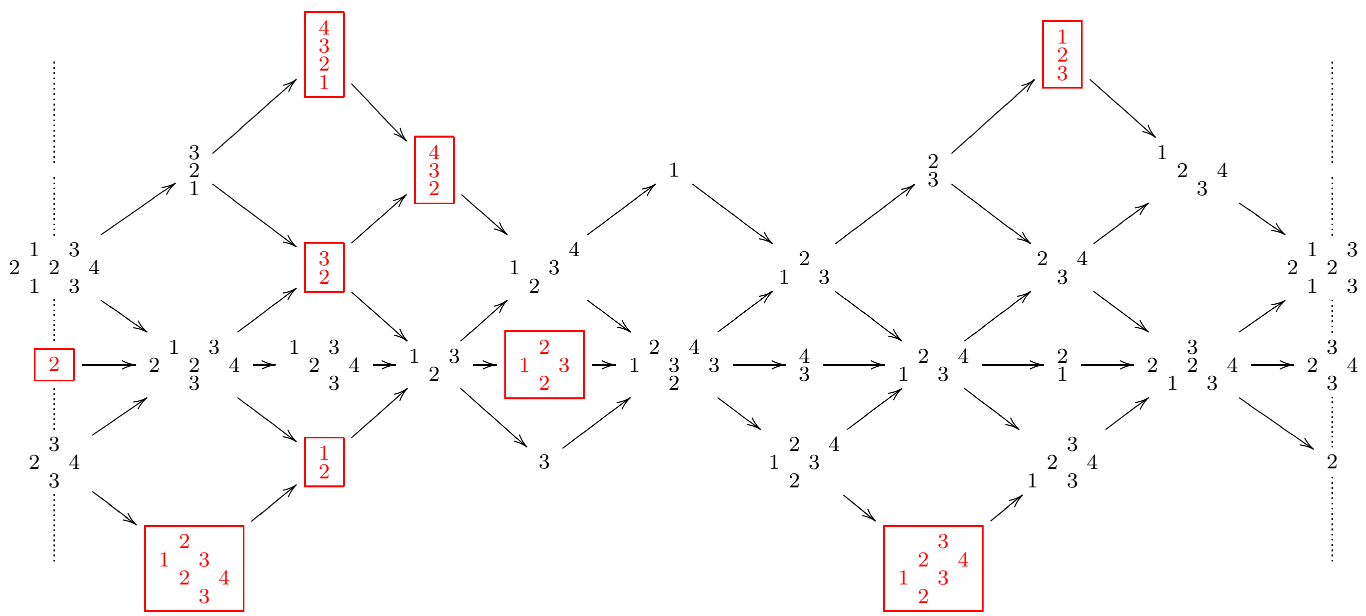


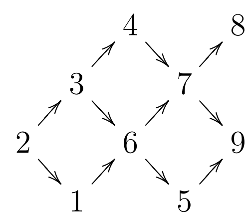
Figure 6.2: Auslander-Reiten quiver of the category $\mathcal{C}_M$

In figure 6.1 we can see the Auslander-Reiten quiver of the category $\mathsf{mod}\, kQ$. The indecomposables of $add(M)$ are framed and the indecomposables of $add(H)$ are marked in red.

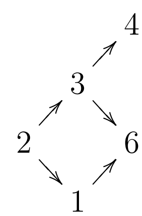
The category $\mathcal{C}_M$ is then a subcategory of $\mathsf{mod}\, \Lambda(A_4)$. Figure 6.2 shows its Auslander-Reiten quiver. The corresponding stable category is triangle equivalent to $\mathcal{C}_{D_5}$. The direct summands of the canonical maximal rigid object are marked in red. Let $\pi_Q : \mathsf{mod}\, \Lambda \rightarrow$

$\mathsf{mod}\, kQ$ be the canonical projection. The $\Lambda$-module $\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}$ is a projective-injective of $\mathcal{C}_M$, and its image under $\pi_Q$ is $2 \oplus \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$ which is the direct sum of a $\tau$-orbit in $add(M)$.

The quiver of the category $\mathcal{M} = add(M)$ is the following:




The endomorphism algebra $A = \mathsf{End}_{kQ}(\overline{M})$ is defined by the following quiver with the mesh relations.



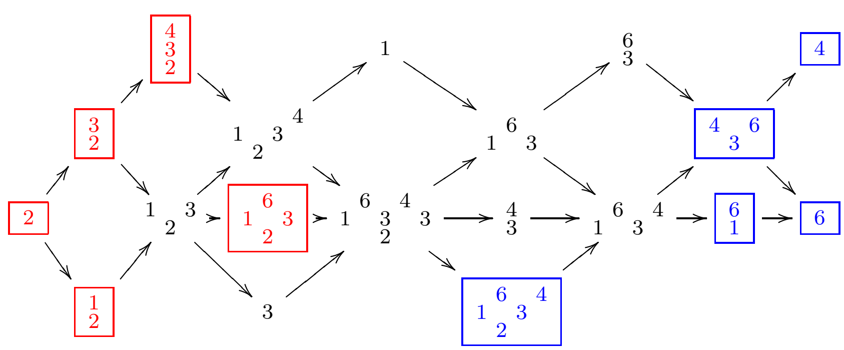



Figure 6.3: Auslander-Reiten quiver of the category $\mathsf{mod}\, A$

Figure 6.2.4 shows the Auslander-Reiten of $\mathsf{mod}\, A$. The indecomposable projectives are marked in red and the indecomposable injectives are marked in blue.

The algebra $A$ can be seen as the endomorphism algebra of a tilting module in $\mathsf{mod}\, kD_5$, so the derived category $\mathcal{D}^b(\mathsf{mod}\, A)$ is equivalent to the category $\mathcal{D}^b(\mathsf{mod}\, kD_5)$. Figure 6.4 shows its Auslander-Reiten quiver. The object $X$ is the following complex:

$$\cdots \longrightarrow 0 \longrightarrow \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 6 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \longrightarrow 0 \longrightarrow \cdots$$

whose homology is non zero in degree $-1$ and $0$.

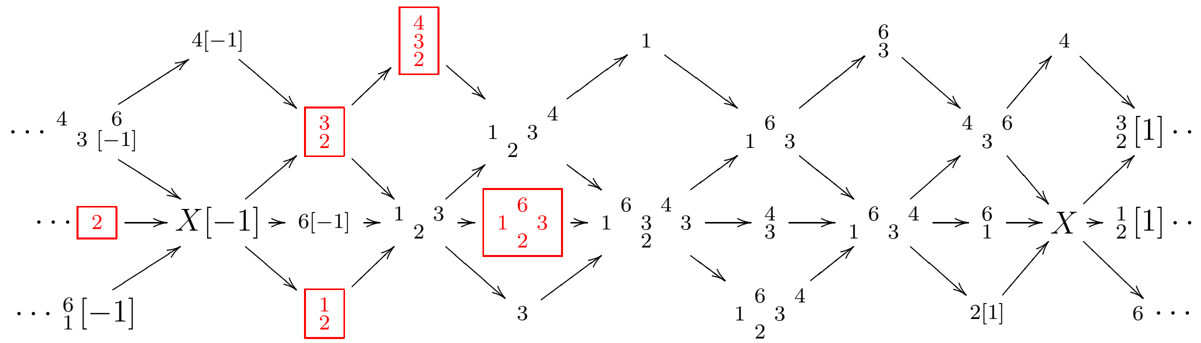



Figure 6.4: Auslander-Reiten quiver of the derived category $\mathcal{D}^b(\mathsf{mod}\, A)$

Now let $X = {}_2$ be an indecomposable of $\overline{\mathcal{M}} = add(\overline{M})$. Then $\tau^{-1}X$ is ${}^{1}{}_{2}{}^{3} \in \mathcal{M}$. The triangle $X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X[1]$ of property 1 of proposition 6.1.1 is the following:

$$2 \longrightarrow {}^{1}{}_{2}{}^{3}{}^{4} \longrightarrow 1 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \longrightarrow 2\,[1]\,.$$

The exact sequence

$$0 \longrightarrow X^{\wedge}_{|\mathcal{M}} \longrightarrow H^{\wedge}_{0|\mathcal{M}} \longrightarrow H^{\wedge}_{1|\mathcal{M}} \longrightarrow (\tau^{-1}X)^{\vee}_{|\mathcal{M}}/\mathsf{proj}\, B \longrightarrow 0$$

of $\mathsf{mod}\,\mathcal{M}$ is then

$$0 \longrightarrow 2^{\wedge} \longrightarrow 7^{\wedge} \longrightarrow 8^{\wedge} \oplus 9^{\wedge} \longrightarrow 6^{\vee}/\mathsf{proj}\, B \longrightarrow 0$$

that is to say the sequence

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix} & & 7 & \\ & 6 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 9 & & \\ 5 & & 7 & \\ & 6 & & 4\\ & & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}8\\7\\6\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 9 & & 8\\ 5 & & 7 & \\ & 6 & & \end{smallmatrix} \longrightarrow 0\,.$$

Note that in this case, since $B = \mathsf{End}_{kQ}(T)$ is hereditary, $(\tau^{-1}X)^{\vee}_{|\mathcal{M}}/\mathsf{proj}\, B$ is isomorphic to $(\tau^{-1}X)^{\vee}_{|\mathcal{M}}$. Its image under $F$ in $\mathsf{mod}\,\Lambda$ is

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix} & & 3 & \\ & 2 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4\\ & & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 2 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} = F\Big(\begin{smallmatrix} & 6 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}\Big) \longrightarrow 0\,.$$

Thus we get an isomorphism in $\underline{\mathcal{C}}_M$ between $2\,[2] = F(2^{\wedge})[2]$ and $\begin{smallmatrix} & 2 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} = F\Big(\begin{smallmatrix} & 6 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}\Big) = F(2^{\vee}_{|\mathcal{M}})$.

Now let $X$ be the indecomposable object $\begin{smallmatrix}3\\2\end{smallmatrix}$ of $\overline{\mathcal{M}}$. Then $\tau^{-1}X$ is ${}^{1}{}_{2}{}^{3}{}^{4}$ which is in $\mathcal{M}$. The triangle $X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X[1]$ in $\mathcal{D}^b(\mathsf{mod}\, kQ)$ is the following:

$$\begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow {}^{1}{}_{2}{}^{3}{}^{4} \oplus 3 \longrightarrow 1 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix}[1].$$

The exact sequence $0 \longrightarrow X^{\wedge}_{|\mathcal{M}} \longrightarrow H^{\wedge}_{0|\mathcal{M}} \longrightarrow H^{\wedge}_{1|\mathcal{M}} \longrightarrow (\tau^{-1}X)^{\vee}_{|\mathcal{M}} \longrightarrow 0$ of $\mathsf{mod}\,\mathcal{M}$ is then

$$0 \longrightarrow 3^{\wedge} \longrightarrow 7^{\wedge} \oplus 5^{\wedge} \longrightarrow 8^{\wedge} \oplus 9^{\wedge} \longrightarrow 7^{\vee} \longrightarrow 0$$

that is to say the sequence

$$0 \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix} & & 7 & \\ & 6 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \oplus \begin{smallmatrix}5\\6\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 9 & & \\ 5 & & 7 & \\ & 6 & & 4\\ & & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}8\\7\\6\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}8 & & 9\\ & 7 & \end{smallmatrix} \longrightarrow 0\,.$$

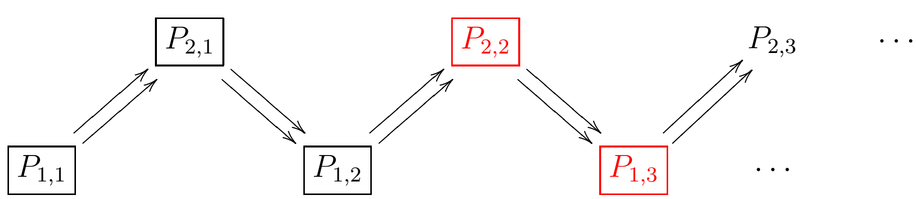
Its image under $F$ in $\mathsf{mod}\,\Lambda$ is

$$0 \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix} & & 3 & \\ & 2 & & 4\\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4\\ & & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}2 & & 4\\ & 3 & \end{smallmatrix} = F(\begin{smallmatrix}2 & & 4\\ & 3 & \end{smallmatrix}) \longrightarrow 0\,.$$

Thus there is an isomorphism in $\underline{\mathcal{C}}_M$ between $\begin{smallmatrix}3\\2\end{smallmatrix}[2] = F(3^{\wedge}_{|\overline{\mathcal{M}}})[2]$ and $\begin{smallmatrix}2 & & 4\\ & 3 & \end{smallmatrix} = F(\begin{smallmatrix}2 & & 4\\ & 3 & \end{smallmatrix}) = F(3^{\vee}_{|\overline{\mathcal{M}}})$.

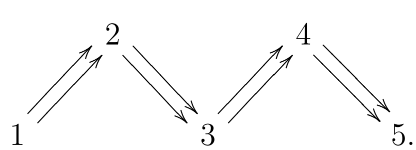
### 6.2.5 Example: Non-Dynkin case

Now take $Q = \tilde{A}_2 = 1 \rightrightarrows 2$. Thus the postprojective component of the Auslander-Reiten quiver of $\mathsf{mod}\,kQ$ is




where $P_{1,1} = 1$, $P_{2,1} = \begin{smallmatrix} & 2 & \\ 1 & & 1\end{smallmatrix}$, $P_{1,2} = \begin{smallmatrix} & 2 & & 2 & \\ 1 & & 1 & & 1\end{smallmatrix}$, $P_{2,2} = \begin{smallmatrix} & 2 & & 2 & & 2 & \\ 1 & & 1 & & 1 & & 1\end{smallmatrix}$, etc.

Put $M = P_{1,1} \oplus P_{2,1} \oplus P_{1,2} \oplus P_{2,2} \oplus P_{1,3}$, then $\overline{M}$ is the direct sum $P_{1,1} \oplus P_{2,1} \oplus P_{1,2}$ and $H$ is $P_{2,2} \oplus P_{1,3}$. Thus the algebra $\mathsf{End}_{kQ}(M)$ is defined by the following quiver with the mesh relations:




The indecomposable projectives of the algebra $\mathsf{End}_{kQ}(M)$ are given by

$$P_1 = 1, \quad P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 1\end{smallmatrix}, \quad P_3 = \begin{smallmatrix} & & 3 & & \\ & 2 & & 2 & \\ 1 & & 1 & & 1\end{smallmatrix}, \quad P_4 = \begin{smallmatrix} & & & 4 & & & \\ & & 3 & & 3 & & \\ & 2 & & 2 & & 2 & \\ 1 & & 1 & & 1 & & 1\end{smallmatrix} \quad \text{and} \quad \mathrm{P}_5 = \begin{smallmatrix} & & & & 5 & & & & \\ & & & 4 & & 4 & & & \\ & & 3 & & 3 & & 3 & & \\ & 2 & & 2 & & 2 & & 2 & \\ 1 & & 1 & & 1 & & 1 & & 1\end{smallmatrix}\,.$$

The projective-injectives of the category $\mathcal{C}_M$ are the modules $F(P_4) = \begin{smallmatrix} &&&2&&& \\ &&1&&1&& \\ &2&&2&&2& \\ 1&&1&&1&&1 \end{smallmatrix}$ and $F(P_5) = \begin{smallmatrix} &&&&1&&&& \\ &&&2&&2&&& \\ &&1&&1&&1&& \\ &2&&2&&2&&2& \\ 1&&1&&1&&1&&1 \end{smallmatrix}$.

Let $X = {\scriptstyle 1}$ be an object in $\overline{\mathcal{M}}$. Then we have the short exact sequence:

$$0 \longrightarrow {\scriptstyle 1} \longrightarrow P^4_{2,2} \longrightarrow P^3_{1,3} \longrightarrow 0$$

But we have $X^{\wedge}_{|\mathcal{M}} = X^{\wedge}_{|\overline{\mathcal{M}}} = {\scriptstyle 1}$, $(\tau^{-1}X)^{\vee}_{|\mathcal{M}} = \begin{smallmatrix} 5&&5&&5 \\ &4&&4& \\ &&3&& \end{smallmatrix}$ and $X^{\vee}_{|\overline{\mathcal{M}}} = \begin{smallmatrix} 3&&3&&3 \\ &2&&2& \\ &&1&& \end{smallmatrix}$. Therefore the following sequence is exact in $\mathsf{mod}\,\mathcal{M}$

$$0 \longrightarrow {\scriptstyle 1} \longrightarrow P^4_4 \longrightarrow P^5_5 \longrightarrow \begin{smallmatrix} 5&&5&&5 \\ &4&&4& \\ &&3&& \end{smallmatrix} \longrightarrow 0\,.$$

For $X = 2$, in the same way, we have the following exact sequence

$$0 \longrightarrow \begin{smallmatrix} &2& \\ 1&&1 \end{smallmatrix} \longrightarrow P^3_4 \longrightarrow P^2_5 \longrightarrow \begin{smallmatrix} 5&&5 \\ &4& \end{smallmatrix} \longrightarrow 0$$

Those exact sequences in $\mathcal{C}_M$ give isomorphisms in $\underline{\mathcal{C}}_M$ between $F(X^{\wedge}_{|\overline{\mathcal{M}}})[2]$ and $F(X^{\vee}_{|\overline{\mathcal{M}}})$.

