Mémoire d'habilitation à diriger des recherches

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Catégories triangulées, équivalences et modèles topologiques

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Introduction

L'objet de ce mémoire est de présenter les résultats principaux de mes recherches depuis ma thèse de doctorat. Mon domaine de recherche est la théorie des représentations, et je m'intéresse en particulier à l'étude des catégories triangulées. Je vais maintenant essayer d'expliquer pourquoi ces catégories apparaissent naturellement en théorie des représentations.

Un des objectifs de la théorie des représentations est le suivant. Etant donné une algèbre Λ sur un corps k, on souhaite comprendre tous les modules sur Λ, ainsi que les morphismes entre eux. La famille $\mathrm{Mod}\Lambda$ de tous ces modules et de leurs morphismes forme ce qui s'appelle une *catégorie*. Elle fait partie des catégories dites *abéliennes*. Sans entrer dans les détails précis de la définition, disons que ce sont des catégories dans lesquelles les notions de noyau, image, conoyau, et donc de suites exactes jouent un rôle fondamental.

Etant donné un Λ-module M, il est souvent utile d'en donner une *présentation*, c'est-à-dire de le voir comme le conoyau d'un morphisme $d_0 : P_1 \to P_0$ entre modules projectifs (qui sont une généralisation des modules libres). Le morphisme d_0 n'étant en général pas injectif, on en vient à considérer un complexe de Λ-modules projectifs

$$P_\bullet(M) := \ \cdots \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0 \longrightarrow \cdots$$

tel que pour tout $n \geq 0$ on a $\mathrm{Ker} d_n = \mathrm{Im} d_{n+1}$ et où M est le conoyau du morphisme d_0. Dans ce cas, l'homologie du complexe $P_\bullet(M)$ est nulle en tout degré sauf en degré 0 où elle est isomorphe à M. Un tel complexe $P_\bullet(M)$ est appelé une *résolution projective* de M: de manière un peu raccourcie, on peut comprendre le module P_0 comme engendré par un ensemble de générateurs pour M, le module P_1 comme engendré par les relations entre ces générateurs, le module P_2 comme engendré par les relations entre ces relations, etc... Par ailleurs, une propriété des résolutions projectives est la suivante : tout morphisme de modules $f : M \to N$ donne lieu à un morphisme de complexes $P_\bullet(M) \to P_\bullet(N)$. Malheureusement, si tout module admet bien une résolution projective, celle-ci n'est unique qu'à homotopie de complexes près, et de même pour le morphisme induit $P_\bullet(M) \to P_\bullet(N)$. Il devient alors naturel de travailler dans la catégorie homotopique $\mathcal{K}^{-,\mathrm{b}}(\mathrm{Proj}\Lambda)$, dont les objets sont des complexes de projectifs bornés à droite, et dont les morphismes sont les morphismes de complexes modulo homotopie. La catégorie des Λ-modules $\mathrm{Mod}\Lambda$ peut alors se voir comme une sous-catégorie pleine de $\mathcal{K}^{-,\mathrm{b}}(\mathrm{Proj}\Lambda)$. Si par ailleurs l'algèbre Λ est de dimension finie, tout Λ-module de dimension finie admet une résolution formée de modules projectifs de dimension finie. Cette résolution sera de plus bornée à gauche si la dimension globale de Λ est finie. On aura donc dans ce cas une inclusion

$$\mathrm{mod}\,\Lambda \subset \mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda),$$

où $\mathrm{mod}\,\Lambda$ désigne la catégorie des Λ-modules de dimension finie (qui est aussi abélienne), et $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ la catégorie homotopique des complexes bornés de modules projectifs de dimension finie. Malheureusement, les notions de noyau, de conoyau ou d'image ne sont pas bien définies dans la catégorie $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$: cette catégorie n'est pas abélienne. Elle hérite malgré tout des belles propriétés de la catégorie $\mathrm{mod}\,\Lambda$ en ayant la structure de ce qu'on appelle une *catégorie triangulée.* Par exemple, toute suite exacte

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

dans mod Λ donne lieu à ce qu'on appelle un *triangle*

$$P_\bullet(X) \longrightarrow P_\bullet(Y) \longrightarrow P_\bullet(Z) \longrightarrow P_\bullet(X)[1]$$

où $P_\bullet(X)[1]$ est le complexe $P_\bullet(X)$ décalé d'un cran vers la gauche. Les triangles de $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ vérifient des propriétés proches de celles des suites exactes courtes de mod Λ. De plus la catégorie $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ contient de nombreuses informations homologiques sur l'algèbre Λ. En particulier on a des isomorphismes pour tout $n \in \mathbb{Z}$

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)}(P_\bullet(X), P_\bullet(Y)[n]) \simeq \mathrm{Ext}^n_\Lambda(X, Y). \tag{0.1}$$

fonctoriels en X et en Y. Par ailleurs, dans l'idée de mieux "voir" la catégorie mod Λ dans la catégorie homotopique, il est souvent plus aisé de donner une autre description de la catégorie $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$. On considère la catégorie dont les objets sont cette fois des complexes bornés de Λ-modules de dimension finie (qui ne sont donc plus nécessairement projectifs), et dont les morphismes sont les morphismes de complexes modulo homotopie, où l'on inverse formellement les quasi-isomorphismes (i.e. les morphismes induisant un isomorphisme dans l'homologie). Le Λ-module X peut alors se voir comme un complexe concentré en degré zéro :

$$\cdots \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow \cdots .$$

La projection naturelle $P_\bullet(X) \to X$ est alors un quasi-isomorphisme et devient donc un isomorphisme dans cette catégorie. Cette catégorie, notée $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ est appelée la *catégorie dérivée bornée*. Dans le cas où Λ est de dimension finie et de dimension globale finie, elle coincide avec la catégorie $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$. Du fait des isomorphismes (0.1), elle est un invariant homologique très intéressant de l'algèbre Λ. Une question naturelle et difficile de théorie des représentation est de déterminer si deux algèbres données ont des catégories dérivées équivalentes. Celle-ci a donné naissance dans les années 80 à ce qu'on appelle la *théorie du basculement* (tilting), omniprésente en théorie des représentations depuis lors [AHK07].

Les catégories triangulées apparaissent aussi naturellement en théorie des représentations dans la construction de la *catégorie stable* décrite par Happel dans [Hap88]. Si Λ est une algèbre auto-injective, les Λ-modules projectifs sont aussi injectifs. Ils jouent donc un rôle très particulier dans la catégorie mod Λ. Il devient alors intéressant de considérer la catégorie stable $\underline{\mathrm{mod}}\Lambda$, où l'on quotiente les espaces de morphismes par les morphismes se factorisant par des projectifs-injectifs. Cette procédure 'régularise' en quelque sorte la catégorie mod Λ en 'éliminant' ces objets particuliers que sont les projectifs-injectifs. Happel montre dans [Hap88] que cette catégorie a aussi une structure de catégorie triangulée. Cette construction se généralise aux algèbres Iwanaga-Gorenstein (le cas auto-injectif correspondant à la dimension Gorenstein 0), où l'on considère non plus tous les Λ-modules, mais certains modules appelés *modules de Cohen-Macaulay*. On obtient ainsi une catégorie triangulée $\underline{\mathrm{CM}}(\Lambda)$, appelée parfois *catégorie des singularités*. Cette catégorie est en effet nulle si et seulement si l'algèbre Λ est homologiquement lisse, autrement dit de dimension globale finie.

Partant de ces deux types de constructions de catégories triangulées (et de leurs généralisations), on peut en construire d'autres, en utilisant la notion de quotient de Verdier d'une catégorie triangulée par une sous-catégorie triangulée pleine. Presque toutes les catégories triangulées de théorie des représentations sont construites de cette manière.

Mon travail de recherche peut se résumer en deux fils conducteurs, tous deux allant dans le sens de mieux comprendre ces catégories:

1. trouver des équivalences entre certaines catégories triangulées ;

2. décrire explicitement les objets indécomposables et morphismes de certaines catégories triangulées.

La première question est source de nombreux travaux dans le domaine. Mentionnons déjà trois résultats fondamentaux très classiques :

- le premier, dû à Happel [Hap88], permet de comprendre la catégorie $\mathcal{D}^{\mathrm{b}}(\Lambda)$ comme une catégorie stable graduée $\underline{\mathrm{mod}}^{\mathbb{Z}}\mathbf{T}(\Lambda)$, via l' extension triviale de Λ par son dual $\mathrm{Hom}_k(\Lambda, k)$;
- le deuxième, dû à Buchweitz [Buc87] et Rickard [Ric89], permet de comprendre la catégorie stable $\underline{\mathrm{CM}}(\Lambda)$ d'une algèbre Iwanaga-Gorenstein comme le quotient de $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ par la sous-catégorie pleine $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$;
- le troisième donne un critère permettant de montrer qu'une catégorie triangulée est équivalente à $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ via l'existence d'objets *basculants*. Ce dernier résultat, dû à de multiples auteurs selon la généralité de l'énoncé [Hap87, Ric89b, Kel94], a donné lieu à la théorie du basculement.

La deuxième question est en général sans espoir. En effet la plupart des algèbres sont de type *sauvage*, et l'on ne peut même pas espérer décrire leur catégorie de modules de dimension finie. Cependant, pour certaines algèbres particulières Λ, les catégories $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ sont bien connues et ont été largement étudiées dans la littérature. C'est notamment le cas pour les algèbres de chemins sur un carquois acyclique (ou algèbres héréditaires) dans le cas où le carquois est de type Dynkin, ou Dynkin étendu [Hap87]. C'est aussi le cas pour une classe d'algèbres introduites par Assem et Skowronski [AS87] appelées algèbres *aimables*, ou encore pour les algbres *quasi-aimables* (skew-gentle) introduites par Geiss et de la Peña [GePe99]. Toutes ces algèbres sont *dociles* (tame) et dérivée-dociles. Il a ainsi été possible grâce à de nombreux travaux de décrire les objets indécomposables des catégories $\mathrm{mod}\,\Lambda$ et $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ ainsi que les espaces de morphismes en termes combinatoires à partir du carquois à relation définissant Λ [BR87, BM03, ALP16, CB00, BMM03]. D'autres informations concernant par exemple le carquois d'Auslander-Reiten de ces catégories ont aussi pu être obtenues [BR87, AAG08, GePe99].

Une troisième famille de catégories triangulées joue un rôle important dans ce mémoire, il s'agit de la *catégorie d-amassée* $\mathcal{C}_d(\Lambda)$ construite à partir d'une algèbre Λ de dimension globale $\leq d$ et vérifiant une certaine condition de finitude (τ_d-finie) [BMRRT06, Ami, Guo11]. Ces catégories sont construites comme enveloppe triangulée d'une catégorie d'orbites de la catégorie dérivée $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. Elles peuvent aussi s'interpréter comme le quotient de catégories dérivées de certaines algèbres différentielles graduées (DG). Ces catégories ont la particularité d'avoir une symétrie appelée *d-Calabi-Yau*, et d'avoir un ensemble d'objets appelés *amas-basculants* (cluster-tilting) dont la combinatoire (dans le cas $d = 2$) a des similarités avec la combinatoire des algèbres amassées (cluster). Ces propriétés combinatoires sont notamment encodées dans l'algèbre d'endomorphismes $\Pi_{(d+1)}(\Lambda)$ d'un objet amas-basculant initial appelée l'algèbre $(d+1)$-*préprojective* de Λ. Dans le cas $d = 2$, la catégorie amassée $\mathcal{C}_2(\Lambda)$ est très proche de la catégorie $\mathrm{mod}\,\Pi_2(\Lambda)$. Ainsi lorsque $\Pi_2(\Lambda)$ est une algèbre aimable ou quasi-aimable, il devient possible d'obtenir de nombreux renseignements sur la catégorie amassée associée.

Partant d'une algèbre Λ de dimension finie sur un corps k, et de dimension globale finie $\leq d$, on peut alors lui associer les trois objets suivants:

- sa catégorie dérivée bornée $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$;
- sa catégorie d-amassée $\mathcal{C}_d(\Lambda)$;
- son algèbre $(d+1)$-préprojective $\Pi_{d+1}(\Lambda)$,

qui sont les principaux objets d'étude de ce mémoire.

Ce travail est divisé en deux parties principales.

- La première porte sur les différentes propriétés algébriques de ces trois constructions et les liens entre elles.

- La deuxième partie quant à elle, décrit comment on peut associer un objet topologique à certaines de ces catégories triangulées (amassées ou dérivées) et d'établir un dictionnaire entre les propriétés algébriques de la catégorie et les propriétés topologiques de l'objet associé.

Passons maintenant à une description plus précise de la première partie. Le premier chapitre se concentre sur l'algèbre $(d+1)$-préprojective et ses différentes propriétés. La construction dans le cas $d = 1$ est classique et dûe à Gelfand et Ponomarev [GP79]. Elle a été intensément étudiée dans la littérature [Rin98, ES98I, BBK02, CBH98, RV89]. La construction a été généralisée par Iyama et Oppermann [IO11] dans le cadre de la théorie d'Auslander-Reiten supérieure. Un des objectifs de ce chapitre consiste à montrer que la construction pour un $d \geq 2$ satisfait des propriétés similaires à la construction classique, au moins pour certaines algèbres Λ dites *d-héréditaires* [HIO14]. Je m'intéresse ici en particulier aux propriétés homologiques satisfaites par l'algèbre préprojective. Ces propriétés sont notamment décrites dans les articles [Ami14b, AIR15, AO15, Dug12]. Notons que dans le cas $d = 1$, les algèbres préprojectives ont aussi été très largement étudiées pour les liens très étroits qu'elles entretiennent avec les algèbres amassées. Ces questions ne seront pas abordées dans ce mémoire, nous renvoyons à [GLS13, Lec11] pour des articles de survol sur le sujet.

Le deuxième chapitre se concentre sur les catégories d-amassées d'algèbres τ_d-finies, et à leur lien avec certaines catégories de singularités d'algèbres Iwanaga-Gorenstein. En effet, pour certaines de ces algèbres R, comme par exemple certaines singularités isolées de dimension d, la catégorie $\underline{\mathrm{CM}}(R)$ est d-Calabi-Yau et contient des objets d-amas-basculants. On cherche ici à construire une équivalence triangulée

$$\mathcal{C}_d(\Lambda) \simeq \underline{\mathrm{CM}}(R).$$

Ce chapitre contient des résultats des articles [Ami09, ART11, AIRT12, AIR15, Ami14a] ainsi que plusieurs de leurs généralisations [IO13, Kim17, Han].

Le troisième chapitre se concentre quant à lui sur la catégorie dérivée des algèbres τ_2-finies. Le cas $d = 2$ est spécialement intéressant car la combinatoire des algèbres amassées, notamment la mutation des carquois à potentiel, y joue un rôle primordial. Ce chapitre traite de différentes collaborations avec Steffen Oppermann [AO13a, AO14, AO13b]. Il s'agit ici d'interpréter la combinatoire amassée et la théorie de l'amas-basculement non plus dans la catégorie amassée $\mathcal{C}_2(\Lambda)$, mais dans la catégorie dérivée $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$. Nous introduisons la notion de *mutation graduée*, qui nous permet entre autres choses, d'obtenir de nouveaux critères combinatoires pour déterminer si deux algèbres de dimension globale ≤ 2 sont dérivée-équivalentes.

La deuxième partie de ce mémoire porte sur la notion de modèle topologique pour une catégorie triangulée. L'idée est d'attacher un objet topologique (ou géométrique) à une catégorie triangulée qui 'encoderait' la catégorie. L'idéal étant que cet objet détermine entièrement la catégorie. Lorsqu'on a un tel modèle, il devient intéressant d'essayer de traduire les différentes propriétes de la catégorie en terme de cet objet topologique.

Le premier chapitre de cette partie décrit de tels modèles dans le cadre amassé. En particulier, un premier exemple de modèle topologique provient de la catégorie amassée associée au carquois à potentiel d'une surface triangulée. Le tout premier exemple remonte à un article de Caldero, Chapoton et Schiffler [CCS06] où les auteurs donnent une description de la catégorie amassée de type A en termes de diagonales de polygones. Plus généralement, partant de la catégorie amassée associée à une surface triangulée, différents travaux [ABCP10, BZ11, QZ17, CS17, CPS19] ont permis d'interpréter les objets indécomposables et les espaces de morphismes de cette catégorie en termes de courbes et d'intersections sur la surface. De plus, en combinant les résultats [KY11, LF09, BZ11, QZ17], on peut montrer que deux catégories amassées d'une surface triangulée sont équivalentes si et seulement si les surfaces correspondantes sont homéomorphes. Le modèle topologique naturel de la catégorie amassée est donc ici la surface (et non plus la surface munie d'une triangulation). Mon travail en collaboration avec Pierre-Guy Plamondon [AP] s'est concentré sur le cas où la surface admet des points marqués à l'intérieur. En interprétant ces points marqués non pas comme des pointures, mais comme des points orbifold, et en construisant un revêtement double de cet orbifold, nous avons pu décrire les objets indécomposables de la catégorie amassée associée en termes des courbes sur la surface orbifold. Ce travail a fortement utilisé les résultats de l'article [RR85] qui étudie les algèbres tordues par des groupes finis. On peut ainsi comprendre la catégorie amassée associée à une surface avec des points marqués à l'intérieur, comme la catégorie amassée associée à son revêtement double tordue par le groupe $\mathbb{Z}/2\mathbb{Z}$.

Ce cadre amassé peut se raffiner pour obtenir un modèle topologique pour la catégorie dérivée de certaines algèbres, dites *algèbres de coupes de surface* (surface cut algebras). À une surface triangulée munie d'une certaine *coupe admissible*, on associe une algèbre τ_2-finie Λ dont la catégorie amassée $\mathcal{C}_2(\Lambda)$ est équivalente à la catégorie amassée de la surface triangulée. En utilisant la mutation graduée introduite dans [AO14], nous montrons dans [AG16, Ami16, ALP20] que le modèle topologique naturel de la catégorie $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ est la surface $\mathcal{S}$ (éventuellement orbifold) à points marqués munie d'un certain $H^1(\mathcal{S},\mathbb{Z})$-espace affine, qui peut s'interpréter comme une classe d'homotopie de champ de droites sur la surface.

Le deuxième chapitre de cette partie sort du cadre de la combinatoire amassée pour se concentrer sur la catégorie dérivée des algèbres aimables et quasi-aimables. Ainsi dans [OPS], les auteurs associent à toute algèbre aimable Λ une surface à points marqués munie d'une dissection (qui généralise la notion de triangulation), et décrivent les objets indécomposables de la catégorie $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ ainsi que les morphismes en termes topologiques. Cette description est particulièrement intéressante car elle possède des liens avec certaines catégories de Fukaya partiellement enroulées [LP, HKK]. Dans la prépublication [APS] (voir aussi [Opp]), nous associons un champ de droites à toute surface munie d'une dissection, et montrons que la surface munie de la classe d'homotopie du champ de droites est un invariant dérivé complet pour l'algèbre aimable Λ. Ceci répond à une conjecture de [HKK] dans le cas non gradué. La construction de cet invariant utilise très fortement la théorie du basculement.

Nous généralisons ce résultat dans ma dernière prépublication avec Thomas Brüstle [AB] au cas des algèbres quasi-aimables, en considérant cette fois non pas une surface munie d'un champ de droites, mais une surface orbifold munie d'un champ de droites, ainsi que son revêtement double. Ce modèle topologique permet par ailleurs d'obtenir une description des objets indécomposables en termes de courbes sur la surface orbifold associée. Je donne dans l'appendice de ce mémoire une preuve alternative au résultat de [LSV], qui n'utilise pas la description combinatoire des objets de la catégorie dérivée donnée dans [BMM03].

Liste des principaux travaux présentés dans ce mémoire

I. Algèbres préprojectives supérieures, catégories amassées et dérivées

1. *The ubiquity of the generalized cluster categories*, with Gordana Todorov and Idun Reiten, Advances in Mathematics, Vol. 226. (2011), pages 3813-3849.
2. *Preprojective algebras and c-sortable words*, with Osamu Iyama, Gordana Todorov and Idun Reiten, Proceedings of the London Mathematical Society Volume 104 Part 3 (2012), pages 513-539.
3. *The image of the derived category in the cluster category*, with Steffen Oppermann, International Mathematical Research Notices, Volume 2013, Issue 4, pages 733-760 (28 pages).
4. *Singularity categories, Preprojective algebras and orthogonal decompositions*, Algebras, Quivers and Representations, The Abel Symposium 2011, Buan, Reiten, Solberg Editors, Springer, pages 1-11.
5. *Cluster equivalence and graded derived equivalence*, with Steffen Oppermann, Documenta Math. 19 (2014) pages 1155-1206.
6. *Stable categories of Cohen-Macaulay modules and cluster categories*, with Osamu Iyama et Idun Reiten, American Journal of Math, Volume 137, Number 3, (2015), pages 813-857.
7. *Higher preprojective algebras and stably Calabi-Yau properties*, with Steffen Oppermann, Math Research Letters, Volume 21, Number 4 (2015) pages 1-31.

II. Catégories triangulées et modèles topologiques

1. *Algebras on acyclic cluster type: tree type and type* $\widetilde{A}$, with Steffen Oppermann, Nagoya Mathematical Journal, Volume 211 (2013), pages 1-50 (50 pages).
2. *Derived invariants for surface algebras*, with Y. Grimeland, Journal of Pure and Applied Algebra 220 (2016) pp. 3133-3155.
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7. *Derived equivalences between skew-gentle algebras using orbifolds*, with Thomas Brüstle, submitted.

Contents

Notation and convention

We fix an algebraically closed field k. All algebras, categories and functors are k-linear (unless otherwise stated). We denote by $D = \mathrm{Hom}_k(-,k)$ the k duality.

Categories

By a triangulated category we mean a k-linear category with finite dimensional Hom-spaces (unless otherwise stated) with a Serre duality, denoted by $\mathbb{S}$. The shift functor is denoted by $[1]$ and the bifunctor $\mathrm{Hom}_{\mathcal{T}}(-,-[n])$ by $\mathrm{Ext}^n_{\mathcal{T}}(-,-)$. For $d \geq 0$, we define the autoequivalence $\mathbb{S}_d := \mathbb{S}[-d]$.

The following definition is fundamental in this memoir.

Definition 0.1. A (Hom-finite) triangulated category is called *d-Calabi-Yau* (d-CY for short) if there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq D\,\mathrm{Hom}_{\mathcal{C}}(Y, X[d]),$$

so in other words, the functor $\mathbb{S}_d$ is isomorphic to the identity.

For $\mathcal{A}$ an additive k-category, we denote by $\mathcal{K}(\mathcal{A})$ its homotopy category, by $\mathcal{D}(\mathcal{A})$ its derived category, and by $\mathrm{per}\,\mathcal{A}$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ closed under direct summands generated by objects in $\mathcal{A}$.

Algebras and modules

For Λ a k-algebra, we denote by $\mathrm{Mod}\Lambda$ the category of right Λ-modules, by $\mathrm{mod}\,\Lambda$ the subcategory of finitely generated Λ-modules, and by $\mathrm{fd}\Lambda$ the category of finite dimensional Λ-modules. We denote by Λ^{op} the opposite algebra, and $\Lambda^{\mathrm{e}} := \Lambda^{\mathrm{op}} \otimes_k \Lambda$ the envelopping algebra. The bimodule duality $\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(-,\Lambda^{\mathrm{e}})$ is denoted by $(-)^{\vee} :=$. We denote by $\mathcal{D}^{\mathrm{b}}(\Lambda)$ the bounded derived category of $\mathrm{mod}\,\Lambda$.

An algebra Λ is called Iwanaga-Gorenstein if the projective dimension of the module $D\Lambda$ and the injective dimension of the module Λ are finite. In that case, these dimensions coincide and is called the Gorenstein dimension of Λ. For Λ an Iwanaga-Gorenstein algebra, we denote by

$$\mathrm{CM}(\Lambda) := \{M \in \mathrm{mod}\,\Lambda |\ \mathrm{Ext}^i_{\Lambda}(M,\Lambda) = 0\ \forall i > 0\}$$

the category of (maximal) Cohen-Macaulay Λ-modules.

For a graded algebra $\Lambda = \bigoplus_{n\in\mathbb{Z}} \Lambda_n$, we denote by $\mathrm{Mod}^{\mathbb{Z}}\Lambda$ (resp. $\mathrm{mod}^{\mathbb{Z}}\,\Lambda$, resp. $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$) the category of (resp. finitely generated, resp. Cohen-Macaulay) graded Λ-modules. For a graded module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, we denote by $M(1)$ the graded module where $M(1)_n := M_{n-1}$. Note that the algebra Λ^{e} inherits of a natural grading, and we often consider it as a $\mathbb{Z}$-graded algebra. We may also consider Λ as a DG-algebra with zero differential. In this case, we denote by $\mathcal{D}(\Lambda^{DG})$ the derived category of the DG-algebra Λ.

Quivers

For a quiver Q, we denote by Q_0 its set of vertices, Q_1 its set of arrows, and by $s,t : Q_1 \to Q_0$ the source and target maps. The path algebra is denoted by kQ. Composition of arrows is from right to left as functions. For any $i \in Q_0$, we denote by e_i the trivial path at vertex i.

A potential W on a quiver is given by a class in $kQ/[kQ,kQ]$. We define the partial derivative ∂_a with respect to an arrow a as a map $\partial_a : kQ/[kQ,kQ] \to kQ$ defined by

$$\partial_a(a_1 \dots a_r) = \sum_{i=1}^{r} a_{i+1} \dots a_r a_1 \dots a_{i-1},$$

and extended by k-linearity.

The Jacobian algebra $\mathrm{Jac}(Q,W)$ of the quiver with potential (Q,W) (QP for short) is defined as

$$\mathrm{Jac}(Q,W) := \widehat{kQ} \big/ \overline{\langle \partial_a W, a \in Q_1 \rangle},$$

where $\widehat{kQ}$ is the completion of the path algebra kQ, and $\overline{\langle \partial_a W, a \in Q_1 \rangle}$ is the closure of the ideal generated by $\partial_a W$ for any $a \in Q_1$.

The complete Ginzburg DG algebra $\widehat{\Gamma} = \widehat{\Gamma}_{Q,W}$ of (Q,W) is defined as follows.

Let $\bar{Q}$ be the graded quiver whose vertices set is that of Q and whose arrows set contains

- for every arrow $\alpha : i \to j$ in Q, an arrow $\alpha : i \to j$ of degree 0;
- for every arrow $\alpha : i \to j$ in Q, an arrow $\bar{\alpha} : j \to i$ of degree -1; and
- for every vertex i of Q, a loop $t_i : i \to i$ of degree -2.

Then, as a graded algebra, $\widehat{\Gamma}$ is the complete path algebra of $\bar{Q}$, that is, for every integer m,

$$\widehat{\Gamma}^m = \prod_{w \text{ path of degree } m} kw.$$

The differential of $\widehat{\Gamma}$ is the continuous map defined as follows on arrows, and extended by linearity and the Leibniz rule: for any arrow α of Q, $d(\alpha) = 0$ and $d(\bar{\alpha}) = \partial_\alpha S$, and for any vertex i of Q, $d(t_i) = e_i\big(\sum_{\alpha \in Q_1}(\alpha\bar{\alpha} - \bar{\alpha}\alpha)\big)e_i$.

Note that in this memoir, we consider quivers with potential associated with a triangulated surface with non-empty boundary, and the non completed Ginzburg or Jacobian algebra is isomorphic to the completed one.

Part I

Preprojective algebras, cluster categories and derived categories

Chapter 1

Higher preprojective algebras

The notion of (higher) preprojective algebra is central in my work. The aim of this chapter is to describe different properties satisfied by these algebras, and in which context they naturally appear. Let us start with the definition.

Definition 0.1. [IO13] Let Λ be an algebra of global dimension $\leq d$, then the $(d+1)$-*preprojective algebra* of Λ is defined to be the tensor algebra over Λ of the Λ-bimodule $\operatorname{Ext}^d_{\Lambda^e}(\Lambda, \Lambda^e)$

$$\Pi_{d+1}(\Lambda) := \mathrm{T}_\Lambda \operatorname{Ext}^d_{\Lambda^e}(\Lambda, \Lambda^e).$$

The classical notion of a preprojective algebra of a quiver (the case $d = 1$) was introduced by Gelfand and Ponomarev [GP79]. Already in this set up we can make the following observations:

- seen as a kQ-module, $\Pi_2(kQ)$ is the direct sum of all preprojective kQ-modules[1] (that justifies the name) [Rin98]
- the behaviour of $\Pi_2(kQ)$ changes completely depending whether Q is Dynkin or not;
- the algebra $\Pi_2(kQ)$ has a presentation in term of quiver and relations that can easily be contructed from Q;
- when Q is of extended Dynkin type, and when k has characteristic 0, the preprojective algebra $\Pi_2(kQ)$ is Noetherian. Moreover, it can be related with the finite subgroup G of $\mathrm{SL}_n(k)$ corresponding to the Dynkin type of Q: it is a non commutative resolution of the invariant polynomial ring $k[X, Y]^G$, [Rei87, RV89]

The aim of this chapter is to show how all these properties generalize for general d.

We can first be a bit more precise concerning the second item: in the Dynkin case, the preprojective algebra is finite dimensional and its bimodule projective resolution is periodic [ES98I, ES98II, BBK02]; whereas in the non Dynkin case, the preprojective algebra is infnite dimensional and its bimodule projective resolution has a certain symmetry, it is bimodule 2-Calabi-Yau [CB00].

In a first section, we relate the $(d+1)$-preprojective algebra to the derived category of Λ and certain endofunctors $\mathbb{S}_d$ and τ_d that are higher generalizations of Auslander-Reiten translate for the derived and for the module category. We also explain how the preprojective algebra can be viewed as the H^0 of a certain negatively graded DG algebra $\mathbf{\Pi}_{(d+1)}(\Lambda)$ called the *derived preprojective algebra*. Furthermore, we explain that the $(d+1)$-preprojective algebra of Λ, when finite-dimensional, appear naturally in the construction of the d-cluster category of Λ [Ami, Guo11].

[1] Preprojective modules are modules that are direct summands of direct sums of $\tau^{-n}(kQ)$ where τ is the Auslander-Reiten translation in $\operatorname{mod} kQ$.

In a second section, we discuss the presentation of the preprojective algebra by a quiver with relations. In the classical case $d = 1$, this description relies on reflection functors, and we show how these reflection functors can be "seen" in the module or derived category of the preprojective algebras. We further explain how the presentation of the preprojective algebra as a quiver with "symmetric" relations can be generalized in the case $d = 2$. Some generalizations in the case $d \geq 3$ have also been studied with the notion of (higher) Jacobian algebras [Kel11, BSW10, IG].