## 6.3 Relation with categories $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$

### 6.3.1 Results of Buan, Iyama, Reiten and Scott

Let us recall some results of Buan, Iyama, Reiten and Scott in [BIRS07]. Let $Q$ be a finite connected quiver without oriented cycles and $\Lambda$ the associated preprojective algebra. We denote by $\{1,\ldots,n\}$ the set of vertices of $Q$. For a vertex $i$ of $Q$, we denote by $\mathcal{I}_i$ the ideal $\Lambda(1-e_i)\Lambda$ of $\Lambda$. We denote by $W$ the *Coxeter group* associated to the quiver $Q$. The group $W$ is defined by the generators $1,\ldots,n$ and the relations:

- $i^2 = 1$ for all $i$ in $\{1,\ldots,n\}$;
- $ij = ji$ if there are no arrows between the vertices $i$ and $j$;
- $iji = jij$ if there is exactly one arrow between $i$ and $j$.

Let $w = i_1 i_2 \ldots i_r$ be a $W$-reduced word. For $m \leq r$, let $\mathcal{I}_{w_m}$ be the the following ideal:

$$\mathcal{I}_{w_m} = \mathcal{I}_{i_m} \ldots \mathcal{I}_{i_2}\mathcal{I}_{i_1}.$$

For simplicity we will denote $\mathcal{I}_{w_r}$ by $\mathcal{I}_w$. The category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ is the subcategory of f.l.$\Lambda$ generated by the sub-$\Lambda$-modules of $\Lambda/\mathcal{I}_w$. Buan, Iyama, Reiten and Scott proved the following:

**Theorem 6.6** (Buan-Iyama-Reiten-Scott [BIRS07])**.** *The category* $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ *is a Frobenius category and its stable category* $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ *is* 2*-Calabi-Yau. The object* $\bigoplus_{m=1}^{r} e_{i_m}\Lambda/\mathcal{I}_{w_m}$ *is a cluster-tilting object.*

Note that this theorem is written only for non Dynkin quiver in [BIRS07], but the Dynkin case is an easy consequence of theorem II.2.8 and corollary II.3.5 of [BIRS07].

### 6.3.2 Construction of a reduced word

Let $Q$ be a finite connected quiver without oriented cycles, and $\Lambda$ the associated preprojective algebra. Let $T$ be a preinjective tilting $kQ$-module, and $B = \mathsf{End}_{kQ}(T)$ its endomorphism algebra. As previously, we define the $B$-modules $H_i = \mathsf{Hom}_{kQ}(T, I_i)$ where the $I_i$ are the indecomposable injectives of $\mathsf{mod}\,kQ$, and $\mathcal{M} = \{X \in \mathsf{mod}\,B\ /\mathsf{Ext}^1_B(X, H) = 0\}$ where $H = \bigoplus_i H_i$.

Let us order the indecomposables $X_1, \ldots, X_N$ of $\mathcal{M}$ in such a way: if the morphism space $\mathsf{Hom}_B(X_i, X_j)$ does not vanish, $i$ is smaller than $j$. It is possible since $Q$ has no oriented cycles.

By proposition 6.1.1, for $X_i \in \mathcal{M}$ there exists a unique $q \geq 0$ such that $\tau_B^{-q} X_i \simeq H_{\varphi(i)}$ for a certain integer $\varphi(i)$. So we get a function $\varphi : \{1, \ldots, N\} \to \{1, \ldots, n\}$. Let $w$ be the word $\varphi(1)\varphi(2)\ldots\varphi(N)$.

**Proposition 6.3.1.** *The word $w$ is $W$-reduced.*

*Proof.* The proof is in several steps:

*Step 1: For two integers $i < j$ in $\{1, \ldots, N\}$, we have $\varphi(i) = \varphi(j)$ if and only if there exists a positive integer $p$ such that $X_i = \tau_B^p X_j$.*

*Step 2: The element $w$ of the Coxeter group does not depend on the order on the indecomposables of $\mathcal{M}$.*

Let $i$ be in $\{1, \ldots, N-1\}$. Assume there is an arrow $\varphi(i) \to \varphi(i+1)$ in $Q$. We show that there is an arrow $X_i \to X_{i+1}$ in the Auslander-Reiten quiver of $\mathcal{M}$. By proposition 6.1.1, there exist positive integers $p$ and $q$ such that $X_i = \tau_B^q H_{\varphi(i)}$ and $X_{i+1} = \tau_B^p H_{\varphi(i+1)}$. By hypothesis, there is an arrow between $H_{\varphi(i)}$ and $H_{\varphi(i+1)}$. Thus we want to show that $p$ is equal to $q$.

Suppose that $p \geq q+1$, then since $\mathcal{M}$ is closed under $\tau_B$, the objects $\tau_B^q H_{\varphi(i+1)}$ and $\tau_B^{q+1} H_{\varphi(i+1)}$ are non zero and are in $\mathcal{M}$. Let $l$ be the integer in $\{1, \ldots, N\}$ such that $X_l = \tau_B^{q+1} H_{\varphi(i+1)}$. We have an arrow

$$X_i = \tau_B^q H_{\varphi(i)} \to \tau_B^q H_{\varphi(i+1)} = \tau_B^{-1} X_l.$$

Thus, by the property of the AR-translation, there is an arrow

$$X_l \to X_i.$$

Thus $i$ should be strictly greater than $l$. But by step 1, and the hypothesis $p \geq q+1$, we have $i+1 \leq l$. This a contradiction.

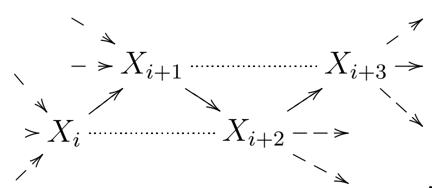
The cases $q \geq p+1$, and $\varphi(i+1) \to \varphi(i)$ in $Q$ can be solved in the same way.

*Step 3: It is not possible to have $\varphi(i) = \varphi(i+1)$.*

Suppose we have $\varphi(i) = \varphi(i+1)$. By step 1 there exists a positive integer $p$ such that $X_i = \tau_B^p X_{i+1}$. Suppose that $p$ is $\geq 2$, then $\tau_B X_{i+1} = \tau_B^{-p+1} X_i$ is in $\mathcal{M}$, it is isomorphic to an $X_k$ for an integer $k$ with $\varphi(k) = \varphi(i)$. But $k$ must be strictly greater than $i$ and strictly smaller than $i+1$ which is clearly impossible. Thus $p$ is equal to 1. There should exist an $X_l$ in $\mathcal{M}$ such that $\mathsf{Hom}(X_i, X_l) \neq 0$ and $\mathsf{Hom}(X_l, X_{i+1}) \neq 0$. Thus $l$ must be strictly between $i$ and $i+1$ which is impossible.

*Step 4: It is not possible to have $\varphi(i) = \varphi(i+2)$ and $\varphi(i+1) = \varphi(i+3)$ with exactly one arrow in $Q$ between $\varphi(i)$ and $\varphi(i+1)$.*

In this case we have, by step 1, $X_i = \tau_B^p X_{i+2}$ and $X_{i+1} = \tau_B^q X_{i+3}$. By the same argument as in step 3, $p$ and $q$ have to be equal to 1. Thus the AR quiver of $\mathcal{M}$ has locally the following form:




There is only one arrow with tail $X_{i+2}$. Indeed, suppose there is an $X_k$ with an arrow $X_k \to X_{i+2}$. Thus there is an arrow $\tau_B X_{i+2} = X_i \to X_k$ and $k$ must be strictly between $i$ and $i+2$. By the same argument, there is only one arrow with tail $X_{i+3}$, one arrow with source $X_i$ and one arrow with source $X_{i+1}$. Thus we have the following AR sequences in $\mathsf{mod}\,B$:

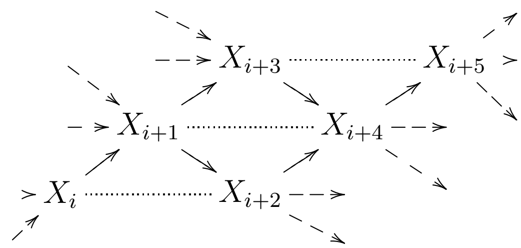
$$0 \to X_i \to X_{i+1} \to X_{i+2} \to 0 \quad \text{and} \quad 0 \to X_{i+1} \to X_{i+2} \to X_{i+3} \to 0$$

which is clearly impossible.

*Step 5: There is no subsequence of type jkjlkl in $w$ with an arrow between $j$ and $k$ and an arrow between $k$ and $l$*

Suppose we have $\varphi(i) = \varphi(i+2) = j$, $\varphi(i+1) = \varphi(i+4) = k$ and $\varphi(i+3) = \varphi(i+5) = l$. As previously, we have $X_i = \tau_B X_{i+2}$, $X_{i+1} = \tau_B X_{i+4}$ and $X_{i+3} = \tau_B X_{i+5}$. There is an arrow $X_{i+1} \to X_{i+2}$ so there is an arrow $X_{i+2} \to X_{i+4}$. There is an arrow $X_{i+3} \to X_{i+4}$.

Thus there is an arrow $X_{i+1} \to X_{i+3}$. Finally, the AR quiver of $\mathcal{M}$ locally looks like:




As in step 4, it is easy to check that there are only two arrows with tail $X_{i+4}$, one arrow with tail $X_{i+5}$ and $X_{i+2}$, one arrow with source $X_{i+3}$ and $X_i$ and two arrows with source $X_{i+1}$. Thus we have the 3 following AR sequences in $\mathsf{mod}\, B$:

$$0 \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+2} \longrightarrow 0 \qquad 0 \longrightarrow X_{i+3} \longrightarrow X_{i+4} \longrightarrow X_{i+5} \longrightarrow 0$$

$$\text{and} \qquad 0 \longrightarrow X_{i+1} \longrightarrow X_{i+3} \oplus X_{i+2} \longrightarrow X_{i+4} \longrightarrow 0$$

Therefore we get the following equalities:

$$\begin{aligned} \dim_k X_{i+1} &= \dim_k X_i + \dim_k X_{i+2} \\ \dim_k X_{i+4} &= \dim_k X_{i+3} + \dim_k X_{i+5} \\ \dim_k X_{i+3} + \dim_k X_{i+2} &= \dim_k X_{i+1} + \dim_k X_{i+4} \end{aligned}$$

which imply that $X_i$ and $X_{i+5}$ are zero.

By the second step, we know that using the relation of commutativity is the same as changing the order on the indecomposables of $\mathcal{M}$. Moreover we just saw that locally we can not reduce the word $w$. Thus it is reduced.

□

### 6.3.3 Image of the orbit-cluster-tilting object

Let $F : \mathsf{mod}\, \mathcal{M} \to \text{f.l.}\Lambda$ be the functor constructed in section 6.1.2.

**Proposition 6.3.2.** *For $i = 1, \ldots, N$, we have an isomorphism in f.l.$\Lambda$:*

$$F(X_i^\wedge) \simeq e_{\varphi(i)} \Lambda / \mathcal{I}_{w_i}$$

*where $w_i$ is the word $\varphi(1) \cdots \varphi(i)$.*

*Proof.* The functor $F$ is right exact and sends the simple functor $S_{X_i}$ onto the simple $S_{\varphi(i)}$. Since $F(X_i^\wedge)$ surjects onto $F(S_{X_i})$, there is a morphism $e_{\varphi(i)}\Lambda \to F(X_i^\wedge)$. Explicitly, we will take the morphism given in this way:

The object $X_i$ is of the form $\tau_B^q H_{\varphi(i)}$ for a $q \geq 0$. If $j$ is in $\{1, \ldots, n\}$, the vector space $e_{\varphi(i)}\Lambda e_j$ is isomorphic to $\prod_{p\geq 0} \mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^p I_j, I_{\varphi(i)})$ where $I_j$ is the injective indecomposable module of $\mathsf{mod}\, kQ$ corresponding to the vertex $j$. Let $f$ be a morphism in $\mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^p I_j, I_{\varphi(i)})$, then $\tau_{\mathcal{D}}^q(f)$ is a morphism in $\mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^{p+q} I_j, \tau_{\mathcal{D}}^q I_{\varphi(i)})$, and then $P(\tau_{\mathcal{D}}^q f) = \tau_B^q P(f)$ is a morphism in $\mathcal{M}$ from $\tau_B^{p+q} H_j$ to $\tau_B^q H_{\varphi(i)} = X_i$, so is in $F(X_i^\wedge)e_j$.

*Step 1: The morphism $e_{\varphi(i)}\Lambda \to F(X_i^\wedge)$ vanishes on the ideal $\mathcal{I}_{w_i}$.*

A word $j_1 j_2 \cdots j_r$ will be called a *subword* of $w_i$ if there exist integers $1 \leq l_1 < l_2 < \cdots < l_r \leq i$ such that $j_1 j_2 \cdots j_r = \varphi(l_1)\varphi(l_2)\cdots\varphi(l_r)$. Then it is easy to check that the vector space $e_{\varphi(i)}\mathcal{I}_{w_i} e_j$ is generated by the paths from $j$ to $\varphi(i)$ such that there exists a factorization

$$j \rightsquigarrow j_1 \rightsquigarrow j_2 \rightsquigarrow \cdots \rightsquigarrow j_r \rightsquigarrow \varphi(i)$$

with $j j_1 j_2 \cdots j_r \varphi(i)$ not a subword of $w_i$.

Let $f$ be a morphism $\tau_{\mathcal{D}}^p I_j \to I_{\varphi(i)}$ in $\mathcal{I}(kQ)$ given by such a path. Assume that the image $P(\tau_{\mathcal{D}}^q f)$ of $f$ in $F(X_i^\wedge)$ is non zero. Let

$$\tau_{\mathcal{D}}^p I_j \xrightarrow{f_0} \tau_{\mathcal{D}}^{p_1} I_{j_1} \xrightarrow{f_1} \tau_{\mathcal{D}}^{p_2} I_{j_2} \xrightarrow{f_2} \cdots \longrightarrow \tau_{\mathcal{D}}^{p_r} I_{j_r} \xrightarrow{f_r} I_{\varphi(i)}$$

be the factorization of $f$ given by the above factorization of the path. Then $P(\tau_{\mathcal{D}}^q f)$ is equal to the composition

$$\tau_B^{p+q} H_j \longrightarrow \tau_B^{p_1+q} H_{j_1} \longrightarrow \tau_B^{p_2+q} H_{j_2} \longrightarrow \cdots \longrightarrow \tau_B^{p_r+q} H_{j_r} \longrightarrow \tau_B^q H_{\varphi(i)} = X_i\ .$$

Since $P(\tau_{\mathcal{D}}^q f)$ is not zero, all morphisms $P(\tau_{\mathcal{D}}^q f_l)$ are not zero, and all objects $\tau_B^{p_l+q} H_{j_l}$ are non zero. Thus the objects $\tau_B^{p_l+q} H_{j_l}$ are of the form $X_{h_l}$ with $h_0 < h_1 < \cdots < h_r < i$. Furthermore, we have $\varphi(h_l) = j_l$. Thus $j j_1 \cdots j_r \varphi(i) = \varphi(h_0)\varphi(h_1)\cdots\varphi(h_r)\varphi(i)$ is a subword of $w_i$. This contradiction shows that the image of $f$ in $F(X_i^\wedge)$ must be zero.

*Step 2: The morphism $e_{\varphi(i)}\Lambda \to F(X_i^\wedge)$ is surjective.*

Let $f$ be a morphism $\tau_B^{p+q} H_j \to \tau_B^q H_{\varphi(i)} = X_i$ in $\mathcal{M}$. Hence $\tau_B^{-q} f$ is a morphism $\tau_B^p H_j \to H_{\varphi(i)}$ in $\mathcal{M}$. Since $P$ is full (cf. proposition 6.1.4), there exists a morphism $g : \tau_{\mathcal{D}}^p I_i \to I_{\varphi(i)}$ such that $P(g) = \tau_B^{-q} f$. Thus we have $P(\tau_{\mathcal{D}}^q g) = \tau_B^q P(g) = f$.

*Step 3: The morphism $e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i} \to F(X_i^\wedge)$ is injective.*

Let $f$ be a non zero morphism $\tau_{\mathcal{D}}^p I_j \to I_{\varphi(i)}$ in $\mathcal{I}(kQ)$ such that $P(\tau_{\mathcal{D}}^q f)$ is zero. By lemma 6.1.1, we can assume that there exists a factorization of $\tau_{\mathcal{D}}^q f$ of the form

$$\tau_{\mathcal{D}}^{q+p} I_j \xrightarrow{h} Y \xrightarrow{g} \tau_{\mathcal{D}}^q I_{\varphi(i)}$$

with $Y$ indecomposable and $P(Y)=0$. The object $Y$ is of the form $\tau_{\mathcal{D}}^h I_l$ with $h \geq q$ and we have $\tau_B^h H_l = 0$.