In a third section, we introduce the notion of *d-hereditary algebras*, through *d-representation finite* algebras (d-RF) and *d-representation infinite* algebras (d-RI), that are higher analogues of the dichotomy Dynkin/not Dynkin cases [HIO14]. Certain behaviour of the derived category generalize in that case, as certain properties of the d-preprojective algebra.

In a fourth section, we show how the properties of the bimodule projective resolution of the preprojective algebras generalize in particular in the setting of d-RI and d-RF. This relies on the works [AIR15] and [AO15] (see also [Ami14b]).

Finally in a fifth section, we show how certain Noetherian preprojective algebras can be seen as non-commutative resolutions of certain Gorenstein algebras (see [AIR15]).

Motivating example

In the rest of the chapter, we will consider only preprojective algebras $\Pi_{d+1}(\Lambda)$ where Λ is finite dimensional. However we start with the example of the polynomial ring, which gives an idea of the behaviour of certain higher preprojective algebras (typically the $(d+1)$-preprojective algebras of d-RI algebras).

Denote $R = k[x_1, \ldots, x_d]$. For any $0 \leq \ell \leq d$ define the set

$$I_d^\ell := \{(i_1, i_2, \cdots, i_\ell), i_1 < i_2 < \ldots < i_\ell, i_j \in \{1, \ldots, d\}\}.$$

Consider the following complex of R-bimodules:

$$0 \longrightarrow P_d \xrightarrow{\partial_d} \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

where $P_\ell := (R \otimes R)^{I_d^\ell}$ and where

$$\partial_\ell((1 \otimes 1)_{i_1 \ldots i_\ell}) = \sum_{s=1}^{\ell} (-1)^{s+1} (x_{i_s} \otimes 1 - 1 \otimes x_{i_s})_{i_1 \ldots \hat{i}_s \ldots i_\ell}.$$

This complex of R-bimodules gives a projective resolution of R as an R-bimodule. Moreover, by applying the functor $(-)^\vee := \mathrm{Hom}_{R^e}(-, R^e)$ and using the canonical pairing between I_d^ℓ and $I_d^{d-\ell}$, one obtains an isomorphism of complexes

$$P_\bullet^\vee[d] \simeq P_\bullet.$$

As a consequence we obtain an isomorphism of R-bimodules

$$R \simeq \mathrm{Ext}^d_{R^e}(R, R^e),$$

hence one has $\Pi_{d+1}(R) \simeq k[x_1, \ldots, x_{d+1}]$.

1 Preprojective algebras and the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$

In the rest of this work, we will concentrate on the case where Λ is a finite dimensional algebra. In that case we have an isomorphism of Λ-bimodules

$$\mathrm{Ext}^d_{\Lambda^e}(\Lambda, \Lambda^e) \simeq \mathrm{Ext}^d_\Lambda(D\Lambda, \Lambda), \text{ (see for instance [IG, Lemma 2.9])}.$$

1.1 The functors τ_d and $\mathbb{S}_d$

Definition 1.1 (Keller, [Kel11]). Let Λ be an algebra of global dimension $\leq d$. The *derived* $(d+1)$*-preprojective algebra* of Λ is defined to be the tensor DG-algebra

$$\mathbf{\Pi}_{d+1}(\Lambda) := \mathrm{T}_\Lambda \Theta$$

where Θ is a cofibrant replacement of the DG-bimodule $\mathbf{R}\mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda^e)[d]$.

Since the algebra Λ has global dimension $\leq d$, then we have a canonical isomorphism

$$H^0(\mathbf{\Pi}_{d+1}(\Lambda)) \simeq \Pi_{d+1}(\Lambda).$$

For Λ of global dimension $\leq d$, we denote by $\mathbb{S} := - \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} D\Lambda$ the Serre functor of the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$, and by $\mathbb{S}_d := \mathbb{S} \circ [-d]$ its composition with the d power of the inverse of the shift. The functor $\mathbb{S}_d$ is an auto-equivalence, we denote by $\mathbb{S}_d^{-1}$ its inverse, and for $\ell \geq 0$ by $\mathbb{S}_d^{-\ell}$ the ℓ-power of the inverse. We also denote by $\tau_d := H^0(\mathbb{S}_d) : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda$. In the case $d = 1$, $\mathbb{S}_1$ is the Auslander-Reiten translate in the derived category, while τ_1 is the Auslander-Reiten translate in the module category.

The behaviour of these two functors have important impact on the properties of the preprojective algebras. Here is a first observation.

Proposition 1.2. *Let Λ be a finite dimensional algebra of global dimension $\leq d$. We have isomorphisms*

$$\mathbf{\Pi}_{d+1}(\Lambda) \simeq \bigoplus_{\ell \geq 0} \mathbb{S}_d^{-\ell}(\Lambda) \text{ in } \mathcal{D}(\mathrm{Mod}\Lambda); \quad \text{and } \Pi_{d+1}(\Lambda) \simeq \bigoplus_{\ell \geq 0} \tau_d^{-\ell}\Lambda \text{ in } \mathrm{Mod}\Lambda.$$

This leads to the following

Definition 1.3. [Ami, Guo11] An algebra of global dimension $\leq d$ is said to be τ_d*-finite* if the preprojective algebra $\Pi_{d+1}(\Lambda)$ is finite dimensional. This is equivalent to the fact that the functor τ_d is nilpotent.

1.2 Cluster-tilting objects

Here we recall how finite dimensional d-preprojective algebras can be seen as endomorphism algebras of d-cluster-tilting object in certain triangulated categories.

Definition 1.4. [Iya07a, KR07] Let $\mathcal{C}$ be a Hom-finite abelian, or triangulated category. A full subcategory $\mathcal{U} \subset \mathcal{C}$ is called *d-cluster-tilting* if $\mathcal{U}$ is functorially finite [2] and if

$$\begin{aligned} \mathcal{U} &= \{X \in \mathcal{C}, \text{ such that } \mathrm{Ext}^i_{\mathcal{C}}(U, X) = 0 \ \forall i = 1, \ldots, d-1, \ \forall U \in \mathcal{U}\} \\ &= \{X \in \mathcal{C}, \text{ such that } \mathrm{Ext}^i_{\mathcal{C}}(X, U) = 0 \ \forall i = 1, \ldots, d-1, \ \forall U \in \mathcal{U}\} \end{aligned}$$

An object $U \in \mathcal{C}$ is called *d-cluster-tilting* if $\mathrm{add}U$ is d-cluster-tilting.

The following definition has been given in my thesis [Ami] and [Ami09] for the case $d = 2$, and generalized for any d in Guo's thesis [Guo11].

Definition 1.5. [Ami, Guo11] The *(generalized) d-cluster category* of a τ_d-finite algebra of global dimension $\leq d$ is defined as the triangulated hull

$$\mathcal{C}_d(\Lambda) := (\mathcal{D}^{\mathrm{b}}(\Lambda)/\mathbb{S}_d)_\Delta,$$

as defined in [Kel05] (see also Appendix of [IO13]).

[2] A subcategory $\mathcal{U}$ of $\mathcal{C}$ is called functorially finite if any object in $\mathcal{C}$ has left and right approximations by objects in $\mathcal{U}$.

The d-cluster category comes naturally with a triangle functor $\pi : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{C}_d(\Lambda)$.

Theorem 1.6 (Amiot [Ami09] Guo [Guo11])**.** *Let Λ be a τ_d-finite algebra. Then the d-cluster category of Λ is d-Calabi-Yau and the object $\pi(\Lambda)$ is a d-cluster-tilting object such that*

$$\mathrm{End}_{\mathcal{C}_d(\Lambda)}(\pi(\Lambda)) \simeq \Pi_{d+1}(\Lambda).$$

2 Description in terms of quivers with relations

2.1 Classical case $d = 1$

Presentation of $\Pi_2(kQ)$

Let Q be an acyclic quiver. Then the algebra kQ is a finite dimensional algebra of global dimension ≤ 1.

We define the double quiver $\overline{Q}$ from Q by adding for each arrow $a \in Q_1$ an arrow a^* in the opposite direction.

Theorem 2.1 (Ringel [Rin98])**.** *Let Q be an acyclic quiver. Then there is an isomorphism of $\mathbb{Z}$-graded algebras*

$$\Pi_2(kQ) \simeq k\bar{Q}/\langle \sum_{a\in Q_1} [a, a^*]\rangle,$$

where the $\mathbb{Z}$-grading on the RHS is induced by a grading on $\bar{Q}$ assigning degree 0 to arrows $a \in Q_1$ and degree 1 to arrows a^*, $a \in Q_1$.

Note that preprojective algebras have been first introduced and studied by Gelfand and Ponomarev in [GP79] using the definition with the double quiver.

We give here an alternative idea of the argument for this result using projective bimodule resolutions (which is different from the proof in [Rin98]). The minimal projective Λ-bimodule resolution of $\Lambda = kQ$ is given as follows:

$$0 \longrightarrow \bigoplus_{a\in Q_1} \Lambda e_{t(a)} \otimes e_{s(a)}\Lambda \xrightarrow{\partial} \bigoplus_{i\in Q_0} \Lambda e_i \otimes e_i\Lambda \longrightarrow 0\ ;$$

where ∂ is defined as

$$\partial(e_{t(a)} \otimes e_{s(a)}) = (a \otimes e_{s(a)})_{s(a)} - (e_{t(a)} \otimes a)_{t(a)}.$$

Applying $(-)^\vee$ to this complex we obtain:

$$0 \longrightarrow \bigoplus_{i\in Q_0} \Lambda e_i \otimes e_i\Lambda \xrightarrow{\partial^\vee} \bigoplus_{a\in Q_1} \Lambda e_{s(a)} \otimes e_{t(a)}\Lambda \longrightarrow 0\ ;$$

where

$$\partial^\vee(e_i \otimes e_i) = \sum_{a,t(a)=i} (a \otimes e_i)_a - \sum_{b,s(b)=i} (e_i \otimes b)_b.$$

Define the algebra $\Pi := k\bar{Q}/\langle\sum_{a\in Q_1}[a, a^*]\rangle$, as above, and define the Λ-subbimodule of Π as $E = \bigoplus_{a\in Q_1} \Lambda a^*\Lambda$. Then one checks that E is the cokernel of $\partial^\vee$ via the map $e_{s(a)} \otimes e_{t(a)} \mapsto a^*$. Therefore we have an isomorphism of Λ-bimodules

$$E \simeq \mathrm{Ext}^1_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}}).$$

Example 2.2. Let Q be the quiver $1 \xrightarrow{a} 2$. Then the preprojective algebra of Q is presented by the quiver $1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2$ with the relations $aa^* = a^*a = 0$.

Preprojective algebras and reflection functors

This description in term of quiver with relations gives another point of view on the category $\operatorname{mod}\Pi_2(kQ)$. It contains $\operatorname{mod} kQ'$ where Q' is any acyclic orientation of the underlying graph of Q. In particular, if Q and Q' are related by a reflection at a sink i of Q, then there is an equivalence [APR79, BGP73]

$$R_i : \operatorname{mod} kQ/[e_i DkQ] \xrightarrow{\sim} \operatorname{mod} kQ'/[e_i kQ'],$$

and moreover there is an isomorphism $\Pi_2(kQ) \simeq \Pi_2(kQ')$. This equivalence is encoded in the category $\operatorname{mod}\Pi_2(kQ)$ via certain tilting objects, as described in the following result.

Theorem 2.3. *[AIRT12, Cor 2.12] Let Q be an acyclic quiver, and i be a sink in Q. Denote by $I_i := \Pi/\Pi(1-e_i)\Pi$ where $\Pi := \Pi_2(kQ)$ is the preprojective algebra of Q. Then there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{mod} kQ/[e_iDkQ] & \xrightarrow{R_i} & \operatorname{mod} kQ'/[e_ikQ'] \\ \downarrow & & \downarrow \\ \operatorname{mod}\Pi_2(kQ) & \xrightarrow{-\otimes_\Pi I_i} & \operatorname{mod}\Pi_2(kQ) \end{array}.$$

If moreover Q is not Dynkin, then I_i is a tilting object in $\mathcal{D}(\mathrm{Mod}\Pi)$ and we have

$$\begin{array}{ccc} \operatorname{mod} kQ/[e_iDkQ] & \xrightarrow{R_i} & \operatorname{mod} kQ'/[e_ikQ'] \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{b}}(\mathrm{f.l.}\Pi) & \xrightarrow[-\overset{\mathbf{L}}{\underset{\Pi}{\otimes}} I_i]{\sim} & \mathcal{D}^{\mathrm{b}}(\mathrm{f.l.}\Pi) \end{array}.$$

2.2 Case $d \geq 2$

Case $d = 2$ and Jacobian algebras

If $\Lambda = kQ/I$ is a basic algebra of global dimension ≤ 2, let us choose a basis of $\{\rho_\ell\}$ of the spaces $\mathrm{Ext}^2_\Lambda(S_i, S_j)$ for any i, j vertices of Q. The set of $\{\rho_\ell\}$ is a set of generators of the ideal I. Then we define a quiver $\bar{Q}$ from Q and I by adding a new arrow $a_\ell : i \to j$ for any $\rho_\ell \in \mathrm{Ext}^2(S_i, S_j)$, and define W as

$$W := \sum_\ell \rho_\ell a_\ell.$$

Then we have the following:

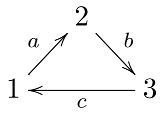
Theorem 2.4. *[Kel11] Let $\Lambda = kQ/I$ be a finite dimensional algebra of global dimension ≤ 2. Let $\bar{Q}$ and W defined as above. Then there is an isomorphism of $\mathbb{Z}$-graded algebras*

$$\Pi_3(\Lambda) \simeq k\bar{Q}/\langle \partial_a W, a \in Q_1 \rangle.$$

where the grading on the RHS is induced from a grading on $\bar{Q}$ assigning degree 0 to any arrow $a \in Q_1$, and degree 1 to any new arrow a_ℓ.

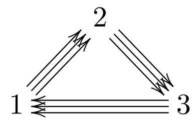
More precisely, Keller constructs a morphism of DG algebras $\Gamma_{(\bar{Q},W)} \to \mathbf{\Pi}_3(\Lambda)$ which is a quasi-isomorphism, where $\Gamma_{(\bar{Q},W)}$ is the Ginzburg DG algebra associated to the quiver with potential $(\bar{Q}, W)$ [Kel11, Thm 6.3]. Since the Ginzburg algebra is negatively graded, and since the Jacobian algebra of $(\bar{Q}, W)$ is the H^0 of the Ginzburg algebra, the theorem above is a consequence of this quasi-isomorphism. A converse of this result has been shown by Van den Bergh in [V15].

Example 2.5. Let Λ be the algebra presented by the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with the relation $ba = 0$. The preprojective algebra of Λ is presented by the following quiver with relations:



$$ba = cb = ac = 0.$$

Example 2.6. Let Λ be the algebra presented by the quiver $1 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 2 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 3$ with the commutativity relations. The preprojective algebra of Λ is presented by the following quiver with the commutativity relations:



.

Case $d \geq 3$

The situation becomes much more complicated for higher d. However in case where the algebra is d-hereditary (see next section for definition) and Koszul, one has a description in terms of higher Jacobian algebras [BSW10, IG, Thi], see also [V15, VV16].

3 d-hereditary algebras

3.1 Definition

The motivation of the introduction of d-hereditary algebras comes from the following observation due to Iyama, which generalizes the case $d = 1$ due to Happel [Hap88].

Lemma 3.1. *[Iya11, Lemma 5.2] Let Λ be an algebra of global dimension $\leq d$. If $X \in \mathcal{D}^{\mathrm{b}}(\Lambda)$ is such that $\mathrm{H}^i(X) = 0$ for $i \in \mathbb{Z} \setminus d\mathbb{Z}$ then X is isomorphic to its homology $\bigoplus_{j \in \mathbb{Z}} \mathrm{H}^j(X)$.*

In general however, if Λ has finite global dimension ≥ 2, the homology of the indecomposable objects can be spread in many degrees. But at least the subcategory

$$\mathcal{D}^{d\mathbb{Z}} := \mathrm{add}\{X \in \mathcal{D}^{\mathrm{b}}(\Lambda), \mathrm{H}^i(X) = 0 \text{ for } i \in \mathbb{Z} \setminus d\mathbb{Z}\}$$

behaves as a higher analogue of the derived category of a hereditary algebra, indeed it is equivalent to copies of the module category $\mathrm{mod}\,\Lambda$.

The d-hereditary algebras are algebras where we have a control of the homology of the $\mathbb{S}_d$-orbit of Λ. More precisely we have the following definition:

Definition 3.2. [HIO14] A finite dimensional algebra Λ is said to be *d-hereditary* if it has global dimension d and if for any $\ell \in \mathbb{Z}$, $\mathbb{S}_d^{\ell}\Lambda \in \mathcal{D}^{d\mathbb{Z}}$.

By Proposition 1.2 the derived $(d+1)$-preprojective algebra is isomorphic to the direct sum of $\mathbb{S}_d^{-\ell}\Lambda$ with $\ell \geq 0$, hence for a d-hereditary algebra, the derived preprojective algebra is isomorphic to its homology. But this isomorphism is an isomorphism of Λ-module and not a DG algebra one, so this remark does not a priori imply that the derived preprojective algebra is formal. However one could hope to have more control on the corresponding preprojective algebra, especially when noticing that its H^0 is the $(d+1)$-preprojective algebra.

With the definition above, one observes two main different behaviours for d-hereditary algebras ([HIO14, Lemma 3.6]): if P is an indecomposable projective Λ-module, then

- either there exists n such that $\mathbb{S}_d^{-n}(P) \in \mathrm{add}(D\Lambda)$,
- or for any n, $\mathbb{S}_d^{-n}P \in \mathrm{mod}\,\Lambda$.

This observation leads to the following definition:

Definition 3.3. [IO11, HIO14] Let Λ be a finite dimensional algebra of global dimension d. Then

- Λ is said *d-representation finite* (d-RF) if $\mathcal{U} = \mathcal{U}[d]$ where $\mathcal{U} = \mathsf{add}\{\mathbb{S}_d^{-p}\Lambda,\ p \in \mathbb{Z}\}$;
- Λ is said *d-representation infinite* (d-RI) if $\mathcal{U}^+ \subset \operatorname{mod}\Lambda$ where $\mathcal{U}^+ = \mathsf{add}\{\mathbb{S}_d^{-p}\Lambda,\ p \in \mathbb{N}\}$;

For the case $d = 1$, using the description of the derived category $\mathcal{D}^{\mathrm{b}}(kQ)$ one immediately observes that

$$kQ \text{ is 1-RF } \Leftrightarrow Q \text{ is Dynkin } \Leftrightarrow kQ \text{ is representation-finite}$$

$$kQ \text{ is 1-RI } \Leftrightarrow Q \text{ is non Dynkin } \Leftrightarrow kQ \text{ is representation-infinite.}$$

In general, as shown in [HIO14], any d-hereditary algebra which is indecomposable as a ring is either d-RF or d-RI. Moreover, one easily verifies

$$\Lambda\ d\text{-RF} \Rightarrow \Lambda\ \tau_d\text{-finite.}$$

$$\Lambda\ d\text{-RI} \Rightarrow \Lambda\ \tau_d\text{-infinite .}$$

The link between d-hereditary algebras and $(d+1)$-preprojective algebras is given by the following characterization.

Proposition 3.4. *Let Λ be an algebra of global dimension d. Then we have the following equivalences*

- Λ *is d-RI if and only* $\mathbf{\Pi}_{(d+1)}(\Lambda)$ *is concentrated in (homological) degree* 0*, that is the projection*
$$p : \mathbf{\Pi}_{d+1}(\Lambda) \to \mathrm{H}^0(\mathbf{\Pi}_{d+1}(\Lambda)) = \Pi_{d+1}(\Lambda)$$
is a quasi-isomorphism.
- Λ *is d-RF if and only if* $\Pi_{d+1}(\Lambda)$ *is a finite dimensional self-injective algebra [IO13, Cor 3.4].*

Note that in the case where Λ is d-RI, the derived $(d+1)$-preprojective algebra is formal.

Example 3.5. The algebra presented by the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with the relation $ba = 0$ is 2-RF.

The algebra presented by the quiver $1 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 2 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 3$ with the commutativity relations is 2-RI.

3.2 d-Auslander algebras

The d-RF algebras can also be caracterized by the existence of a d-cluster-tilting object in their module category.

Theorem 3.6. *[Iya11] Let Λ be an algebra of global dimension $\leq d$. Then it is d-RF if and only if there exists a d-cluster-tilting object U in* $\operatorname{mod}\Lambda$*. This object is moreover unique and is isomorphic to* $\Pi_{d+1}(\Lambda)$ *as a Λ-module.*

In that case, the endomorphism algebra $\operatorname{End}_\Lambda(\Pi_{d+1}(\Lambda))$ of the d-cluster-tilting object is not a higher preprojective algebra. It is called the *d-Auslander algebra* of the d-RF algebra Λ.

The concept of d-Auslander algebras has been generalized by Iyama to higher Krull dimension. More generally, if U is a d-cluster-tilting object in a category $\mathrm{CM}\,\Gamma$ where Γ is a Cohen-Macaulay ring over an Artin algebra R, then $\operatorname{End}_\Gamma(U)$ is also called d-Auslander algebra. In general however, there might be more than one cluster-tilting object.

3.3 Constructing d-hereditary algebras

There are different ways to construct d-hereditary algebras. One can construct inductively on d using tensor products [HI11, HIO14]. The d-hereditary algebras (and d-Auslander algebras) of type A have been entirely described in [IO11].

From a d-hereditary algebra, one can also construct new d-hereditary algebras using an operation called d-APR tilt [IO11] which is the higher analogue of the Auslander-Platzeck-Reiten tilt introduced in [APR79]. A natural question arising here is whether an analogue of Theorem 2.3 is true for $d \geq 2$: does the d-APR tilt have an interpretation in the derived category of the preprojective algebra of a d-hereditary algebra ?

4 Calabi-Yau properties

We investigate here the different Calabi-Yau properties satisfied by the $(d+1)$-preprojective algebra.

4.1 Motivations

The motivation for the results of this section follow from the following two results.

The first one is the classical case $d = 1$.

Theorem 4.1. *[ES98I, ES98II, CB00, BBK02] Let Q be a finite quiver without oriented cycles. Then*

- *if Q is Dynkin, the preprojective algebra $\Pi_2(kQ)$ is selfinjective and the stable category $\underline{\mathrm{mod}}\Pi_2(kQ)$ is 2-Calabi-Yau.*
- *if Q is not Dynkin, then $\Pi_2(kQ)$ has global dimension 2 and the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{fd}\,\Pi_2(kQ))$ of finite dimensional $\Pi_2(kQ)$-modules is 2-Calabi-Yau.*

The second one concerns the derived $(d+1)$-preprojective algebra.

Theorem 4.2. *[Kel11] Let Λ be a finite dimensional algebra. Then the derived $(d+1)$-preprojective algebra $\mathbf{\Pi} := \mathbf{\Pi}_{(d+1)}(\Lambda)$ of Λ is bimodule $(d+1)$-Calabi-Yau, that is $\mathbf{\Pi}$ is homologically smooth and we have an isomorphism*

$$\boldsymbol{R}\mathrm{Hom}_{\mathbf{\Pi}^{\mathrm{e}}}(\mathbf{\Pi}, \mathbf{\Pi}^{\mathrm{e}})[d+1] \simeq \mathbf{\Pi} \text{ in } \mathcal{D}(\mathbf{\Pi}^{\mathrm{e}}).$$

One result in my collaboration with Iyama and Reiten [AIR15], and the main result in one of my work with Oppermann [AO15] aim at generalizing these results for general d-hereditary algebras.

4.2 d-RI case

The homological results presented in Theorem 4.1 concern some triangulated categories attached to preprojective algebras and not the algebras themselves. In order to have a higher analogue, which could be also seen as a characterization of preprojective algebras, one should enhance the Calabi-Yau property at the level of the graded algebra itself as follows:

Definition 4.3. Let $\Gamma = \sum_{\ell \in \mathbb{Z}} \Gamma_\ell$ be a $\mathbb{Z}$-graded algebra with $\dim_k \Gamma_\ell < \infty$ for all ℓ. The algebra is said to be (1)-*twisted bimodule* $(d+1)$-*Calabi-Yau* (or bimodule $(d+1)$-Calabi-Yau of Gorenstein parameter 1) if the following two conditions are satisfied

- Γ is homologically smooth (that is $\Gamma \in \mathrm{per}\,\Gamma^{\mathrm{e}}$);

- there is an isomorphism $\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma, \Gamma^{\mathrm{e}})[d+2] \simeq \Gamma(1)$ in $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}}\Gamma^{\mathrm{e}})$.

  Here (1) is the degree shift in the category $\mathrm{Mod}^{\mathbb{Z}}\Gamma^{\mathrm{e}}$, where Γ^{e} is considered as a $\mathbb{Z}$-graded algebra.

Note that a (1)-twisted bimodule $(d+1)$-Calabi-Yau algebra seen as a DG algebra with zero differential, is $(d+1)$-bimodule Calabi-Yau. Indeed the isomorphism above implies

$$\mathbf{R}\mathrm{Hom}_\Gamma(\Gamma, \Gamma^{\mathrm{e}})[d+1] \simeq \Gamma \text{ in } \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}}\Gamma^{\mathrm{e}})\big/(1) \circ [-1], \text{ so in } \mathcal{D}((\Gamma^{\mathrm{e}})^{DG})$$

.

Moreover we have:

Proposition 4.4. *[Gin, Kel08] Let Γ be a bimodule $(d+1)$-Calabi-Yau graded algebra, then $\mathcal{D}^{\mathrm{b}}(\mathrm{fd}\,\Gamma)$ is a $(d+1)$-Calabi-Yau triangulated category.*

The next result gives then a complete homological characterization of $(d+1)$-preprojective algebra of d-RI algebras.

Theorem 4.5. *[Kel11, HIO14, MM11, AIR15] Let $\Gamma = \bigoplus_{i\geq 0}\Gamma_i$ be a graded algebra with finite dimensional degree zero part $\Lambda := \Gamma_0$. Then the following are equivalent*

(1) Λ is d-RI and $\Gamma \simeq \Pi_{d+1}(\Lambda)$ as graded algebras;

(2) Γ is (1)-twisted bimodule $(d+1)$-Calabi-Yau.

The implication (1) ⇒ (2) follows from what was explained before. The preprojective algebra of a d-RI algebra is quasi-isomorphic to the derived preprojective algebra. Then, applying [Kel11, Thm 4.8], we get the result (see also [HIO14]).

The implication (2) ⇒ (1) was shown independently in [MM11, Thm 4.8] and [AIR15, Thm 3.4].

Let us mention a few words about the proof (2) ⇒ (1) given in [AIR15]. The idea here is to consider the projective minimal resolution $P_\bullet$ of Γ as a graded Γ-bimodule. It satisfies

$$P_\bullet^\vee[d+1] \simeq P_\bullet(1) \text{ in } \mathcal{C}(\mathrm{proj}^{\mathbb{Z}}\,\Gamma^{\mathrm{e}}), \tag{4.1}$$

where $(-)^\vee = \mathrm{Hom}_{\Gamma^{\mathrm{e}}}(-, \Gamma^{\mathrm{e}})$. It is generated in degree 0 and 1, and hence we can 'split' each term P_i of the complex $P_\bullet$ into $P_i \simeq P_i^0 \oplus P_i^1(-1)$ and show that there is an exact sequence

$$0 \longrightarrow P_\bullet^0 \longrightarrow P_\bullet \longrightarrow P_\bullet^1(-1) \longrightarrow 0 \text{ in } \mathcal{C}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}\,\Gamma^{\mathrm{e}}).$$

From (4.1), we deduce $(P_\bullet^1)^\vee[d] \simeq P_\bullet^0$. Then using the fact that $\Lambda \underset{\Gamma}{\otimes} P_\bullet^0 \simeq \Gamma$ in $\mathcal{D}(\Lambda^{\mathrm{op}} \otimes \Gamma)$ and the above short exact sequence, we deduce a triangle

$$\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}}) \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Gamma(-1) \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}}) \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Gamma(-1)[1] \text{ in } \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}}(\Lambda^{\mathrm{op}}\otimes\Gamma)).$$

It permits us to construct an isomorphism

$$\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}})^{\otimes \ell} \simeq \Gamma_\ell \text{ in } \mathcal{D}(\Lambda^{\mathrm{e}}) \text{ for any } \ell \in \mathbb{Z},$$

and so the desired algebra isomorphism.

4.3 d-RF case

The next result is the d-RF analogue of Theorem 4.5. It can be seen as a generalization of the Dynkin case of Theorem 4.1, and gives a homological characterization of the preprojective algebras of d-RF algebras.

Definition 4.6. Let $\Gamma = \bigoplus_{\ell\in\mathbb{Z}} \Gamma_\ell$ be a finite dimensional $\mathbb{Z}$-graded algebra. Then Γ is said to be (1)-*twisted stably bimodule* $(d+1)$-*Calabi-Yau* if there is an isomorphism

$$\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma, \Gamma^{\mathrm{e}})[d+2] \simeq \Gamma(1) \text{ in } \underline{\mathrm{mod}}^{\mathbb{Z}}\Gamma^{\mathrm{e}}.$$

One easily checks the following.

Proposition 4.7. *[AO15, Thm 2.12] If Γ is (1)-twisted stably bimodule $(d+1)$-Calabi-Yau, and self-injective then the category $\underline{\mathrm{mod}}\Gamma$ is $(d+1)$-Calabi-Yau.*

The next theorem is a complete analogue of Theorem 4.5 for d-RF algebras.

Theorem 4.8. *[Dug12, AO15] Let $\Gamma = \bigoplus_{i\geq 0} \Gamma_i$ be a finite dimensional graded algebra. Denote by Λ its degree zero part. Then the following are equivalent*

1. *Λ is d-RF, has global dimension d and $\Gamma \cong \Pi_{d+1}(\Lambda)$ as graded algebras;*
2. *Γ is selfinjective and (1)-twisted stably bimodule $(d+1)$-Calabi-Yau.*

The implication (1) ⇒ (2) is shown in [Dug12, Thm 3.2], while the implication (2) ⇒ (1) is shown in [AO15, Thm 3.1].

The idea of the proof (2) ⇒ (1) is similar to the one of Theorem 4.5, by computing the cohomology spaces of the triangle

$$\Lambda \overset{\mathbf{L}}{\underset{\Gamma}{\otimes}} \Lambda \longrightarrow \Gamma \overset{\mathbf{L}}{\underset{\Gamma}{\otimes}} \Lambda \longrightarrow \Gamma_{>0} \overset{\mathbf{L}}{\underset{\Gamma}{\otimes}} \Lambda \longrightarrow \Lambda \overset{\mathbf{L}}{\underset{\Gamma}{\otimes}} \Lambda[1] \text{ in } \mathcal{D}(\Gamma^{\mathrm{op}} \otimes \Lambda)$$

It has also been shown in [IG] and [Dug12] that the bimodule projective resolution of the $(d+1)$-preprojective algebra of a d-RF algebra has a certain periodicity. This generalizes the case $d = 1$ treated in [ES98I, ES98II].