The morphism $g$ is a sum of compositions of irreducible morphisms between indecomposables. Let

$$\tau_{\mathcal{D}}^h I_l \xrightarrow{g_0} Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \longrightarrow Y_s \xrightarrow{g_s} \tau_{\mathcal{D}}^q I_{\varphi(i)}$$

be such a summand of $g$. The objects $Y_k$, $1 \leq k \leq s$ are indecomposable and so are of the form $\tau_{\mathcal{D}}^{r_k} I_{j_k}$, and the morphisms $g_k$, $0 \leq k \leq s$ are irreducible. We will show that the word $lj_1j_2\dots j_s\varphi(i)$ is not a subword of $w_i$. Without loss of generality, we may assume that for $1 \leq k \leq s$, $P(Y_k)$ is not zero, so there exist integers $l_k$ such that $P(Y_k) = X_{l_k}$. Since the morphisms $g_k$ are irreducible, $P(g_k)$ does not vanish, and we have $1 \leq l_1 < l_2 < \cdots < l_s < i$. The word $j_1j_2\dots j_s\varphi(i)$ is equal to the word $\varphi(l_1)\varphi(l_2)\cdots\varphi(l_s)\varphi(i)$, so $j_1j_2\dots j_s\varphi(i)$ is a subword of $w_i$.

*Substep 1: The sequence $1 \leq l_1 < l_2 < \cdots < l_s < i$ is the maximal element of the set $\{1 \leq i_1 < i_2 < \cdots < i_s < i_{s+1} \leq i \quad | \quad \varphi(i_1) = j_1, \dots, \varphi(i_s) = j_s, \varphi(i_{s+1}) = \varphi(i)\}$ for the lexicographic order.*

We prove by decreasing induction that $l_k$ is the maximal integer with $l_k < l_{k+1}$ and $\varphi(l_k) = j_k$. For $k = s+1$ it is obvious. Now suppose there exists an integer $i_k$ such that $\varphi(l_k) = \varphi(i_k) = j_k$ and $l_k < i_k < l_{k+1}$. Thus by step 1 of proposition 6.3.1, there exists an integer $r \geq 1$ such that $X_{l_k} = \tau_B^r X_{i_k}$. The morphism $P(g_k): X_{l_k} \to X_{l_{k+1}}$ is irreducible, so there exists a non zero irreducible morphism $X_{l_{k+1}} \to \tau_B^{-1} X_{l_k}$. The object $\tau_B^{-1} X_{l_k}$ is in $\mathcal{M}$ since $X_{l_k}$ and $\tau_B^{-r} X_{l_k} = X_{i_k}$ are in $\mathcal{M}$. It is of the form $X_t$, and we have $l_{k+1} < t$. Since $r$ is $\geq 1$, by step 1 of proposition 6.3.1, $t$ is $\leq i_k$. This implies $l_{k+1} < i_k$ which is a contradiction.

*Substep 2: $l$ does not belong to the set $\{\varphi(1), \varphi(2), \dots, \varphi(l_1 - 1)\}$.*

Suppose that there exists an integer $1 \leq k \leq N$ such that $\varphi(k)$ is equal to $l$. Thus there exists an integer $r \geq 0$ such that $X_k$ is equal to $\tau_B^r H_l$. Since $\tau_B^h H_l = P(\tau_{\mathcal{D}}^h I_l)$ is zero, $r$ is $\leq h-1$.

Since the morphism $g_0 : \tau_{\mathcal{D}}^h I_l \to Y_1$ is an irreducible morphism of $\mathcal{I}(kQ)$, there exists an irreducible morphism $Y_1 \to \tau_{\mathcal{D}}^{h-1} I_l$ in $\mathcal{I}(kQ)$. Thus there exists an irreducible morphism $\tau_{\mathcal{D}}^{r-h+1} Y_1 \to \tau_{\mathcal{D}}^r I_l$ in $\mathcal{I}(kQ)$. The object $P(\tau_{\mathcal{D}}^r I_l) = \tau_B^r H_l = X_k$ is not zero and lies in $\mathcal{M}$, so the object $P(\tau_{\mathcal{D}}^{r-h+1} Y_1) = \tau_B^{r-h+1} X_{l_1}$ is not zero and lies in $\mathcal{M}$ since $\mathcal{M}$ is stable by kernel. Thus there is an irreducible morphism $\tau_B^{r-h+1} X_{l_1} = X_t \to X_k$ in $\mathcal{M}$. Therefore $t$ has to be $< k$. Moreover since $r-h+1 \leq 0$, by step 1 of proposition 6.3.1, $l_1$ is $\leq s$. Finally we get $l_1 < k$.

Then combining substep 1 and substep 2, we can prove that $lj_1j_2\dots j_s\varphi(i)$ can not be a subword of $w_i$. Indeed, assume $lj_1j_2\dots j_s\varphi(i)$ is a subword of $w_i$. Thus there exist $1 \leq i_0 < i_1 < \dots < i_s < i_{s+1} \leq i$ such that $\varphi(i_0)\varphi(i_1)\dots\varphi(i_{s+1}) = lj_1j_2\dots j_s\varphi(i)$. In particular, the word $j_1 j_2 \ldots j_s \varphi(i)$ is a subword of $w_i$ and $1 \leq i_1 < \ldots < i_s < i_{s+1} \leq i$ is in the set of substep 1. Thus by substep 1, $i_1$ has to be $\leq l_1$. By substep 2, $i_0$ can not exist.

□

### 6.3.4 Endomorphism algebra of the cluster-tilting object

**Lemma 6.3.1.** *Let $X_i$ and $X_j$ be indecomposables of $\mathcal{M}$. Then we have an isomorphism of vector space*

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq \bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p X_j, X_i).$$

*Proof. Case 1: $j \geq i$*
Then by [BIRS07] (lemma II.1.14) we have an isomorphism

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i} e_{\varphi(j)}.$$

By proposition 6.3.2, this is isomorphic to the space

$$\bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i).$$

By definition of the function $\varphi$, there exists some $q \geq 1$ such that $X_j = \tau_B^q H_{\varphi(j)}$. Thus we can write the sum

$$\bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) = \bigoplus_{p=1}^{q} \mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p X_j, X_i)$$

Since $j \geq i$, there is no morphism from $\tau_B^{-p} X_j$ to $X_i$ for $p \geq 1$, and the first summand is zero. Therefore we get the result.

*Case 2: $j < i$*
Then by [BIRS07] (lemma II.1.14) we have an isomorphism

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)}(\mathcal{I}_{\varphi(i)} \ldots \mathcal{I}_{\varphi(j+1)}/\mathcal{I}_{w_i}) e_{\varphi(j)}.$$

By proposition 6.3.2, this space is a subspace of the space

$$\bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) \simeq \bigoplus_{p\geq 1} \mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p\geq 0} \mathcal{M}(\tau_B^p X_j, X_i).$$

*Step 1: If $f$ is a non zero morphism in $\mathcal{M}(\tau_B^{-p}X_j, X_i)$ with $p \geq 1$ then $f$ is not in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)} \dots \mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.*

Let $X_{l_0}$ be the indecomposable $\tau_B^{-p}X_j$. Since $p \geq 1$ then $l_0$ is $\leq j+1$. The morphism is a sum of composition of the form

$$X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_i$$

with the $X_{l_k}$ indecomposables. Since $f$ is not zero, we have $j+1 \leq l_0 < l_1 < \dots < l_r < i$. Thus the word $\varphi(l_0)\varphi(l_1)\dots\varphi(l_r)\varphi(i)$ is a subword of $\varphi(j+1)\varphi(j+2)\dots\varphi(i)$. Since it holds for each factorization of $f$, the morphism $f$ is not in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)} \dots \mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.

*Step 2: If $f$ is a morphism in $\mathcal{M}(\tau_B^{p}X_j, X_i)$ with $p \geq 0$ then $f$ is in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)} \dots \mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.*

Let $X_{l_0}$ be the indecomposable $\tau_B^{p}X_j$. Since $p$ is $\geq 0$, we have $l_0 \leq j$. Let us show that if $f$ is a composition of irreducible morphisms

$$X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_{l_{r+1}} = X_i$$

then the word $\varphi(l_0)\varphi(l_1)\cdots\varphi(l_r)\varphi(i)$ is not a subword of $\varphi(j+1)\varphi(j+2)\dots\varphi(i)$.

We have $l_0 < l_1 < \cdots < l_r < i$. Since $l_0$ is $< j+1$, and $i$ is $\leq j+1$, there exists $1 \leq k \leq r+1$ such that $l_{k-1} < j+1 \leq l_k$. Therefore $\varphi(l_k)\dots\varphi(l_r)\varphi(i)$ is a subword of $\varphi(j+1)\varphi(j+2)\dots\varphi(i)$, and the sequence $l_k < l_{k+1} < \cdots < l_r < i$ is the maximal element of the set

$$\{j+1 \leq i_k < \cdots < i_{r+1} \leq i \quad | \quad \varphi(i_k) = \varphi(l_k), \dots, \varphi(i_r) = \varphi(l_r), \varphi(i_{r+1}) = \varphi(i)\}$$

for the lexicographic order (exactly for the same reasons as in substep 1 of proposition 6.3.2). Now we can prove exactly as in substep 2 of proposition 6.3.2 that $\varphi(l_{k-1})$ does not belong to the set $\{\varphi(j+1), \dots, \varphi(l_k - 1)\}$. Thus $\varphi(l_{k-1})\varphi(l_k)\dots\varphi(l_r)\varphi(i)$ can not be a subword of $\varphi(j+1)\varphi(j+2)\dots\varphi(i)$.

Finally, let $f = f_1 + f_2$ be a morphism in

$$\bigoplus_{p \geq 0} \mathcal{M}(\tau_B^{p}H_{\varphi(j)}, X_i) \simeq \bigoplus_{p \geq 1} \mathcal{M}(\tau_B^{-p}X_j, X_i) \oplus \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^{p}X_j, X_i).$$

By step 2, $f_2$ is in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)} \dots \mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$. By step 1, the morphism $f$ is in $e_{\varphi(i)}\mathcal{I}_{\varphi(i)} \dots \mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$ if and only if $f-1$ is zero. Thus we get an isomorphism

$$\mathsf{Hom}_{\Lambda}(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^{p}X_j, X_i).$$

□

**Corollary 6.7.** *If $X_i$ and $X_j$ are indecomposables of $\overline{\mathcal{M}}$, then we have*

$$\mathsf{Hom}_{\underline{Sub}\Lambda/\mathcal{I}_w}(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{X_j}\tilde{A}e_{X_i}.$$

*Proof.* The proof is exactly the same as the proof of corollary 6.4. □

### 6.3.5 Construction of the triangle equivalence

Finally, we can formulate the following theorem:

**Theorem 6.8.** *The functor $F \circ i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to f.l.\Lambda$ induces an algebraic triangle equivalence between $\mathcal{C}_{\overline{\mathcal{M}}}$ and $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$.*

*Proof.* For an indecomposable $X$ in $\mathcal{M}$, we have seen that $F(X^\wedge)$ is in $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. Thus $F$ induces a triangulated functor $F : \mathcal{D}^b(\mathsf{mod}\,\mathcal{M}) \to \mathcal{D}^b(\mathsf{Sub}\,\Lambda/\mathcal{I}_w)$. Thus we have the following composition:

$$\mathcal{D}^b(\mathsf{mod}\,\overline{\mathcal{M}}) \simeq \mathcal{D}^b(\mathsf{mod}\,A) \xrightarrow{i_*} \mathcal{D}^b(\mathsf{mod}\,\mathcal{M}) \xrightarrow{F} \mathcal{D}^b(\mathsf{Sub}\,\Lambda/\mathcal{I}_w) \longrightarrow \underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$$

with the endofunctor $? \overset{L}{\otimes}_A DA[-2]$ acting on $\mathcal{D}^b(\mathsf{mod}\,A)$.

As previously, the functor $F \circ i_*$ is triangulated algebraic. It is isomorphic to the functor $? \overset{L}{\otimes}_A \underline{R}$ where $\underline{R} = F \circ i_*(A)$ is a cluster-tilting object of $\underline{\mathsf{Sub}}_\Lambda/\mathcal{I}_w$. Let $X$ be an indecomposable of $\overline{\mathcal{M}}$. By proposition 6.1.5, we have the following exact sequence functorial in $X$:

$$0 \longrightarrow F \circ i_*(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F \circ i_*(X^\vee) \longrightarrow 0$$

with $H_0$ and $H_1$ in $add(H)$.

By lemma 6.3.2, we know that

$$\Lambda/\mathcal{I}_w \simeq \bigoplus_{i=1}^{n} \Lambda/\mathcal{I}_w e_i \simeq \bigoplus_{i=1}^{n} F(X_{N-i}^\wedge) \simeq \bigoplus_{i=1}^{n} F(H_i^\wedge)$$

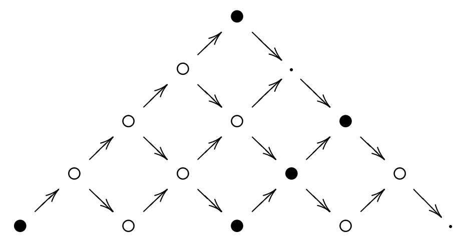
Thus $F(H_0^\wedge)$ and $F(H_1^\wedge)$ are projective-injective in $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. Now we conclude as in the proof of theorem 6.5. We get a triangle algebraic functor $F : \mathcal{C}_A \to \underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ which sends the orbit-cluster-tilting object $A$ onto the cluster-tilting object $T = \overline{\bigoplus_{i=1}^{N}} e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}$. Moreover, by corollary 6.7 it induces an equivalence between the subcategories $add(A)$ and $add(T)$. Thus by lemma 5.4.1 this is an algebraic equivalence. □

### 6.3.6 Example: Dynkin case

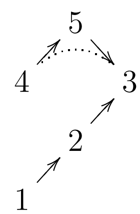
Let $Q$ be $A_5$ with the following orientation:

$$Q : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5\,,$$

and $T$ the following tilting module:



Then the algebra $B = \mathsf{End}_{kQ}(T)$ is given by the quiver:




with the relation given by the dotted line. On figure 6.5 we can see the Auslander-Reiten quiver of the category $\mathsf{mod}\, B$, the subcategory $\mathcal{M}$ is formed by the framed modules.

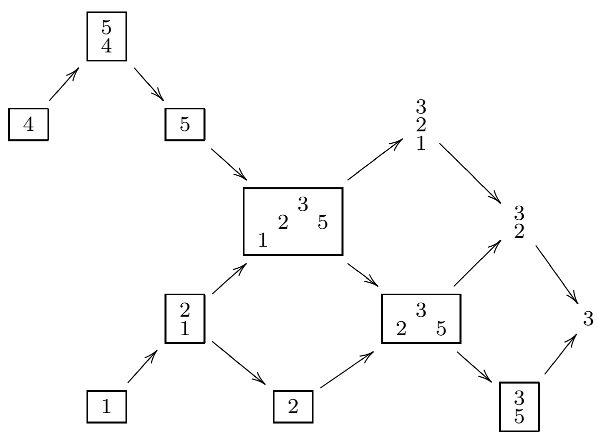
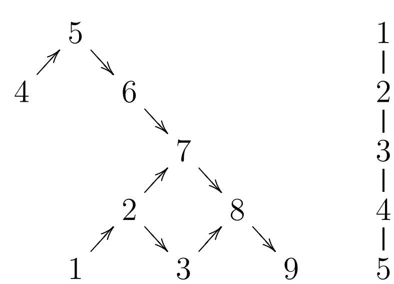


Figure 6.5: Category $\mathcal{M} \subset \mathsf{mod}\, B$

Now put an order on the indecomposables of $\mathcal{M}$, respecting the morphisms. For example:



The indecomposable projectives of $\mathsf{mod}\,\mathcal{M}$ are the following:

$$1,\quad \begin{smallmatrix}2\\1\end{smallmatrix},\quad \begin{smallmatrix}3\\2\end{smallmatrix},\quad 4,\quad \begin{smallmatrix}5\\4\end{smallmatrix},\quad \begin{smallmatrix}6\\5\end{smallmatrix},\quad \begin{smallmatrix}&&7&&\\&2&&6&\\1&&&&5\end{smallmatrix}\quad \begin{smallmatrix}&&&8&\\&&7&&3\\&6&&2&\\5&&&&\end{smallmatrix},\quad \begin{smallmatrix}9\\8\\7\\6\\5\end{smallmatrix}$$

The projective-injectives of $\mathsf{mod}\,\Lambda(A_5)$ have the following form:

$$\begin{smallmatrix}1\\2\\3\\4\\5\end{smallmatrix},\quad \begin{smallmatrix}&2&&&\\1&&3&&\\&2&&4&\\&&3&&5\\&&&4&\end{smallmatrix},\quad \begin{smallmatrix}&&3&&\\&2&&4&\\1&&3&&5\\&2&&4&\\&&3&&\end{smallmatrix},\quad \begin{smallmatrix}&&&4&\\&&3&&5\\&2&&4&\\1&&3&&\\&2&&&\end{smallmatrix},\quad \begin{smallmatrix}5\\4\\3\\2\\1\end{smallmatrix}$$

With the ordering we put on the quiver of $\mathcal{M}$, we get the word $w = 545212345$ of the Coxeter group of $A_5$. Then it is easy to compute:

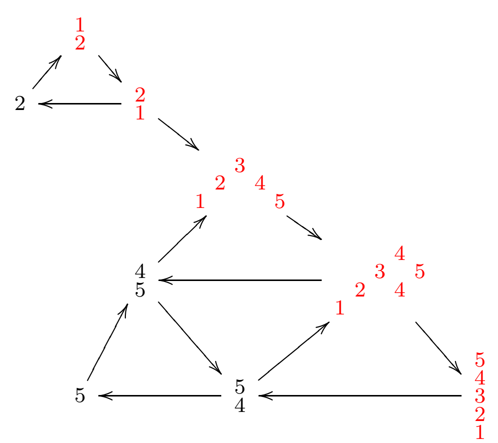
$$\Lambda/\mathcal{I}_w = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&&3&&\\&2&&4&\\1&&&&5\end{smallmatrix} \oplus \begin{smallmatrix}&&&4&\\&&3&&5\\&2&&4&\\1&&&&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\\2\\1\end{smallmatrix}$$

The rigid maximal object associated to this word $w$ is the following:

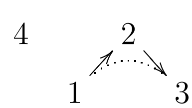
$$R = 5 \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&&3&&\\&2&&4&\\1&&&&5\end{smallmatrix} \oplus \begin{smallmatrix}&&&4&\\&&3&&5\\&2&&4&\\1&&&&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\\2\\1\end{smallmatrix}$$

One can see easily, that this is the image of the projectives of $\mathcal{M}$ through the morphism $F : \mathsf{mod}\,\mathcal{M} \to f.l.\Lambda$.

Then it is not difficult to compute the Auslander-Reiten quiver of $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. It is shown here in figure 6.3.6. The indecomposables of the cluster-tilting object are framed, and the projective-injectives are framed in red. We can easily check that the stable category $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ is equivalent to $\mathcal{C}_{A_3} \times \mathcal{C}_{A_1}$ and that the endomorphism algebra of the tilting object has the form:



Moreover, the category $\overline{\mathcal{M}}$ has the following quiver:



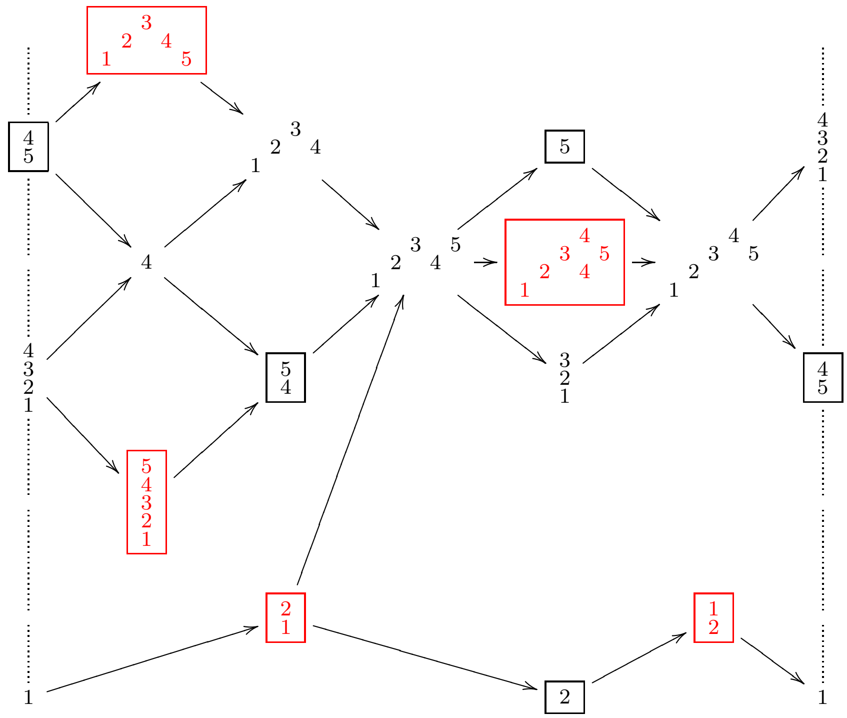
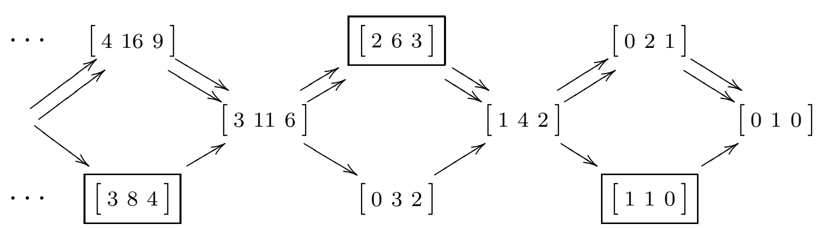


Figure 6.6: Auslander-Reiten quiver of the Frobenius category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$

Thus $\overline{\mathcal{M}}$ is equivalent to the direct sum of $k$ with the endomorphism algebra of a tilting module $T'$ of $\mathsf{mod}\,kA_3$ with the usual orientation. Thus the derived category of $\overline{\mathcal{M}}$ is equivalent to $\mathcal{D}^b(\mathsf{mod}\,k) \times \mathcal{D}^b(\mathsf{mod}\,A_3)$. Finally, the cluster category of $\overline{\mathcal{M}}$ is equivalent $\mathcal{C}_{A_1} \times \mathcal{C}_{A_3}$. This confirms theorem **??**.

### 6.3.7 Example: Non-Dynkin case

Let $Q$ be the following quiver: $1 \longrightarrow 2 \leftleftarrows 3$. The preinjective component of $modkQ$ looks as follows:

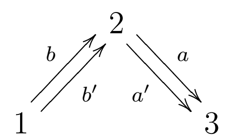



Here we denote the $kQ$-modules by their dimension vector in order to lighten the writing. For example the module $[\,1\;4\;2\,]$ has the following decomposition series: ${}^{2}{}_{3}{}^{2}{}_{1}{}^{2}{}_{3}{}^{2}$.

If we mutate the tilting object $[\,2\;6\;3\,] \oplus [\,1\;4\;2\,] \oplus [\,1\;1\;0\,]$ in the direction $[\,1\;4\;2\,]$, we stay in the preinjective component. We get the tilting object:

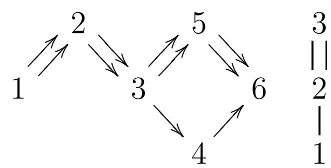
$$T = \left[\,2\ 6\ 3\,\right] \oplus \left[\,3\ 8\ 4\,\right] \oplus \left[\,1\ 1\ 0\,\right].$$

The algebra $B = \mathsf{End}_{kQ}(T)$ is given by the quiver:




with the relation $\quad ba + b'a' = 0.$

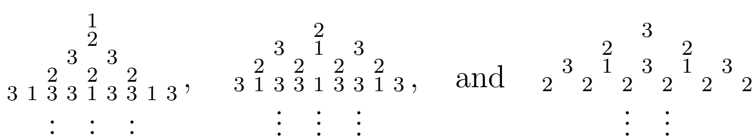
The category $\mathcal{M} \subset \mathsf{mod}\, B$, has the following Auslander-Reiten quiver:



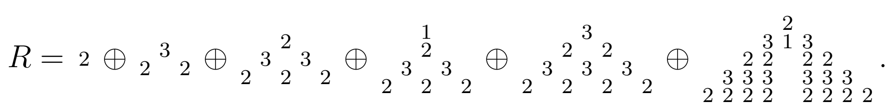

The projective indecomposables of $\mathsf{mod}\,\mathcal{M}$ are the followings:

$$\begin{bmatrix} & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & & 0 & \end{bmatrix},\quad \begin{bmatrix} & 1 & & 0 & \\ 2 & & 0 & & 0 \\ & & & 0 & \end{bmatrix},\quad \begin{bmatrix} & 2 & & 0 & \\ 3 & & 1 & & 0 \\ & & & 0 & \end{bmatrix},\quad \begin{bmatrix} & 2 & & 0 & \\ 3 & & 1 & & 0 \\ & & & 1 & \end{bmatrix},\quad \begin{bmatrix} & 3 & & 1 & \\ 4 & & 2 & & 0 \\ & & & 0 & \end{bmatrix},\quad \begin{bmatrix} & 6 & & 2 & \\ 8 & & 4 & & 1 \\ & & & 1 & \end{bmatrix}$$

The associated word is $w = 232132$. The projectives of the preprojective algebra associated to $Q$ have the following composition series:



The maximal rigid object of the category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ associated to the writing of $w = 232132$ is



The last three summands are the projective-injectives of the Frobenius category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. If we write these modules with their dimension vectors we get:

$$R = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 8 \\ 13 \\ 1 \end{bmatrix}$$

and it is easy to check that this module corresponds to the projection of $T$.

Now take the module $X = 1$ in $\mathcal{M}$. It corresponds to the module $\left[\,3\ 8\ 4\,\right]$ in $\mathsf{mod}\, kQ$. We have the injective resolution in $\mathsf{mod}\, kQ$:

$$0 \longrightarrow \left[\,3\ 8\ 4\,\right] \longrightarrow \left[\,0\ 2\ 1\,\right]^4 \oplus \left[\,1\ 1\ 0\,\right]^3 \longrightarrow \left[\,0\ 1\ 0\,\right]^3 \longrightarrow 0$$

Thus the short exact sequence in $\mathcal{M}$ $0 \longrightarrow X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow 0$ is the following:

$$0 \longrightarrow 1 \longrightarrow 4^3 \oplus 5^4 \longrightarrow 6^3 \longrightarrow 0$$

Therefore, the sequence $0 \longrightarrow X^\wedge \longrightarrow H_0^\wedge \longrightarrow H_1^\wedge \longrightarrow (\tau^{-1}X)^\vee/\mathsf{proj}\, B \longrightarrow 0$ in $\mathsf{mod}\, \mathcal{M}$ becomes:

$$0 \longrightarrow \left[\begin{smallmatrix} & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & & 0 & \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} & 2 & & 0 & \\ 3 & & 1 & & 0 \\ & & & 1 & \end{smallmatrix}\right]^3 \oplus \left[\begin{smallmatrix} & 3 & & 1 & \\ 4 & & 2 & & 0 \\ & & & 0 & \end{smallmatrix}\right]^4 \longrightarrow \left[\begin{smallmatrix} & 6 & & 2 & \\ 8 & & 4 & & 1 \\ & & & 1 & \end{smallmatrix}\right]^3 \longrightarrow \left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 3 \\ & & & 0 & \end{smallmatrix}\right] \longrightarrow 0$$

where $\left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 3 \\ & & & 0 & \end{smallmatrix}\right]$ is the quotient of $(\tau_B^{-1}1)^\vee = 3^\vee = \left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 4 \\ & & & 1 & \end{smallmatrix}\right]$ by the projectives. Applying the projection functor we get the exact sequence in $\mathsf{mod}\, \Lambda$:

$$0 \longrightarrow \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 2 \\ 4 \\ 1 \end{smallmatrix}\right]^3 \oplus \left[\begin{smallmatrix} 4 \\ 6 \\ 0 \end{smallmatrix}\right]^4 \longrightarrow \left[\begin{smallmatrix} 8 \\ 13 \\ 1 \end{smallmatrix}\right]^3 \longrightarrow \left[\begin{smallmatrix} 2 \\ 4 \\ 0 \end{smallmatrix}\right] \longrightarrow 0$$

The $\overline{\mathcal{M}}$-module $1^\vee_{|\overline{\mathcal{M}}}$ is $\left[\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & & & 0 \end{smallmatrix}\right] = \begin{smallmatrix} 3 & & 3 & & 3 \\ & 2 & & 2 & \\ & & 1 & & \end{smallmatrix}$. We have $F \circ i_*(1^\vee_{|\overline{\mathcal{M}}}) = \left[\begin{smallmatrix} 2 \\ 4 \\ 0 \end{smallmatrix}\right]$. By the exact sequence above, there is an isomorphism in $\underline{\mathsf{Sub}}\, \Lambda/\mathcal{I}_w$ between $F \circ i_*(1^\wedge_{|\overline{\mathcal{M}}})$ and $F \circ i_*(1^\vee_{|\overline{\mathcal{M}}})[2]$.

# Chapter 7

# Cluster-tilting object

Let $k$ be a field and $A$ a differential graded (=dg) $k$-algebra. We denote by $A^e$ the dg algebra $A^{op} \otimes A$. We denote by $\mathcal{D} = \mathcal{D}A$ the derived category of dg $A$-modules and by $\mathcal{D}^b A$ its full subcategory formed by the dg $A$-modules whose homology is of finite total dimension over $k$. We write $\mathsf{per}\, A$ for the category of perfect dg $A$-modules, *i.e.* the smallest triangulated subcategory of $\mathcal{D}A$ containing $A$ and stable under taking direct summands. The suspension functors of all these categories will be denoted by [1].

Suppose that $A$ has the following properties:

- $A$ is homologically smooth (*i.e.* the object $A$, viewed as an $A^e$-module, is perfect);
- for each $p > 0$, the space $H^p A$ is zero;
- the space $H^0 A$ is finite dimensional;
- $A$ is bimodule 3-Calabi-Yau, *i.e.* there is an isomorphism in $\mathcal{D}(A^e)$

$$R\mathsf{Hom}_{A^e}(A, A^e) \simeq A[-3].$$

Since $A$ is homologically smooth, the category $\mathcal{D}^b A$ is a subcategory of $\mathsf{per}\, A$ (see [Kel08a], lemma 4.1). We denote by $\pi$ the canonical projection functor $\pi : \mathsf{per}\, A \to \mathcal{C} = \mathsf{per}\, A/\mathcal{D}^b A$. Moreover, by the same lemma, there is a bifunctorial isomorphism

$$D\mathsf{Hom}_{\mathcal{D}}(L, M) \simeq \mathsf{Hom}_{\mathcal{D}}(M, L[3])$$

for all objects $L$ in $\mathcal{D}^b A$ and all objects $M$ in $\mathsf{per}\, A$. We call this property the *Calabi-Yau property.*

The aim of this chapter is to show the following result:

**Theorem 7.1.** *Let $A$ be a dg $k$-algebra with the above properties. Then the category $\mathcal{C} = \mathsf{per}\, A/\mathcal{D}^b A$ is $\mathsf{Hom}$-finite and 2-Calabi-Yau. Moreover, the object $\pi(A)$ is a cluster-tilting object. Its endomorphism algebra is isomorphic to $H^0 A$.*

# 7.1 t-structure on $\mathsf{per}\, A$

The main tool of the proof of theorem 7.1 is the existence of a canonical $t$-structure in $\mathsf{per}\, A$.

## 7.1.1 Construction of a $t$-structure on $\mathcal{D}A$

Let $\mathcal{D}_{\leq 0}$ be the full subcategory of $\mathcal{D}$ whose objects are the dg modules $X$ such that $H^pX$ vanishes for all $p > 0$.

**Lemma 7.1.1.** *The subcategory $\mathcal{D}_{\leq 0}$ is an aisle in the sense of Keller-Vossieck [KV88].*

*Proof.* The canonical morphism $\tau_{\leq 0}A \to A$ is a quasi-isomorphism of dg algebras. Thus we can assume that $A^p$ is zero for all $p > 0$. The full subcategory $\mathcal{D}_{\leq 0}$ is stable under $X \mapsto X[1]$ and under extensions. We claim that the inclusion $\mathcal{D}_{\leq 0} \hookrightarrow \mathcal{D}$ has a right adjoint. Indeed, for each dg A-module $X$, the dg A-module $\tau_{\leq 0}X$ is a dg submodule of $X$, since $A$ is concentrated in negative degrees. Thus $\tau_{\leq 0}$ is a well-defined functor $\mathcal{D} \to \mathcal{D}_{\leq 0}$. One can check easily that $\tau_{\leq 0}$ is the right adjoint of the inclusion.

□

**Proposition 7.1.1.** *Let $\mathcal{H}$ be the heart of the $t$-structure, i.e. $\mathcal{H}$ is the intersection $\mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. We have the following properties:*

*(i) The functor $H^0$ induces an equivalence from $\mathcal{H}$ onto $\mathsf{Mod}\, H^0A$.*

*(ii) For all $X$ and $Y$ in $\mathcal{H}$, we have an isomorphism $\mathsf{Ext}^1_{H^0A}(X,Y) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[1])$.*

Note that it is not true for general $i$ that $\mathsf{Ext}^i_{\mathcal{H}}(X,Y) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[i])$.