4.4 τ_d-finite case: beyond the d-RF case

In general the finite dimensional preprojective algebras are not selfinjective but their behaviour is still similar to the one of the preprojective algebras of d-RF algebras. In the case $d = 2$, Keller and Reiten proved in [KR07] that the algebras $\Pi_3(\Lambda)$ are Iwanaga-Gorenstein of dimension ≤ 1. Hence the correct analogue Calabi-Yau triangulated category is given by the stable category of maximal Cohen-Macaulay Π-modules. Indeed they proved in [KR07] that the category $\underline{\mathsf{CM}}\, \Pi_3(\Lambda)$ is 3-Calabi-Yau.

These results were the motivation for the following characterization of finite dimensional preprojective algebras.

Theorem 4.9. *[AO15, Thm 3.1] Let $\Gamma = \bigoplus_{i\geq 0} \Gamma_i$ be a (non trivially) graded finite dimensional algebra. Denote by Λ its degree zero part. Assume that*

(a) Γ is Iwanaga-Gorenstein of dimension $\leq d-1$;

(b) there is an isomorphism $\boldsymbol{R}\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma, \Gamma^{\mathrm{e}})[d+2] \cong \Gamma(1)$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\,\Gamma^{\mathrm{e}})/\,\mathrm{per}^{\mathbb{Z}}\,\Gamma^{\mathrm{e}}$.

(c) $\mathrm{Ext}^i_{\Gamma^{\mathrm{e}}}(\Gamma, \Gamma^{\mathrm{e}}(j)) = 0$ for any $i \geq 1$ and any $j \leq -1$.

Then Λ has global dimension d and $\Gamma \simeq \Pi_{d+1}(\Lambda)$ as graded algebras.

Here property (b) is again an algebraic (and graded) enhancement of the $(d+1)$-Calabi-Yau property of the category $\underline{\mathsf{CM}}\,\Gamma$ (see [AO15, Thm 2.12]).

We also show in [AO15] that these properties are satisfied by finite dimensional preprojective algebras in the case $d=2$ and $d=3$ using the description of the preprojective algebra in term of quivers with relations as described in Section 2.

5 Non commutative resolutions

5.1 Preprojective algebras as d-Auslander algebras

The following result in [AIR15] states that certain Noetherian $(d+1)$-preprojective algebras can be seen as d-Auslander algebras of Iwanaga-Gorenstein algebras. This is related with the last item of the introduction concerning the 2-preprojective algebras of extended Dynkin type.

Theorem 5.1. *[AIR15, Thm 2.2] Let Λ be a d-RI algebra such that the corresponding preprojective algebra $\Pi := \Pi_{d+1}(\Lambda)$ is noetherian. Assume that there exists an idempotent $e \in \Pi$ such that $\Pi/\Pi e\Pi$ is finite dimensional. Then we have the following:*

- *the algebra $R = e\Pi e$ is Iwanaga-Gorenstein of dimension $(d+1)$;*
- *we have an isomorphism $\boldsymbol{R}\mathrm{Hom}_{R^{\mathrm{e}}}(R, R^{\mathrm{e}})[d+1] \simeq R(1)$ in $\mathcal{D}(\mathrm{mod}^{\mathbb{Z}} R^{\mathrm{e}})$, hence the category $\underline{\mathrm{CM}}(R)$ is d-Calabi-Yau;*
- *the category $\mathrm{CM}(R)$ has a d-cluster-tilting object Πe and we have an isomorphism of algebras $\mathrm{End}_R(\Pi e) \simeq \Pi$, so in other words, Π is the d-Auslander algebra of R.*

Note that here R is not perfect as a bimodule, so R is not $(d+1)$-bimodule Calabi-Yau.

This theorem has fruitful connections with non commutative algebraic geometry when R is a local ring. In that case, Π is the endomorphism algebra of a Cohen-Macaulay module M (thus reflexive in the sense that $M \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$), and has global dimension equal to the dimension of R. Hence Π is a *non commutative crepant resolution of R.*

5.2 Auslander-Mckay correspondence

Let us illustrate the above result in the case of polynomial skew-group algebras.

Let S be the polynomial ring $k[x_0, \ldots, x_d]$ over an algebraically closed field k of characteristic zero, and G be a finite subgroup of $\mathsf{SL}_{d+1}(k)$ acting freely on $k^{d+1}\backslash\{0\}$. The group G acts on S in a natural way. The invariant ring S^G is known to be a Gorenstein isolated singularity of Krull dimension $(d+1)$.

We denote by SG the skew group algebra: it is defined as the vector space by $S \underset{k}{\otimes} kG$ with multiplication induced by

$$(P \otimes g)(Q \otimes h) := Pg(Q) \otimes gh.$$

By classical results of Auslander [Aus86, Yos90], SG is Morita equivalent to $\mathrm{End}_{SG}(S)$. Moreover by [Iya07a, Thm 2.5] S is a d-cluster-tilting object in the category $\mathrm{CM}(R)$.

Case $d=1$

The link between this setup and preprojective algebra is also classical in the case $d=1$ and due to Reiten and Van den Bergh.

Theorem 5.2. *[RV89] Let $G \subset \mathsf{SL}_2(k)$ be a finite subgroup acting on $S = k[x,y]$. Then the skew-group algebra SG is Morita equivalent to the preprojective algebra $\Pi_2(kQ)$ where Q is an orientation of the extended Dynkin graph associated to G via the Mckay correspondence.*

These results can be reinterpreted as the case $d = 1$ of Theorem 5.1: First observe that $\Lambda = kQ$ is 1-RI if and only if Q is not Dynkin. Moreover if Q is not Dynkin, the corresponding preprojective algebra is noetherian if and only if Q is extended Dynkin [BGL87]. Denote by e the extended vertex, and by $\underline{Q}$ the corresponding Dynkin quiver Q/e. Then we have $\underline{\Pi} = \Pi_2(k\underline{Q})$, so $\Pi/\Pi e\Pi$ is finite-dimensional. Therefore we obtain:

- $R = e\Pi e$ is Iwanaga-Gorenstein of dimension 2;
- the category $\underline{\mathrm{CM}}(R)$ is 1-Calabi-Yau;
- the category $\mathrm{CM}(R)$ has a 1-cluster-tilting object (equivalently is representation finite) Πe, and we have $\mathrm{End}_R(\Pi e) \simeq \Pi$.

Moreover the extended vertex e in Theorem 5.2 corresponds to the summand S^G in S, so we obtain an isomorphism
$$R = e\Pi e \simeq \mathrm{End}_{S^G}(S^G) \simeq S^G.$$

Case $d \geq 2$

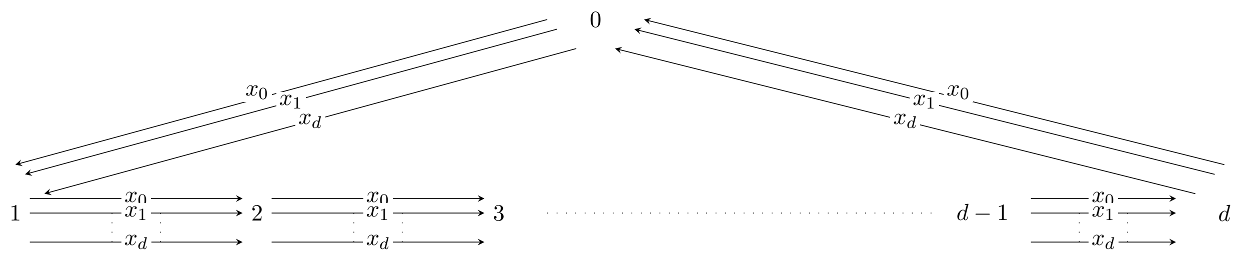
Using Theorem 4.5, we can generalize Theorem 5.2 for certain cyclic groups.

Corollary 5.3. *[AIR15] Let G be cyclic subgroup of $\mathsf{SL}_{d+1}(k)$ of order n generated by $g = \mathrm{diag}(\zeta^{a_0}, \ldots, \zeta^{a_d})$ where ζ is a primitive n-root of 1, with $\gcd(a_i, n) = 1$. Then there exists a d-RI algebra Λ such that the skew-group algebra SG is isomorphic to $\Pi_{d+1}(\Lambda)$, where $S = k[x_0, \ldots, x_d]$.*

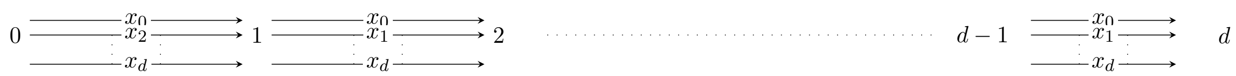
To prove this theorem, one uses the isomorphism $SG \simeq \mathrm{End}_R(S)$ (since G is abelian, the Morita equivalence comes from an isomorphism).

Using a suitable grading on R, one describes explicitely the algebra $\mathrm{End}_R(S)$ by a quiver with relations, and deduces a graded bimodule projective resolution $P_\bullet$ that satisfies $P_\bullet^\vee[d+1] \simeq P(1)$.

Example 5.4. Let $G = \mathrm{diag}(\zeta, \ldots, \zeta)$ where ζ is a primitive $d+1$-root of 1. Then one shows that the algebra SG is isomorphic to the path algebra



with relations $x_i x_j = x_j x_i$ for any $i \neq j$. It is the $(d+1)$-preprojective algebra of the d-Beilinson algebra Λ given by the quiver :



with the relations $x_i x_j = x_j x_i$.

The algebra $R = e\Pi e$ is isomorphic to the invariant ring S^G, thus is the subalgebra of S generated by monomials of degree $d+1$ which is the $(d+1)$-Veronese algebra.

Remark 5.5. Unfortunately Corollary 5.3 does not generalize to any finite subgroup of $\mathsf{SL}_d(k)$ as shown in [Thi].

5.3 Dimer and toric varieties

We give here another application of Theorem 5.1.

Let Γ be a bipartite graph (or a dimer model) on the torus. As described in [Bro12], one can associate a quiver with potential to such graph: the quiver Q is the dual of the graph Γ, where the faces of Q corresponding to white vertices are oriented clockwise, and faces of Q corresponding to black vertices are oriented counterclockwise. The potential is the difference between 'white' faces of Q and 'black' faces of Q.

If the bipartite graph is consistent in the sense of [Bro12] (see also [Boc11, Dav11]), then the algebra $\Pi(\Gamma) := \mathrm{Jac}(Q, W)$ is a non commutative crepant resolution over its center which is of the form $e\Pi e$, where e is any vertex in the quiver Q. Moreover its center $e\Pi e$ is the coordinate ring of a Gorenstein affine toric threefold. Moreover the coordinate ring of any Gorenstein affine toric threefold can be obtained in this way.

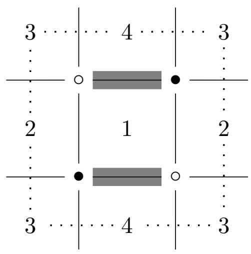
The data of a perfect matching (or a dimer configuration) on the graph Γ induces a grading on Q for which the potential W is homogeneous of degree 1, so induces a grading on Π.

Using again explicit graded bimodule resolution of Π, one obtains the following:

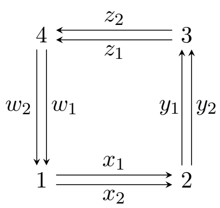
Corollary 5.6. *Let Γ be a consistent dimer model on the torus, and D be a dimer configuration. Denote by Π the corresponding graded Jacobian algebra. If the degree zero part Λ of Π is finite dimensional, then Λ is 2-RI, and Π is the 3-preprojective algebra of Λ.*

If moreover there exists a primitive idempotent e such that $\Pi/\Pi e\Pi$ is finite-dimensional, Theorem 5.1 implies that the center $e\Pi e$ is d-representation finite (as CM-ring), and that Π is a NCCR of $e\Pi e$.

Example 5.7. Let Γ and D be given by the following picture.



The associated Jacobian algebra Π is presented by the quiver



with potential $W = w_1z_1y_1x_1 + w_2z_2y_2x_2 - w_1z_2y_1x_2 - w_2z_1y_2x_1$.

Then the perfect matching D corresponds to $\{w_1, w_2\}$. Thus the algebra Λ of Π is given by the quiver

$$1 \overset{x_1}{\underset{x_2}{\rightrightarrows}} 2 \overset{y_1}{\underset{y_2}{\rightrightarrows}} 3 \overset{z_1}{\underset{z_2}{\rightrightarrows}} 4 \quad \text{with relations } z_1y_1x_1 = z_1y_2x_1 \text{ and } z_2y_1x_2 = z_2y_2x_2.$$

The center R of this algebra is the semigroup algebra $R = \mathbb{C}[\mathbb{Z}^3 \cap \sigma^\vee]$ where $\sigma^\vee$ is the positive cone

$$\sigma^\vee = \{\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 + \lambda_4 n_4, \lambda_i \geq 0\}, \quad n_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, n_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, n_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, n_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The algebra R is the homogenous coordinate algebra of $\mathbb{P}^1 \times \mathbb{P}^1$.

Chapter 2

Cluster categories and Cohen-Macaulay modules

Cluster categories associated with τ_2-finite algebras were constructed in my thesis [Ami]. The aim was to generalize the notion of cluster category associated with an acyclic quiver introduced by Buan, Marsh, Reineke, Reiten and Todorov in their seminal paper [BMRRT06]. The idea was to construct a category in which cluster combinatorics appeared naturally. An instance of such categories is given by 2-Calabi-Yau triangulated categories with cluster-tilting objects as shown in [BIRS09], and the main result of my thesis was the fact that cluster categories associated with τ_2-finite algebras are 2-Calabi-Yau with cluster-tilting objects. This construction was then generalized by Guo [Guo11] from 2 to $d \geq 2$.

Other examples of d-Calabi-Yau triangulated categories with d-cluster-tilting objects arise naturally in representation theory, especially as stable categories of Cohen-Macaulay modules (or singularity categories) $\underline{\mathrm{CM}}\Gamma$ of some Iwanaga-Gorenstein algebras Γ. It is then natural to ask the following:

When are these categories d-cluster categories ?

This is the question we address in this chapter, and for which the papers [Ami09, ART11, AIRT12, AIR15] gave some answers.

We also refine the question in the graded setting. By construction as triangulated hull of an orbit category the d-cluster category can be seen as an ungraded version of the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$, where the grading is given by the functor $\mathbb{S}_d$.

Then, given an Iwanaga-Gorenstein algebra Γ such that $\underline{\mathrm{CM}}(\Gamma)$ is d-Calabi-Yau, it becomes natural to ask whether there exists a grading on Γ so that we have equivalences:

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\Lambda) & \xrightarrow{\sim} & \underline{\mathrm{CM}}^{\mathbb{Z}}(\Gamma) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{forget}} \\ \mathcal{C}_d(\Lambda) & \xrightarrow{\sim} & \underline{\mathrm{CM}}(\Gamma). \end{array} \tag{0.1}$$

The typical strategy to answer these questions can be summarized in the following steps:

1. First one finds a d-cluster-tilting object in $\underline{\mathrm{CM}}(\Gamma)$ so that the algebra $\underline{\mathrm{End}}_\Gamma(T)$ is isomorphic to the d-preprojective algebra of some algebra Λ. The results of the previous chapter (especially Theorems 4.8, 4.5 and 5.1) are then very useful.

2. The second step requires to construct a triangle functor $F : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \operatorname{per}\Gamma$ that induces a functor $\bar{F} : \mathcal{C}_d(\Lambda) \longrightarrow \operatorname{per}\Gamma/\mathcal{D}^{\mathrm{b}}(\Gamma) = \underline{\mathrm{CM}}(\Gamma)$. In order to do this, one may apply the universal property of the d-cluster category (see Theorem 1.1 in the next section)

3. The third step consists in showing that the functor $\bar{F}$ is an equivalence. For this one can apply the following

   Proposition 0.1. *[KR07, IY08] Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories with d-cluster-tilting subcategories $\mathcal{U}$ and $\mathcal{U}'$. If there exists a triangle functor $F : \mathcal{T} \to \mathcal{T}'$ inducing an equivalence $\mathcal{U} \simeq \mathcal{U}'$ then F is an equivalence.*

4. Finally, to deal with the graded version, one needs to find a good grading on Γ. It generally comes from the grading on $\underline{\mathrm{End}}_\Gamma(T) \simeq \Pi_{(d+1)}(\Lambda)$. Then either one uses a graded analogue of Theorem 1.1, or one shows the existence of a tilting object in the category $\underline{\mathrm{CM}}^{\mathbb{Z}}(\Gamma)$. But in this last case, one also needs to show the commutativity of the above diagram.

The plan of this chapter is as follows. We first recall general results on the construction of the d-cluster category, its universal property, and its alternative description as a quotient of triangulated categories. In a second section, we consider the case where Γ is a preprojective algebra of a d-RF algebra. We give some generalisation in Section 3 in the case $d = 2$ for algebras Γ_w associated with elements in the Coxeter group of a quiver. In the last section we consider the case where Γ is given by $e\Pi e$ where Π is the $(d+1)$-preprojective algebra associated with a d-RI algebra as in Theorem 5.1.

1 Cluster categories as quotient of triangulated categories

We refer to [Ami11] for a detailed construction of the generalized cluster category, and for motivation for this construction.

1.1 Universal property

By construction, the d-cluster category of a τ_d-finite algebra Λ satisfies an universal property that will be essential in this chapter.

Theorem 1.1. *[Kel05, Section 9.6][IO13, Thm A20] Let Γ be a DG algebra, and $\mathcal{T}$ be a thick subcategory of $\mathcal{D}(\Gamma)$. Let Λ be a τ_d-finite algebra of global dimension $\leq d$. Assume there exists an object $M \in \mathcal{D}(\Lambda^{\mathrm{op}} \otimes \Gamma)$ and a morphism*

$$M \longrightarrow \boldsymbol{R}\mathrm{Hom}_\Lambda(D\Lambda, \Lambda) \overset{\boldsymbol{L}}{\underset{\Lambda}{\otimes}} M[d] \quad \text{in } \mathcal{D}(\Lambda^{\mathrm{op}} \otimes \Gamma)$$

(or a morphism $\boldsymbol{R}\mathrm{Hom}_\Lambda(D\Lambda, \Lambda) \overset{\boldsymbol{L}}{\underset{\Lambda}{\otimes}} M[d] \longrightarrow M$) whose cone lies in $\mathcal{T}$ when viewed as an object in $\mathcal{D}(\Gamma)$. Then there is a triangle functor $\mathcal{C}_d(\Lambda) \to \mathcal{D}(\Gamma)/\mathcal{T}$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\Lambda) & \xrightarrow{-\overset{\boldsymbol{L}}{\underset{\Lambda}{\otimes}} M} & \mathcal{D}(\Gamma) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathcal{C}_d(\Lambda) & \longrightarrow & \mathcal{D}(\Gamma)/\mathcal{T} \end{array}$$

This universal property permits to exhibit alternative constructions of the d-cluster category using DG-algebras.

1.2 Using higher trivial extensions algebras

The first one uses an higher analogue of the trivial extension of Λ.

Definition 1.2. [IG] Let Λ be a finite dimensional algebra of finite global dimension. The $(d+1)$-*trivial extension* of the algebra Λ is defined to be the $\mathbb{Z}$-graded algebra

$$\mathbf{T}_{(d+1)}\Lambda := \Lambda \oplus D\Lambda(-d-1),$$

where the multiplication is given as $(a,f).(b,g) = (ab, ag+fb)$. Hence, forgetting the grading, $\mathbf{T}_{(d+1)}(\Lambda)$ is the usual trivial extension of Λ.

The graded algebra $\mathbf{T}_{(d+1)}\Lambda$ can be viewed as a DG-algebra with zero differential. And we have the following result.

Proposition 1.3. *[Ami, Section 7.3] Let Λ be a finite dimensional algebra of global dimension $\leq d$ which is τ_d-finite, then there is an equivalence of triangulated categories :*

$$\mathcal{C}_d(\Lambda) \simeq {}^{\operatorname{thick}_{\mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})}(\Lambda)}\diagup_{\operatorname{per}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})}$$

where $\mathbf{T}_{(d+1)} = \mathbf{T}_{(d+1)}(\Lambda)$ and where $\operatorname{thick}_{\mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})}(\Lambda)$ is the thick subcategory in $\mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})$ generated by the object Λ viewed as an object in $\mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})$.

This description can be understood as follows. Since Λ has finite global dimension, the projection map $p : \mathbf{T}_{(1)} \to \Lambda$ induces an equivalence of triangulated categories by [Hap88]:

$$\mathcal{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \underline{\operatorname{mod}}^{\mathbb{Z}}(\mathbf{T}_1) \simeq {}^{\mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}}\mathbf{T}_{(1)})}\diagup_{\operatorname{per}^{\mathbb{Z}}\mathbf{T}_{(1)}}.$$

The Serre functor of the category $\underline{\operatorname{mod}}^{\mathbb{Z}}(\mathbf{T}_{(1)})$ is $\nu\circ[-1]$, and since we have $D(\mathbf{T}_{(1)}) \simeq \mathbf{T}_{(1)}(1)$ as graded $\mathbf{T}_{(1)}$-bimodules, we obtain an equivalence

$$\mathcal{D}^{\mathrm{b}}(\Lambda)\diagup_{\mathbb{S}_d} \xrightarrow{\sim} \underline{\operatorname{mod}}^{\mathbb{Z}}(\mathbf{T}_{(1)})\diagup_{[-(d+1)]\circ(1)}.$$

Now one can check that there is an equivalence

$$\mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}}\mathbf{T}_{(1)})\diagup_{[-(d+1)\circ(1)]} \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}}\mathbf{T}_{(d+1)}))\diagup_{[-1]\circ(1)}.$$

Since the triangulated hull of $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}}\mathbf{T}_{(d+1)})\diagup_{[-1]\circ(1)}$ is the category $\mathcal{D}^{\mathrm{b}}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})$ (see [KY16, Thm 1.3]), we obtain an embedding of triangulated categories

$$\mathcal{C}_d(\Lambda) \hookrightarrow {}^{\mathcal{D}^{\mathrm{b}}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})}\diagup_{\operatorname{per}\mathbf{T}^{\mathrm{DG}}_{(d+1)}}.$$

Note that the equivalence of Proposition 1.3 can also be constructed using the universal property applied to the restriction functor $\mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})$ of the natural projection $\mathbf{T}_{(d+1)} \to \Lambda$, since we have a triangle in $\mathcal{D}(\Lambda^{\mathrm{op}} \otimes \mathbf{T}^{\mathrm{DG}}_{(d+1)})$

$$\mathbb{S}_d\Lambda[-1] \longrightarrow \mathbf{T}_{(d+1)} \longrightarrow \Lambda \longrightarrow \mathbb{S}_d(\Lambda).$$

This was the argument used in [Ami] to prove the equivalence.

1.3 Using the higher derived preprojective algebras

The other description of the d-cluster category uses the derived $(d+1)$-preprojective algebra of Λ.

Proposition 1.4. *[Ami09, Thm 4.10][Guo11] Let Λ be a finite dimensional algebra of global dimension $\leq d$ which is τ_d-finite, then there is an equivalence of triangulated categories :*

$$\mathcal{C}_d(\Lambda) \simeq {}^{\operatorname{per}\mathbf{\Pi}}\!/\!_{\mathcal{D}_{\mathrm{fd}}(\mathbf{\Pi})},$$

where $\mathbf{\Pi} = \mathbf{\Pi_{d+1}}(\Lambda)$ is the derived $(d+1)$-preprojective algebra of Λ and where $\mathcal{D}_{\mathrm{fd}}$ is the subcategory of $\mathcal{D}(\mathbf{\Pi})$ of objects of finite dimensional total cohomology.

Note that here, the DG algebra $\mathbf{\Pi}_{d+1}(\Lambda)$ is infinite dimensional, so $\mathbf{\Pi}$ is not in $\mathcal{D}_{\mathrm{fd}}$. Moreover, $\mathbf{\Pi}$ is homologically smooth, that is $\mathbf{\Pi} \in \operatorname{per}\mathbf{\Pi}^{\mathrm{e}}$ so we have an inclusion $\mathcal{D}_{\mathrm{fd}}(\mathbf{\Pi}) \subset \operatorname{per}\mathbf{\Pi}$.

The construction of the functor comes also here from the universal property (Theorem 1.1) applied with $\Gamma = \mathbf{\Pi}$, and $M = \mathbf{\Pi}$. Then there is a triangle in $\mathcal{D}(\Lambda^{\mathrm{op}} \otimes \mathbf{\Pi})$

$$\Lambda \longrightarrow \mathbf{\Pi} \longrightarrow \Theta \otimes_\Lambda \mathbf{\Pi} \longrightarrow \Lambda[1]$$

where Θ is the cofibrant replacement of the DG-bimodule $\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}})[d]$ as defined in 1.1. Since Λ is finite dimensional, it is in $\mathcal{D}_{\mathrm{fd}}(\mathbf{\Pi})$ and we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{b}}(\Lambda) & \xrightarrow{-\overset{\mathbf{L}}{\underset{\Lambda}{\otimes}}\mathbf{\Pi}} & \operatorname{per}\mathbf{\Pi} \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\mathcal{C}_d(\Lambda) & \xrightarrow{\sim} & {}^{\operatorname{per}\mathbf{\Pi}}\!/\!_{\mathcal{D}_{\mathrm{fd}}(\mathbf{\Pi})}.
\end{array}$$

In fact, it can be related to Proposition 1.3 as follows. On shows that there exists an isomorphism in the homotopy category of DG algebras (cf [Ami, Lemma 7.3.1])

$$\mathbf{R}\mathrm{Hom}_{\mathbf{T}_{(d+1)}}(\Lambda, \Lambda) \simeq \mathbf{\Pi}_{(d+1)}.$$

This makes the DG algebra $\mathbf{\Pi}_{(d+1)}$ a Λ-Koszul dual of the DG algebra $\mathbf{T}_{d+1}$. Therefore the functor $\mathbf{R}\mathrm{Hom}_{\mathbf{T}_{(d+1)}}(\Lambda, -)$ induces the following diagram

$$\begin{array}{ccc}
\operatorname{per}(\mathbf{T}^{\mathrm{DG}}_{d+1}) & \xrightarrow{\sim} & \mathcal{D}_{\mathrm{fd}}(\mathbf{\Pi}_{(d+1)}) \\
\downarrow & & \downarrow \\
\operatorname{thick}_{\mathcal{D}(\mathbf{T}^{\mathrm{DG}}_{(d+1)})}(\Lambda) & \xrightarrow{\sim} & \operatorname{per}\mathbf{\Pi}_{(d+1)}.
\end{array}$$

2 Preprojective algebra of RF-algebras

2.1 Case of 1-RF algebras

The first instance of an equivalence $\underline{\mathrm{CM}}(\Gamma) \simeq \mathcal{C}_d(\Lambda)$ was given in my thesis for the stable module category of a preprojective algebra of Dynkin type. Such categories are 2-Calabi-Yau (see Theorem 4.1) and have cluster-tilting objects (see [GLS06]).

Theorem 2.1. *Let Q be a Dynkin quiver. There exists a τ_2-finite algebra $\underline{\Lambda}$ of global dimension ≤ 2 together with a triangle equivalence*

$$\mathcal{C}_2(\underline{\Lambda}) \simeq \underline{\mathrm{mod}}\Pi_2(kQ).$$

In this result, the algebras $\underline{\Lambda}$ and kQ can be related as follows: $\underline{\Lambda}$ is the stable Auslander algebra of kQ, that is

$$\underline{\Lambda} := \underline{\mathrm{End}}_{kQ}(M),$$

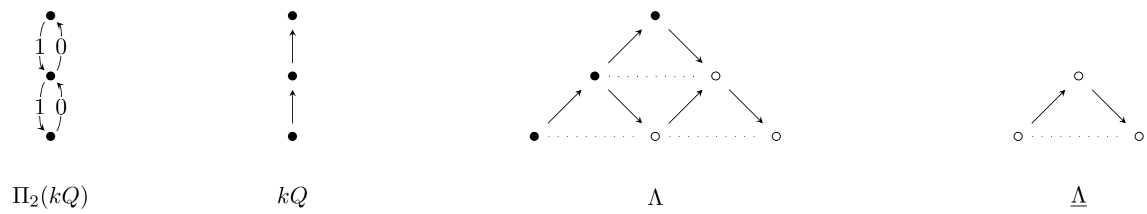
where M is a generator of the category $\mathrm{mod}\, kQ$.

More precisely, we have the following diagram

$$\Pi_2(kQ) \underset{\text{2-preprojective algebra}}{\overset{\text{degree zero part}}{\rightleftarrows}} kQ \underset{e(-)e}{\overset{\text{Auslander algebra}}{\rightleftarrows}} \Lambda \xrightarrow{\quad -/\langle e\rangle \quad} \underline{\Lambda}$$

Here the idempotent e corresponds to the projective objects in $\mathrm{mod}\, kQ$.

Example 2.2. Let Q be the quiver $1 \to 2 \to 3$. Then the above algebras $\Pi_2(kQ)$, Λ and $\underline{\Lambda}$ are given as follows. The idempotent e corresponds to black dots.



2.2 General d and graded version

Theorem 2.1 has been generalized by Iyama and Oppermann in the context of higher Auslander theory.