*Proof.* (i) We may assume that $A^p = 0$ for all $p > 0$. Then we have a canonical morphism $A \to H^0A$. The restriction along this morphism yields a functor $\Phi : \mathsf{Mod}\, H^0A \to \mathcal{H}$ such that $H^0 \circ \Phi$ is the identity of $\mathsf{Mod}\, H^0A$. Thus the functor $H^0 : \mathcal{H} \to \mathsf{Mod}\, H^0A$ is full and essentially surjective. Moreover, it is exact and an object $N \in \mathcal{H}$ vanishes if and only if $H^0N$ vanishes. Thus if $f : L \to M$ is a morphism of $\mathcal{H}$ such that $H^0(f) = 0$, then $\mathsf{Im} H^0(f) = 0$ implies that $H^0(\mathsf{Im} f) = 0$ and $\mathsf{Im} f = 0$, so $f = 0$. Thus $H^0 : \mathcal{H} \to \mathsf{Mod}\, H^0A$ is also faithful.

(ii) By section 3.1.7 of [BBD82] there exists a triangle functor $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}$ which yields for $X$ and $Y$ are in $\mathcal{H}$ and for $n \leq 1$ an isomorphism (remark (ii) section 3.1.17 p.85)

$$\mathsf{Hom}_{\mathcal{D}\mathcal{H}}(X, Y[n]) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[n]).$$

Applying this for $n = 1$ and using (i), we get the result.

□

### 7.1.2 Hom-finiteness

**Proposition 7.1.2.** *The category* $\mathsf{per}\, A$ *is* $\mathsf{Hom}$*-finite.*

**Lemma 7.1.2.** *For each $p$, the space $H^p A$ is finite dimensional.*

*Proof.* By hypothesis, $H^p A$ is zero for $p > 0$. We prove by induction on $n$ the following statement: The space $H^{-n}A$ is finite dimensional, and for all $p \geq n+1$ the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite dimensional for each $M$ in $\mathsf{mod}\, H^0 A$.

For $n = 0$, the space $H^0 A$ is finite dimensional by hypothesis. Let $M$ be in $\mathsf{mod}\, H^0 A$. Since $\tau_{\leq 0} A$ is ismorphic to $A$, $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq 0}A, M[p])$ is isomorphic $H^0(M[p])$, and so is zero for $p \geq 1$.

Suppose that the property holds for $n$. Form the triangle:

$$(H^{-n}A)[n-1] \longrightarrow \tau_{\leq -n-1}A \longrightarrow \tau_{\leq -n}A \longrightarrow (H^{-n}A)[n]$$

Let $p$ be an integer $\geq n+1$. Applying the functor $\mathsf{Hom}_{\mathcal{D}}(?, M[p])$ we get the long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p]) \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A, M[p]) \to \mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p]) \to \cdots .$$

By induction the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite dimensional.

Since $M[p]$ is in $\mathcal{D}^b A$ we can apply the Calabi-Yau property. If $p$ is $\geq n+3$, we have isomorphisms:

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[p-n+1]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M[p-n-2], H^{-n}A).\end{aligned}$$

Since $p-n-2$ is $\geq 1$, this space is zero.

If $p = n+2$, we have the following isomorphisms.

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+2]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[3]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M, H^{-n}A) \\ &\simeq D\mathsf{Hom}_{H^0 A}(M, H^{-n}A).\end{aligned}$$

The last isomorphism comes from lemma 7.1.1 $(i)$. By induction, the space $H^{-n}A$ is finite dimensional. Thus for $p \geq n+2$, the space $\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p])$ is finite dimensional.

If $p = n+1$ we have the following isomorphisms:

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+1]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[2]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M, H^{-n}A[1]) \\ &\simeq D\mathsf{Ext}^1_{H^0 A}(M, H^{-n}A)\end{aligned}$$

The last isomorphism comes from lemma 7.1.1 $(ii)$. By induction, since $H^{-n}A$ is finite dimensional, the space $\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+1])$ is finite dimensional and so is $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A, M[n+1])$.

Now, look at the triangle

$$\begin{array}{ccccccc} \tau_{\leq -n-2}A & \longrightarrow & \tau_{\leq -n-1}A & \longrightarrow & (H^{-n-1}A)[n+1] & \longrightarrow & (\tau_{\leq -n-2}A)[1]\,. \\ & \overset{0}{\searrow} & \downarrow & \swarrow & & \overset{0}{\swarrow} & \\ & & M[n+1] & & & & \end{array}$$

The spaces $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-2}A, M[n+1])$ and $\mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -n-2}A)[1], M[n+1])$ vanish since $M[n+1]$ is in $\mathcal{D}_{\geq -n-1}$. Thus we have

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A[n-1], M[n+1]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n-1}A)[n+1], M[n+1]) \\ &\simeq \mathsf{Hom}_{\mathcal{D}}(H^{-n-1}A, M) \\ &\simeq \mathsf{Hom}_{H^0A}(H^{-n-1}A, M). \end{aligned}$$

This holds for all finite dimensional $H^0A$-modules $M$. Thus it holds for the compact cogenerator $M = DH^0A$. The space $\mathsf{Hom}_{H^0A}(H^{-n-1}A, DH^0A) \simeq DH^{-n-1}A$ is finite dimensional, and therefore $H^{-(n+1)}A$ is finite dimensional. □

*Proof.* (of proposition 7.1.2) For each integer $p$, the space $\mathsf{Hom}_{\mathcal{D}}(A, A[p]) \simeq H^p(A)$ is finite dimensional by lemma 7.1.2. The subcategory of $(\mathsf{per}\,A)^{op} \times \mathsf{per}\,A$ whose objects are the pairs $(X,Y)$ such that $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ is finite dimensional is stable under extensions and passage to direct factors. □

### 7.1.3 Restriction of the $t$-structure to $\mathsf{per}\,A$

**Lemma 7.1.3.** *For each $X$ in $\mathsf{per}\,A$, there exist integers $N$ and $M$ such that $X$ belongs to $\mathcal{D}_{\leq N}$ and ${}^{\perp}\mathcal{D}_{\leq M}$.*

*Proof.* The object $A$ belongs to $\mathcal{D}_{\leq 0}$. Moreover, since for $X$ in $\mathcal{D}A$, the space $\mathsf{Hom}_{\mathcal{D}}(A, X)$ is isomorphic to $H^0X$, the dg module $A$ belongs to ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus the property is true for $A$. For the same reasons, it is true for all shifts of $A$. Moreover, this property is clearly stable under taking direct summands and extensions. Thus it holds for all objects of $\mathsf{per}\,A$. □

This lemma implies the following result:

**Proposition 7.1.3.** *The $t$-structure on $\mathcal{D}A$ restricts to $\mathsf{per}\,A$.*

*Proof.* Let $X$ be in $\mathsf{per}\,A$, and look at the canonical triangle:

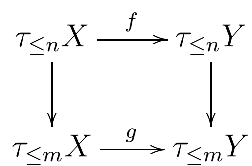
$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{>0}X \longrightarrow (\tau_{\leq 0}X)[1].$$

Since $\mathsf{per}\,A$ is $\mathsf{Hom}$-finite, the space $H^pX \simeq \mathsf{Hom}_{\mathcal{D}}(A, X[p])$ is finite dimensional for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$ by lemma 7.1.3. Thus the object $\tau_{>0}X$ is in $\mathcal{D}^bA$ and so in $\mathsf{per}\,A$. Since $\mathsf{per}\,A$ is a triangulated subcategory, it follows that $\tau_{\leq 0}X$ also lies in $\mathsf{per}\,A$. □

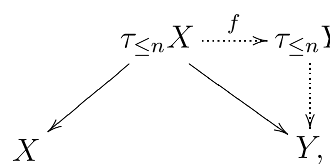
**Proposition 7.1.4.** *Let $\pi$ be the projection $\pi : \mathsf{per}\,A \to \mathcal{C}$. Then for $X$ and $Y$ in $\mathsf{per}\,A$, we have*

$$\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y) = \varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$$

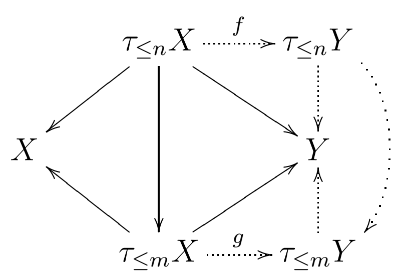
*Proof.* Let $X$ and $Y$ be in $\mathsf{per}\,A$. An element of $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ is an equivalence class of morphisms $\tau_{\leq n}X \to \tau_{\leq n}Y$. Two morphisms $f : \tau_{\leq n}X \to \tau_{\leq n}Y$ and $g : \tau_{\leq m}X \to \tau_{\leq m}Y$ with $m \geq n$ are equivalent if there is a commutative square:




where the vertical arrows are the canonical morphisms. If $f$ is a morphism $f : \tau_{\leq n}X \to \tau_{\leq n}Y$, we can form the following morphism from $X$ to $Y$ in $\mathcal{C}$:

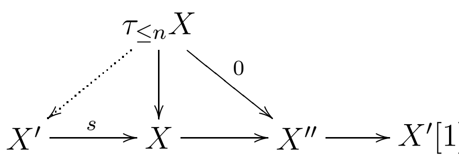



where the morphisms $\tau_{\leq n}X \to X$ and $\tau_{\leq n}Y \to Y$ are the canonical morphisms. This is a morphism from $\pi X$ to $\pi Y$ in $\mathcal{C}$ because the cone of the morphism $\tau_{\leq n}X \to X$ is in $\mathcal{D}^bA$. Moreover, if $f : \tau_{\leq n}X \to \tau_{\leq n}Y$ and $g : \tau_{\leq m}X \to \tau_{\leq m}Y$ are equivalent, there is an equivalence of diagrams:

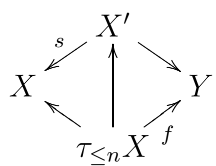



Thus we have a well-defined map from $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ to $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$ which is injective.

Now let $X \leftarrow X' \xrightarrow{s} Y$ be a morphism in $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$. Let $X''$ be the cone of $s$. It is an object of $\mathcal{D}^bA$, and therefore lies in $\mathcal{D}_{>n}$ for some $n \ll 0$. Thus there are no morphisms from $\tau_{\leq n}X$ to $X''$ and there is a factorization:




We obtain an isomorphism of diagrams:




The morphism $f : \tau_{\leq n}X \to Y$ induces a morphism $f' : \tau_{\leq n}X \to \tau_{\leq n}Y$ which lifts the given morphism. Thus the map from $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ to $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$ is surjective. $\square$

## 7.2 Fundamental domain

Let $\mathcal{F}$ be the following subcategory of $\mathsf{per}\, A$:

$$\mathcal{F} = \mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \mathsf{per}\, A.$$

The aim of this section is to show:

**Proposition 7.2.1.** *The projection functor $\pi : \mathsf{per}\, A \to \mathcal{C}$ induces a $k$-linear equivalence between $\mathcal{F}$ and $\mathcal{C}$.*

### 7.2.1 $add(A)$-approximation for objects of the fundamental domain

**Lemma 7.2.1.** *For each object $X$ of $\mathcal{F}$, there exists a triangle*

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1]$$

*with $P_0$ and $P_1$ in $add(A)$.*

*Proof.* For $X$ in $\mathsf{per}\, A$, the morphism

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(A, X) & \to & \mathsf{Hom}_{\mathcal{H}}(H^0A, H^0X) \\ f & \mapsto & H^0(f) \end{array}$$

is an isomorphism since $\mathsf{Hom}_{\mathcal{D}}(A,X) \simeq H^0X$. Thus it is possible to find a morphism $P_0 \to X$, with $P_0$ a free dg $A$-module, inducing an epimorphism $H^0P_0 \twoheadrightarrow H^0X$. Now take $X$ in $\mathcal{F}$ and $P_0 \to X$ as previously and form the triangle

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1].$$

*Step 1: The object $P_1$ is in $\mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -1}$.*

The objects $X$ and $P_0$ are in $\mathcal{D}_{\leq 0}$, so $P_1$ is in $\mathcal{D}_{\leq 1}$. Moreover, since $H^0P_0 \to H^0X$ is an epimorphism, $H^1(P_1)$ vanishes and $P_1$ is in $\mathcal{D}_{\leq 0}$.

Let $Y$ be in $\mathcal{D}_{\leq -1}$, and look at the long exact sequence:

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_0,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_1,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X[-1],Y) \longrightarrow \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[-1],Y)$ vanishes since $X$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$ and $Y$ is in $\mathcal{D}_{\leq -1}$. The object $P_0$ is free, and $H^0Y$ is zero, so the space $\mathsf{Hom}_{\mathcal{D}}(P_0,Y)$ also vanishes. Consequently, the object $P_1$ is in ${}^{\perp}\mathcal{D}_{\leq -1}$.

*Step 2: $H^0P_1$ is a projective $H^0A$-module.*

Since $P_1$ is in $\mathcal{D}_{\leq 0}$ there is a triangle

$$\tau_{\leq -1}P_1 \longrightarrow P_1 \longrightarrow H^0P_1 \longrightarrow (\tau_{\leq -1}P_1)[1].$$

Now take an object $M$ in the heart $\mathcal{H}$, and look at the long exact sequence:

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -1}P_1)[1],M[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(H^0P_1,M[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_1,M[1]) \longrightarrow \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -1}P_1)[1],M[1])$ is zero because $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq -2},\mathcal{D}_{\geq -1})$ vanishes in a $t$-structure. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(P_1,M[1])$ vanishes because $P_1$ is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus $\mathsf{Hom}_{\mathcal{D}}(H^0P_1,M[1])$ is zero. But this space is isomorphic to the space $\mathsf{Ext}^1_{\mathcal{H}}(H^0P_1,M)$ by proposition 7.1.1. This proves that $H^0P_1$ is a projective $H^0A$-module.

*Step 3: $P_1$ is isomorphic to an object of $add(A)$.*

As previously, since $H^0P_1$ is projective, it is possible to find an object $P$ in $add(A)$ and a morphism $P \to P_1$ inducing an isomorphism $H^0P \to H^0P_1$. Form the triangle

$$Q \xrightarrow{u} P \xrightarrow{v} P_1 \xrightarrow{w} Q[1]$$

Since $P$ and $P_1$ are in $\mathcal{D}_{\leq 0}$ and $H^0(v)$ is surjective, the cone $Q$ lies in $\mathcal{D}_{\leq 0}$. But then $w$ is zero since $P_1$ is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus the triangle splits, and $P$ is isomorphic to the direct sum $P_1 \oplus Q$. Therefore we have a short exact sequence:

$$0 \longrightarrow H^0Q \longrightarrow H^0P \longrightarrow H^0P_1 \longrightarrow 0,$$

and $H^0Q$ vanishes. The object $Q$ is in $\mathcal{D}_{\leq -1}$, the triangle splits, and there is no morphism between $P$ and $\mathcal{D}_{\leq -1}$, so $Q$ is the zero object.

□

### 7.2.2 Equivalence between the shifts of $\mathcal{F}$

**Lemma 7.2.2.** *The functor $\tau_{\leq -1}$ induces an equivalence from $\mathcal{F}$ to $\mathcal{F}[1]$*

*Proof. Step 1: The image of the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is in $\mathcal{F}[1]$.*
Recall that $\mathcal{F}$ is $\mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \operatorname{per} A$. Thus $\mathcal{F}[1]$ is $\mathcal{D}_{\leq -1} \cap {}^{\perp}\mathcal{D}_{\leq -3} \cap \operatorname{per} A$. Let $X$ be an object in $\mathcal{F}$. By definition, $\tau_{\leq -1}X$ lies in $\mathcal{D}_{\leq -1}$ and there is a canonical triangle:

$$\tau_{\leq -1}X \longrightarrow X \longrightarrow H^0X \longrightarrow \tau_{\leq -1}X[1] \,.$$

Now let $Y$ be an object in $\mathcal{D}_{\leq -3}$ and form the long exact sequence

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -1}X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}((H^0X)[-1],Y) \longrightarrow \cdots$$

Since $X$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$, the space $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ vanishes. The object $H^0X[-1]$ is of finite total dimension, so by the Calabi-Yau property, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{D}}(H^0X[-1],Y) \simeq D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[2]).$$

But since $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq -3},\mathcal{D}_{\geq -2})$ is zero, the space $\mathsf{Hom}_{\mathcal{D}}((H^0X)[-1],Y)$ vanishes and $\tau_{\leq -1}X$ lies in ${}^{\perp}\mathcal{D}_{\leq -3}$.