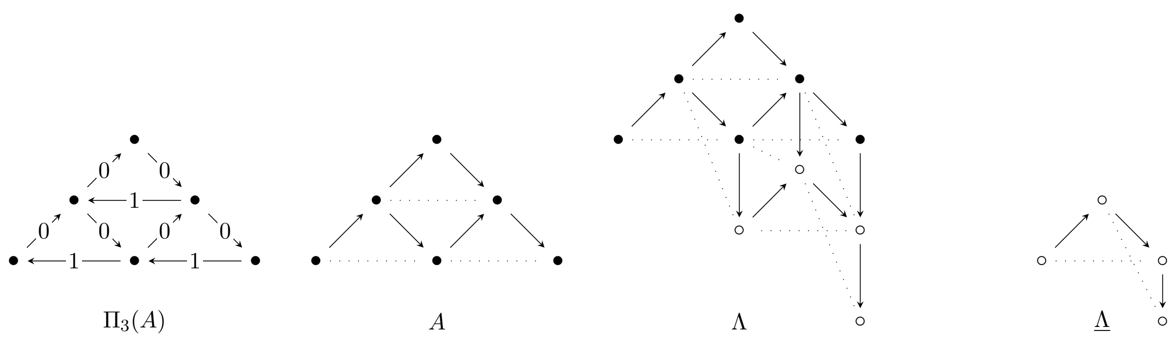
Theorem 2.3. *[IO13] Let A be a $(d-1)$-RF algebra. There exists a τ_d-finite algebra $\underline{\Lambda}$ of global dimension ≤ 2 together with a commutative diagram of triangle functors*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) & \xrightarrow{\quad\sim\quad} & \underline{\mathrm{mod}}^{\mathbb{Z}}\Pi_d(A) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\text{forget}} \\ \mathcal{C}_d(\underline{\Lambda}) & \xrightarrow{\quad\sim\quad} & \underline{\mathrm{mod}}\Pi_d(A). \end{array}$$

Here $\underline{\Lambda}$ and A are related as follows:

$$\Pi_d(A) \underset{\text{d-preprojective algebra}}{\overset{\text{degree zero part}}{\rightleftarrows}} A \underset{e(-)e}{\overset{\text{(d-1)-Auslander algebra}}{\rightleftarrows}} \Lambda \xrightarrow{\quad -/\langle e\rangle \quad} \underline{\Lambda}$$

Example 2.4. Let A be the Auslander algebra of the quiver A_3, it is 2-RF.



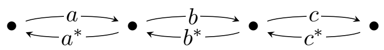
Using Happel's Theorem, the equivalence given in Theorem 2.3 corresponds to a triangle equivalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}}\Pi_d(A) \simeq \underline{\mathrm{mod}}^{\mathbb{Z}}\mathbf{T}_1(\underline{\Lambda})$$

where $\mathbf{T}_1(\underline{\Lambda})$ is the trivial extension of the algebra $\underline{\Lambda}$. The graded algebras $\Pi_d(A)$ and $\mathbf{T}_1(\underline{\Lambda})$ are far from being isomorphic (the rank of their Grothendieck group is not even the same). However, Iyama and Oppermann showed an equivalence

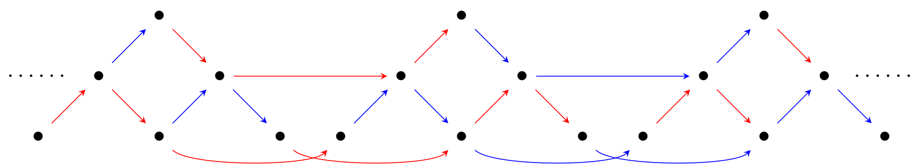
$$\mathrm{proj}^{\mathbb{Z}}\,\Pi_d(A) \simeq \mathrm{proj}^{\mathbb{Z}}\,T(\underline{\Lambda}).$$

This equivalence can be seen as follows in the example where A is the Auslander algebra of the path algebra of the quiver A_3. The trivial extension of the algebra $\underline{\Lambda}$ is given by the quiver:



with relations ba, cb,a^*b^*, b^*c^*
$aa^* - b^*b$, $bb^* - c^*c$

The equivalence $\mathrm{proj}^{\mathbb{Z}}\,\Pi_d(A) \simeq \mathrm{proj}^{\mathbb{Z}}\,\mathbf{T}_1(\underline{\Lambda})$ can be seen in the following picture, considering the $\mathbb{Z}$-covers of the algebras:



Here the $\mathbb{Z}$-action on $\Pi_d(A)$ is given by a horizontal translation, while the $\mathbb{Z}$-action on $\mathbf{T}_1(\underline{\Lambda})$ is given by sending a red fondamental domain to the next red one.

Note that Iyama and Oppermann proved in [IO13] a more general version of Theorem 2.3 for τ_d-finite algebras such that the preprojective algebra $\Pi_d(A)$ has Gorenstein dimension ≤ 1 (or equivalently satisfying a certain vanishing condition called *vosnex property* [AO15, Cor. 4.10]).

3 Coxeter group and CM modules

3.1 General w

In the case of a non Dynkin quiver, the preprojective algebra is not selfinjective anymore. However, one can construct Iwanaga-Gorenstein algebras from it using elements in the Weyl group of Q. The general construction is due to Buan, Iyama, Reiten and Scott [BIRS09], but some particular cases were studied by Geiss Leclerc and Schröer [GLS06].

Let Q be any acyclic quiver, and let W_Q be the Weyl group associated to the graph of Q. It is the free group generated by s_i, $i \in Q_0$ with the relations

- $s_i^2 = 1$;
- $s_i s_j = s_j s_i$ if there is no arrows between i and j;
- $s_i s_j s_i = s_j s_i s_j$ if there is precisely one arrow between i and j.

Denote by $\Pi := \Pi_2(kQ)$ the preprojective algebra of Q, and for $i \in Q_0$, define $\mathcal{I}_i$ to be the two-sided ideal $\Pi(1 - e_i)\Pi$.

Let w be an element in W_Q. For $\mathbf{w} = s_{u_1} \dots s_{u_\ell}$ a reduced expression of w, we define $\Pi_{\mathbf{w}} := \Pi/\mathcal{I}_{u_\ell} \dots \mathcal{I}_{u_1}$. It is shown in [BIRS09] that the algebra $\Pi_w := \Pi_{\mathbf{w}}$ does not depend on the choice of the reduced expression of w, and that this algebra is Iwanaga Gorenstein of Gorenstein dimension ≤ 1. Therefore the restriction functor $\mathrm{mod}\,\Pi_w \longrightarrow \mathrm{mod}\,\Pi$ induces an equivalence of categories $\mathrm{CM}(\Pi_w) \xrightarrow{\sim} \mathrm{Sub}(\Pi_w)$ where $\mathrm{Sub}(\Pi_w)$ is the subcategory of $\mathrm{mod}\,\Pi$ of submodules of Π_w seen as a Π-module. Moreover the stable category $\underline{\mathrm{CM}}(\Pi_w)$ is 2-Calabi-Yau, and each reduced expression $\mathbf{w}$ gives rise to a cluster-tilting object $M_{\mathbf{w}}$ in $\mathrm{CM}(\Pi_w)$ [BIRS09, Thm II.2.8].

Theorem 3.1. *[ART11] Let Q be an acyclic quiver and let w be in W_Q. Then for any $\mathbf{w}$ reduced expression of w, there exists a τ_2-finite algebra $\underline{\Lambda}'_{\mathbf{w}}$ such that we have a triangle equivalence*

$$\underline{\mathrm{CM}}(\Pi_w) \simeq \mathcal{C}_2(\underline{\Lambda}'_{\mathbf{w}}).$$

If Q is Dynkin, and if w is the longest element in W_Q, then the algebra Π_w is the preprojective algebra $\Pi_2(kQ)$. However, Theorem 3.1 is not a generalisation of Theorem 2.1. The algebra $\underline{\Lambda}'_{\mathbf{w}}$ cannot be understood as a stable Auslander algebra.

It is constructed here as follows: to each choice of a reduced expression of w, let $M_{\mathbf{w}} \in \mathrm{CM}(\Pi_w)$ be the cluster-tilting object constructed in [BIRS09]. We fix an orientation of Q compatible with $\mathbf{w}$ in the following sense.

if there exists an arrow $i \to j$, then $t_i < t_j$ where t_k is the last integer such that $s_{t_k} = s_k$ in the reduced expression $\mathbf{w} = s_{u_1} \dots s_{u_\ell}$.

This choice of orientation induces a $\mathbb{Z}$-grading on $\Pi_2(kQ)$ and thus on Π_w. The cluster-tilting object $M_{\mathbf{w}}$ comes from a natural graded object. We define

$$\Lambda'_{\mathbf{w}} := \mathrm{End}^{\mathbb{Z}}_{\Pi_w}(M_{\mathbf{w}}) \quad \text{and} \quad \underline{\Lambda}'_{\mathbf{w}} := \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi_w}(M_{\mathbf{w}}).$$

For this particular choice of grading on Π_w, we obtain a graded analogue of Theorem 3.1.

Theorem 3.2 (Kimura)**.** *[Kim18, Kim17]*

Let Q be an acyclic quiver whose orientation is compatible with a choice of a reduced expression $\mathbf{w}$ of $w \in W_Q$. Then there is commutative diagram of triangle functors

$$\begin{CD} \mathcal{D}^{\mathrm{b}}(\underline{\Lambda}'_{\mathbf{w}}) @>{\sim}>> \underline{\mathrm{CM}}^{\mathbb{Z}}(\Pi_w) \\ @V{\pi}VV @VV{\text{forget}}V \\ \mathcal{C}_2(\underline{\Lambda}'_{\mathbf{w}}) @>{\sim}>> \underline{\mathrm{CM}}(\Pi_w). \end{CD}$$

Remark 3.3. A generalisation of Theorem 2.1 has been shown in [AIRT12] for special elements $\mathbf{w}$ in W_Q called co-c-sortable. In that case, one can construct an algebra $\underline{\Lambda}_{\mathbf{w}}$ as a stable Auslander algebras of a certain torsion class in $\mathrm{mod}\, kQ$. It generalizes Theorem 2.1 in the following sense: in the case where Q is Dynkin, where c is the Coxeter element associated to the orientation of Q, and where $\mathbf{w}$ is a co-c-sortable expression of the longest element in W_Q, we have an isomorphism $\underline{\Lambda} \simeq \underline{\Lambda}_{\mathbf{w}}$.

It is shown in [Ami12] that the two different τ_2-algebras constructed $\underline{\Lambda}'_{\mathbf{w}}$ and $\underline{\Lambda}_{\mathbf{w}}$ are related by a sequence of 2-APR tilts, and so are derived equivalent.

Question

A natural question arising here is to understand the relations between the algebras $\Lambda'_{\mathbf{w}}$ and $\Lambda'_{\mathbf{w}'}$ where the reduced expressions $\mathbf{w}$ and $\mathbf{w}'$ represent the same element in W_Q, so for instance when $\mathbf{w}$ and $\mathbf{w}'$ differ by one braid relation.

3.2 Higher analogues for special w

Let us give some more details on the construction of the algebras $\Lambda_{\mathbf{w}}$ and $\Lambda'_{\mathbf{w}}$ in the following case.

Let Q be a non Dynkin quiver, and c be the Coxeter element compatible with the orientation of Q as above. For $n \geq 2$ we define $\mathbf{w} = c^n$. Then we have $\Pi_w \simeq \Pi/\langle E^{\otimes n}\rangle$ where $E := \mathrm{Ext}^1_{kQ}(DkQ, kQ)$.

For the construction of $\Lambda_{\mathbf{w}}$ as in [AIRT12] we have

$$\Lambda_{\mathbf{w}} = \begin{pmatrix} kQ & E & \cdots & E^{\otimes n} \\ 0 & kQ & \ddots & \vdots \\ & & \ddots & E \\ & & & kQ \end{pmatrix} = \operatorname{End}_{kQ}(\Pi_w)$$

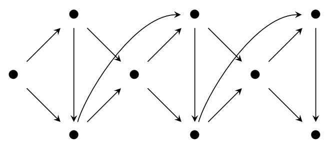
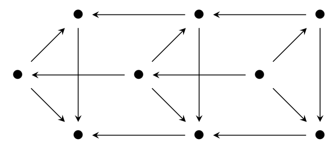
and

$$\underline{\Lambda}_{\mathbf{w}} = \begin{pmatrix} kQ & E & \cdots & E^{\otimes n-1} \\ 0 & kQ & \ddots & \vdots \\ & & \ddots & E \\ & & & kQ \end{pmatrix} = \underline{\operatorname{End}}_{kQ}(\Pi_w).$$

For the construction of $\Lambda'_{\mathbf{w}}$ as in [ART11], we have

$$\Lambda'_{\mathbf{w}} = kQ \underset{k}{\otimes} k\overleftarrow{A}_n \quad \text{and} \quad \underline{\Lambda}'_{\mathbf{w}} = kQ \underset{k}{\otimes} k\overleftarrow{A}_{n-1}.$$

For example, taking Q of type $\widetilde{A}_2$, and $n = 3$ one obtains the following

$\Lambda_{\mathbf{w}} =$  $\Lambda'_{\mathbf{w}} =$ 

If we consider the case $n = 2$, the algebra Π_w is just the trivial extension of kQ by the bimodule $\operatorname{Ext}^1_{kQ}(DkQ, kQ)$. And one gets $\Lambda_{\mathbf{w}} = \Lambda'_{\mathbf{w}} = kQ$. So Theorem 3.1 gives an equivalence

$$\underline{\mathrm{CM}}(kQ \oplus E) \simeq \mathcal{C}_2(kQ);$$

which was already a consequence of [KR08] and [BIRS09]. This has been generalized as follows by Hanihara:

Theorem 3.4. *[Han] Let $d \geq 2$ and A be a $(d-1)$-RI algebra. Then A is τ_d-finite algebra. Denote by Γ the trivial extension of the algebra A by the bimodule $\operatorname{Ext}^{d-1}_A(DA, A)$. Then we have an equivalence of categories*

$$\underline{\mathrm{CM}}(\Gamma) \simeq \mathcal{C}_d(A).$$

Note moreover that $\Pi_d(A)$ is (1)-twisted bimodule d-Calabi-Yau as a graded algebra since A is d-RI. Hence it is bimodule $(d+1)$-Calabi-Yau as a DG algebra. In fact, we even have an isomorphism

$$\mathbf{\Pi}_{(d+1)}(A) \simeq \Pi_d(A)$$

in the homotopy category of DG algebras. Thus, when A is $(d-1)$-RI, we have an equivalence

$$\mathcal{C}_d(A) \simeq {}^{\operatorname{per}(\Pi_d(A)^{DG})}\!\big/_{\mathcal{D}_{\mathrm{fd}}(\Pi_d(A)^{DG})}.$$

These results have a more general analogue in [Han] for (a)-twisted bimodule d-Calabi-Yau graded algebras.

Further directions

All these results leads naturally to the following generalisations.

Let A be a $(d-1)$-RI algebra, and $n \geq 2$. Define the following algebras

- $\Pi_{d,n}(A) := {}^{\Pi_d(A)}\!\big/\!_{\langle E^{\otimes n}\rangle}$ where $E = \mathrm{Ext}_A^{d-1}(DA, A)$;
- $\underline{\Lambda}_{d,A,n} = \begin{pmatrix} A & E & \cdots & E^{\otimes n-1} \\ 0 & A & \ddots & \vdots \\ & & \ddots & E \\ & & & A \end{pmatrix} = \underline{\mathrm{End}}_{kQ}(\Pi_{d,n}(A))$.
- $\underline{\Lambda}'_{d,A,n} = A \underset{k}{\otimes} k\overleftarrow{A}_{n-1}$.

It is then natural to ask the following

Are there triangle equivalences $\underline{\mathrm{CM}}(\Pi_{d,n}(A)) \simeq \mathcal{C}_d(\underline{\Lambda}_{d,A,n}) \simeq \mathcal{C}_d(\underline{\Lambda}'_{d,A,n})$?

Are the algebras $\underline{\Lambda}_{d,A,n}$ *and* $\underline{\Lambda}'_{d,A,n}$ *related by d-APR tilts ?*

We could also try, following [Han], to generalize the situation for higher Gorenstein parameters.

4 CM over isolated singularities

4.1 General result

Another source of examples comes from algebraic geometry: Auslander showed in [Aus78] (see also [Yos90]) that the stable category of (maximal) Cohen-Macaulay modules over a commutative isolated d-dimensional local Gorenstein singularity is $(d-1)$-Calabi-Yau. Using higher analogues of Auslander-Reiten theory, Iyama proved the existence of $(d-1)$-cluster-tilting objects for quotient singularities in [Iya07a]. Similar results have been proved for some three dimensional hypersurface singularities in [BIKR08], (see also [IR08, VV16]).

In this section, we discuss the case where the Iwanaga-Gorenstein algebra R is given as in Theorem 5.1. Under some further assumption on the idempotent e, we obtain the following.

Theorem 4.1. *[AIR15] Let Λ be a finite dimensional d-RI algebra, such that the corresponding $(d+1)$-preprojective algebra $\Pi := \Pi_{d+1}(\Lambda)$ is Noetherian. Asume moreover that there exists an idempotent $e \in \Lambda$ such that*

- *$\underline{\Pi} := \Pi/\Pi e \Pi$ is finite dimensional;*
- *$e\Lambda(1-e) = 0$.*

Then the algebra $\underline{\Lambda} := \Lambda/\Lambda e \Lambda$ is τ_d-finite and we have a diagram of triangle functors

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) & \xrightarrow{\ \sim\ } & \underline{\mathrm{CM}}^{\mathbb{Z}}(R) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\text{forget}} \\ \mathcal{C}_d(\underline{\Lambda}) & \xrightarrow{\ \sim\ } & \underline{\mathrm{CM}}(R) \end{array} ,$$

where $R := e\Pi e$.

We can summarize as follows the relation between R and $\underline{\Lambda}$:

$$R = e\Pi e \underset{e(-)e}{\overset{d\text{-Auslander algebra}}{\rightleftarrows}} \Pi \underset{d+1\text{-preprojective algebra}}{\overset{\text{degree 0 part}}{\rightleftarrows}} \Lambda \xrightarrow{\ -/\langle e\rangle\ } \underline{\Lambda} = \Lambda/\Lambda e \Lambda$$

4.2 Applications

Case $d = 1$

We come back to Subsection 5.2 in Chapter 1, and apply Theorem 4.1 in the case $d = 1$. Then $\Lambda = kQ$ where k is an algebraically closed field of characteristic zero, Q is some extended Dynkin quiver, and e is the extended vertex. We can choose an orientation of Q so that $ekQ(1-e) = 0$ holds. The algebra $\underline{\Lambda}$ is then the path algebra $k\underline{Q}$ of the corresponding Dynkin quiver. The previous diagram becomes

$$k[x,y]^G \underset{e(-)e}{\overset{\text{Auslander algebra}}{\rightleftarrows}} \Pi_2(Q) \underset{\text{preprojective algebra}}{\overset{\text{degree 0 part}}{\rightleftarrows}} kQ \xrightarrow{\;-/\langle e\rangle\;} k\underline{Q}$$

The theorem states that we have triangle equivalences

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(k\underline{Q}) & \xrightarrow{\sim} & \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x,y]^G)\ , \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{forget}} \\ \mathcal{C}_1(k\underline{Q}) & \xrightarrow{\sim} & \underline{\mathrm{CM}}(k[x,y]^G) \end{array}$$

Here the functor $\mathbb{S}_1$ is the AR translation of the derived category, and since $k\underline{Q}$ is hereditary, the 1-cluster category is the orbit category $\mathcal{D}^{\mathrm{b}}(k\underline{Q})/\mathbb{S}_1$ which is k-equivalent to the category of the projective modules over the preprojective algebra $\Pi_2(k\underline{Q})$. So the bottom equivalence (seen as an equivalence of k-categories) is the well-known result due to Reiten and Van den Bergh [Rei87, RV89]. The above equivalence (seen as an equivalence of k-categories) was also already proved in [LP11] and [KST07], since we clearly have $\mathcal{D}^{\mathrm{b}}(k\underline{Q}) \simeq \mathrm{proj}^{\mathbb{Z}}(\Pi_2(k\underline{Q}))$ as k-categories.

Beilinson algebras for general d

For general d, we can come back to Example 5.4 in Chapter 1 with Λ being the d-Beilinson algebra where we take $e = e_0$ which clearly statisfies the hypothesis.

We obtain triangle equivalences

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \simeq \mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) \quad \text{and} \quad \underline{\mathrm{CM}}(R) \simeq \mathcal{C}_d(\underline{\Lambda}),$$

where R is the $(d+1)$-Veronese algebra, and where $\underline{\Lambda}$ is given by the quiver

$$1 \underset{x_d}{\overset{x_0}{\underset{x_2}{\rightrightarrows}}}\!\!\vdots\; 2 \underset{x_d}{\overset{x_0}{\underset{x_1}{\rightrightarrows}}}\!\!\vdots\; 3 \;\cdots\cdots\; d-1 \underset{x_d}{\overset{x_0}{\underset{x_1}{\rightrightarrows}}}\!\!\vdots\; d$$

with the relations $x_i x_j = x_j x_i$.

Dimer models

The situation also applies to dimer models in the setup of Corollary 5.6 in Chapter 1, if there exists an idempotent e such that $\Pi/\Pi e\Pi$ is finite dimensional and with $e\Lambda(1-e) = 0$.

Let us come back to Example 5.7 in Chapter 1. The vertex 1 is a source in the quiver of Λ, so $e_1\Lambda(1-e_1) = 0$. Moreover, the algebra $\underline{\Pi} = \Pi/\Pi e_1 \Pi$ is the path algebra of an acyclic quiver, so it is finite dimensional. Therefore we can apply Theorem 4.1 and we obtain a triangle equivalence $\mathcal{C}_2(\underline{\Lambda}) \simeq \underline{\mathrm{CM}}(R)$ where $\underline{\Lambda}$ is the path algebra of the quiver $2 \rightrightarrows 3 \rightrightarrows 4$, and where R is the homogenous coordinates algebra on $\mathbb{P}^1 \times \mathbb{P}^1$.

4.3 Orlov decomposition and recollement

In this section, we explain how the restriction functor $\mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) \longrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda)$ can be seen as a "non commutative" analogue of Orlov's orthogonal decomposition theorem [Orl09]. We refer to [Ami14a] for more details.

Here is a result which gives a geometric interpretation of the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$.

Theorem 4.2. *[Min12, Thm 3.12] Let Λ be a d-RI algebra, and $\Pi := \Pi_{d+1}(\Lambda)$ be the corresponding $(d+1)$-preprojective algebra. If Π is left graded coherent, then there exists a triangle equivalence*

$$\mathcal{D}^{\mathrm{b}}(\Lambda) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}\Pi)$$

where $\mathrm{qgr}\Pi := \mathrm{mod}^{\mathbb{Z}}\,\Pi/\,\mathrm{fd}^{\mathbb{Z}}\,\Pi$ *is the category of graded tails of* Π.

Now, assume that e is as in Theorem 4.1. Since the algebra $\Pi/\Pi e\Pi$ is finite dimensional, the functor $\mathrm{mod}^{\mathbb{Z}}\,\Pi \to \mathrm{mod}^{\mathbb{Z}}\,R$ which is a right multiplication by e, induces an equivalence

$$\mathrm{qgr}(\Pi) \simeq \mathrm{qgr}R.$$

The second assumption, stating that $e\Lambda(1-e) = 0$ ensures that there is an embedding

$$\mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) \to \mathcal{D}^{\mathrm{b}}(\Lambda).$$

Therefore combining Theorems 4.1 and 4.2 we obtain an embedding $\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \to \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}R)$. If $e\Lambda e = k$, I show in [Ami14a] that this embedding is the one constructed by Orlov in [Orl09] for graded commutative Noetherian rings of Gorenstein parameter 1.

In this case, we even get a recollement

$$\underline{\mathrm{CM}}(R) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{b}}(\mathrm{qgr}R) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{b}}(k)\ .$$

Coming back to the example of the Beilinson algebra in subsection 5.4, we obtain the following: R is the $(d+1)$-Veronese algebra, so we have $\mathrm{Proj}R = \mathrm{Proj}S$, and hence by Serre's result [Ser55], we obtain $\mathcal{D}^{\mathrm{b}}(\mathrm{qgr}R) = \mathcal{D}^{\mathrm{b}}(\mathrm{coh}S) = \mathcal{D}^{\mathrm{b}}(\mathrm{coh}\mathbb{P}^d)$. So Minamoto's result can be understood as a generalisation of Beilinson's result [Bei78]

$$\mathcal{D}^{\mathrm{b}}(\Lambda) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{coh}\mathbb{P}^d).$$

The recollement above becomes

$$\underline{\mathrm{CM}}(R) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{b}}(\mathrm{coh}\mathbb{P}^d) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{b}}(k)\ .$$

Chapter 3

Derived categories as graded cluster categories (Case $d = 2$)

As mentionned in Chapters 1 and 2, the cluster category $\mathcal{C}_d(\Lambda)$ of a τ_d-finite algebra Λ is a d-Calabi-Yau triangulated category with d-cluster-tilting objects. The structure of such categories is very rich and can be summarized as these three (intentionnally imprecise) statements:

- lots of information of the entire category is encoded in a single cluster-tilting object;
- one can construct inductively new cluster-tilting objects from an initial one by performing an operation called *mutation*;
- the situation is especially nice for $d = 2$, since the mutation of cluster-tilting objects is encoded via *mutation of quivers* and *mutation of quivers with potential*.

The leading idea of this chapter is to use the cluster-tilting machinery not for the cluster category $\mathcal{C}_d(\Lambda)$ but rather for the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$. Indeed the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$ can be understood as the graded analogue of $\mathcal{C}_d(\Lambda)$ (where the grading is played by the endofunctor $\mathbb{S}_d$). It has a natural cluster-tilting subcategory $\mathcal{U}_\Lambda^d := \pi^{-1}(\pi\Lambda)$ given by the preimage of the canonical d-cluster-tilting object $\pi(\Lambda)$ in $\mathcal{C}_d(\Lambda)$ by the triangle functor

$$\pi : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{C}_d(\Lambda).$$

Therefore we have

$$\mathcal{U}_\Lambda^d := \mathrm{add}\{\mathbb{S}_d^p\Lambda, p \in \mathbb{Z}\} = \mathrm{add}\,\pi^{-1}(\pi(\Lambda)),$$

and $\mathcal{U}_\Lambda^d$ can be understood as the $\mathbb{Z}$-covering of the $(d+1)$-preprojective algebra $\Pi_{d+1}(\Lambda)$ with its natural $\mathbb{Z}$-grading.

The plan of this chapter is as follows. In a first section, we recall general results of d-cluster-tilting theory in triangulated categories, mainly due to Iyama and Yoshino [IY08]. We then concentrate on the case $d = 2$, and with the link with mutation of quivers and quivers with potential. In a second section, we describe the main results of [AO13a] where we investigate the image of the functor π. The third section is dedicated to the results of [AO14]: we define the notion of graded mutation (which refines the notion of mutation of QPs) in order to encode the cluster-tilting mutation in the derived category. This allows us in particular to describe a new tool that detects whether two τ_2-finite algebras are derived equivalent. Graded mutation is strongly used in Chapter 4.

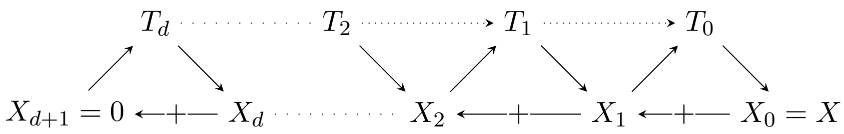
1 Cluster-tilting theory in triangulated categories

Before concentrating on the case $d = 2$, we recall some very general results on d-cluster-tilting theory, mainly due to Iyama and Yoshino in this generality [IY08].

In all the section, we assume that $\mathcal{T}$ is a Hom-finite triangulated category with a Serre functor, that we denote by $\mathbb{S}$. We denote by $\mathbb{S}_d := \mathbb{S}[-d]$ the autoequivalence. A first observation concerning subcategories, is the fact that any d-cluster-tilting subcategory is stable under the functor $\mathbb{S}_d$ ([IY08, Prop 3.4]).

1.1 Approximation $(d+1)$-angles

An interesting aspect of a d-cluster-tilting subcategory is the fact that it plays the role of 'projective-injective" objects in the category $\mathcal{T}$ in the following sense: every object sits in a approximation $(d+1)$-angle with objects in $\mathcal{U}$. More precisely for any $X \in \mathcal{T}$, there exist maps



where each T_i is in $\mathcal{U}$ and where for each $i = 0, \ldots, d-1$

$$X_{i+1} \xrightarrow{g_i} T_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}[1]$$

is a triangle, the map f_i is a left $\mathcal{U}$-approximation and the map g_i is a right $\mathcal{U}$-approximation. In other words we have the equality (cf [IY08, Thm 3.1])

$$\mathcal{U} * \mathcal{U}[1] * \cdots * \mathcal{U}[d-1] = \mathcal{T},$$

where $\mathcal{X} * \mathcal{Y}$ is the full subcategory generated by cones of maps from $\mathcal{Y}[-1]$ to $\mathcal{X}$.

Then, morally at least, the knowledge of a d-cluster-tilting subcategory should be enough to recover the entire category $\mathcal{T}$. This is in fact more complicated, especially because of the fact that cones are not functorial in a triangulated category: the knowledge of a d-cluster-tilting subcategory is enough to understand the objects in $\mathcal{T}$, but not all the morphisms. Proposition 0.1 of Chapter 2 is a consequence of these approximation triangles.

However in general, it is not known whether two triangulated categories having equivalent d-cluster-tilting subcategories are equivalent, except in the case $d = 2$ and where the endomorphism algebra of d-cluster-tilting subcategory is hereditary (see [KR08]). This has recently be generalized by Kalck and Yang in higher Calabi-Yau dimension, in the case where the endomorphism algebra of the d-cluster-tilting object is the path algebra of a tree (see Theorem 5.7 of [KY]) with some extra vanishing conditions and when the characteristic of the field is zero.

1.2 The module category $\operatorname{mod}\mathcal{U}$

We denote by $\operatorname{mod}\mathcal{U}$ the category of finitely presented functors $\mathcal{U}^{\mathrm{op}} \to \operatorname{Mod}k$.

If $\mathcal{U}$ is a d-cluster-tilting subcategory, we have just seen that the category $\mathcal{T}$ can be built from $\mathcal{U}$ by iterated cones. So one can wonder how the pieces $\mathcal{U}[\ell] * \mathcal{U}[\ell+1]$ are related with $\mathcal{U}$. The answer is given by

Proposition 1.1. *[IY08, Cor 6.4] The natural functor*

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \operatorname{mod}\mathcal{U} \\ T & \mapsto & \operatorname{Hom}(-,T)_{|\mathcal{U}} \end{array}$$

induces an equivalence

$$\left(\mathcal{U}[\ell] * \mathcal{U}[\ell+1]\right)\big/_{\mathcal{U}[\ell+1]} \xrightarrow{\sim} \operatorname{mod}\mathcal{U}.$$

The situation is here especially nice for the case $d = 2$ and for a 2-cluster-tilting object T. In that case we obtain an equivalence (see [KR07]).

$$\mathcal{T}\big/_{\operatorname{add} T[1]} \simeq \operatorname{mod}\operatorname{End}_{\mathcal{T}}(T).$$

1.3 Mutation of d-cluster-tilting subcategories

The other important aspect of d-cluster-tilting object is the notion of *mutation*. If we remove an indecomposable summand of a d-cluster-tilting object, (or a $\mathbb{S}_d$-orbit in an indecomposable of a d-cluster-tilting subcategory), there exists a systematic way to replace it to get a new d-cluster-tilting object (resp. subcategory). This can precisely be formulated as follows:

Theorem 1.2. *[IY08, Thms 5.3 and 5.8] Let $\mathcal{U} = \mathcal{U}' \cup \{\mathbb{S}_d^p X, p \in \mathbb{Z}\}$ be a d-cluster-tilting subcategory, with X indecomposable.*

1. *if $d = 2$ then $\mathcal{U}'$ is contained in exactly two 2-cluster-tilting subcategories denoted by $\mathcal{U}$ and $\mu_X(\mathcal{U})$.*
2. *under the condition that X has no loops in $\mathcal{U}$, then $\mathcal{U}'$ is contained in exactly d d-cluster-tilting subcategories denoted by $\mu_X^i(\mathcal{U})$, $i \in \mathbb{Z}/d\mathbb{Z}$.*

The operation μ_X, the *mutation*, can be explicitely described in terms of exchange $(d+1)$-angles.