*Step 2: The functor $\tau_{\leq -1} : \mathcal{F} \to \mathcal{F}[1]$ is fully faithful.*
Let $X$ and $Y$ be two objects in $\mathcal{F}$ and $f : \tau_{\leq -1}X \to \tau_{\leq -1}Y$ be a morphism.

$$\begin{array}{ccccccc} H^0X[-1] & \longrightarrow & \tau_{\leq -1}X & \longrightarrow & X & \longrightarrow & H^0X \\ & & \downarrow{\scriptstyle f} & & \vdots & & \\ H^0Y[-1] & \longrightarrow & \tau_{\leq -1}Y & \xrightarrow{i} & Y & \longrightarrow & H^0Y \end{array}$$

The space $\mathsf{Hom}_{\mathcal{D}}(H^0X[-1],Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[2])$ by the Calabi-Yau property. Since $Y$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$, this space is zero, and the composition $i \circ f$ factorizes through the canonical morphism $\tau_{\leq -1}X \to X$. Therefore, the functor $\tau_{\leq -1}$ is full.

Let $X$ and $Y$ be objects of $\mathcal{F}$ and $f : X \to Y$ a morphism satisfying $\tau_{\leq -1}f = 0$. It induces a morphism of triangles:

$$\begin{array}{ccccccc} H^0X[-1] & \longrightarrow & \tau_{\leq -1}X & \xrightarrow{i} & X & \longrightarrow & H^0X \\ \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & \swarrow & \downarrow \\ H^0Y[-1] & \longrightarrow & \tau_{\leq -1}Y & \longrightarrow & Y & \longrightarrow & H^0Y \end{array}$$

The composition $f \circ i$ vanishes, so $f$ factorizes through $H^0X$. But by the Calabi-Yau property the space of morphisms $\mathsf{Hom}_{\mathcal{D}}(H^0X,Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[3])$ which is zero since $Y$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$. Thus the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is faithful.

*Step 3: The functor $\tau_{\leq -1} : \mathcal{F} \to \mathcal{F}[1]$ is essentially surjective.*
Let $X$ be in $\mathcal{F}[1]$. By the previous lemma there exists a triangle

$$P_1[1] \longrightarrow P_0[1] \longrightarrow X \longrightarrow P_1[2]$$

with $P_0$ and $P_1$ in $add(A)$. Denote by $\nu$ the Nakayama functor on the projectives of $\mathsf{mod}\, H^0A$. Let $M$ be the kernel of the morphism $\nu H^0 P_1 \to \nu H^0 P_0$. It lies in the heart $\mathcal{H}$.

*Substep (i): There is an isomorphism of functors:* $\mathsf{Hom}(?, X[1])_{|\mathcal{H}} \simeq \mathsf{Hom}_{\mathcal{H}}(?, M)$
Let $L$ be in $\mathcal{H}$. Then we have a long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(L, P_0[2]) \to \mathsf{Hom}_{\mathcal{D}}(L, X[1]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_0[3]) \to \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}(L, P_0[2])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(P_0, L[1])$ by the Calabi-Yau property, and vanishes because $P_0$ is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Moreover, we have the following isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) &\simeq D\mathsf{Hom}_{\mathcal{D}}(P_1, L) \\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^0P_1, L) \\ &\simeq \mathsf{Hom}_{\mathcal{H}}(L, \nu H^0 P_1). \end{aligned}$$

Thus $\mathsf{Hom}_{\mathcal{D}}(?, X[1])_{|\mathcal{H}}$ is isomorphic to the kernel of $\mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_1) \to \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0 P_0)$, which is just $\mathsf{Hom}_{\mathcal{H}}(?, M)$.

*Substep (ii): There is a monomorphism of functors:* $\mathsf{Ext}^1_{\mathcal{H}}(?, M) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}$ .
For $L$ in $\mathcal{H}$, look at the following long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, X[2]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[4]) \to \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}(L, P_1[4])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(P_1[1], L)$ which is zero since $P_1[1]$ is in $\mathcal{D}_{\leq -1}$ and $L$ is in $\mathcal{D}_{\geq 0}$. Thus the functor $\mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}$ is isomorphic to the cokernel of $\mathsf{Hom}_{\mathcal{H}}(?, \nu H^0 P_1) \to \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0 P_0)$. By defninition, $\mathsf{Ext}^1_{\mathcal{H}}(?, M)$ is the first homology of a complex of the form:

$$\cdots \longrightarrow 0 \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0 P_1) \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0 P_0) \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, I) \longrightarrow \cdots,$$

where $I$ is an injective $H^0A$-module. Thus we get the canonical injection:

$$\mathsf{Ext}^1_{\mathcal{H}}(?, M) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}.$$

Now form the following triangle:

$$X \longrightarrow Y \longrightarrow M \longrightarrow X[1].$$

*Substep (iii): $Y$ is in $\mathcal{F}$ and $\tau_{\leq -1} Y$ is isomorphic to $X$.*

Since $X$ and $M$ are in $\mathcal{D}_{\leq 0}$, $Y$ is in $\mathcal{D}_{\leq 0}$. Let $Z$ be in $\mathcal{D}_{\leq -2}$ and form the following long exact sequence:

$$\cdots \mathsf{Hom}_{\mathcal{D}}(X[1], Z) \to \mathsf{Hom}_{\mathcal{D}}(M, Z) \to \mathsf{Hom}_{\mathcal{D}}(Y, Z) \to \mathsf{Hom}_{\mathcal{D}}(X, Z) \to \mathsf{Hom}_{\mathcal{D}}(M[-1], Z) \cdots .$$

By the Calabi-Yau property and the fact that $Z[2]$ is in $\mathcal{D}_{\leq 0}$, we have isomorphisms

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(M[-1], Z) & \simeq & D\mathsf{Hom}_{\mathcal{D}}(Z[-2], M) \\ & \simeq & D\mathsf{Hom}_{\mathcal{H}}(H^{-2}Z, M). \end{array}$$

Moreover, since $X$ is in ${}^{\perp}\mathcal{D}_{\leq -3}$, we have

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(X, Z) & \simeq & \mathsf{Hom}_{\mathcal{D}}(X, (H^{-2}Z)[2]) \\ & \simeq & D\mathsf{Hom}_{\mathcal{H}}(H^{-2}Z, X[1]). \end{array}$$

By substep (i) the functors $\mathsf{Hom}_{\mathcal{H}}(?, M)$ and $\mathsf{Hom}_{\mathcal{D}}(?, X[1])_{|\mathcal{H}}$ are isomorphic. Therefore we deduce that the morphism $\mathsf{Hom}_{\mathcal{D}}(X, Z) \to \mathsf{Hom}_{\mathcal{D}}(M[-1], Z)$ is an isomorphism.

Now look at the triangle

$$\tau_{\leq -3}Z \longrightarrow Z \longrightarrow H^{-2}Z[2] \longrightarrow (\tau_{\leq -3}Z)[1]$$

and form the commutative diagram

$$\begin{array}{ccccccc} \mathsf{Hom}_{\mathcal{D}}(M, \tau_{\leq -3}Z) & \longrightarrow & \mathsf{Hom}_{\mathcal{D}}(M, Z) & \longrightarrow & \mathsf{Hom}_{\mathcal{D}}(M, H^{-2}Z[2]) & \longrightarrow & \mathsf{Hom}_{\mathcal{D}}(M, \tau_{\leq -3}Z[1]) \\ \uparrow a & & \uparrow b & & \uparrow c & & \uparrow d \\ \mathsf{Hom}_{\mathcal{D}}(X[1], \tau_{\leq -3}Z) & \to & \mathsf{Hom}_{\mathcal{D}}(X[1], Z) & \to & \mathsf{Hom}_{\mathcal{D}}(X[1], H^{-2}Z[2]) & \to & \mathsf{Hom}_{\mathcal{D}}(X[1], \tau_{\leq -3}Z[1]) \end{array} .$$

By the Calabi-Yau property and the fact that $(\tau_{\leq -3}Z)[-3]$ is in $\mathcal{D}_{\leq 0}$, we have isomorphisms

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(M[-1], \tau_{\leq -3}Z[-1]) & \simeq & D\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -3}Z[-1], M) \\ & \simeq & D\mathsf{Hom}_{\mathcal{H}}(H^{-3}Z, M). \end{array}$$

Since $X$ is in ${}^{\perp}\mathcal{D}_{\leq -3}$, we have

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(X, (\tau_{\leq -3}Z)[-1]) & \simeq & \mathsf{Hom}_{\mathcal{D}}(X, H^{-3}Z[-2]) \\ & \simeq & D\mathsf{Hom}_{\mathcal{H}}(H^{-3}Z, X[1]). \end{array}$$

Now we deduce from substep (i) that $a[-1]$ is an isomorphism.

The space $\mathsf{Hom}_{\mathcal{D}}(X[1], \tau_{\leq -3}Z[1])$ is zero because $X$ is ${}^{\perp}\mathcal{D}_{\leq -3}$. Moreover there are isomorphisms

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(M, H^{-2}Z[2]) & \simeq & D\mathsf{Hom}_{\mathcal{D}}(H^{-2}Z, M[1]) \\ & \simeq & D\mathsf{Ext}^1_{\mathcal{H}}(H^{-2}Z, M). \end{array}$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[1], H^{-2}Z[2])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(H^{-2}Z, X[2])$. And by substep (ii), the morphism $\mathsf{Ext}^1_{\mathcal{H}}(?, M) \to \mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}$ is injective, so $c$ is surjective. Therefore using a weak form of the five-lemma we deduce that $b$ is surjective.

Finally, we have the following exact sequence:

$$\mathsf{Hom}_{\mathcal{D}}(X[1], Z) \twoheadrightarrow \mathsf{Hom}_{\mathcal{D}}(M, Z) \to \mathsf{Hom}_{\mathcal{D}}(Y, Z) \to \mathsf{Hom}_{\mathcal{D}}(X, Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(M[-1], Z)$$

Thus the space $\mathsf{Hom}_{\mathcal{D}}(M, Z)$ is zero, and $Z$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$.

It is now easy to see that there is an isomorphism of triangles:

$$\begin{array}{ccccccc} \tau_{\leq -1}Y & \longrightarrow & Y & \longrightarrow & H^0Y & \longrightarrow & \tau_{\leq -1}Y[1] \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & X[1]. \end{array}$$

□

### 7.2.3 Proof of proposition 7.2.1

*Step 1: The functor $\pi$ restricted to $\mathcal{F}$ is fully faithful.*

Let $X$ and $Y$ be objects in $\mathcal{F}$. By proposition 7.1.1 (iii), the space $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$ is isomorphic to the direct limit $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$. A morphism between $X$ and $Y$ in $\mathcal{C}$ is a diagram of the form

$$X \swarrow \tau_{\leq n}X \searrow Y.$$

The canonical triangle

$$(\tau_{>n}X)[-1] \longrightarrow \tau_{\leq n}X \longrightarrow X \longrightarrow \tau_{>n}X$$

yields a long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(\tau_{>n}X, Y) \to \mathsf{Hom}_{\mathcal{D}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, Y) \to \mathsf{Hom}_{\mathcal{D}}((\tau_{>n}X)[-1], Y) \to \cdots$$

The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{>n}X)[-1], Y)$ is isomorphic to the space $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>n}X)[2])$. The object $X$ is in $\mathcal{D}_{\geq 0}$, hence so is $\tau_{>n}X$, and the space $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>n}X)[2])$ vanishes. For the same reasons, the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{>n}X, Y)$ vanishes. Thus there are bijections

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(X, Y)$$

Therefore, the functor $\pi : \mathcal{F} \to \mathcal{C}$ is fully faithful.

*Step 2: For $X$ in $\mathsf{per}\,A$, there exists an integer $N$ and an object $Y$ of $\mathcal{F}[-N]$ such that $\pi X$ and $\pi Y$ are isomorphic in $\mathcal{C}$.*

Let $X$ be in $\mathsf{per}\, A$. By lemma 7.1.3, there exists an integer $N$ such that $X$ is in ${}^{\perp}\mathcal{D}_{\leq N-2}$. For an object $Y$ in $\mathcal{D}_{\leq N-2}$, the space $\mathsf{Hom}_{\mathcal{D}}((\tau_{>N}X)[-1], Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>N}X)[2])$ and so vanishes. Therefore, $\tau_{\leq N}X$ is still in ${}^{\perp}\mathcal{D}_{\leq N-2}$, and so is in $\mathcal{F}[-N]$. Since $\tau_{>N}X$ is in $\mathcal{D}^bA$, the objects $\tau_{\leq N}X$ and $X$ are isomorphic in $\mathcal{C}$.

*Step 3: The functor $\pi$ restricted to $\mathcal{F}$ is essentially surjective.*
Let $X$ be in $\mathsf{per}\, A$ and $N$ such that $\tau_{\leq N}X$ is in $\mathcal{F}[-N]$. By lemma 7.2.2, $\tau_{\leq -1}$ induces an equivalence between $\mathcal{F}$ and $\mathcal{F}[1]$. Thus since the functor $\pi \circ \tau_{\leq -1} : \mathsf{per}\, A \to \mathcal{C}$ is isomorphic to $\pi$, there exists an object $Y$ in $\mathcal{F}$ such that $\pi(Y)$ and $\pi(X)$ are isomorphic in $\mathcal{C}$. Therefore, the functor $\pi$ restricted to $\mathcal{F}$ is essentially surjective.

**Proposition 7.2.2.** *If $X$ and $Y$ are objects in $\mathcal{F}$, there is a short exact sequence:*

$$0 \longrightarrow \mathsf{Ext}^1_{\mathcal{D}}(X,Y) \longrightarrow \mathsf{Ext}^1_{\mathcal{C}}(X,Y) \longrightarrow D\mathsf{Ext}^1_{\mathcal{D}}(Y,X) \longrightarrow 0.$$

*Proof.* Let $X$ and $Y$ be in $\mathcal{F}$. The canonical triangle

$$\tau_{<0}X \longrightarrow X \longrightarrow \tau_{\geq 0}X \longrightarrow (\tau_{<0}X)[1]$$

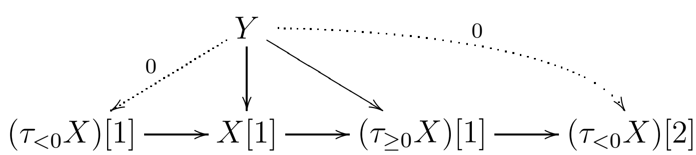
yields the long exact sequence:

$$\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1]) \longleftarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X, Y[1]) \longleftarrow \mathsf{Hom}_{\mathcal{D}}(X, Y[1]) \longleftarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X, Y[1])\ .$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[-1], Y[1])$ is zero because $X$ is in ${}^{\perp}\mathcal{D}_{\leq -2}$ and $Y$ is in $\mathcal{D}_{\leq 0}$. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X, Y[1])$ is zero because of the Calabi-Yau property. Thus this long sequence reduces to a short exact sequence:

$$0 \longrightarrow \mathsf{Ext}^1_{\mathcal{D}}(X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X, Y[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1]) \longrightarrow 0\ .$$

*Step 1: There is an isomorphism $\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y) \simeq D\mathsf{Ext}^1_{\mathcal{D}}(Y,X)$.*
The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y, \tau_{\geq 0}X[1])$ by the Calabi-Yau property.




But since $\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{<0}X)[1])$ and $\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{<0}X)[2])$ are zero, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X[-1], Y) \simeq D\mathsf{Ext}^1_{\mathcal{D}}(Y,X).$$

*Step 2: There is an isomorphism $\mathsf{Ext}^1_{\mathcal{C}}(\pi X, \pi Y) \simeq \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -1}X, Y[1])$.*
By lemma 7.2.2, the object $\tau_{<0}X$ belongs to $\mathcal{F}[1]$ and clearly $Y[1]$ belongs to $\mathcal{F}[1]$. By proposition 7.2.1 (applied to the shifted $t$-structure), the functor $\pi : \mathsf{per}\, A \to \mathcal{C}$ induces an equivalence from $\mathcal{F}[1]$ to $\mathcal{C}$ and clearly we have $\pi(\tau_{<0}X, Y[1]) \xrightarrow{\sim} \pi(X)$. We obtain bijections

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X, Y[1]) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(\pi\tau_{<0}X, \pi Y[1]) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(\pi X, \pi Y[1]).$$

□

### 7.2.4 Proof of the main theorem

*Step 1: The category $\mathcal{C}$ is $\mathsf{Hom}$-finite and 2-Calabi-Yau.*

The category $\mathcal{F}$ is obviously $\mathsf{Hom}$-finite, hence so is $\mathcal{C}$ by proposition 7.2.1. The categories $\mathcal{T} = \mathsf{per}\, A$ and $\mathcal{N} = \mathcal{D}^b A \subset \mathsf{per}\, A$ satisfy the hypotheses of chapter 4. By [Kel08a], thanks to the Calabi-Yau property, there is a bifunctorial non degenerate bilinear form:

$$\beta_{N,X} : \mathsf{Hom}_{\mathcal{D}}(N, X) \times \mathsf{Hom}_{\mathcal{D}}(X, N[3]) \to k$$

for $N$ in $\mathcal{D}^b A$ and $X$ in $\mathsf{per}\, A$. Thus, by chapter 4, there exists a bilinear bifunctorial form

$$\beta'_{X,Y} : \mathsf{Hom}_{\mathcal{C}}(X, Y) \times \mathsf{Hom}_{\mathcal{C}}(Y, X[2]) \to k$$

for $X$ and $Y$ in $\mathcal{C} = \mathsf{per}\, A/\mathcal{D}^b A$. We would like to show that it is non degenerate. Since $\mathsf{per}\, A$ is $\mathsf{Hom}$-finite, by theorem 4.2 and proposition 4.2.1, it is sufficient to show the existence of local $\mathcal{N}$-envelopes. Let $X$ and $Y$ be objects of $\mathsf{per}\, A$. Then by lemma 7.1.3, $X$ is in ${}^{\perp}\mathcal{D}_{\leq N}$. Thus there is an injection

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X, \tau_{>N} Y)$$

and $Y \to \tau_{>N} Y$ is a local $\mathcal{N}$-envelope relative to $X$. Therefore, $\mathcal{C}$ is 2-Calabi-Yau.

Note that the bilinear form $\beta_{X,Y}$ yields a bifunctorial map

$$\eta_{XY} : \mathsf{Ext}^1_{\mathcal{C}}(X, Y) \to D\mathsf{Ext}^1_{\mathcal{C}}(Y, X).$$

One can check that it makes the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ext}^1_{\mathcal{D}}(X', Y') & \longrightarrow & \mathsf{Ext}^1_{\mathcal{C}}(\pi X', \pi Y') & \longrightarrow & D\mathsf{Ext}^1_{\mathcal{D}}(Y', X') & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \eta & & \| & & \\ 0 & \longrightarrow & DD\mathsf{Ext}^1_{\mathcal{D}}(X', Y') & \longrightarrow & D\mathsf{Ext}^1_{\mathcal{C}}(\pi Y', \pi X') & \longrightarrow & D\mathsf{Ext}^1_{\mathcal{D}}(Y', X') & \longrightarrow & 0 \end{array}$$

commutative, where $X'$ and $Y'$ are objects in $\mathcal{F}$ such that $\pi(X') = X$ and $\pi(Y') = Y$.

*Step 2: The object $\pi A$ is a cluster-tilting object of the category $\mathcal{C}$.*

Let $A$ be the free dg $A$-module in $\mathsf{per}\, A$. Since $H^1 A$ is zero, the space $\mathsf{Ext}^1_{\mathcal{D}}(A, A)$ is also zero. Thus by the short exact sequence

$$0 \longrightarrow \mathsf{Ext}^1_{\mathcal{D}}(A, A) \longrightarrow \mathsf{Ext}^1_{\mathcal{C}}(\pi A, \pi A) \longrightarrow D\mathsf{Ext}^1_{\mathcal{D}}(A, A) \longrightarrow 0$$

of proposition 7.2.2, $\pi(A)$ is a rigid object of $\mathcal{C}$. Now let $X$ be an object of $\mathcal{C}$. By proposition 7.2.1, there exists an object $Y$ in $\mathcal{F}$ such that $\pi Y$ is isomorphic to $X$. Now by lemma 7.2.1 , there exists a triangle in $\mathsf{per}\, A$

$$P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow P_1[1]$$

with $P_1$ and $P_0$ in $add(A)$. Applying the triangle functor $\pi$ we get a triangle in $\mathcal{C}$:

$$\pi P_1 \longrightarrow \pi P_0 \longrightarrow X \longrightarrow \pi P_1[1]$$

with $\pi P_1$ and $\pi P_0$ in $add(\pi A)$. If $\mathsf{Ext}^1_{\mathcal{C}}(\pi A, X)$ vanishes, this triangle splits and $X$ is a direct factor of $\pi P_0$. Thus, the object $\pi A$ is a cluster-tilting object in the 2-Calabi-Yau category $\mathcal{C}$.

## 7.3 Application to the cluster category of an algebra of global dimension 2

Let $A$ be a finite dimensional $k$-algebra of global dimension $\leq 2$. We denote by $B$ the dg-algebra $A \oplus (DA)[-3]$, and by $p$ the canonical projection $B \to A$. Let us denote by $\langle A \rangle_B$ the thick subcategory of $\mathcal{D}^b B$ generated by the image of the restriction along $p$. The cluster category associated to $A$ is defined as the quotient $\mathcal{C}_A = \langle A \rangle_B / \mathsf{per}\, B$ (section 7 of [Kel05]). We assume that the functor $\mathsf{Tor}^A_2(?, DA)$ is nilpotent. This is equivalent to the fact that $\mathcal{C}_A$ is $\mathsf{Hom}$-finite by theorem 5.1.

We denote by $\Theta$ a cofibrant resolution of the dg $A$-bimodule $R\mathcal{H}om^{\bullet}_A(DA, A)$. Following [Kel08a] and [Kel08b], we define the 3-derived preprojective algebra as the tensor algebra

$$\Pi_3(A) = T_A(\Theta[2]).$$

The complex $R\mathcal{H}om^{\bullet}_A(DA, A)[2]$ has its homology concentrated in degrees $-2$, $-1$ and $0$, and we have

$$H^{-2}(\Theta[2]) \simeq \mathsf{Hom}_{\mathcal{D}A}(DA, A),\ H^{-1}(\Theta[2]) \simeq \mathsf{Ext}^1_A(DA, A) \text{ and } H^0(\Theta[2]) \simeq \mathsf{Ext}^2_A(DA, A).$$

Thus the homology of the dg algebra $\Pi_3(A)$ vanishes in strictly positive degrees and we have

$$H^0 \Pi_3 A = T_A \mathsf{Ext}^2_A(DA, A) = \tilde{A}.$$

Moreover, by lemma 5.3.1 the nilpotence of the functor $\mathsf{Tor}^A_2(?, DA)$ means that there exists $N$ such that $\mathsf{Ext}^2_A(DA, A)^{\otimes N}$ vanishes. Keller showed that $\Pi_3(A)$ is homologically smooth and bimodule 3-Calabi-Yau [Kel08b]. Thus we can apply theorem 7.1 and we have the following results:

**Corollary 7.2.** *The category $\mathcal{C} = \mathsf{per}\, \Pi_3 A / \mathcal{D}^b \Pi_3 A$ is 2-Calabi-Yau and the free dg module $\Pi_3 A$ is a cluster-tilting object in $\mathcal{C}$.*

The aim of this section is to construct a triangle equivalence between $\mathcal{C}_A$ and $\mathcal{C}$ sending $A$ to $\Pi_3 A$.

Let us recall a theorem of Keller ([Kel94], or theorem 8.5, p.96 [AHHK07]):

**Theorem 7.3.** *[Keller] Let $B$ be dg algebra, and $T$ an object of $\mathcal{D}B$. Denote by $C$ the dg algebra $R\mathcal{H}om^\bullet_B(T,T)$. Denote by $\langle T\rangle_B$ the thick subcategory of $\mathcal{D}B$ generated by $T$. Then the functor $R\mathcal{H}om^\bullet_B(T,?) : \mathcal{D}B \to \mathcal{D}C$ induces an algebraic triangle equivalence*

$$R\mathcal{H}om^\bullet_B(T,?) : \langle T\rangle_B \xrightarrow{\sim} \mathsf{per}\, C.$$

Let us denote by $\mathcal{H}o(dgalg)$ the homotopy category of dg algebras, i.e. the localization of the category of dg algebras at the class of quasi-isomorphisms.

**Lemma 7.3.1.** *In $\mathcal{H}o(dgalg)$, there is an isomorphism between $\Pi_3 A$ and $R\mathsf{Hom}_B(A_B, A_B)$.*

*Proof.* The dg algebra $B$ is $A \oplus (DA)[-3]$. Denote by $X$ a cofibrant resolution of the dg $A$-bimodule $DA[-2]$. Now look at the dg submodule of the bar resolution of $B$ seen as a bimodule over itself (see the proof of theorem 7.1 in [Kel05]):

$$bar(X,B) : \cdots \longrightarrow B\otimes_A X^{\otimes_A 2}\otimes_A B \longrightarrow B\otimes_A X\otimes_A B \longrightarrow B\otimes_A B \longrightarrow 0 \longrightarrow \cdots$$

This is a cofibrant resolution of the dg $B$-bimodule $B$. Thus $A\otimes_B bar(X,B)$ is a cofibrant resolution of the dg $B$-module $A$. Therefore, we have the following isomorphisms

$$\begin{aligned} R\mathcal{H}om^\bullet_B(A_B,A_B) &\simeq \mathcal{H}om^\bullet_B(A\otimes_B bar(X,B), A) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_B(A\otimes_A X^{\otimes_A n}\otimes_A B, A_B) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_A(X^{\otimes_A n}, \mathsf{Hom}_B(B, A_B)_A) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_A(X^{\otimes_A n}, A_A), \end{aligned}$$

where the differential on the last complex is induced by that of $X^{\otimes_A n}$. Note that

$$\begin{aligned} \mathcal{H}om^\bullet_A(X,A) &= R\mathcal{H}om^\bullet_A(DA[-2],A) \\ &= R\mathcal{H}om^\bullet_A(DA,A)[2] = \Theta[2]. \end{aligned}$$

We can now use the following lemma:

**Lemma 7.3.2.** *Let $A$ be a dg algebra, and $L$ and $M$ dg $A$-bimodules such that $M_A$ is perfect as right dg $A$-module. Then there is an isomorphism in $\mathcal{D}(A^{op}\otimes A)$*

$$R\mathcal{H}om^\bullet_A(L,A)\overset{L}{\otimes}_A R\mathcal{H}om^\bullet_A(M,A) \simeq R\mathcal{H}om^\bullet_A(M\overset{L}{\otimes}_A L, A).$$

*Proof.* Let $X$ and $M$ be dg $A$-bimodules. The following morphism of $\mathcal{D}(A^{op}\otimes A)$

$$\begin{aligned} X\overset{L}{\otimes}_A R\mathcal{H}om_A(M,A) &\longrightarrow R\mathcal{H}om_A(M,X) \\ x\otimes\varphi &\mapsto (m\mapsto x\varphi(m)) \end{aligned}$$

is clearly an isomorphism for $M = A$. Thus it is an isomorphism if $M$ is perfect as a right dg $A$-module. Applying this to the right dg $A$-module $R\mathcal{H}om_A(L, A)$, we get an isomorphism of dg $A$-bimodules

$$R\mathcal{H}om_A(L, A) \overset{L}{\otimes}_A R\mathcal{H}om_A(M, A) \simeq R\mathcal{H}om_A(M, R\mathcal{H}om_A(L, A)).$$

Finally, by adjunction we get an isomorphism of dg $A$-bimodules

$$R\mathcal{H}om_A(L, A) \overset{L}{\otimes}_A R\mathcal{H}om_A(M, A) \simeq R\mathcal{H}om_A(M \overset{L}{\otimes}_A L, A).$$

$\square$

Therefore, the dg $A$-bimodule $\mathcal{H}om^{\bullet}_A(X^{\otimes_A n}, A_A)$ is isomorphic to $(\Theta[2])^{\otimes_A n}$, and there is an isomorphism of dg algebras

$$R\mathcal{H}om^{\bullet}_B(A_B, A_B) \simeq \bigoplus_{n \geq 0} (\Theta[2])^{\overset{L}{\otimes}_A n} = \Pi_3(A)$$

because for each $p \in \mathbb{Z}$, the group $H^p(\Theta[2]^{\overset{L}{\otimes}_A n})$ vanishes for all $n \gg 0$. $\square$

By theorem 7.3, the functor $R\mathcal{H}om^{\bullet}_B(A_B, ?)$ induces an equivalence between the thick subcategory $\langle A \rangle_B$ of $\mathcal{D}B$ generated by $A$, and $\mathsf{per}\,\Pi_3(A)$. Thus we get a triangle equivalence that we will denote by $F$:

$$F = R\mathcal{H}om^{\bullet}_B(A_B, ?) : \langle A \rangle_B \xrightarrow{\sim} \mathsf{per}\,\Pi_3 A$$

This functor sends the object $A_B$ of $\mathcal{D}^b B$ onto the free module $\Pi_3 A$ and the free $B$-module $B$ onto $R\mathcal{H}om^{\bullet}_B(A_B, B)$, that is to say onto $A_{\Pi_3 A}$. So $F$ induces an equivalence

$$F : \mathsf{per}\, B = \langle B \rangle_B \xrightarrow{\sim} \langle A \rangle_{\Pi_3 A}.$$

**Lemma 7.3.3.** *The thick subcategory $\langle A \rangle_{\Pi_3 A}$ of $\mathcal{D}\Pi_3 A$ generated by $A$ is $\mathcal{D}^b \Pi_3 A$.*

*Proof.* The algebra $A$ is finite dimensional, therefore $\langle A \rangle_{\Pi_3 A}$ is obviously included in $\mathcal{D}^b \Pi_3 A$. Moreover, the category $\mathcal{D}^b \Pi_3 A$ equals $\langle \mathsf{mod}\, H^0(\Pi_3 A) \rangle_{\Pi_3 A}$ by the existence of the $t$-structure. The dg algebra $\Pi_3 A$ is the tensor algebra $T_A(\theta[2])$ so there is a canonical projection $\Pi_3 A \to A$ which yields a restriction functor $\mathcal{D}^b A \to \mathcal{D}^b(\Pi_3 A)$ respecting the $t$-structure:

$$\begin{array}{ccc} \mathsf{mod}\, H^0 \Pi_3 A = \mathcal{H} & \hookrightarrow & \mathcal{D}^b(\Pi_3 A) \\ \uparrow & & \uparrow \\ \mathsf{mod}\, A & \hookrightarrow & \mathcal{D}^b A \end{array}$$

This restriction functor induces a bijection in the set of isomorphism classes of simple modules because the kernel of the map $H^0(\Pi_3 A) \to A$ is a nilpotent ideal (namely the sum of the tensor powers over $A$ of the bimodule $\mathsf{Ext}^2_A(DA, A)$). Thus each simple of $\mathsf{mod}\, H^0\Pi_3 A$ is in $\langle A\rangle_{\Pi_3 A}$ and we have

$$\langle A\rangle_{\Pi_3 A} \simeq \langle \mathsf{mod}\, H^0(\Pi_3 A)\rangle_{\Pi_3 A} \simeq \mathcal{D}^b\Pi_3 A.$$

□

In conclusion, we have the following commutative square:

$$\begin{array}{ccc} F : \langle A\rangle_B & \xrightarrow{\sim} & \mathsf{per}\,\Pi_3 A \\ \uparrow & & \uparrow \\ \mathsf{per}\, B & \xrightarrow{\sim} & \mathcal{D}^b\Pi_3 A \end{array}$$

Thus $F$ induces a triangle equivalence

$$\mathcal{C}_A = \langle A\rangle_B/\mathsf{per}\, B \xrightarrow{\quad\sim\quad} \mathsf{per}\,\Pi_3 A/\mathcal{D}^b\Pi_3 A = \mathcal{C}$$

sending the object $A$ onto the free module $\Pi_3 A$. By theorem 7.1, $A$ is therefore a cluster-tilting object of the cluster category $\mathcal{C}_A$.

# 7.4 Cluster category for Jacobi-finite quivers with potential

## 7.4.1 Ginzburg dg algebra

Let $k$ be a field and $Q$ a finite quiver. For each arrow $a$ of $Q$, we define the *cyclic derivative with respect to a* $\partial_a$ as the unique linear map

$$\partial_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path $p$ to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path $p$ (where $u$ and $v$ are possibly idempotents $e_i$ associated to a vertex $i$ of $Q$).

An element $W$ of $kQ/[kQ, kQ]$ is called a *potential on* $Q$. It is given by a linear combination of cycles in $Q$.

**Definition 7.4** (Ginzburg)**.** [Gin06](section 4.2) Let $Q$ be a finite quiver and $W$ a potential on $Q$. Let $\widehat{Q}$ be the graded quiver with the same vertices as $Q$ and whose arrows are

- the arrows of $Q$ (of degree 0),

- an arrow $a^* : j \to i$ of degree $-1$ for each arrow $a : i \to j$ of $Q$,
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

The *Ginzburg dg algebra* $\Gamma(Q, W)$ is a dg $k$-algebra whose underlying graded algebra is the graded path algebra $k\widehat{Q}$. Its differential is the unique linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p u dv,$$

for all homogeneous $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\widehat{Q}$:

- $da = 0$ for each arrow $a$ of $Q$,
- $d(a^*) = \partial_a W$ for each arrow $a$ of $Q$,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex $i$ of $Q$ where $e_i$ is the idempotent associated to $i$ and the sum runs over all arrows of $Q$.