The "no loop" condition means that the exchange $(d+1)$-angle associated to X coincides with the AR-$(d+1)$-angle of X in $\mathcal{U}$, that is, there is no summands in $\{\mathbb{S}_d^p X, p \in \mathbb{Z}\}$ appearing in the middle terms of the AR-$(d+1)$-angle on X. For the case $d = 2$, it is equivalent to the fact that the quiver of $\mathcal{U}$ contains no arrows from X to some $\mathbb{S}^p X$ for any p, so in the 2-CY case, it is equivalent to the fact that the quiver of $\mathcal{U}$ contains no loop at X.

1.4 The case $d = 2$

The case $d = 2$ is especially nice since the mutation of cluster-tilting objects can be understood combinatorially.

Quiver mutation

For a 2-cluster-tilting subcategory $\mathcal{U} = \mathcal{U}' \cup \{\mathbb{S}_d^p X, p \in \mathbb{Z}\}$ where X is indecomposable summand, the category $\mu_X(\mathcal{U})$ can be computed from $\mathcal{U}$ as $\mathcal{U}' \cup \{\mathbb{S}_d^p X^L, p \in \mathbb{Z}\}$ where X^L appears in a triangle

$$X \xrightarrow{u} B \xrightarrow{v} X^L \xrightarrow{w} X[1]$$

where u is a minimal left $\mathcal{U}'$-approximation.

It can also be computed as $\mathcal{U}' \cup \{\mathbb{S}_d^p X^R, p \in \mathbb{Z}\}$ where X^R appears in a triangle

$$X^R \xrightarrow{u'} B' \xrightarrow{v'} X \xrightarrow{w} X^R[1]$$

where v' is a minimal right $\mathcal{U}'$-approximation.

As a direct consequence, we obtain that there exists $p \in \mathbb{Z}$ such that $X^L \simeq \mathbb{S}_d^p(X^R)$. The situation is then easier in the 2-Calabi-Yau setting, where X^R and X^L coincide.

These two special triangles are called *exchange triangles*. They have been first described by Buan, Marsh, Reineke, Reiten and Todorov (see [BMRRT06, Proposition 6.9]) for cluster categories. The corresponding exchange short exact sequences in module categories over a preprojective algebra of Dynkin type appeared also in the work of Geiss, Leclerc and Schröer (see [GLS06, Lemma 5.1]). The general statement is due to Iyama and Yoshino [IY08, Theorem 5.3].

This recursive process of mutation of cluster-tilting objects, especially in the context of 2-Calabi-Yau categories is closely related to the notion of mutation of quivers defined by Fomin and Zelevinsky [FZ02] which was the original motivation of cluster categorification.

Theorem 1.3 (Buan-Iyama-Reiten-Scott [BIRS09]). *Let $\mathcal{C}$ be a* Hom*-finite 2-CY triangulated category with cluster-tilting object T. Let T_i be an indecomposable direct summand of T, and denote by T' the cluster-tilting object $\mu_{T_i}(T)$. Denote by Q_T (resp. $Q_{T'}$) the Gabriel quiver of the endomorphism algebra* $\operatorname{End}_{\mathcal{C}}(T)$ *(resp. $Q_{T'}$).Assume that there are no loops and no 2-cycles at the vertex i of Q_T (resp. $Q_{T'}$) corresponding to the indecomposable T_i (resp. T_i^*). Then we have*

$$Q_{T'} = \mu_i(Q_T),$$

where μ_i is the Fomin-Zelevinsky quiver mutation.

We illustrate this result by the following diagram.

$$\begin{array}{ccc} \underset{\text{cluster-tilting}}{T} & \overset{\text{IY-mutation}}{\longleftarrow\cdots\cdots\longrightarrow} & \underset{\text{cluster-tilting}}{T'} \\ \downarrow & & \downarrow \\ Q_T & \overset{\text{FZ-mutation}}{\longleftarrow\cdots\cdots\longrightarrow} & Q_{T'} \end{array}$$

The corresponding results have been first shown in the setting of cluster categories in [BMR08] and in the setting of preprojective algebras of Dynkin type in [GLS06].

These results have been generalized for d-cluster categories of acyclic type (see [BT09, ZZ09, Wra09]). The combinatorics of this process is much more technical and can be encoded via mutation of coloured quivers.

Mutation of quiver with potential

Let (Q, W) be a Jacobi-finite quiver with potential, that is a quiver with potential such that the corresponding Jacobian algebra is finite dimensional. The Ginzburg DG algebra $\Gamma_{(Q,W)}$ is a homologically smooth bimodule 3-Calabi-Yau DG-algebra which is negatively graded [Kel11]. The construction of the 2-cluster category can be applied to obtain a generalized cluster category

$$\mathcal{C}_{(Q,W)} := {}^{\operatorname{per}\Gamma}\!/\!_{\mathcal{D}_{\mathrm{fd}}\Gamma}$$

which is 2-Calabi-Yau and has cluster-tilting objects [Ami]. We refer to [Ami11] for more details on the generalized cluster category associated with quiver with potential and Ginzburg DG algebras. Note that in a recent paper[KY], Kalck and Yang have shown that in characteristic zero, any triangulated d-Calabi-Yau category with a d-cluster-tilting is equivalent to a quotient ${}^{\operatorname{per}\Gamma}\!/\!_{\mathcal{D}_{\mathrm{fd}}\Gamma}$ for some DG algebra Γ with similar properties as a higher analogue of a Ginzburg algebra.

A notion of mutation of QPs is defined in [DWZ08]. The link with generalized categories is given in the following theorem.

Theorem 1.4. *[KY11] Let (Q, W) be a Jacobi-finite reduced QP, and i a vertex of Q such that there is no loops, nor 2-cycles at i in Q. Denote by $(Q', W') = \mu_i(Q, W)$ the mutation of the QP (Q, W) at vertex i in the sense of Derksen-Weyman-Zelevinsky. Then there is a triangle equivalence*

$$\mathcal{C}_{(Q,W)} \simeq \mathcal{C}_{\mu_i(Q,W)}$$

sending the canonical cluster-tilting object $\Gamma_{(Q,W)}$ on $\mu_i(\Gamma_{(Q',W')})$.

When combining this result with Theorem 2.4, we obtain two consequences for the cluster category of a τ_2-finite algebras.

Corollary 1.5. *Let Λ be a τ_2-finite algebra. Denote by $(\bar{Q}, W)$ the QP associated to $\Pi_3(\Lambda)$ as in Theorem 2.4. Let T be a cluster-tilting object in $\mathcal{C} := \mathcal{C}_2(\Lambda)$ obtained from $\pi(\Lambda)$ by a sequence of mutations $s = i_1, \ldots, i_n$. If for each $0 \leq \ell \leq n$ there is no 2-cycle at i_ℓ in the quiver of* $\operatorname{End}_{\mathcal{C}}(T^{\ell-1})$*, where the T^ℓ are the iterated mutate of $\pi(\Lambda)$, then there is an isomorphism*

$$\operatorname{End}_{\mathcal{C}}(T) \simeq \operatorname{Jac}(\mu_s(\bar{Q}, W)).$$

Note that a similar result has been proved by Buan, Iyama, Reiten and Smith in [BIRS11] in the setup of a 2-Calabi-Yau category with a cluster-tilting object having a Jacobian endomorphism algebra. It is more general in that sense, but they need in the proof a technical condition (the glueing condition) which may be complicated to check in examples.

Corollary 1.6. *Let Λ and Λ' be τ_2-finite algebras. Denote by $(\bar{Q}, W)$ (resp. $(\bar{Q}', W')$) the QP associated to $\Pi_3(\Lambda)$ (resp. $\Pi_3(\Lambda')$) as in Theorem 2.4. If there exists a sequence of mutation (with no 2-cycles on each mutated vertex of the sequence of mutated quivers), then there is an equivalence*

$$\mathcal{C}_2(\Lambda) \simeq \mathcal{C}_2(\Lambda').$$

One important remark about this result is the fact that the converse statement is not known. It is an open question to know whether the cluster-tilting graph is connected for a category $\mathcal{C}_2(\Lambda)$. It is true for Λ hereditary (see [BMRRT06] and [HU03]). Some counter-examples have been discovered for cluster categories associated with some QP [Pla11], but these are not cluster categories associated with a τ_2-finite algebra.

2 The image of the functor π

2.1 A conjecture

The aim of the article [AO13a] is to understand the image of the functor $\pi : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{C}_2(\Lambda)$. In particular we are interested to understand when the functor π is dense. Indeed the construction of the triangulated hull of an orbit category is difficult to manipulate in practice, since it uses DG-enhancement. The cluster category becomes much simpler for computation when it is equal to the orbit category $\mathcal{D}^{\mathrm{b}}(\Lambda)/\mathbb{S}_2$. By [Kel05], it is the case when the algebra Λ is piecewise hereditary, that is when the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is equivalent to some category $\mathcal{D}^{\mathrm{b}}(\mathcal{H})$ where $\mathcal{H}$ is a hereditary category. In the paper [AO13a], we aim to prove the converse statement.

Coming back to Proposition 1.1 applied for the cluster-tilting subcategory $\mathcal{U} := \mathcal{U}_\Lambda^2$ of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ we obtain the following commutative diagram, where Π is the 3-preprojective algebra of Λ, and $\operatorname{mod}^{\mathbb{Z}} \Pi$ is the category of graded modules over Π (which can be identified to the category $\operatorname{mod} \mathcal{U}$ by the above remark).

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\Lambda) & \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(\mathcal{U},-)} & \operatorname{mod}^{\mathbb{Z}} \Pi \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\text{forget}} \\ \mathcal{C}_2(\Lambda) & \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\pi\Lambda,-)} & \operatorname{mod} \Pi \end{array} .$$

We first make the following observation :*An object $X \in \mathcal{C}_2(\Lambda)$ is in the image if and only if its image $\operatorname{Hom}_{\mathcal{C}}(\pi\Lambda, X) \in \operatorname{mod} \Pi$ is a gradable module.*

The conjecture leading the work in [AO13a] becomes then as follows:

Conjecture 2.1. *Let Λ be a τ_2-finite algebra, and Π be the 3-preprojective algebra of Λ. Then Λ is piecewise hereditary if and only if any Π-module is gradable.*

A very similar conjecture has been already stated by Skowronski [Sko] concerning the trivial extension algebra, instead of the 3-preprojective algebra.

2.2 Properties of objects in the image

We first discuss different properties satisfied by the objects in the image of π.

Proposition 2.2. *Let Λ be a τ_2-finite algebra. Let X be an idecomposable object in $\mathcal{C}_2(\Lambda)$.*

1. *X is in the image of π if and only if all objects in its AR-component are;*

2. *if X is not in the image of π, then there exists a one parameter family $(X_\alpha)_{\alpha\in k^\times}$ of objects in $\mathcal{C}_2(\Lambda)$ which are not in the image and such that for any α, $X_\alpha \simeq X_\beta$ for finitely many β.*

Using these results, we prove the direct direction of the conjecture in different cases:

- when some objects satisfy a certain fractional CY property (see [AO13a, Thm 6.1])
- when the quiver of Λ has an oriented cycle [AO13a, Thm 7.1]);
- and for surface cut algebras (see Proposition 2.13 in Chapter 4).

2.3 Cluster-tilting objects

From 2. of Proposition 2.2, we can deduce the following.

Proposition 2.3. *Let Λ be a τ_2-finite algebra of global dimension ≤ 2. Let X be an indecomposable object in $\mathcal{C}_2(\Lambda)$ which is a summand of a cluster-tilting object, then X is in the image of π. As a consequence the functor π induces a bijection*

$$\{2\text{-cluster-tilting objects in } \mathcal{C}_2(\Lambda)\} \longleftrightarrow \{2\text{-cluster-tilting subcat. in } \mathcal{D}^{\mathrm{b}}(\Lambda)\}.$$

In the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$ we can also construct approximation triangles (see subsection 1.1), and exchange triangles (see subsection 1.4), and their image through π are approximation and exchange triangles respectively.

As a consequence, cluster-tilting combinatorics of the category $\mathcal{C}_2(\Lambda)$ is encoded in the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$.

Note that it is not completely clear that Proposition 2.3 holds for general d.

3 Mutation in derived categories

3.1 Recognition theorem

As we have seen in Proposition 0.1 in Chapter 2, to construct an equivalence between two triangulated categories with a cluster-tilting subcategory, one needs to construct a functor which restricts to an equivalence on the cluster-tilting subcategories. Constructing such a functor can be in general difficult.

The key result in [AO14] is the fact that the existence of such a functor is not needed if one of the category is $\mathcal{D}^{\mathrm{b}}(\Lambda)$ with Λ a τ_2-finite algebra.

Theorem 3.1. *[AO14, Thm 3.5] Let $\mathcal{T}$ be a* Hom*-finite algebraic triangulated category with a Serre functor $\mathbb{S}$ and with a cluster-tilting subcategory $\mathcal{V}$. If there exists a τ_2-finite algebra Λ together with an equivalence $f : \mathcal{U}_\Lambda \simeq \mathcal{V}$ commuting with the action of $\mathbb{S}_2 := \mathbb{S}[-2]$, then there exists an equivalence $F : \mathcal{D}^{\mathrm{b}}(\Lambda) \simeq \mathcal{T}$.*

The proof of this theorem uses strongly the existence of approximation triangles. We introduce the category Mor$\mathcal{V}$ of radical morphisms in $\mathcal{V}$. This category is not exactly the category $\mathcal{T}$ since the cones are not functorial in general, however, the cone map Mor$\mathcal{V} \to \mathcal{T}$ induces a bijection on objects. This is enough in this setup to prove that the object $f(\Lambda) \in \mathcal{V}$ is a tilting object in the category $\mathcal{T}$.

3.2 Graded mutation

The graded category $\mathcal{U}_\Lambda \subset \mathcal{D}^{\mathrm{b}}(\Lambda)$ is the graded covering of the graded algebra $\Pi_3(\Lambda)$, which is by Theorem 2.4, isomorphic to a graded Jacobian algebra.

Given an indecomposable summand T_i in a cluster-tilting subcategory $\mathcal{V} \subset \mathcal{D}^{\mathrm{b}}(\Lambda)$, taking right or left $\mathcal{V} \setminus \mathrm{add}\{\mathbb{S}_2^p T_i, p \in \mathbb{Z}\}$-approximation maps gives two different replacement for T_i:

$$T_i \xrightarrow{u} B \xrightarrow{v} T_i^L \xrightarrow{w} T_i[1] \quad \text{and} \quad T_i^R \xrightarrow{u'} B' \xrightarrow{v'} T_i \xrightarrow{w'} T_i^R[1]$$

which are in the same $\mathbb{S}_2$-orbit. We aim in [AO14] to answer the following question: *If $\mathcal{V}$ is the covering of some graded Jacobian algebra, is the new category $\mu_i(\mathcal{V})$ still the covering of a graded Jacobian algebra ?*

We need for that a few definitions.

Definition 3.2. A graded QP (Q, W, d) is a QP (Q, W) together with a map $d : Q_1 \to \mathbb{Z}$ such that W is homogenous of degree 1.

Definition 3.3. Let (Q, W, d) be a graded QP, and $i \in Q_0$ such that there is no loop, nor 2-cycle at i in Q. We define *the left mutation* $\mu_i^L(Q, W, d) = (Q', W', d')$ of the graded QP (Q, W, d) as follows:

- replace each arrow $a : i \to j$ in Q by an arrow $a^* : j \to i$ and put $d'(a^*) := -d(a)$
- replace each arrow $b : j \to i$ in Q by an arrow $b^* : i \to j$ and put $d'(b^*) := 1 - d(b)$
- for each composition $j \xrightarrow{a} i \xrightarrow{b} k$, add an arrow $[ba] : j \to k$ and put $d([ba]) := d(a) + d(b)$;
- the unchanged arrows keep the same grading;
- the potential $W' = W^* + [W]$ is defined as in [DWZ08].

We can dually define right mutation μ_i^R, by exchanging the degree formula of the first 2 items.

One immediately observes that W' is homogenous of degree 1, so (Q', W', d') is a graded QP. Similary to [DWZ08], one can define reduced and trivial graded QPs, and prove that any graded QP is graded right equivalent to direct sum of a reduced and a trivial QP. Therefore the reduced part of $\mu_i^L(Q, W, d)$ is still a graded QP.

Here is a graded analogue of Corollary 1.5 which gives an answer to the above question.

Proposition 3.4. *[AO14, Thm 6.12] Let Λ be a τ_2-finite algebra. Denote by $(\bar{Q}, W, d)$ the QP associated to $\Pi_3(\Lambda)$ as in Theorem 2.4. Let T be an object in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ obtained from Λ by a sequence of left/right mutations $i_1, \ldots, i_n$. If for each $0 \leq \ell \leq n$ there is no 2-cycle at i_ℓ in the quiver of $\mathrm{End}_{\mathcal{C}}(T^{\ell-1})$, where the T^ℓ are the iterated mutate of $\pi(\Lambda)$, then there is an isomorphism of graded algebras*

$$\bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(T, \mathbb{S}_2^{-p} T)) \simeq \mathrm{Jac}(\mu_s^{L,R}(\bar{Q}, W, d)).$$

3.3 Graded mutation and derived equivalences

Combining Proposition 3.4 together with Theorem 3.1, we obtain the graded analogue of Corollary 1.6.

Corollary 3.5. *[AO14, Cor 6.14] Let Λ_1 and Λ_2 be τ_2-finite algebras of global dimension ≤ 2. Denote by (Q^1, W^1, d^1) and (Q^2, W^2, d^2) the corresponding graded quivers with potential. If there exists a sequence a left/right mutation from (Q^1, W^1, d^1) to (Q^2, W^2, d^2) (with no 2-cycle on each mutated vertex of the sequence of mutated quivers), then there is a triangle equivalence*

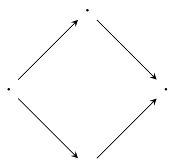
$$\mathcal{D}^{\mathrm{b}}(\Lambda_1) \simeq \mathcal{D}^{\mathrm{b}}(\Lambda_2).$$

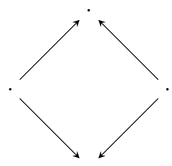
If moreover (Q^1, W^1) is mutation acyclic (that is mutation equivalent to an acyclic quiver), then the converse holds.

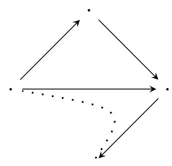
Here again, the obstruction for the converse direction comes from the open question about the connectedness of the cluster-tilting graph. Note that if a quiver Q is acyclic, then the mutation at i of the QP $(Q, 0)$, when i is a source or a sink, is $(R_i(Q), 0)$ where R_i is the reflection of the quiver Q at i. So Corollary 3.5 can also be understood as a generalisation of the following well-known result:

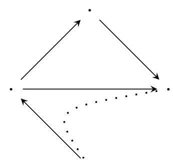
Theorem 3.6. *[Hap87] Let Q and Q' be two acyclic quivers. Then there exists a derived equivalence $\mathcal{D}^{\mathrm{b}}(kQ) \simeq \mathcal{D}^{\mathrm{b}}(kQ')$ if and only if Q and Q' can be related by a sequence of reflections.*

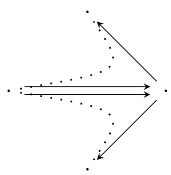
Example 3.7. For example, we can classify the τ_2-finite algebras whose cluster category is of type $\widetilde{\mathbb{A}}_{2,2}$. They belong in two different derived equivalence classes. The first class contain eight non isomorphic algebras given as follows.

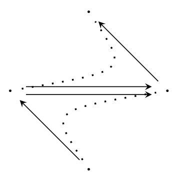

Λ_1

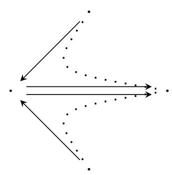

Λ_2

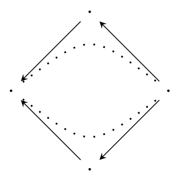

Λ_3


Λ_4

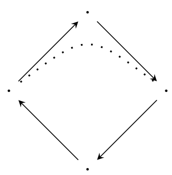

Λ_5

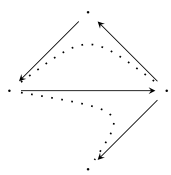

Λ_6

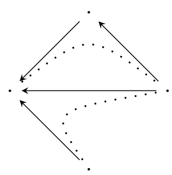

Λ_7


Λ_8

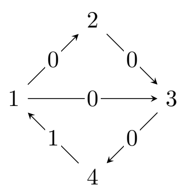
The second class contains three non isomorphic algebras as follows:


Λ_9

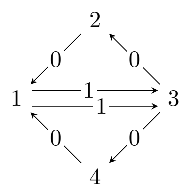

Λ_{10}


Λ_{11}

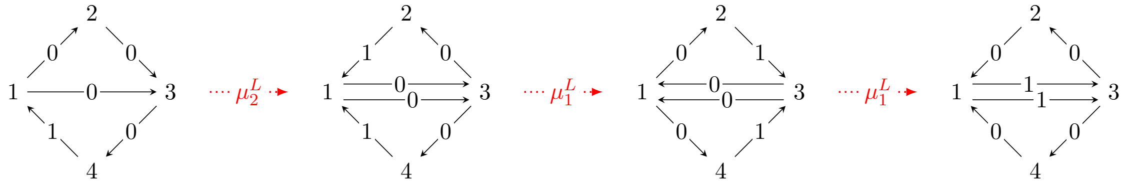
To check that the algebras Λ_3 and Λ_8 are derived equivalent, one needs to exhibit a sequence of graded mutations between the graded quivers



and



Below is one example of such a sequence:



3.4 Cluster equivalence and graded derived equivalence

Another question we are considering in [AO14] is the following : *if Λ_1 and Λ_2 are τ_2-finite algebras, such that the associated QP are linked by a sequence of mutation as in Corollary 1.6, what can be said about the categories $\mathcal{D}^{\mathrm{b}}(\Lambda_1)$ and $\mathcal{D}^{\mathrm{b}}(\Lambda_2)$?*

To start with, let us consider the situation where the sequence of mutations is empty, that is when there is an isomorphism of algebras

$$\Pi := \Pi_3(\Lambda_1) \simeq \Pi_3(\Lambda_2).$$

Then we get two different gradings on the algebra Π. Assume that these two gradings can be lifted to a $\mathbb{Z}^2$-grading (d^1, d^2) on the algebra Π and thus on the quiver of Π (this is a priori not always possible, two $\mathbb{Z}$-gradings on an algebra do not give rise in general to a $\mathbb{Z}^2$-grading on it). Then the degree map d^2 (resp. d^1) induces a grading on the degree 0 part of Π with respect to d^1 (resp. d^2), that is on Λ_1 (resp. Λ_2). Then using a graded analogue of Theorem 3.1, we prove in the last section of [AO14] that we have a triangle equivalence

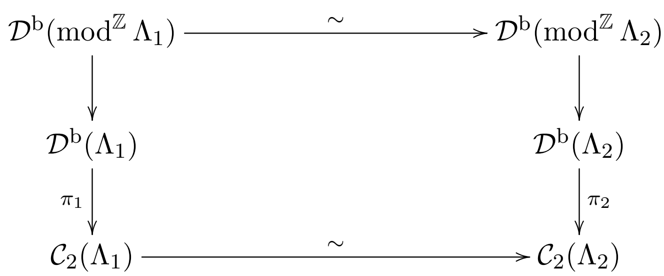
$$\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_1) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_2).$$

Coming back to the general situation, assume that the QPs (Q^1, W^1) and (Q^2, W^2) can be related by a sequence of mutations (without 2-cycles on each mutated vertex). Using graded mutations, we obtain two different gradings on the preprojective $\Pi_3(\Lambda_1)$, one given by the tensor algebra grading, and one given by mutating the graded QP (Q^2, W^2, d^2). Once again, we have to assume that this grading lifts to a $\mathbb{Z}^2$-grading on the quiver of $\Pi_3(\Lambda_1)$ (this condition is called *compatibility condition* in [AO14]). This $\mathbb{Z}^2$-grading gives a $\mathbb{Z}$-grading on Λ_1. Using a notion of $\mathbb{Z}^2$-graded mutation of $\mathbb{Z}^2$-graded QPs (which is a straitforward generalisation of the left/right mutation), we obtain a $\mathbb{Z}^2$-grading on (Q^2, W^2), and then a $\mathbb{Z}$-grading on Λ_2 (so in other words the compatibility condition is symmetric in Λ_1 and Λ_2). Theorem 8.7 of [AO14] states that we have a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_1) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_2)$.

We can also try to understand the meaning of these different gradings through the derived equivalences. The graded shift functor (1) in the category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_1)$ is equivalent to the functor $\mathbb{S}_2^{-1} \circ (-1)$ in the category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_2)$. These isomorphism of functors can be lifted at the DG category level, hence we obtain a triangle equivalence

$$\mathcal{D}^{\mathrm{b}}(\Lambda_1) \simeq \left(\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} \Lambda_2) \big/ \mathbb{S}_2(1) \right)_{\Delta}.$$

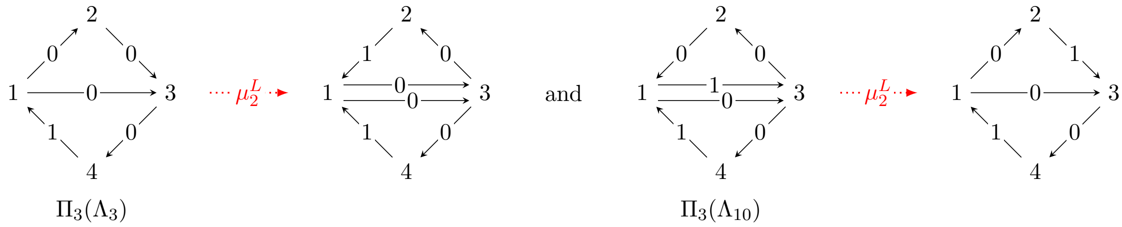
This implies that this equivalence is compatible with the equivalence constructed by Keller and Yang in Corollary 1.6, that is we have the following commutative diagram :



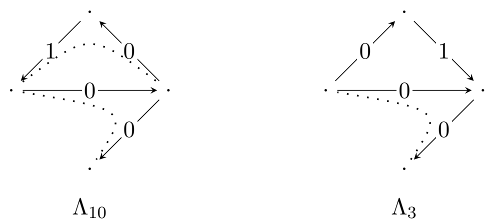
Acyclic case

The situation is particularly nice when the QP associated to Λ is mutation equivalent to an acyclic quiver. Indeed, in this case, we automatically obtain that the QP is rigid, so the condition on 2-cycles does not need to be checked. Morover the compatibility condition is also automatically satisfied, since two $\mathbb{Z}$-gradings on a quiver induce a $\mathbb{Z}^2$-grading on it.

Example 3.8. Coming back to example 3.7, we can apply the result below for the algebras Λ_3 and Λ_{10}. By the graded mutations



we obtain the following gradings on Λ_{10} and Λ_3:



Our theorem states that we have an equivalence $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\,\Lambda_{10}) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\,\Lambda_3)$.

Moreover, if the quiver is a tree, one easily shows that any two $\mathbb{Z}$-gradings on a tree can be related by a sequence of left/right mutations. As a corollary, we obtain the following

Corollary 3.9. *[AO13b] Let Λ be a τ_2-finite algebra such that the QP associated to $\Pi_3(\Lambda)$ is mutation equivalent to a quiver Q whose underlying graph is a tree. Then Λ is derived equivalent to kQ.*

The first non trivial case beyond the tree type arises in type $\widetilde{\mathbb{A}}_n$, where two gradings on a quiver are not necessary linked by a sequence of left/right mutations. In that case, we prove that the equivalent classes of gradings on a quiver of type $\widetilde{\mathbb{A}}_n$ are parametrized by $\mathbb{Z}$. As we will see in the next part of this memoir, this parameter has different interpretations, using either the AG invariant of [AAG08], or a geometric interpretation using line fields on the annulus, and winding numbers.

Part II

Topological models for triangulated categories

Chapter 4

Triangulated marked surfaces

As seen in the previous chapters, cluster combinatorics appear naturally in triangulated categories with cluster-tilting objects. But they also appear naturally in topology through flips and triangulations of surfaces. More precisely, following [FST08] (resp. [LF09]) one can associate to any triangulated surface a quiver (respectively a QP), and the operation of flipping an arc of the triangulation corresponds to the mutation of the associated quiver (resp. QP). It becomes then quite natural to investigate whether one can link more directly cluster categories with the surface. This is the main idea of topological model: one constructs a dictionnary between categories and surfaces, and interprets algebraic operations in term of topological operations.

In the first section of this Chapter, we concentrate on cluster categories associated to triangulated surfaces with non-empty boundary. We first recall results in the case where all marked points are located on the boundary of the surface (the *unpunctured case*). In this case, the cluster category is very close to the module category of a gentle algebra. Therefore one can obtain a topological description of indecomposable objects of the category [ABCP10, BZ11], and of morphisms [CS17, CPS19]. We then focus on the *punctured case* which is treated in a joint work with Pierre-Guy Plamondon [AP]. We use the skew-group algebra construction to see the punctured surface as an orbifold with 2-folded cover. This permits us to obtain a complete description of the indecomposable objects in topological terms.

In a second part of this chapter, we show that any cluster category of a surface can be interpreted as a 2-cluster category of a τ_2-finite algebra, that we call *surface cut algebra*. We apply results on graded mutation in [AO14] (see Chapter 3) in order to get derived invariants of surface cut algebras. The annulus case is treated in [AO13b], the unpunctured case in [AG16] and [Ami16], and the punctured case in [ALP20].