The strictly positive homology of this dg algebra clearly vanishes. Moreover B. Keller showed the following result:

**Theorem 7.5** (Keller)**.** *[Kel08b] Let $Q$ be a finite quiver and $W$ a potential on $Q$. Then the Ginzburg dg algebra $\Gamma(Q, W)$ is homologically smooth and bimodule 3-Calabi-Yau.*

### 7.4.2 Jacobian algebra

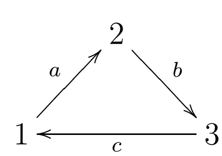
**Definition 7.6.** Let $Q$ be a finite quiver and $W$ a potential on $Q$. The *Jacobian algebra* $J(Q, W)$ is the zeroth homology of the Ginzburg algebra $\Gamma(Q, W)$. This is the quotient algebra

$$kQ/\langle \partial_a W, a \in Q_1 \rangle$$

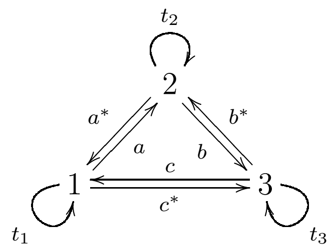
where $\langle \partial_a W, a \in Q_1 \rangle$ is the two-sided ideal generated by the $\partial_a W$.

Remark: We follow the terminology of H. Derksen, J. Weyman and A. Zelevinsky ([DWZ07] definition 3.1).

*Exemple.* 1. Let $Q$ be the following quiver



with the potential $W = acb$. Then the quiver $\widehat{Q}$ is given by

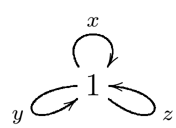



The differential of the Ginzburg dg algebra satisfies

$$\begin{aligned} d(a^*) &= cb, & d(t_1) &= cc^* - a^*a \\ d(b) &= ac, & d(t_2) &= aa^* - b^*b \\ d(c^*) &= ba & d(t_3) &= bb^* - c^*c. \end{aligned}$$

The Jacobi algebra $J(Q, W)$ is then $kQ/\langle ba, ac, cb\rangle$ and so is finite dimensional.

2. Let $Q$ be the quiver




with the potential $W = xyz - xzy$. Then the Jacobi algebra $J(Q, W)$ is isomorphic to the algebra

$$k\langle x, y, z\rangle/(xy - yx, xz - zx, yz - zy) \simeq k[x, y, z]$$

and so is infinite dimensional.

In recent works, B. Keller [Kel08b] and A. Buan, O. Iyama, I. Reiten and D. Smith [BIRS08] have shown independently the following result:

**Theorem 7.7** (Keller, Buan-Iyama-Reiten-Smith)**.** *Let $T$ be a cluster-tilting object in the cluster category $\mathcal{C}_Q$ associated to an acyclic quiver $Q$. Then there exists a quiver potential $(Q', W)$ such that $\mathsf{End}_{\mathcal{C}_Q}(T)$ is isomorphic to $J(Q', W)$.*

### 7.4.3 Jacobi-finite quiver potentials

The quiver potential $(Q, W)$ is called *Jacobi-finite* if the Jacobian algebra $J(Q, W)$ is finite dimensional.

**Definition 7.8.** Let $(Q, W)$ be a Jacobi-finite quiver potential. Denote by $\Gamma$ the Ginzburg dg algebra $\Gamma(Q, W)$. Let $\mathsf{per}\,\Gamma$ be the smallest thick subcategory of $\mathcal{D}\Gamma$ which contains $\Gamma$ and $\mathcal{D}^b\Gamma$ the full subcategory of $\mathcal{D}\Gamma$ of the dg $\Gamma$-modules whose homology is of finite total dimension. The *cluster category* $\mathcal{C}_{(Q,W)}$ associated to $(Q, W)$ is defined as the quotient of triangulated categories $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$.

Combining theorem 7.1 and theorem 7.5 we get the result:

**Theorem 7.9.** *Let $(Q,W)$ be a Jacobi-finite quiver potential. Then the cluster category $\mathcal{C}_{(Q,W)}$ associated to $(Q,W)$ is* Hom*-finite and 2-Calabi-Yau. Moreover the image $T$ of the free module $\Gamma$ in the quotient $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$ is a cluster-tilting object. Its endomorphim algebra is isomorphic to the Jacobian algebra $J(Q,W)$.*

As a direct consequence of this theorem we get the corollary:

**Corollary 7.10.** *Each finite dimensional Jacobi algebra $\mathcal{J}(Q,W)$ is 2-Calabi-Yau-tilted in the sense of I. Reiten (cf. [Rei07]), i.e. it is the endomorphism algebra of some cluster-tilting object of a 2-Calabi-Yau cateogry.*

**Definition 7.11.** Let $(Q,W)$ and $(Q',W')$ be two quiver potentials. A *triangular extension* between $(Q,W)$ and $(Q',W')$ is a quiver potential $(\bar{Q},\bar{W})$ where

- $\bar{Q}_0 = Q_0 \cup Q'_0$;
- $\bar{Q}_1 = Q_1 \cup Q'_1 \cup \{a_i, i \in I\}$, where for each $i$ in the finite index set $I$, the source of $a_i$ is in $Q_0$ and the tail of $a_i$ is in $Q'_0$;
- $\bar{W} = W + W'$.

**Proposition 7.4.1.** *Denote by $\mathcal{JF}$ the class of Jacobi-finite quiver potentials. Then $\mathcal{JF}$ satisfies the properties:*

1. *it contains all acyclic quivers (with potential 0);*
2. *it is stable under quiver potential mutation defined in [DWZ07];*
3. *it is stable under triangular extensions.*

*Proof.* 1. This is obvious since the Jacobi algebra $J(Q,0)$ is isomorphic to $kQ$.

2. This is corollary 6.6 of [DWZ07].

3. Let $(Q,W)$ and $(Q',W')$ be two quiver potentials in $\mathcal{JF}$ and $(\bar{Q},\bar{W})$ a triangular extension. Let $\bar{Q}_1 = Q_1 \cup Q'_1 \cup F$ be the set of arrows of $\bar{Q}$. Then we have

$$k\bar{Q} = kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ$$

where $R$ is the semi-simple algebra $kQ_0$ and $R'$ is $kQ'_0$. Let $\bar{W}$ be the potential $W+W'$ associated to the triangular extension. If $a$ is in $Q_1$, then $\partial_a\bar{W} = \partial_a W$, if $a$ is in $Q'_1$ then $\partial_a \bar{W} = \partial_a W'$ and if $a$ is in $F$, then $\partial_a \bar{W} = 0$. Thus we have isomorphisms

$$\begin{aligned} J(\bar{Q},\bar{W}) &= k\bar{Q}/\langle \partial_a \bar{W}, a \in \bar{Q}_1\rangle \\ &\simeq kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/\langle \partial_a W, a \in Q_1, \partial_b W', b \in Q'_1\rangle \\ &\simeq kQ'/\langle \partial_b W', b \in Q'_1\rangle \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/\langle \partial_a W, a \in Q_1\rangle \\ &\simeq J(Q',W') \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R J(Q,W). \end{aligned}$$

Thus if $J(Q',W')$ and $J(Q,W)$ are finite dimensional, $J(\bar{Q},\bar{W})$ is finite dimensional since $F$ is finite.

□

In a recent work [KY08], B. Keller and D. Yang proved the following:

**Theorem 7.12** (Keller-Yang)**.** *Let $(Q,W)$ be a Jacobi-finite quiver potential. Assume that $Q$ has no loops nor two-cyles. Then for each vertex $i$ of $Q$, there is a derived equivalence*

$$\mathcal{D}\Gamma(\mu_i(Q,W)) \simeq \mathcal{D}\Gamma(Q,W),$$

*where $\mu_i(Q,W)$ is the mutation of $(Q,W)$ at the vertex $i$ in the sense of [DWZ07].*

Remark: in fact Keller and Yang proved this theorem in a more general setting. This also true if $(Q,W)$ is not Jacobi-finite, but then there is a derived equivalence between the completions of the Ginzburg dg algebras.

Combining this theorem with theorem 7.9 and some results of [BIRS08], we get the corollary:

**Corollary 7.13.** *1. If $Q$ is an acyclic quiver, and $W=0$, the cluster category $\mathcal{C}_{(Q,W)}$ is canonically equivalent to the cluster category $\mathcal{C}_Q$.*

*2. Let $Q$ be an acyclic quiver and $T$ a cluster-tilting object of $\mathcal{C}_Q$. If $(Q',W)$ is the quiver potential associated with the cluster-tilted algebra $\mathsf{End}_{\mathcal{C}_Q}(T)$ (cf. [Kel08b], [BIRS08]), then the cluster category $\mathcal{C}_{(Q,W)}$ is triangle equivalent to the cluster category $\mathcal{C}_{Q'}$.*

*Proof.* 1. The cluster category $\mathcal{C}_{(Q,0)}$ is a 2-Calabi-Yau category with a cluster-tilting object whose endomorphism algebra is isomorphic to $kQ$. Thus by [KR07], this category is triangle equivalent to $\mathcal{C}_Q$.

2. In a cluster category, all cluster-tilting objects are mutation equivalent. Thus by results of [BIRS08], $(Q,W)$ is mutation equivalent to $(Q',0)$. Moreover, $(Q,W)$ and $Q'$ have no loops nor two-cycles. Thus, the theorem of Keller and Yang [KY08] applies and we have an equivalence

$$\mathcal{D}\Gamma(Q,W) \simeq \mathcal{D}\Gamma(Q',0).$$

Thus the categories $\mathcal{C}_{(Q,W)}$ and $\mathcal{C}_{(Q',0)}$ are triangle equivalent. By 1. we get the result.

□

# Bibliography


[ABS06] I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, preprint (2006), arXiv:math. RT/0601537.

[AHHK07] L. Angeleri-Hügel, D. Happel, and H. Krause (eds.), *Handbook of Tilting Theory*, London Mathematical society, vol. 332, Cambridge University press, 2007.

[Ami07] C. Amiot, *On the structure of triangulated categories with finitely many indecomposables*, Bull. Soc. Math. France **135** (2007), 435–474.

[AR86] M. Auslander and I. Reiten, *McKay quivers and extended Dynkin diagrams*, Trans. Amer. Math. Soc. **293** (1986), no. 1, 293–301.

[Asa99] H. Asashiba, *The derived equivalence classification of representation-finite self-injective algebras*, J. Algebra **214** (1999), no. 1, 182–221.

[BBD82] A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[BBK02] S. Brenner, M. C. R. Butler, and A. D. King, *Periodic algebras which are almost Koszul*, Algebr. Represent. Theory **5** (2002), no. 4, 331–367.

[BES07] J. Białkowski, K. Erdmann, and A. Skowroński, *Deformed preprojective algebras of generalized Dynkin type*, Trans. Amer. Math. Soc. **359** (2007), no. 6, 2625–2650 (electronic).

[BIRS07] A. B. Buan, O. Iyama, I. Reiten, and J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, preprint (2007), arXiv:math. RT/0701557.

[BIRS08] A. B. Buan, O. Iyama, I. Reiten, and D. Smith, *Mutation of cluster-tilting objects and potentials*, preprint (2008), arXiv:math. RT/08043813.

[BMR+06] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[BMR07] A. B. Buan, R. Marsh, and I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 323–332 (electronic).

[BMR08] ______, *Cluster mutation via quiver representations*, Comment. Math. Helv. **83** (2008), no. 1, 143–177.

[BMRT07] A. B. Buan, R. Marsh, I. Reiten, and G. Todorov, *Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3049–3060 (electronic), With an appendix coauthored in addition by P. Caldero and B. Keller.

[Boc07] R. Bocklandt, *Graded Calabi-Yau algebras of dimension 3*, preprint (2007), arXiv:math. RA/0603558.

[BS06] J. Białkowski and A. Skowroński, *Calabi-Yau stable module categories of finite type*, preprint (2006), http://www.mat.uni.torun.pl/preprints/.

[CC06] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616.

[CCS06] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters ($A_n$ case)*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1347–1364 (electronic).

[CK06] P. Caldero and B. Keller, *From triangulated categories to cluster algebras. II*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 983–1009.

[CK08] ______, *From triangulated categories to cluster algebras*, Invent. Math. **172** (2008), 169–211.

[CR] J. Chuang and R. Rouquier, *preprint in preparation*.

[Die87] E. Dieterich, *The Auslander-Reiten quiver of an isolated singularity*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 244–264.

[DWZ07] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, preprint (2007), arXiv:math. RA/07040649.

[FK07] C. Fu and B. Keller, *On cluster algebras with coefficients and 2-Calabi-Yau categories*, preprint (2007), arXiv:math. RT/07103152.

[FZ02] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic).

[FZ03] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[FZ07] ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164.

[Gin06] V. Ginzburg, *Calabi-Yau algebras*, preprint (2006), arXiv:math. AG/0612139.

[GL] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, pp. 265–297.

[GLS06a] C. Geiß, B. Leclerc, and J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (to appear) (2006), arXiv:math. RT/0609138.

[GLS06b] ———, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632.

[GLS07a] ———, *Auslander algebras and initial seeds for cluster algebras*, J. London Math. Soc. (2) **75** (2007), no. 3, 718–740.

[GLS07b] ———, *Cluster algebra structures and semi-canonical bases for unipotent groups*, preprint (2007), arXiv:math. RT/0703039.

[Hap87] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[Hap88] ———, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[Hap01] ———, *A characterization of hereditary categories with tilting object*, Invent. Math. **144** (2001), no. 2, 381–398.

[Hel68] A. Heller, *Stable homotopy categories*, Bull. Amer. Math. Soc. **74** (1968), 28–63.

[HJ06a] T. Holm and P. Jørgensen, *Cluster categories and selfinjective algebras: type $A$*, preprint (2006), arXiv:math. RT/0610728.

[HJ06b] ———, *Cluster categories and selfinjective algebras: type $D$*, preprint (2006), arXiv:math. RT/0612451.

[HPR80a] D. Happel, U. Preiser, and C. M. Ringel, *Binary polyhedral groups and Euclidean diagrams*, Manuscripta Math. **31** (1980), no. 1-3, 317–329.

[HPR80b] ———, *Vinberg's characterization of Dynkin diagrams using subadditive functions with application to D*Tr*-periodic modules*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 280–294.

[HR02] D. Happel and I. Reiten, *A characterization of the hereditary categories derived equivalent to some category of coherent sheaves on a weighted projective line*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 643–651 (electronic).

[IY06] O. Iyama and Y. Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, preprint (2006), arXiv:math. RT/0607736.

[Kel91] B. Keller, *Derived categories and universal problems*, Comm. Algebra **19** (1991), no. 3, 699–747.

[Kel94] ———, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[Kel05] ———, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581 (electronic).

[Kel06] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[Kel08a] ———, *Calabi-Yau triangulated categories*, preprint (2008), http://www.math.jussieu.fr/~keller/publ/KellerCYtriangCat.pdf.

[Kel08b] ———, *Deformed CY-completions and their duals*, in preparation (2008).

[KR06] B. Keller and I. Reiten, *Acyclic Calabi-Yau categories*, preprint (2006), arXiv:math. RT/0610594.

[KR07] ———, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

[KS06] M. Kontsevich and Y. Soibelman, *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I*, preprint (2006), arXiv:math. RA/0606241.

[KV87] B. Keller and D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 6, 225–228.

[KV88] ———, *Aisles in derived categories*, Bull. Soc. Math. Belg. **40** (1988), 239–253.

[KY08] B. Keller and D. Yang, *Quiver mutation and derived equivalences*, in preparation (2008).

[MRZ03] R. Marsh, M. Reineke, and A. Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4171–4186 (electronic).

[Nee01] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[Pal] Y. Palu, *On algebraic Calabi-Yau categories*, Ph.D. in preparation.

[Pal08] ———, *Grothendieck group and generalized mutation rule for 2-Calabi–Yau triangulated categories*, preprint (2008), arXiv:math. RT/08033907.

[Rei07] I. Reiten, *Calabi-Yau categories*, Talk at the meeting: Calabi-Yau algebras and $N$-Koszul algebras (CIRM 2007).

[Rie80a] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. **55** (1980), no. 2, 199–224.

[Rie80b] ———, *Representation-finite self-injective algebras of class $A_n$*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 449–520.

[Rie83a] ———, *Many algebras with the same Auslander-Reiten quiver*, Bull. London Math. Soc. **15** (1983), no. 1, 43–47.

[Rie83b] ———, *Representation-finite self-injective algebras of class $D_n$*, Compositio Math. **49** (1983), no. 2, 231–282.

[Rin84] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984.

[Rin98] ———, *The preprojective algebra of a quiver*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 467–480.

[Rin06] ———, *Hereditary triangulated categories*, Comp. Math. (2006), (to appear).

[RVdB02] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366 (electronic).

[Tab07] G. Tabuada, *On the structure of Calabi-Yau categories with a cluster tilting subcategory*, Doc. Math. **12** (2007), 193–213 (electronic).

[Tep] M. Tepetla, *Ph.D. in preparation.*

[Ver77] J.-L. Verdier, *Catégories dérivées. Quelques résultats*, Lect. Notes Math. **569** (1977), 262–311.

[Ver96] ______, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, edited and with a note by Georges Maltsiniotis.

[XZ02] J. Xiao and B. Zhu, *Relations for the Grothendieck groups of triangulated categories*, J. Algebra **257** (2002), no. 1, 37–50.

[XZ05] ______, *Locally finite triangulated categories*, J. Algebra **290** (2005), no. 2, 473–490.

[Yos90] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990.