1 Cluster categories from triangulated surfaces

1.1 Definition

A *marked surface* $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ is an oriented surface $\mathcal{S}$ with non-empty boundary, together with a finite set of marked points $\mathbb{M}$ on the boundary such that there is at least one marked point on each boundary component, and a finite set of marked points $\mathbb{P}$ in the interior of $\mathcal{S}$, called the *punctures*. A curve on the boundary of $\mathcal{S}$ intersecting marked points only on its endpoints is called a *boundary segment*. An *arc* on $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ is the homotopy class of a curve on $\mathcal{S}$, with endpoints in $\mathbb{M} \cup \mathbb{P}$ and without selfintersection (except for the endpoints), which is not homotopic to a boundary segment. Two arcs are *admissible* if there exists some representative that do not cross in the interior of the surface. A *triangulation* is a maximal collection of pairwise admissible arcs. It cuts out the surface into triangles.

To each triangulation Δ, one can associate a quiver with potential (Q^Δ, W^Δ). It is given as follows when Δ does not contain any self-folded triangle:

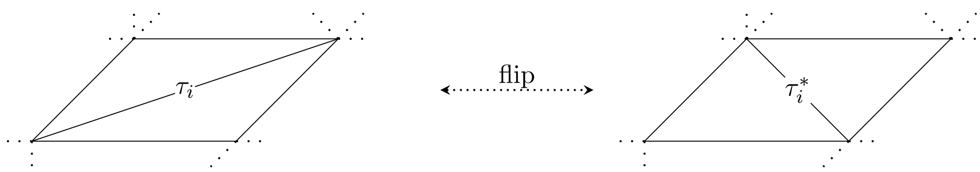
- the vertices of Q^Δ are in bijection with the set of arcs of Δ.
- for each consecutive arcs i and j in counterclockwise direction around a marked point, one puts an arrow $i \to j$.

For each internal triangle τ of Δ, there exists a 3-cycle $c_\tau \in kQ^\Delta/[kQ^\Delta, kQ^\Delta]$. For each puncture $p \in \mathbb{P}$, there exists a cycle (of length the valency of p in the triangulation) denoted by $z_p \in kQ^\Delta/[kQ^\Delta, kQ^\Delta]$. The potential W^Δ is defined in [LF09] by

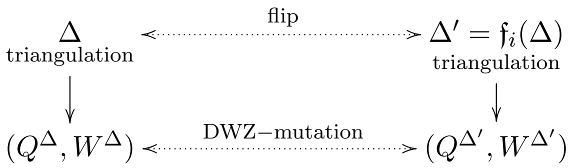
$$W^\Delta := \sum_{\tau \text{ int. triangle}} c_\tau - \sum_{p\in\mathbb{P}} z_p.$$

This QP has been shown to be Jacobi-finite and non degenerate by Labardini-Fragoso in [LF09]. Therefore, one can associate a cluster category to each triangulated surface $(\mathcal{S}, \mathbb{M}, \mathbb{P}, \Delta)$ (see Subsection 1.4 of Chapter 3), that we denote $\mathcal{C}_\Delta$. We also denote by T_Δ the corresponding canonical cluster-tilting object $\mathbf{\Gamma}_{(Q^\Delta, W^\Delta)}$.

Given an arc i in a triangulation Δ (which is not the self-folded arc of a self-folded triangle), one can define a new triangulation by flipping the arc i and replacing it by the only other one making a new triangulation.



In [LF09], Labardini-Fragoso showed that the QP associated with the flip of Δ at the arc i is the Derksen-Weyman-Zelevinsky mutation of the QP (Q^Δ, W^Δ) [DWZ08] up to right equivalence.



Therefore, combining this fact with Theorem 1.4 and the fact that any two triangulations can be linked by a sequence of flips (see [FST08]) we obtain

Corollary 1.1. *Let Δ and Δ' be two triangulations of the marked surface $(\mathcal{S}, \mathbb{M}, \mathbb{P})$, then*

$$\mathcal{C}_\Delta \simeq \mathcal{C}_{\Delta'}.$$

1.2 The unpunctured case

In the case where $\mathbb{P} = \emptyset$, the situation is particularly nice, since the Jacobian algebra associated to any triangulation is gentle (see section 1.1 of Chapter 5 for definition). The module category over a gentle algebra is particularly well known [BR87], [CB00], hence using the equivalence

$$\mathcal{C}_\Delta/\operatorname{add}(T_\Delta[1]) \xrightarrow{\sim} \operatorname{mod}\operatorname{Jac}(Q^\Delta, W^\Delta),$$

a description of indecomposable objects, and of spaces of morphisms have been obtained in [ABCP10, BZ11, CS17, CPS19, CPS].

Denote by $\pi_1(\mathcal{S}, \mathbb{M})$ the groupoid of homotopy classes of curves in $\mathcal{S}$ with endpoints in $\mathbb{M}$ which are not homotopic to a boundary segment. Denote by $\pi_1^{\text{free}}(\mathcal{S})$ the set of non contractible closed curves in $\mathcal{S}$ up to free homotopy.

One has the following description of indecoposable objects:

Theorem 1.2. *[ABCP10, BZ11] Let k be an algebraically closed field*[1]*. Let $(\mathcal{S}, \mathbb{M}, \Delta)$ be a triangulated surface without punctures, and denote by $\mathcal{C}_\Delta$ the corresponding cluster k-category. Then the isomorphism classes of indecomposable objects of $\mathcal{C}_\Delta$ are in bijection with*

- $\{\{\gamma, \gamma^{-1}\}, \gamma \in \pi_1(\mathcal{S}, \mathbb{M}), \gamma \neq 1_M, M \in \mathbb{M}\}$*;*
- $\pi_1^{\text{free}}(\mathcal{S}) \times k^*/\sim$*, where the equivalence relation $\sim$ is given by $([\gamma], \lambda) \sim ([\gamma^{-1}], \lambda^{-1})$.*

This bijection restricts to a bijection

$$T : \{\text{arcs on the surface } (\mathcal{S}, \mathbb{M})\}/\text{htp} \longleftrightarrow \{\text{indecomposable rigid objects in } \mathcal{C}_\Delta\}/\text{isom.},$$

in which each arc of Δ is sent to the object $(T\Delta)_i := e_i\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is the Ginzburg DG algebra associated to the QP (Q^Δ, W^Δ). The intersection number between two arcs coincide with the dimension of Ext^1 between the corresponding objects [BZ11], this bijection induces a bijection between the set of triangulations of $(\mathcal{S}, \mathbb{M})$ and the set of isoclasses of cluster-tilting objects in $\mathcal{C}_\Delta$. Moreover the flip coincide with Iyama-Yoshino mutation in the following sense:

$$\begin{array}{ccc} \underset{\text{triangulation}}{\Delta'} & \overset{\text{flip at } i}{\longleftrightarrow} & \underset{\text{triangulation}}{\Delta'' = \mathfrak{f}_i(\Delta')} \\ \downarrow T & & \downarrow T \\ \underset{\text{cluster-tilting}}{T_{\Delta'}} & \overset{\text{IY-mutation at } T_i}{\longleftrightarrow} & \underset{\text{cluster-tilting}}{T_{\Delta''}} \end{array}$$

An interesting question here, is about what happen when we combine Corollary 1.1 with Theorem 1.2. More precisely we ask the following question: *Are the bijections constructed in Theorem 1.2 independent of the choice of Δ?* This question is more complicated than expected since the triangle equivalence constructed in Corollary 1.1, depends both on the choice of a sequence of flips linking Δ to Δ', and on the choice of a right equivalence map between the mutated QP. The reduction process is in general not unique and not even canonical. If we denote by $T^\Delta : \pi_1(\mathcal{S}, \mathbb{M}) \to \text{Obj}(\mathcal{C}_\Delta)$ and $B^\Delta : \pi_1^{\text{free}}(\mathcal{S}) \times k^*/\sim \to \text{Obj}(\mathcal{C}_\Delta)$ the two bijections of Theorem 1.2, then we have the following:

Proposition 1.3 (Appendix in [CS17])**.** *Let Δ and Δ' be two triangulations of a marked surface $(\mathcal{S}, \mathbb{M})$. Then for any sequence $\mathbf{s}$ of flips relating Δ to Δ' there exists a triangle equivalence*

$$\Phi_{\mathbf{s}} : \mathcal{C}_\Delta \longrightarrow \mathcal{C}_{\Delta'},$$

such that the bijections $T^{\Delta'} \circ \Phi_{\mathbf{s}}$ and T^Δ coincide.

But, one can find a sequence of mutation $\mathbf{s}$ such that the bijections $B^{\Delta'} \circ \Phi_{\mathbf{s}}$ and B^Δ do not coincide. Typically, if i is an arc of Δ, we may have $B^\Delta \circ \Phi_{\mathfrak{f}_i\mathfrak{f}_i} \neq B^\Delta$ (cf Example 1.3.2 in Appendix [CS17]).

1.3 The punctured case using $\mathbb{Z}_2$-action

Tagged arcs

In the case $\mathbb{P} \neq \emptyset$, the situation is more complicated. First of all, a triangulation with self-folded triangles cannot be flipped at a self-folded side in the usual sense. To overcome this situation, Fomin, Shapiro and Thurston introduced the notion of tagged arcs, and tagged triangulations in [FST08]. Then combining the results in [LF09], [Pla11], and [QZ17] we obtain a bijection

$$T : \{\text{tagged arcs on the surface } (\mathcal{S}, \mathbb{M})\}/\text{htp} \longleftrightarrow \{\text{indecomposable rigid objects in } \mathcal{C}_\Delta\}/\text{isom.},$$

[1]This result has an analogue for non necessarily algebraically closed field, using the classification of indecomposable $k[X]$-modules

that induces a bijection

$$T : \{\text{tagged triangulations of } (\mathcal{S}, \mathbb{M})\}/\text{htp} \longleftrightarrow \{\text{cluster-tilting objects in } \mathcal{C}_\Delta\}/\text{isom.},$$

which commutes with flip/mutation in the sense

$$\begin{array}{ccc}
\underset{\text{tagged triangulation}}{\Delta'} & \overset{\text{flip at } i}{\longleftrightarrow} & \underset{\text{tagged triangulation}}{\Delta'' = \mathfrak{f}_i(\Delta')} \\
T \downarrow & & \downarrow T \\
\underset{\text{cluster-tilting}}{T_{\Delta'}} & \overset{\text{IY-mutation at } T_i}{\longleftrightarrow} & \underset{\text{cluster-tilting}}{T_{\Delta''}}
\end{array}$$

This bijection has been extended in [QZ17] to any tagged curve linking two marked points to describe a certain subset of objects called strings in $\mathcal{C}_\Delta$ when Δ satisfies certain properties.

One consequence of the bijection between tagged triangulations and cluster-tilting objects is the following.

Corollary 1.4. *Let $(\mathcal{S}_1, \mathbb{M}_1, \mathbb{P}_1, \Delta_1)$ and $(\mathcal{S}_2, \mathbb{M}_2, \mathbb{P}_2, \Delta_2)$ be two marked surfaces. Then the cluster categories $\mathcal{C}_{\Delta_1}$ and $\mathcal{C}_{\Delta_2}$ are equivalent if and only if there exists a homeomorphism $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ preserving orientation and marked points.*

Proof. The "if" part is a consequence of Corollary 1.1 and the fact that any two triangulations are related by a sequence of flips.

Let us write an explicit proof of the "only if part". First note that we can assume that Δ_1 is a valency ≥ 3 triangulation. Indeed if it is not, by flipping it, we can obtain one together with a triangle equivalence between the two corresponding cluster categories.

Denote by T_1 (resp. T_2) the canonical cluster-tilting object in $\mathcal{C}_{\Delta_1}$ (resp. $\mathcal{C}_{\Delta_2}$.) Assume that there exists a triangle equivalence $F : \mathcal{C}_{\Delta_1} \to \mathcal{C}_{\Delta_2}$. The object $T_1' := F(T_1)$ is a cluster-tilting object in $\mathcal{C}_{\Delta_2}$. Using the bijection written above, it corresponds to a tagged triangulation Δ_1' in $\mathcal{S}_2$. Denote by Δ_1'' the untagged triangulation from Δ_1' (that is, we remove all eventuel taggings of the arcs of Δ_1' to obtain an ideal triangulation). Since F is an equivalence we have the following isomorphisms

$$\text{Jac}(Q^{\Delta_1}, W^{\Delta_1}) \simeq \text{End}_{\mathcal{C}_1}(T_1) \simeq \text{End}_{\mathcal{C}_2}(T_1') \simeq \text{Jac}(Q^{\Delta_1'}, W^{\Delta_1'}) \simeq \text{Jac}(Q^{\Delta_1''}, W^{\Delta_1''}).$$

Therefore the Gabriel quivers of these algebras are isomorphic. Since Δ_1 is a valency ≥ 3-triangulation, the Gabriel quivers are Q^{Δ_1} and $Q^{\Delta_1''}$. We can now apply Proposition 8.5 in [BS] (we refer to [ALP20, Prop 3.15] to include all surfaces), and we obtain a homeomorphism $\mathcal{S}_1 \to \mathcal{S}_2$, sending Δ_1 to Δ_1''.

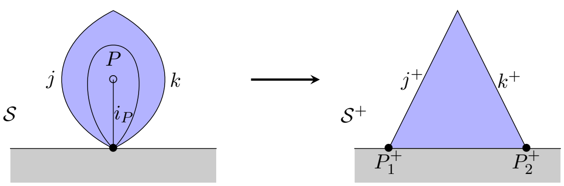
□

Unfolding the surface with punctures

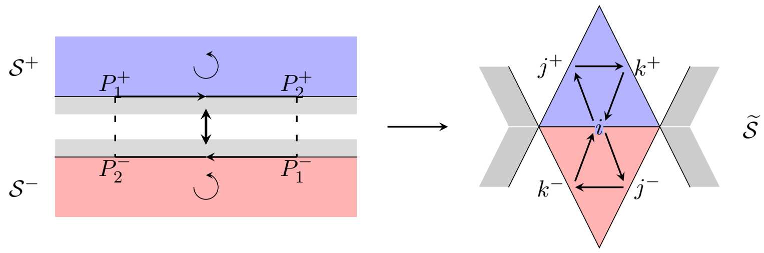
In the joint work [AP] we study the case of the cluster category of a triangulated surface with punctures, in the case where the base field has characteristic $\neq 2$. When the triangulation satisfies the property that any puncture is in a self-folded triangle, the associated Jacobian algebra is skew-gentle (see section 2 of Chapter 5 for definition). This class of algebras has been introduced by Geiss and de la Peña [GePe99] using the construction of skew-group algebras of Reiten and Riedtmann [RR85].

Let $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ be a punctured surface and Δ be a triangulation such that all punctures are in a self-folded triangle. From this data, we construct a new triangulated surface without punctures together with a homeomorphism of order 2 as follows:

- Each puncture P belongs to an self-folded arc i_P. We cut the surface $\mathcal{S}$ along each self-folded side i_P and, obtain a surface $\mathcal{S}^+$, with new boundary segments $[P_1^+, P_2^+]$ corresponding to the arc i_P.



- We then glue to $\mathcal{S}^+$ a copy of itself $\mathcal{S}^-$ along the segments $[P_1^+, P_2^+]$ and $[P_1^-, P_2^-]$, and obtain a new surface $\widetilde{\mathcal{S}}$ with marked points $\widetilde{\mathbb{M}} := \mathbb{M}^+ \cup \mathbb{M}^- \cup \{P_1^+ = P_2^-, P_1^- = P_2^+, P \in \mathbb{P}\}$.

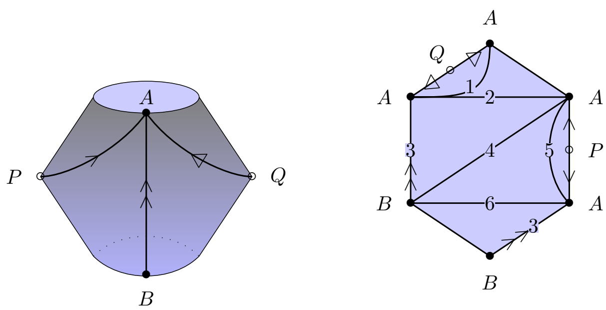


Proposition 1.5. *1. [AP, Thm 3.5] The surface $(\widetilde{\mathcal{S}}, \widetilde{\mathbb{M}})$ constructed above is a surface with marked points, and without punctures. The collection of arcs $\widetilde{\Delta} := \{\tau^+, \tau^-, \tau \in \Delta\} \cup \{[P_1^+, P_2^+], p \in \mathbb{P}\}$ is a triangulation of $\widetilde{\mathcal{S}}$.*

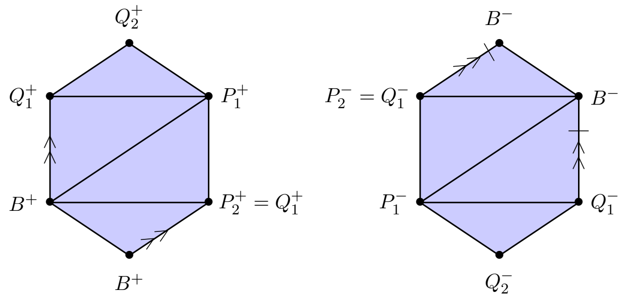
2. [AP, Prop. 3.9] There is a homeomorphism $\sigma : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}}$ of order 2 that exchanges the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$. It has exactly $|\mathbb{P}|$ fixed points, one in each segment $[P_1^+, P_2^+]$, and fixes globally the triangulation $\widetilde{\Delta}$.

3. [AP, Cor. 3.10] The natural projection $\widetilde{\mathcal{S}}/\sigma \to \mathcal{S}$ is a 2-folded cover, with branched points $\mathbb{P}$. It induces a structure of orbifold for $\mathcal{S}$ with orbifold points $\mathbb{P}$.

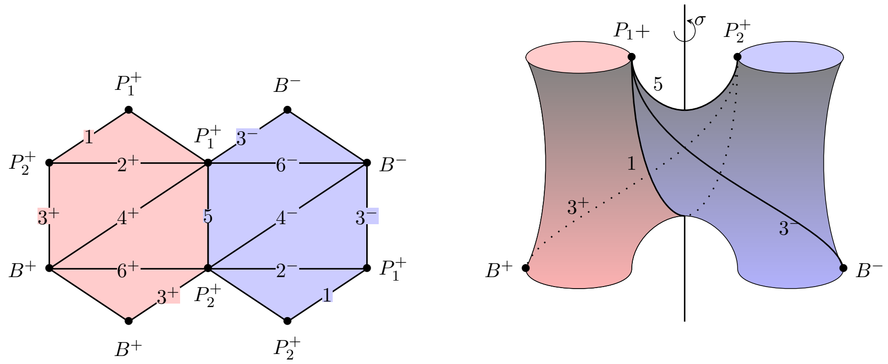
Example 1.6. Let $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ be a cylinder with two punctures $\mathbb{P} = \{P, Q\}$ and two marked points $\mathbb{M} = \{A, B\}$, and with the following triangulation Δ. Cutting the surface along the folded sides and along an arc $[A, B]$, we obtain $\mathcal{S}$ as the following polygon with identifications of sides.



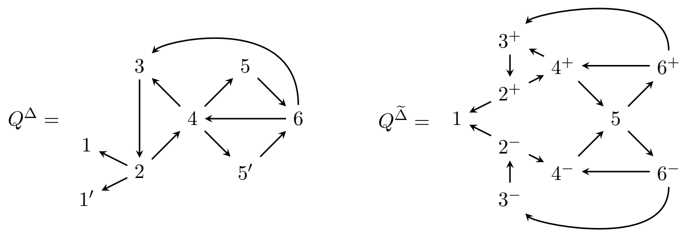
The surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ are then given by the following polygons with identifications:



Hence the surface $\widetilde{\mathcal{S}}$ is a sphere with four boundary components and is given by the following polygon with identifications:



The quivers Q^Δ and $Q^{\widetilde{\Delta}}$ have the following shape:



Indecomposable objects

The action of σ on the surface $\widetilde{\mathcal{S}}$ induces an action of the group $\mathbb{Z}_2$ on $Q^{\widetilde{\Delta}}$ preserving the potential $W^{\widetilde{\Delta}}$. From this, we deduce an $\mathbb{Z}_2$-action on the Ginzburg DG algebra $\mathbf{\Gamma}_{\widetilde{\Delta}}$ and on the corresponding cluster category $\mathcal{C}_{\widetilde{\Delta}}$. Following Reiten and Riedtmann's definition [RR85], we can form the skew-group DG algebra $\mathbf{\Gamma}_{\widetilde{\Delta}}\mathbb{Z}_2$ as follows:

- as a graded vector space, we set $\mathbf{\Gamma}_{\widetilde{\Delta}}\mathbb{Z}_2 := \mathbf{\Gamma}_{\widetilde{\Delta}} \underset{k}{\otimes} k\mathbb{Z}_2$;
- the multiplication is induced by the rule $(\gamma \otimes g).(\gamma' \otimes g') = \gamma g(\gamma') \otimes gg'$;
- the differential is given by $d(\gamma \otimes g) = d(\gamma) \otimes g$.

The algebraic link between the triangulated surfaces $(\mathcal{S}, \Delta)$ and $(\widetilde{\mathcal{S}}, \widetilde{\Delta})$ is given as follows:

Theorem 1.7. *[AP, Thm 2.6] The skewgroup DG algebra $\mathbf{\Gamma}_{\widetilde{\Delta}}\mathbb{Z}_2$ is Morita equivalent to $\mathbf{\Gamma}_\Delta$.*

Adapting the results of Reiten and Riedtmann to the DG-setting, we then obtain the following result :

Theorem 1.8. *[AP, Cor 3.6] There exists a triangle functor $F : \mathcal{C}_{\widetilde{\Delta}} \to \mathcal{C}_\Delta$ which induces a bijection between the isomorphism classes of indecomposable objects in $\mathcal{C}_\Delta$ and the set*

$$\{\sigma - \textit{invariant indec. in } \mathcal{C}_{\widetilde{\Delta}}\} \times \mathbb{Z}_2 \cup \{\sigma - \textit{orbits of non } \sigma - \textit{invariant indec. in } \mathcal{C}_{\widetilde{\Delta}}\}.$$

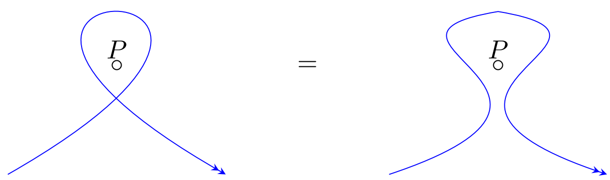
Combining this result together with Theorem 1.2, and translating curves on $\widetilde{\mathcal{S}}$ in term of curves on the orbifold $\mathcal{S}$ we obtain the following description of the indecomposable objects in $\mathcal{C}_\Delta$:

Theorem 1.9 (Corollaries 5.10 and 5.19 in [AP])**.** *Let $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ be a marked surface with non-empty boundary and possibly with punctures. Let Δ be a triangulation of $\mathcal{S}$ such that each puncture belongs to a self-folded triangle and such that no triangle shares a side with two self-folded triangles. Then the indecomposable objects of the cluster category $\mathcal{C}_\Delta$ are in bijection with the following sets:*

1. $\{\{\gamma, \gamma^{-1}\} \mid \gamma \in \pi_1^{\mathrm{orb}}(\mathcal{S}, \mathbb{M}), \gamma \neq \gamma^{-1}\}$,
2. $\{\gamma \in \pi_1^{\mathrm{orb}}(\mathcal{S}, \mathbb{M}) | \gamma = \gamma^{-1}, \gamma \neq 1_M, M \in \mathbb{M}\} \times \mathbb{Z}_2$,
3. $\left\{[\gamma] \in \pi_1^{\mathrm{orb,free}}(\mathcal{S}) |\ [\gamma] \neq [\gamma^{-1}]\right\} \times k^*/\sim$,
4. $\left\{[\gamma] \in \pi_1^{\mathrm{orb,free}}(\mathcal{S}) |\ \gamma^2 \neq 1 \textit{ and } [\gamma] = [\gamma^{-1}]\right\} \times k^* \backslash \{\pm 1\}/\sim$,
5. $\left\{[\gamma] \in \pi_1^{\mathrm{orb,free}}(\mathcal{S}) |\ \gamma^2 \neq 1 \textit{ and } [\gamma] = [\gamma^{-1}]\right\} \times (\mathbb{Z}_2)^2$,

where $\sim$ is the equivalence relation given by $([\gamma], \lambda) \sim ([\gamma^{-1}], \lambda)$.

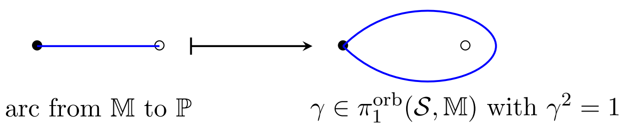
In this theorem, the set $\pi_1^{\mathrm{orb}}(\mathcal{S}, \mathbb{M})$ is the quotient of $\pi_1(\mathcal{S} \setminus \mathbb{P}, \mathbb{M})$ by the equivalence relation given by



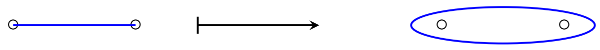
where P is a puncture. The set $\pi_1^{\mathrm{orb,free}}(\mathcal{S})$ is the set of conjugacy classes of the fundamental orbifold group $\pi_1^{\mathrm{orb}}(\mathcal{S})$.

The tagged arcs as described in [FST08] can be recovered from this description as follows:

- The set of arcs with both endpoints in $\mathbb{M}$ injects in the set 1. of Theorem1.9. Note that the condition $\gamma \neq \gamma^{-1}$ excludes the arcs cutting out a once punctured monogon (which are also excluded in [FST08]).
- The set of tagged arcs with one endpoint in $\mathbb{M}$ and one endpoint in $\mathbb{P}$ injects in the set 2. in Theorem 1.9. Indeed to each arc from $\mathbb{M}$ to $\mathbb{P}$, one can associate the closed curve from $\mathbb{M}$ to $\mathbb{M}$ surrounding the puncture as in the following picture. The taggings of the arc corresponds here to $\mathbb{Z}_2$.

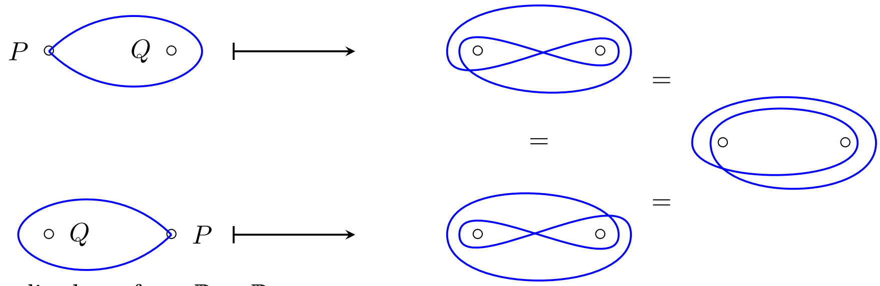


- The set of tagged arcs with both endpoints in $\mathbb{P}$ injects in the set 5. in Theorem 1.9. To each arc from $\mathbb{P}$ to $\mathbb{P}$, one can associate a closed curve surrounding the two punctures as follows:



  The four different taggings of such an arc correspond to $(\mathbb{Z}_2)^2$. Note that the condition $\gamma^2 \neq 1$ in the sets 4. and 5. excludes the curves that surround exactly one puncture.

  It is however dangerous to try to translate the sets 1., 2. and 5. in terms of generalized tagged arcs. The relation given by the orbifold fundamental group may have surprising identifications. For example, if we consider a "generalized" tagged arc that would cut out a once puncture Q monogon, with endpoint P in $\mathbb{P}$ (recall that these are not consider as tagged arcs in [FST08]), then using the previous map, it is sent to a closed curve surrounding twice the puncture P. But using the orbifold relation, it is the same as the generalized tagged arc from Q to Q surrounding P.



Further directions

A natural continuation of this work would concern morphisms spaces, and Auslander-Reiten quivers. The triangulated functors linking $\mathcal{C}_\Delta$ and $\mathcal{C}_{\tilde{\Delta}}$ behave nicely with respect to irreductible morphisms and Auslander-Reiten triangles. It should be then possible to try to describe dimension of morphisms spaces in terms of intersection numbers, and one should be able to deduce the shape of the Auslander-Reiten quiver of $\mathcal{C}_\Delta$ from the one of $\mathcal{C}_{\tilde{\Delta}}$.

2 Derived categories of surface cut algebras

In this section, we regard the cluster category of a triangulated surface as a cluster category associated with a τ_2-finite algebra Λ. The derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$, as a graded analogue of the cluster category $\mathcal{C}_2(\Lambda)$, inherits a topological model from the one of the cluster category. The idea is here to interpret the graded mutation (see Section 3.3 and Corollary 3.5) in term of the topological model.

2.1 Surface cut algebras

The aim of this section is to give an answer to the following question:

Given a triangulated surface $(\mathcal{S}, \mathbb{M}, \mathbb{P}, \Delta)$*, can we build a* τ_2*-finite algebra* Λ *such that* $\mathcal{C}_2(\Lambda)$ *is equivalent to* $\mathcal{C}_\Delta$ *?*

The construction of the QP associated to a global dimension ≤ 2 algebra Λ explained in Theorem 2.4 leads to the following definitions.

Definition 2.1. An *admissible cut* on (Q, W) is a map $d : Q_1 \to \{0, 1\}$ such that W is homogenous of degree 1 and such that any arrow of degree 1 belongs to a term of the potential.

Definition 2.2. Let Δ be a triangulation of a marked surface $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ such that any puncture has valency at most 3. Let d be an admissible cut of the QP (Q^Δ, W^Δ). The degree zero subalgebra $\Lambda(\Delta, d) := \big(\mathrm{Jac}(Q^\Delta, W^\Delta, d)\big)_0$of the graded Jacobian algebra $\mathrm{Jac}(Q^\Delta, W^\Delta, d)$ is called the *surface cut algebra* associated to Δ and d.

In case of an unpunctured surface, the situation is particularly easy since an admissible cut is the choice of one arrow in any triangle in the potential. It becomes easy to prove the following:

Proposition 2.3. *[AG16] Let $(\mathcal{S}, \mathbb{M})$ be a marked surface without punctures, and Δ a triangulation. Then for any admissible cut d, the surface cut algebra $\Lambda(\Delta, d)$ is τ_2-finite algebra. Moreover there is an equivalence of categories*

$$\mathcal{C}_2(\Lambda) \simeq \mathcal{C}_{(Q^\Delta, W^\Delta)}.$$

The situation is more complicated in the punctured case. First it is not clear that admissible cuts always exists. Indeed the potential involves not only internal triangles, but also oriented cycles around punctures. So one cannot choose randomly an arrow in each oriented triangle. It is possible to construct triangulations for which no admissible cut exists [ALP20, Prop 6.9]. Moreover, even if such a cut does exist, the degree zero subalgebra of the corresponding graded Jacobian algebra has not always global dimension ≤ 2. However we can prove the following:

Proposition 2.4. *[ALP20] Let $(\mathcal{S}, \mathbb{M}, \mathbb{P})$ be a marked surface with punctures. Then there exists a triangulation Δ and an admissible cut d such that the corresponding surface cut algebra $\Lambda(\Delta, d)$ is a τ_2-finite algebra. In that case there is an equivalence of categories*

$$\mathcal{C}_2(\Lambda) \simeq \mathcal{C}_{(Q^\Delta, W^\Delta)}.$$

2.2 Derived equivalence and graded mutation

The aim is now to apply Corollary 3.5 to surface cut algebras of global dimension ≤ 2, in order to get a topological criterion that determines when two surface cut algebras are derived equivalent. The first thing to observe is the fact that all cluster-tilting objects in the cluster category $\mathcal{C}_\Delta$ are related by sequences of mutation. Therefore we obtain the following

Corollary 2.5. *Let $\Lambda_1 := \Lambda(\Delta_1, d_1)$ and $\Lambda_2 := \Lambda(\Delta_2, d_2)$ be two surface cut algebras of global dimension ≤ 2. Then the algebras Λ_1 and Λ_2 are derived equivalent if and only if one can pass from $(Q^{\Delta_1}, W^{\Delta_1}, d_1)$ to $(Q^{\Delta_2}, W^{\Delta_2}, d_2)$ by a sequence of graded mutations.*

The next step is to try to find a good topological invariant of this graded mutation. By Corollary 1.4, we already know that the topological data of the marked surface is an invariant. All the missing information should then be given by the degree maps. Therefore, a good invariant should be given by d, but forgetting the triangulation.

It can be done as follows: Given a 1-homogenous degree map $d : Q_1^\Delta \to \mathbb{Z}$, one can construct a well-defined map $d : \pi_1^{\mathrm{free}}(\mathcal{S}) \to \mathbb{Z}$. Indeed, the quiver Q^Δ is a deformation retract of the surface $\mathcal{S}$. The degree of a loop γ counts algebraically the degree of each arrow along γ.

Then we prove the following result which can be considered as the graded version of Corollary 1.4.

Theorem 2.6. *[AG16, Thm 3.12][ALP20, Thm 5.3] Let $(\mathcal{S}_1, \mathbb{M}_1, \mathbb{P}_1, \Delta_1)$ and $(\mathcal{S}_2, \mathbb{M}_2, \mathbb{P}_2, \Delta_2)$ be two triangulated surfaces. Let $\Lambda_1 := \Lambda(\Delta_1, d_1)$ and $\Lambda_2 := \Lambda(\Delta_2, d_2)$ be two surface cut algebras of global dimension ≤ 2. Then the following are equivalent:*

1. *Λ_1 and Λ_2 are derived equivalent;*

2. *there exists a homeomorphism $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ preserving orientation, marked points, and punctures such that the maps $d_2 \circ \Phi = d_1$ as maps $\pi_1^{\text{free}}(\mathcal{S}_1) \to \mathbb{Z}$.*

One key ingredient in the proof of this result is the fact that the map $d : \pi_1^{\text{free}}(\mathcal{S}) \to \mathbb{Z}$ is invariant under graded mutation ([AG16, Lemma 2.14] and [ALP20, Lemma 3.13]). Moreover, if two different gradings given on the same triangulation induce the same map $\pi_1^{\text{free}}(\mathcal{S}) \to \mathbb{Z}$, one shows that they are equivalent as gradings, using a CW-complex associated to the surface. It follows from [AO14] that two equivalent gradings can be related by a sequence of graded flips.

The rest of the proof is a graded analogue of the proof of Corollary 1.4, so each step has a graded analogue that has to be checked. For most of them, it is just a technical verification.

For instance, one has to check the following "commutativity" for Δ any tagged triangulation up to right graded equivalence:

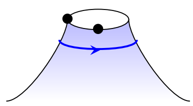
$$\begin{array}{ccc}
\underset{\text{graded tagged triangulation}}{(\Delta, d)} & \overset{\text{graded flip}}{\longleftrightarrow} & \underset{\text{graded tagged triangulation}}{(\Delta', d') = (\mathfrak{f}_i(\Delta), \mu_i^L(d))} \\
\downarrow & & \downarrow \\
(Q^{\Delta}, W^{\Delta}, d) & \overset{\text{graded DWZ-mutation}}{\longleftrightarrow} & (Q^{\Delta'}, W^{\Delta'}, d')
\end{array}$$

This step is easy to check in the case of an unpunctured surface, since the process of reduction is quite natural in this case. It is much more involved in the case of punctures, especially when the triangulations Δ or Δ' involve self-folded triangles (cf [ALP20, Thm 4.1]).

2.3 Geometric interpretation of the degree map

Link with the AG-invariant

One observation is the following: for each boundary component B_i of $\mathcal{S}$, denote by m_i the number of marked points on this component, and denote by c_i the closed curve surrounding the boundary B_i, with the boundary B_i on the left.



Then, any homeomorphism Φ as in theorem 2.6 should send a curve c_i of $\mathcal{S}_1$ to a curve c_j of $\mathcal{S}_2$ with $m_i = m_j$. As a consequence we obtain the following

Corollary 2.7. *The collection of pairs $(d(c_i), m_i)_i$ for a surface cut algebra of global dimension ≤ 2 is a derived invariant.*

In fact, this invariant was already well-known for the case where the surface $\mathcal{S}$ has no punctures. It is closely related with the AG-invariant introduced by Avella-Alaminos and Geiss in [AAG08] for gentle algebras. It is interesting to see that this invariant was introduced by a careful computation of some components of the Auslander-Reiten quiver. It can be understood using fractional CY properties of certain objects in the derived category. Note that in the case of annulus, the interpretation of the pair $(m, d(c))$ in terms of AR-quiver, and fractional CY-properties has been also described in [AO13b].

A natural question arising here is as follows:

Let Λ be a surface cut algebra associated to a surface with punctures. Is there an interpretation of the invariant $(d(c_i), m_i)$ in term of fractional Calabi-Yau properties of some objects in the derived category $\mathcal{D}^{\text{b}}(\Lambda)$?

H^1-affine space

One first observation, is the fact that the map $d : \pi_1^{\mathrm{free}}(\mathcal{S}) \to \mathbb{Z}$ is a $\mathrm{H}^1(\overline{\mathcal{S}\cup\mathbb{P}}, \mathbb{Z})$-affine space. More precisely we have the following

Lemma 2.8. *[AG16, Cor 2.8][ALP20, Cor 3.14] Let (Δ, d) and (Δ', d') be two graded valency ≥ 2-triangulations of $(\mathcal{S}, \mathbb{M}, \mathbb{P})$. Then the map $d - d' : \pi_1^{\mathrm{free}}(\mathcal{S}) \to \mathbb{Z}$ factors through a well-defined map*

$$[d - d'] : \mathrm{H}_1(\overline{\mathcal{S}\cup\mathbb{P}}, \mathbb{Z}) \to \mathbb{Z}.$$

The fact that it factors into a map $\mathrm{H}_1(\mathcal{S}, \mathbb{Z}) \to \mathbb{Z}$ comes from the fact that on any internal triangle, the sum of the degree of the arrows is constant equal to 1. The fact that, when there are punctures it factors through $\mathrm{H}_1(\overline{\mathcal{S}\cup\mathbb{P}}, \mathbb{Z}) \to \mathbb{Z}$ comes from the fact that any curve surrounding a puncture has degree exactly 1.

As a consequence, to apply Theorem 2.6, it is sufficient to check the equality between $d_2 \circ \Phi$ and d_1 on a set of closed curves which is a basis when seen in $\mathrm{H}_1(\overline{\mathcal{S}\cup\mathbb{P}}, \mathbb{Z})$.

As a consequence, we obtain that the AG-invariant is a complete derived invariant in the genus zero case.

Corollary 2.9. *Let $(\mathcal{S}_1, \mathbb{M}_1, \mathbb{P}_1, \Delta_1)$ and $(\mathcal{S}_2, \mathbb{M}_2, \mathbb{P}_2, \Delta_2)$ be two triangulated surfaces of genus 0. And let $\Lambda_1 := \Lambda(\Delta_1, d_1)$ and $\Lambda_2 := \Lambda(\Delta_2, d_2)$ be two surface cut algebras of global dimension ≤ 2. Then the following are equivalent:*

1. *Λ_1 and Λ_2 are derived equivalent;*
2. *they have the same AG-invariant.*

In particular, in the case where the surface is a disc, then the derived category is entirely determined by the number of marked points, and the number of punctures. This was already completely clear in the unpunctured case, but not so immediate in the case of surface cut algebras coming from the punctured disc.

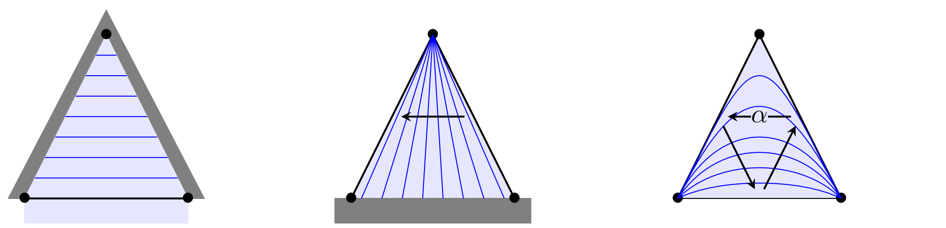
Degree map as a winding number

In fact, it is also possible to interpret these maps as winding numbers associated to certain line fields. Assume now that the surface $\mathcal{S}$ is smooth oriented surface with punctures, and boundary components. A *line field* on $\mathcal{S}$ is a section $\eta : \mathcal{S} \to \mathbb{P}(T\mathcal{S})$ of the projectivized tangent bundle $\mathbb{P}(T\mathcal{S}) \longrightarrow \mathcal{S}$. The set of homotopy classes of line fields $\mathrm{LF}(\mathcal{S})$ on $\mathcal{S}$ is known to also be a $\mathrm{H}^1(\mathcal{S}, \mathbb{Z})$-affine space (see [Chi72] for example). The map

$$\mathrm{LF}(\mathcal{S}) \times \mathrm{LF}(\mathcal{S}) \longrightarrow \mathrm{H}^1(\mathcal{S}, \mathbb{Z})$$

is given by $(\eta, \eta') \mapsto w_\eta - w_{\eta'}$ where w_η is the winding number map with respect to η. We refer to [APS, section 1] for more details.

To a triangulation (with smooth representative arcs) with an admissible cut, we can associate a line field on $\mathcal{S}$ whose corresponding foliation is as follows on each triangle of Δ:



two boundary sides   one boundary side   no boundary side and $d(\alpha) = 1$

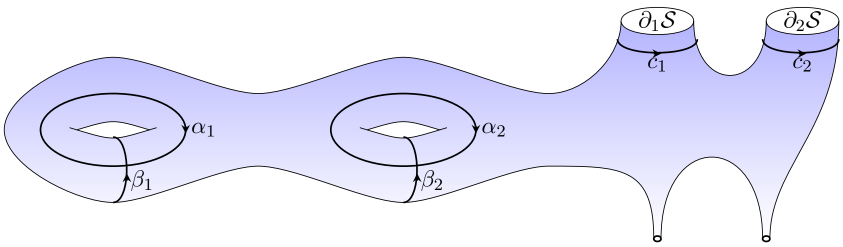
Then the winding number w coincides with the degree map d as a map $\pi_1^{\text{free}}(\mathcal{S}) \to \mathbb{Z}$. Theorem 2.6 can be reformulated as follows

Theorem 2.10. *[AG16, Thm 3.12][ALP20, Thm 5.3] Let $(\mathcal{S}_1, \mathbb{M}_1, \mathbb{P}_1, \Delta_1)$ and $(\mathcal{S}_2, \mathbb{M}_2, \mathbb{P}_2, \Delta_2)$ be two triangulated surfaces. Let $\Lambda_1 := \Lambda(\Delta_1, d_1)$ and $\Lambda_2 := \Lambda(\Delta_2, d_2)$ be two surface cut algebras of global dimension ≤ 2, and denote by η_1 and η_2 the corresponding line fields. Then the following are equivalent:*

1. *Λ_1 and Λ_2 are derived equivalent;*
2. *there exists a diffeomorphism $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ preserving orientation, marked points, and punctures such that the line fields η_1 and $\Phi_*(\eta_2)$ are homotopic.*

Furthermore a careful use of Theorem 3.4 in [LP] giving invariant of mapping class group orbits of homotopy classes of line field of a surface permits to deduce a complete derived invariant in terms of winding numbers of closed curves on the surface $\mathcal{S}$.

More precisely, for a marked surface $(\mathcal{S}, \mathbb{M}, \mathbb{P})$, we denote by b the number of boundary components, by g its genus, by $m(i)$ the number of marked points on the boundary component $\partial_i \mathcal{S}$, and p the number of punctures. Let $\mathcal{B} = \{c_1, \ldots, c_b\}$ be a set of simple closed curves such that for any j, c_j is homotopic to the boundary component $\partial_j \mathcal{S}$. Denote by $\mathcal{S}^{\text{comp}}$ the compactified surface obtained from $\mathcal{S}$ by adding closed discs on each boundary component, and filling the punctures. Let $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a set of closed simple curves whose image in $\mathrm{H}_1(\mathcal{S}^{\text{comp}})$ is a symplectic basis with respect to the intersection form.



Theorem 2.11. *([APS, Thm 8.5] for the unpunctured case) Let $(\mathcal{S}_1, \mathbb{M}_1, \mathbb{P}_1, \Delta_1)$ and $(\mathcal{S}_2, \mathbb{M}_2, \mathbb{P}_2, \Delta_2)$ be two triangulated surfaces. And let $\Lambda_1 := \Lambda(\Delta_1, d_1)$ and $\Lambda_2 := \Lambda(\Delta_2, d_2)$ be two surface cut algebras of global dimension ≤ 2.*

Let $\mathcal{B}_1$ and $\mathcal{G}_1$ (resp. $\mathcal{B}_2$ and $\mathcal{G}_2$) be sets of simple closed curves on $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) as above. Then Λ_1 and Λ_2 are derived equivalent if and only if the following numbers coincide

1. *$g_1 = g_2$, $b_1 = b_2$, $\sharp\mathbb{M}_1 = \sharp\mathbb{M}_2$, $p_1 = p_2$;*
2. *there exists a permutation $\sigma \in \mathfrak{S}_b$ such that for any $i = 1, \ldots, b_1$ we have $m_1(\sigma(i)) = m_2(i)$ and $d_1(c^1_{\sigma(i)}) = d_2(c^2_i)$;*
3. *and if the genus $g = g_1 = g_2$ is ≥ 1, one of the following holds:*
   (a) *for $g = 1$, we have*
   $$\gcd\{d_1(\gamma), d_1(c) + 2, \gamma \in \mathcal{G}_1, c \in \mathcal{B}_1\} = \gcd\{d_2(\gamma), d_2(c) + 2, \gamma \in \mathcal{G}_2, c \in \mathcal{B}_2\}$$
   (b) *for $g \geq 2$, one the following occurs*
      i. *there exist γ in $\mathcal{G}_1 \cup \mathcal{B}_1$ and γ' in $\mathcal{G}_2 \cup \mathcal{B}_2$ such that $d_1(\gamma)$ and $d_2(\gamma')$ are odd, or*
      ii. *for any γ in $\mathcal{G}_1 \cup \mathcal{B}_1$, for any γ' in $\mathcal{G}_2 \cup \mathcal{B}_2$, the numbers $d_1(\gamma)$ and $d_2(\gamma')$ are even and there exists $c \in \mathcal{B}_1$ with $d_1(c) = 0 \mod 4$, or*

iii. for any γ *in* $\mathcal{G}_1 \cup \mathcal{B}_1$ *and* $\gamma' \in \mathcal{G}_2 \cup \mathcal{B}_2$*, the numbers* $d_1(\gamma)$ *and* $d_2(\gamma')$ *are even, for any* $c \in \mathcal{B}_1$ *we have* $d_1(c) = 2 \mod 4$ *and*

$$\sum_{i=1}^{g} \frac{1}{2}(d_1(\alpha_i^1)+1)(d_2(\beta_i^1)+1) = \sum_{i=1}^{g} \frac{1}{2}(d_2(\alpha_i^2)+1)(d_2(\beta_i^2)+1) \mod 2.$$

Note that this result is stated in [APS] in the case of an unpunctured surface, which is completely devoted to gentle algebras. However it is still true for punctured surfaces, the proof is exactly the same. The number computed in *3. (b) iii* is the Arf invariant of some quadratic form on $\mathrm{H}_1(\mathcal{S}^{\mathrm{comp}}, \mathbb{Z}_2)$.

In the case of the unpunctured torus with one boundary component, the invariant of 3. (a) was already introduced in [Ami16]. I did not notice the link of the degree map with line fields and winding numbers at that time, so the method used to prove this invariant was down to earth, but self-contained.

2.4 Description of the category

The unpunctured case

Given a surface cut algebra Λ, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\Lambda)/\mathcal{U}_\Lambda[1] & \xrightarrow{\quad\sim\quad} & \mathrm{mod}^{\mathbb{Z}}\,\Pi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{forget}} \\ \mathcal{C}_2(\Lambda)/\,\mathrm{add}\,\pi(\Lambda)[1] & \xrightarrow{\quad\sim\quad} & \mathrm{mod}\,\Pi \end{array},$$

where the 3-preprojective algebra $\Pi = \Pi_3(\Lambda)$ is isomorphic to the graded Jacobian $\mathrm{Jac}(\Delta, d)$. Therefore indecomposable objects of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ coincide with graded modules over the graded Jacobian algebra $\mathrm{Jac}(\Delta, d)$ together with one copy of the projectives.

In the unpunctured case, the algebra $\mathrm{Jac}(\Delta)$ is gentle, and so is the covering of the graded algebra $\mathrm{Jac}(\Delta, d)$. Hence by [BR87], one has a description of the objects of the covering in terms of strings and bands. Copying what is done in [ABCP10] and [BZ11], we can deduce a description of the objects of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ in terms of curves on the surface. Under the forgetful functor, any string in $\mathrm{mod}^{\mathbb{Z}}\,\Pi$ gives a string in $\mathrm{mod}\,\Pi$, and any preimage of a string in $\mathrm{mod}\,\Pi$ gives a $\mathbb{Z}$-family of strings in $\mathrm{mod}^{\mathbb{Z}}\,\Pi$. For the bands, the situation is a bit different, since not all bands in $\mathrm{mod}\,\Pi$ are gradable. The only gradable bands are the one corresponding of a curve of degree zero. Finally we obtain the following classification:

Theorem 2.12. *Let* $(\mathcal{S}, \mathbb{M}, \Delta)$ *be a triangulated surface without punctures, and* $\Lambda = \Lambda(\Delta, d)$ *be a surface cut algebra. Then the isomorphism classes of indecomposable objects of* $\mathcal{D}^{\mathrm{b}}(\Lambda)$ *are in bijection with*

- $\{\{\gamma, \gamma^{-1}\}, \gamma \in \pi_1(\mathcal{S}, \mathbb{M}), \gamma \neq 1_M, M \in \mathbb{M}\} \times \mathbb{Z}$*;*
- $\{([\gamma], \lambda) \in \pi_1^{\mathrm{free}}(\mathcal{S}) \times k^* \text{ with } d(\gamma) = 0\}/\!\sim\; \times \mathbb{Z}$*, where the equivalence relation* $\sim$ *is given by* $([\gamma], \lambda) \sim ([\gamma^{-1}], \lambda^{-1})$*.*

Note that here the description of objects is very close to the one in [OPS] for derived categories of general gentle algebras (see Theorem 1.8 in Chapter 5). However the bijection is completely different. Indeed if X is the string object corresponding to $(\{\gamma, \gamma^{-1}\}, n)$, then the object corresponding to $(\{\gamma, \gamma^{-1}\}, n+1)$ is $\mathbb{S}_2(X)$, while in [OPS] it corresponds to $X[1]$. The bijection given in [OPS] is much more useful, since it is very explicit, it is easy to write the complex from the data of the graded curve.

From this classification of objects we can also deduce the following

Proposition 2.13. *Let Λ be a surface cut algebra. Then the functor $\pi : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{C}_2(\Lambda)$ is dense if and only if Λ is piecewise hereditary.*

Proof. From the description above, we deduce that the functor π is dense if and only if any closed curve on the surface has degree 0. It is clearly true for the case of the disc. For the annulus, it is true if and only if the generator of $\pi_1(\mathcal{S})$ has degree zero, which is exactly the case where Λ is derived equivalent to a hereditary algebra of type $\widetilde{\mathbb{A}}_n$. Finally if the surface is not a disc or an annulus, we can use the formula (cf [AG16, Prop 2.9])

$$\sum_i d(\partial_i \mathcal{S}) = 4g - 4 + 2b > 0$$

Hence there exists a boundary component such that the curve $\partial_i \mathcal{S}$ has non zero degree. This gives us immediately a band object in $\mathcal{C}_\Delta$ which is not in the image of π. $\square$

The annulus case

In order to get information on the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$, one can also use results in Section 3.4 in Chapter 3. The situation is particulary nice in the annulus case. Let Λ be a surface cut algebra associated to an annulus with m_1 and m_2 marked points on each boundary components. Denote by $w := d(c_1)$ the degree of c_1. Then using the results in [AO14] (see subsection 3.4 in Chapter 3), one can show that we have

$$\mathcal{D}^{\mathrm{b}}(\Lambda) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} H) / \mathbb{S}_2(1),$$

where H is given by the following graded quiver

When $w = 0$, the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is equivalent to $\mathcal{D}^{\mathrm{b}}(k\widetilde{A}_n)$, so is well-known. For $w \neq 0$, we obtain that the covering of the graded algebra (H, d) is of type $\mathbb{A}_\infty$. By a careful description of the autoequivalences (1) and $\mathbb{S}_2$ of the derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} H)$ we obtain the following

Proposition 2.14. *[AO13b, Cor 5.5] Let Λ a surface cut algebra associated to an annulus, with $w \neq 0$. Then the AR quiver of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ contains exactly $3|w|$ connected components, $|w|$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ and $2|w|$ of type $\mathbb{Z}\mathbb{A}_\infty$.*

2.5 Further directions

Description of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ in the punctured case

A natural question arising here is the generalization of Theorem 2.12 in the punctured case using the description of the indecomposable objects in the cluster category in Theorem 1.9. This is however not an easy corollary of these two results, in particular because the Jacobian algebra used in Theorem 1.9 comes from a triangulation Δ with self-folded triangles while a surface cut algebra as defined in [ALP20] come from a triangulation Δ' of valency ≥ 3. Given a curve on $\mathcal{S}$, the way to associate an object in $\mathcal{C}_2(\Lambda)$ is not direct at all. One should first lift the curve in $\widetilde{\mathcal{S}}$, then associate an object in $\mathcal{C}_{\widetilde{\Delta}}$, that we send to an object in $\mathcal{C}_\Delta$. One should then apply a sequence of flips/mutation to pass from Δ to Δ' in order to obtain an object in

$\operatorname{mod}\Pi_3(\Lambda)$. Therefore the first step to answer this question would be to find a "shortcut" for this procedure. Furthermore, one should also understand the gradable objects in $\operatorname{mod}\Pi_3(\Lambda)$, one could for instance use Proposition 2.2 and results in [AO13b].

Auslander-Reiten quiver of $\mathcal{D}^{\mathrm{b}}(\Lambda)$

Another natural question could be to try to generalize the strategy used for the annulus case to deduce properties for the derived category of a surface cut algebra for a surface more complicated than an annulus. Given a surface cut algebra Λ, if we can write an equivalence between $\mathcal{D}^{\mathrm{b}}(\Lambda)$ and the triangulated hull of an orbit category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} A)/\mathbb{S}_2(1)$ where A is an algebra whose derived category is well understood, then one could obtain information for the category $\mathcal{D}^{\mathrm{b}}(\Lambda)$.

Chapter 5

Derived categories of gentle and skew-gentle algebras

As we have seen in the previous chapter, the cluster category and the derived category of a surface cut algebra have topological interpretations. One can understand the objects in terms of (graded) curves, certain dimensions of spaces of morphisms in term of intersection numbers, etc... Moreover the combinatorics of flips and graded flips permit to relate these algebras with each other. In the case where the surface does not have any punctures, these algebras belong to a well-known and well-studied class called *gentle algebras*.

Using the same kind of ideas, Opper, Plamondon and Schroll described the derived category of a gentle algebra in geometric terms in [OPS] : they translate the combinatorial description given in [BM03] in topological terms. Thereby they obtain results very similar to the ones of the previous section. In particular the objects can be interpreted in terms of graded curves, and the morphisms spaces in terms of intersections of curves. It becomes so very natural to try to generalize Theorem 2.10 for gentle algebras.

In the first section of this chapter, we concentrate on the derived category of gentle algebras. We first recall results in [OPS]. We further explain how the degree can be interpreted as the data of a line field on the surface. Finally we expose the main result of the paper [APS] which is completely similar to Theorem 2.10 of Chapter 4. An interesting point is the fact that the proof is completely different: it does not use a concept analogue to 'flip' or 'mutation'.

In a second part, we expose the results in [AB]. This combines the ideas of [AP] together with [APS] : we develop a topological model for the derived category of skew-gentle algebra seen as skew-group algebras of a gentle algebra. This permits us to obtain a topological interpretation of different kind of derived equivalence between skew-gentle algebras.

1 Gentle algebras

Definition 1.1. A *gentle pair* is a pair (Q, I) given by a quiver Q and a subset I of paths of length 2 in Q such that

- for each $i \in Q_0$, there are at most two arrows with source i, and at most two arrows with target i;
- for each arrow $\alpha : i \to j$ in Q_1, there exists at most one arrow β with target i such that $\beta\alpha \in I$ and at most one arrow β' with target i such that $\beta'\alpha \notin I$;
- for each arrow $\alpha : i \to j$ in Q_1, there exists at most one arrow β with source j such that $\alpha\beta \in I$ and at most one arrow β' with source j such that $\alpha\beta' \notin I$.
- the algebra $A(Q, I) := kQ/I$ is finite dimensional.

An algebra is *gentle* if it admits a presentation $A = kQ/I$ where (Q, I) is a gentle pair.

1.1 Topological model

Definition

A *marked surface* $(\mathcal{S}, M_\bullet, P_\bullet)$ is the data of

- an orientable closed smooth surface $\mathcal{S}$ with non-empty boundary, that is a compact closed smooth surface from which some open discs are removed;
- a finite set of marked points $M_\bullet$ on the boundary, such that there is at least one marked point on each boundary component;
- a finite set $P_\bullet$ of marked points in the interior of $\mathcal{S}$.

The points in $M_\bullet$ and $P_\bullet$ are called marked points. A curve on the boundary of $\mathcal{S}$ intersecting marked points only on its endpoints is called a *boundary segment*.

An $\bullet$-*arc* on $(\mathcal{S}, M_\bullet, P_\bullet)$ is a curve $\gamma : [0,1] \to \mathcal{S}$ such that $\gamma_{|(0,1)}$ is injective and $\gamma(0)$ and $\gamma(1)$ are marked points[1]. Each arc is considered up to isotopy (fixing endpoints).

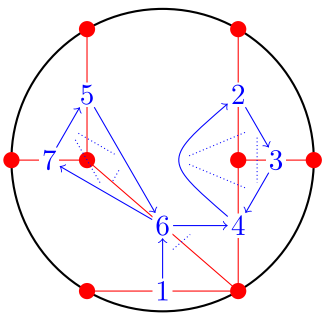
Definition 1.2. A $\bullet$-*dissection* is a collection $D = \{\gamma_1, \dots, \gamma_s\}$ of $\bullet$-arcs cutting $\mathcal{S}$ into polygons with exactly one side being a boundary segment.

Two dissected surfaces $(\mathcal{S}, M_\bullet, P_\bullet, D)$ and $(\mathcal{S}', M'_\bullet, P'_\bullet, D')$ are called diffeomorphic if there exists an orientation preserving diffeomorphism $\Phi : \mathcal{S} \to \mathcal{S}'$ such that $\Phi(M_\bullet) = M'_\bullet$, $\Phi(P_\bullet) = P'_\bullet$, and $\Phi(D) = D'$.

Following [OPS], one can associate to the dissection D a quiver Q, together with a subset of quadratic monomial relations I, such that the algebra $A(D) := A(Q, I)$ is a gentle algebra.

- The vertices of Q are in bijection with $\{i\ \bullet\text{-}arc\}$
- Given i and j $\bullet$-arcs in D, there is one arrow $i \xrightarrow{\alpha} j$ in Q whenever the arcs i and j have a common endpoint $\bullet$ and when i is immediately followed by the arc j in the counterclockwise order around $\bullet$;
- If i, j, and k have a common endpoint, and are consecutive arcs following the counterclockwise order around $\bullet$, then we have $\beta\alpha \in I$, where α (resp. β) is the arrow corresponding to to the angle $j \to i$ (resp. $k \to j$).

  Example 1.3. Below is an example of a dissected surface together with the associated quiver Q and relations I (marked with dots).



Proposition 1.4. *[OPS, BC] The assignment $D \mapsto A(D)$ induces a bijection*

$$\left\{ \begin{array}{c} (\mathcal{S}, M, P, D) \\ \text{dissected surface} \end{array} \right\}_{/\text{ diffeo}} \longleftrightarrow \left\{ \begin{array}{c} A(Q, I) \\ \text{gentle algebra} \end{array} \right\}_{/\text{iso}}$$

This bijection was also described in [BC], where the authors use the geometrical model in order to get a description of indecomposable objects in the module category.

[1] Note that here we allow arcs homotopic to a boundary segment

Link with the model of surface cut algebras

From this proposition it is not very difficult to give a precise link with the topological model of surface cut algebras developped in the previous chapter.

Proposition 1.5. *Let Λ be a gentle algebra. Then Λ is a surface cut algebra associated with an unpunctured surface if and only if Λ has global dimension ≤ 2 and is τ_2-finite.*

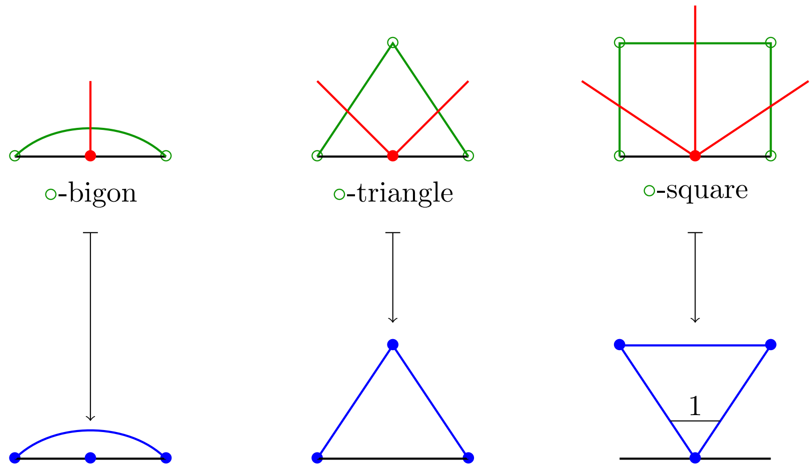
Proof. One direction is proved in Proposition 2.4. Let Λ be a τ_2-finite gentle algebra with global dimension ≤ 2. Denote by $(\mathcal{S}, \mathbb{M}_{\bullet}, \mathbb{P}_{\bullet}, D)$ the corresponding dissection. We first construct the dual dissection: we fix a finite set $\mathbb{M}_{\circ}$ of green points on the boundary of $\mathbb{S}$ such that each boundary segment contains exactly one green point. Then, we define a collection of arcs with endpoints in $\mathbb{M}_{\circ}$ such that each $\circ$-arc intersects exactly one $\bullet$-arc and vice versa. This dual dissection is uniquely defined up to isotopy of $\circ$-arcs fixing the endpoints. This dual dissection cuts out the surface into polygons which either have exactly one side being a boundary $\circ$-segment, or have exactly one $\bullet$ in the interior.

Since Λ has global dimension ≤ 2, then there is no subpath of length 3 in Q

$$i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} \ell$$

with $\beta\alpha$ and $\gamma\beta$ in I. In terms of the dissected surface, this is equivalent to the fact that the $\circ$-dissection cuts out the surface into n-gons with exactly one side being a boundary segment, with $n = 2, 3$ or 4. From this observation, we can do the following construction:

- for each $\circ$-bigon, we keep the two $\circ$-points as marked points, and add a marked point in between on the boundary segment, and keep the $\circ$-arc;
- for each $\circ$-triangle, we keep the three $\circ$-point as marked points, and the corresponding two $\circ$-arcs;
- and for each $\circ$-square, we identify the boundary segment into a marked point, and we declare the corresponding angle of degree 1.



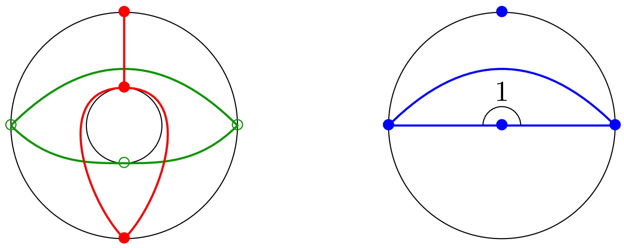
We obtain this way a triangulated surface together with a degree map of degree 1 on each internal triangle. The question is now to check that the obtained marked surface has non-empty boundary, which is not completely clear since the process may identify boundary segment into a point.

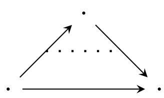
If for each boundary component, there is at least one $\circ$-boundary segment which belongs to a $\circ$-bigon, or to a $\circ$-triangle, then each marked point of the new surface lies on the boundary, so we obtain a unpunctured triangulated surface with an admissible cut. So the algebra Λ is a surface cut algebra of an unpunctured surface.

Now assume that there is one boundary component such that each boundary $\circ$-segment belongs to a $\circ$-quadrilateral. Then one boundary component becomes a puncture in the previous construction. We want to show that the corresponding algebra is not τ_2-finite. First note that to obtain the QP associated to $\Pi_3(\Lambda)$, one associates an arrow for each angle of degree 1 for each internal triangle, and the potential is given by the sum of 3-cycles associated to each internal triangle. (It is therefore different from the Labardini-Fragoso potential.) However, the puncture created by the previous construction yields a cycle in the quiver of strictly positive degree, which is not zero, and whose powers never vanish. Hence, the algebra $\Pi_3(\Lambda)$ is infinite dimensional.

□

Example 1.6. Let Λ be the gentle algebra associated with the following dissected surface:

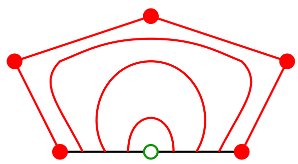


Then Λ is given by the following quiver  The preprojective algebra $\Pi_3(\Lambda)$ has a 2-cycle of degree 1 whose powers never vanish. Therefore Λ is not τ_2-finite.

Gradings and line fields

We fix a finite set of green points $M_\circ$ on the boundary of $\mathcal{S}$ such that each boundary segment contains exactly one point in $M_\circ$.

We define a line field η_D on $\mathcal{S} \setminus (\partial \mathcal{S} \cup P_\bullet)$, that is, a section of the projectivized tangent bundle $\mathbb{P}(T\mathcal{S}) \to \mathcal{S}$. The line field is tangent along each arc of D and is defined up to homotopy in each polygon cut out by D, by the following foliation:



Definition 1.7. Let $\gamma : (0,1) \to \mathcal{S}$ be non contractible smooth curve. Assume that γ does not contain any contractible loops, that γ intersects transversally the dissection D, and that γ does not intersect an arc twice in succession. A *grading* on γ is a map $\mathbf{n} : \gamma(0,1) \cap D \to \mathbb{Z}$ satisfying:

$$\mathbf{n}(\gamma(t_{i+1})) = \mathbf{n}(\gamma(t_i)) + w_\eta(\gamma_{|_{[t_i,t_{i+1}]}}),$$

if $\gamma(t_i)$ and $\gamma(t_{i+1})$ are two consecutive intersections of γ with D. More concretely, on $[t_i, t_{i+1}]$, the curve γ intersects one polygon cut out by D, and we have

$$\mathbf{n}(\gamma(t_{i+1})) = \mathbf{n}(\gamma(t_i)) + 1$$

if the boundary segment of the polygon is on the left of the curve $\gamma_{|_{[t_i,t_{i+1}]}}$, and

$$\mathbf{n}(\gamma(t_{i+1})) = \mathbf{n}(\gamma(t_i)) + 1$$

if the boundary segment lies on the right.

If $(\gamma, \mathbf{n})$ and $(\gamma', \mathbf{n}')$ are two graded curves, such that γ is regular homotopic to γ', and such that $\mathbf{n}(\gamma(t_1)) = \mathbf{n}'(\gamma'(t'_1))$, then their grading coincide, in the sense that for any i we have

$$\gamma(t_i), \gamma(t'_i) \text{ lie on the same arc of } D \text{ and } \mathbf{n}(\gamma(t_i)) = \mathbf{n}'(\gamma'(t'_i)).$$

A graded $\circ$-curve is a pair $(\gamma, \mathbf{n})$ where γ is a non contractible curve with endpoints in $M_\circ$, considered up to homotopy fixing endpoints, and where $\mathbf{n}$ is a grading on it. Denote by $\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$ the set of graded $\circ$-curves.

Since we have

$$\begin{aligned}
\mathbf{n}(\gamma^{-1}(1-t_i)) &= \mathbf{n}(\gamma(t_i)) \\
&= \mathbf{n}(\gamma(t_{i+1})) - w_\eta(\gamma_{|[t_i,t_{i+1}]}) \\
&= \mathbf{n}(\gamma^{-1}(1-t_{i+1)})) + w_\eta(\gamma^{-1}_{|[1-t_{i+1},1-t_i]})
\end{aligned}$$

the equivalence relation $\gamma \sim \gamma^{-1}$ on $\pi_1(\mathcal{S}, M_\circ)$ induces an equivalence relation $(\gamma, \mathbf{n}) \sim (\gamma^{-1}, \mathbf{n})$ on $\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$.

Let $\gamma : [0,1] \to \mathcal{S}$ be a non contractible closed curve on $\mathcal{S}$ that intersects transversally the dissection D. One easily sees that it admits a grading if and only if its winding number with respect to the line field η is 0. Denote by $\pi_1^{\mathrm{gr,free}}(\mathcal{S})$ the set of non contractible graded closed curves, up to free homotopy.

One of the main result in [OPS] is the following

Theorem 1.8. *[OPS] Let Λ be a gentle algebra and $(\mathcal{S}, M_\bullet, P_\bullet, D)$ the associated dissected surface. Then there is a bijection between indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ and the following sets*

1. *$\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ) \sim$ where $(\gamma, \mathbf{n}) \sim (\gamma^{-1}, \mathbf{n})$;*
2. *$\pi_1^{\mathrm{gr,free}}(\mathcal{S}) \times k^*/\sim$, where $\sim$ is defined as $(\gamma, \mathbf{n}, \lambda) \sim (\gamma^{-1}, \mathbf{n}, \lambda^{-1})$.*

Since a grading of a curve is entirely determined by the choice of the first number $\mathbf{n}(\gamma(t_0))$, one recovers a result really similar to Theorem 2.12. However, here the description is much more explicit: if the graded curve $(\gamma, \mathbf{n})$ intersects the arc i, with corresponding degree equal to q, then the corresponding complex of projectives A-modules $P_{(\gamma,\mathbf{n})}$ has the projective P_i in homological position q. Moreover, changing the degree by one is equivalent to shifting the complex in the derived category.

For example, to each arc i of the $\bullet$-dissection, there exists a unique (up to isotopy) $\circ$-arc γ_i that intersects exactly once the arc i. Then the object $P_{(\gamma_i,0)}$ is the stalk complex P_i concentrated in degree 0. Hence the object corresponding to the dual dissection with degree 0, is the stalk complex concentrated in degree 0.

Note also that in case where Λ has infinite global dimension, there are other objects in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ which are not in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$. These can also be interpreted in terms of graded curves (see [OPS]).

1.2 Derived invariant

Geometric interpretation of derived equivalence

With this geometric model we manage to describe explicitely all tilting objects in the derived category of a gentle algebra.

More precisely we prove the following

Theorem 1.9. *[APS] Let $(\mathcal{S}, M_\bullet, P_\bullet, D)$ be a dissected surface and $A = A(D)$ be the corresponding gentle algebra.*

1. *If T is a basic tilting object in $\mathcal{D}^b(A)$, then there exists a collection of graded arcs $\{(\gamma_i, \mathbf{n}_i), i \in I\}$ such that*

   (a) $T \simeq \bigoplus_{i\in I} P_{(\gamma_i,\mathbf{n}_i)}$;

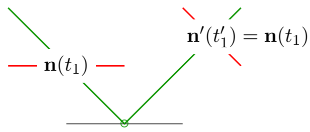
   (b) $\{\gamma_i, i \in I\}$ *is the dual of a* $\bullet$*-dissection denoted by* D_T;

   (c) *we have an isomorphism of algebras* $\mathrm{End}_{\mathcal{D}^b(A)}(T) \simeq A(D_T)$;

   (d) *for any* $\delta \in \pi_1(\mathcal{S})$, *we have* $w_D(\delta) = w_{D_T}(\delta)$.

2. *Let $\{\gamma_i, i \in I\}$ be the dual of a $\bullet$-dissection D'. If for any $\delta \in \pi_1(\mathcal{S})$ we have $w_D(\delta) = w_{D'}(\delta)$, then there exists a grading $\mathbf{n}_i$ for any $i \in I$ such that $\bigoplus_{i\in I} P_{(\gamma_i,\mathbf{n}_i)}$ is a tilting object in $\mathcal{D}^b(A)$.*

Here point (d) comes from the fact that if γ and γ' are two $\circ$-arcs that intersect on the boundary, then there is no extension between the objects $P_{(\gamma,\mathbf{n})}$ and $P_{(\gamma',\mathbf{n}')}$ if and only if their degree coincide on the first $\bullet$-arc intersected on the boundary where they meet.



Therefore here we only use the Ext-vanishing property for tilting object. We do not use any analogue of mutation (like silting mutation for instance).

This permits to prove a result which is completely similar to Theorem 2.10 in Chapter 4. Note that this result has been proved independently by Opper in [Opp].

Corollary 1.10. *[APS, Thm 4.1][Opp, Thm B] Let A and A' be gentle algebras associated respectively to dissected surfaces $(\mathcal{S}, M_\bullet, P_\bullet, D)$ and $(\mathcal{S}', M'_\bullet, P'_\bullet, D')$. Denote by η (resp. η') the line field defined from the dissection D (resp. D'). Then the following are equivalent:*

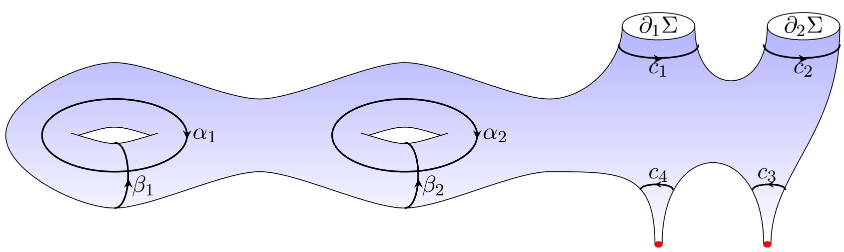
1. *the algebras A and A' are derived equivalent;*

2. *there exists a diffeomorphism $\Phi : \mathcal{S} \to \mathcal{S}'$ preserving orientation and marked points such that $\Phi^*(\eta)$ and η' are homotopic.*

Numerical derived invariant

We can then again use the numerical description of the mapping class group orbit of homotopy classes of line fields to deduce a complete numerical derived invariant.

For a surface $\mathcal{S}$ of genus g with b boundary components and p punctures, denote by $\mathcal{B} = \{c_1, \dots, c_{b+p}\}$ a set of simple closed curves such that for $j = 1, \dots b$, the curve c_j is homotopic to the boundary component $\partial_j\mathcal{S}$ (being on the left of the curve), and so that c_{b+k} is homotopic to a circle around the k-th puncture for $k = 1, \dots, p$. Let denote $\overline{\mathcal{S}}$ the closed surface with empty boundary obtained by adding closed discs to each bounday component. Let $\mathcal{G} = \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$be a set of closed simple curves, such that their image in $H_1(\overline{\mathcal{S}}, \mathbb{Z})$ is a symplectic basis (with respect to the intersection form).



Theorem 1.11. *[APS, Thm 6.4]*

Let A and A' be two gentle algebras with associated dissected surfaces $(S, M_\bullet, P_\bullet, D)$ and $(S', M_\bullet', P_\bullet', D')$ respectively. Let $\mathcal{G} = \{\alpha_1, \ldots, \beta_g\}$, $\mathcal{B} = \{c_1, \ldots, c_{b+p}\}$ (resp. $\mathcal{G}' = \{\alpha'_1, \ldots, \beta'_{g'}\}$, $\mathcal{B}' = \{c'_1, \ldots, c'_{b'+p'}\}$) subsets of simple closed curves on $S \setminus P_\bullet$ (resp. $S' \setminus P'_\bullet$) as before. Then the algebras A and A' are derived equivalent if and only if the following numbers coincide:

1. $g = g'$, $b = b'$, $\sharp M_\bullet = \sharp M'_\bullet$, $\sharp P_\bullet = \sharp P'_\bullet$;
2. *there exists a permutation $\sigma \in \mathfrak{S}_{b+p}$ such that the number of marked points on $\partial_{\sigma i}\mathcal{S}$ and $\partial_i\mathcal{S}'$ are the same and such that $w_\eta(c_{\sigma(j)}) = w_{\eta'}(c'_j)$, for any $j = 1, \ldots, b$;*
3. *for $g = g' \geq 1$ one of the following holds*
   - (a) *for $g = g' = 1$, we have*

$$\gcd\{w_\eta(\gamma), w_\eta(c) + 2, \gamma \in \mathcal{G}, c \in \mathcal{B}\} = \gcd\{w_{\eta'}(\gamma'), w_{\eta'}(c') + 2, \gamma' \in \mathcal{G}', c' \in \mathcal{B}'\}$$

   - (b) *for $g = g' \geq 2$ one the following occurs:*
     - i. *there exist $\gamma \in \mathcal{G} \cup \mathcal{B}$ and $\gamma' \in \mathcal{G}' \cup \mathcal{B}'$ such that $w_\eta(\gamma)$ and $w_{\eta'}(\gamma')$ are odd, or*
     - ii. *for any $\gamma \in \mathcal{G} \cup \mathcal{B}$ and $\gamma' \in \mathcal{G}' \cup \mathcal{B}'$, the numbers $w_\eta(\gamma)$ and $w_{\eta'}(\gamma')$ are even and there exists an i with $w_{\eta'}(c_i) = 0 \mod 4$, or*
     - iii. *for any $\gamma \in \mathcal{G} \cup \mathcal{B}$ and $\gamma' \in \mathcal{G}' \cup \mathcal{B}'$, the numbers $w_\eta(\gamma)$ and $w_{\eta'}(\gamma')$ are even and, for any $i = 1, \ldots, b + p$ we have $w_\eta(c_i) = 2 \mod 4$ and*

$$\sum_{i=1}^{g}(\frac{1}{2}w_\eta(\alpha_i) + 1)(\frac{1}{2}w_\eta(\beta_i) + 1) = \sum_{i=1}^{g}(\frac{1}{2}w_{\eta'}(\alpha'_i) + 1)(\frac{1}{2}w_{\eta'}(\beta'_i) + 1) \mod 2$$

1.3 Further direction

Recollements

The aim is here to interpret certain recollements of derived categories of gentle algebras in a topological way. This is a work in progress with Pierre-Guy Plamondon.

Let $(\mathcal{S}, M_\bullet)$ be a marked surface equipped with a $\bullet$-dissection D, and A be the corresponding gentle algebra. To a sub-collection of arcs δ of D, on can associate

- a sub-collection δ^* of the $\circ$-dissection dual to D ;
- an idempotent e of the algebra A.

The idea is here to try to interpret geometrically the following recollement

$$\mathcal{D}^{\mathrm{b}}(A)/\mathrm{thick(eA)} \leftrightarrows \mathcal{D}^{\mathrm{b}}(A) \leftrightarrows \mathcal{D}^{\mathrm{b}}(eAe)$$

First, using results due to Kalck and Yang [KY16, Theorem 1.3], one can re-interpret it as follows

$$\mathcal{D}^{\mathrm{b}}(\widetilde{A}/\widetilde{A}e\widetilde{A}) \leftrightarrows \mathcal{D}^{\mathrm{b}}(A) \leftrightarrows \mathcal{D}^{\mathrm{b}}(eAe)$$

where $\widetilde{A}$ is a cofibrant replacement of the algebra A, that is a path algebra over a DG quiver negatively graded whose H^0 is isomorphic to A. The aim would be to show that the category $\mathcal{D}^{\mathrm{b}}(eAe)$ is obtained from the surface $\mathcal{S}$ by contracting all the arcs of $D \setminus \delta$, and to show that the category $\mathcal{D}^{\mathrm{b}}(\widetilde{A}/\widetilde{A}e\widetilde{A})$ is obtained from $\mathcal{S}$ by contracting all the arcs in δ^* (getting then a graded gentle algebra seen as a DG algebra with zero differential).

The idea would be to interpret these classical recollements in representation theory as "recollements" of surfaces.

Graded gentle algebras

This project is a work in progress with T. Brüstle, P.G. Plamondon and S. Schroll.

To a marked surface $(\mathcal{S}, M_{\bullet})$ (called a surface with stops in [HKK]) equipped with a line field, one can associate a A_∞-category whose objects are given by graded arcs. This category is the partially wrapped Fukaya category. A collection of graded arcs $\mathcal{A}$ forming a dissection give then rise to a subcategory of this A_∞-category, on which it is possible to compute higher multiplication using Floer homology. This is the strategy used by Haiden, Kontsevich and Katzarkov [HKK] to show that such a collection gives rise to a formal object (that is an object whose endomorphism DG algebra admits a minimal model without higher multiplication) which also is generator. One thus obtains equivalences between the triangulated categories

$$\mathcal{D}(\mathrm{End}^*(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}) \simeq \mathrm{H}^0(\mathrm{Tw}(\mathcal{A})) \simeq \mathrm{H}^0(\mathrm{Tw}Fuk(S, M_{\bullet}, \eta)).$$

Since the object $\mathcal{A}$ is formal, the algebra $\mathrm{End}^*(\mathcal{A})$ is quasi-isomorphic to a DG algebra with zero differential. It is moreover a gentle algebra.

The aim here is to attack the converse. Given an object T in the derived category of a gentle algebra (graded or not) that is a formal generator, is T given by a collection of graded arcs corresponding to a dissection ? We know the answer when the algebra is ungraded, and when the formal generator is a tilting object. In this situation, tilting theory provides a very powerful and efficient tool to compute derived equivalences, whereas the general case one needs to use the very technical A_∞-machinery. We have started to try to generalize this result for any formal generator first in the case where the gentle algebra is ungraded. Indeed in this case, we have a complete description of the indecomposable objects in the derived category. (This description probably generalizes in the graded case, but for the moment the argument is not completely understood.) One needs then to show that if T is not given by a dissection, then it is not a formal generator. Showing non generation is in general not really difficult. But showing non formality for an A_∞-algebra B is in general a very challenging task. Indeed, one needs to show that any minimal model on $H^*(B)$ does not admit any higher multiplication, and already computing one minimal model may be really technical. One strategy in our partial results uses higher Massey product in the cohomology of B, and seems rather efficient.

Silting mutation

Now that we have a description of silting objects in the derived category of a gentle algebra, we can try to understand silting mutation in terms of dissections and line fields. This has been done in [CS] using the description of morphisms between indecomposable in the derived category. Now a natural question is the following

Is the silting graph connected for gentle algebras ?

Moreover, silting mutation has been described combinatorially in terms of graded quiver by Oppermann in [Opp17]. This could be useful to determine that the DG-endomorphism algebra of a silting object in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is formal, that is is isomorphic to its graded homology ring.

Questions about the numerical invariant

One natural question concerning Theorem 1.11 is the following: *given a gentle quiver* (Q, I) *can we compute algorithmically its numerical invariant ?* The topological data of the surface are easy to computed from (Q, I). Moreover, any closed curve on the surface can be seen as a certain walk in the quiver, and then computing its winding number is also easy. The walks associated to the curves c_i are easy to describe (and where already describe in [AAG08] through the AG-invariant). But an algorithm computing the walks associated to curves in $\mathcal{G}$ is much more involved to compute. This would be very useful to have an algorithm to get these numbers, it could be then possible to add it in the programm QPA for instance, or to the applet dedicated to gentle algebras [G]. This would be of great help for the community of representation of quivers. The description of such an algorithm is a project with Francis Lazarus. Indeed similar results have been described in the article [LPVV01].

Another question about these numerical invariants is about their algebraic interpretation. As mentionned before the AG-invariant has an interpretation in terms of a fractional CY properties of certain indecomposable objects. The other numerical invariants computed in Theorem 1.11 3. should also have an interpretation in term of the category. But the answer is here quite mysterious.

One way to handle this question could be the study of the orbit category $\mathcal{D}^{\mathrm{b}}(\Lambda)/[p]$ for any positive integer p and its triangulated hull. One could expect that, in analogy with the cluster category setup, the "band objects" of the triangulated hull of $\mathcal{D}^{\mathrm{b}}(\Lambda)/[1]$ would be in bijection with $\pi_1^{\mathrm{free}}(\mathcal{S})\times k^*/\sim$. The gcd appearing in 3. (a) could be the smallest integer p such that the functor

$$(\mathcal{D}^{\mathrm{b}}(\Lambda)/[p])_{\Delta}\diagup_{[1]} \longrightarrow (\mathcal{D}^{\mathrm{b}}(\Lambda)/[1])_{\Delta}$$

is dense. The interpretation of the Arf invariant appearing in 3.(b) (iii), is for now completely mysterious.

2 Skew-gentle algebras

In the paper [AB] we provide a topological model for the derived category of skew-gentle algebras over a base field of characteristic $\neq 2$. The idea is to use the structure of skew-group algebra of the skew-gentle algebras and to use Opper-Plamondon-Schroll model together with a $\mathbb{Z}_2$-action very similar to the one described in [AP] (see section 1.3 of Chapter 4). This permits us to find a geometric interpretation of derived equivalence between skew-gentle algebras.

Definition 2.1. A *skew-gentle triple* (Q, I, Sp) is the data of a quiver Q, a subset I of paths of length two in Q, and a subset Sp of loops in Q (called '*special loops*) such that $(Q, I \amalg \{e^2, e\in \mathrm{Sp})$ is a gentle pair. In this case, the algebra $\bar{A}(Q,I,\mathrm{Sp}) := kQ/\langle I\amalg\{e^2-e, e\in\mathrm{Sp}\rangle$, is called a *skew-gentle algebra.*

Every gentle algebra is skew-gentle, and the topological model for skew-gentle algebra is a generalisation of the topological model for gentle algebras.

2.1 Skew-gentle algebras and orbifolds

Definition 2.2. A $\mathbb{Z}_2$-dissected surface $(\mathcal{S}, M_{\color{red}\bullet}, P_{\color{red}\bullet}, D, \sigma)$ is the data of:

- a dissected surface $(\mathcal{S}, M_{\color{red}\bullet}, P_{\color{red}\bullet}, D)$ as in Definition 1.2.
- an orientation preserving diffeomorphism $\sigma : \mathcal{S}\to\mathcal{S}$ of order 2 that preserves globally marked points and the dissection D and such that σ has finitely many fixed points which are all in $\mathcal{S}\setminus P_{\color{red}\bullet}$.

Following Proposition 1.4, one can easily deduce from a $\mathbb{Z}_2$-dissected surface, a gentle algebra together with an action of $\mathbb{Z}_2$ by automorphisms.

Moreover, from a $\mathbb{Z}_2$-dissected surface, one can do the quotient $\bar{\mathcal{S}}=\mathcal{S}/\sigma$ which has a structure of orbifold, where orbifold points $X_{\color{red}\times}$ correspond to fixed points of σ. The image $\bar{D} := \pi(D)$ of D under the projection $\pi : \mathcal{S}\to\mathcal{S}/\sigma$ is a ${\color{red}\bullet}$-dissection of $(\bar{\mathcal{S}}\setminus X, \bar{M}_{\color{red}\bullet}, \bar{P}_{\color{red}\bullet}\cup X_{\color{red}\times})$, where each ${\color{red}\times}$ in X is the endpoint of exactly one arc. We call such a dissection a ${\color{red}\times}$-*dissection.*

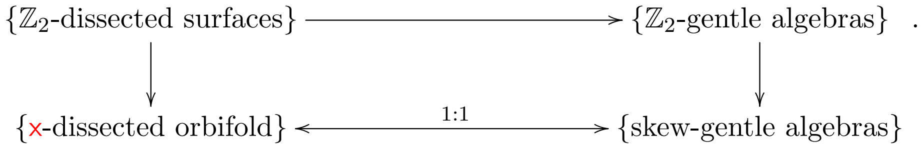
We show the following in [AB].

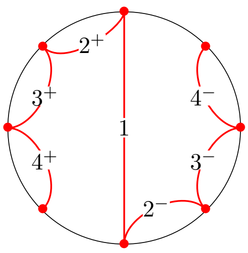
Proposition 2.3. *Let $(\mathcal{S}, M_{\color{red}\bullet}, P_{\color{red}\bullet}, D, \sigma)$ be a $\mathbb{Z}_2$-dissected surface, and A the corresponding gentle algebra.*

1. *the skew-group algebra $A\mathbb{Z}_2$ is Morita equivalent to a skew-gentle algebra $\bar{A}$;*
2. *every skew-gentle algebra arises in this way;*
3. *the assignment $(\mathcal{S}, M_\bullet, P_\bullet, D, \sigma) \mapsto \bar{A}$ is not injective. However, it induces a bijection (except for a few exceptional cases)*

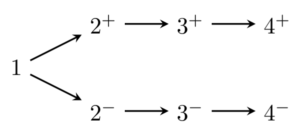
$$(\mathcal{S}, M_\bullet, P_\bullet, D, \sigma) \longrightarrow \left\{(\bar{\mathcal{S}} \setminus X, \bar{M}_\bullet, \bar{P}_\bullet, X_\times, \bar{D}),\ \times\text{-dissected orbifold}\right\}.$$

This can be summarized into the following diagram :

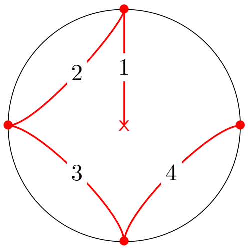




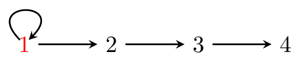
$\mathbb{Z}_2$-dissected surface



$\mathbb{Z}_2$-gentle algebra



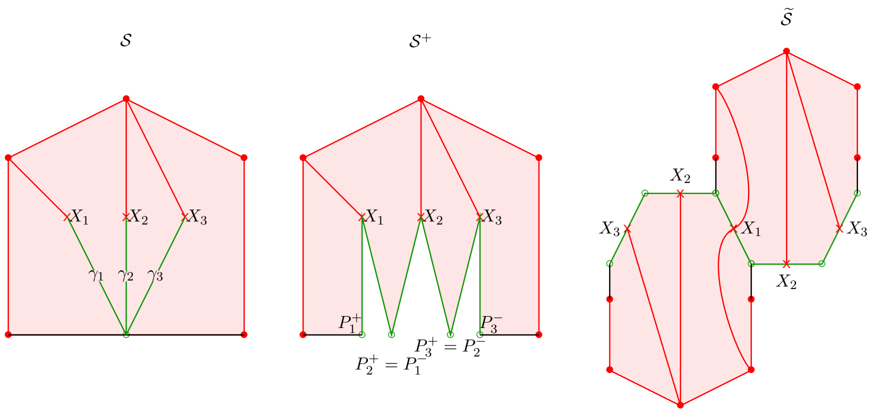
×-dissected orbifold



skew-gentle algebra

(the special vertex is in red)

A way to see point (2) in Proposition 2.3 is to associate a ×-dissected orbifold to $\bar{A}$, and then to "unfold" it in a very similar way as in [AP]. This process can be summarized in the following picture



This procedure provides to any ×-dissected surface a "prefered" $\mathbb{Z}_2$-dissected surface. This prefered $\mathbb{Z}_2$-dissection has also an algebraic meaning : there is a natural $\mathbb{Z}_2$-action on any skew-gentle algebra $\bar{A}$ which consists of the exchange of the idempotents e and $1-e$ for any special loop e. With this $\mathbb{Z}_2$-action the algebra $\bar{A}\mathbb{Z}_2$ is Morita equivalent to a gentle algebra, whose $\mathbb{Z}_2$-dissected surface is the "prefered" one obtained by unfolding the orbifold.

2.2 Indecomposable objects

Let $\bar{A}$ be a skew-gentle algebra and A be its corresponding $\mathbb{Z}_2$-gentle algebra (the prefered one described above). The action of $\mathbb{Z}_2$ on A induces an action of $\mathbb{Z}_2$ on the derived category $\mathcal{D}^{\mathrm{b}}(A)$. Following again Reiten and Riedtmann [RR85], we get functors

$$\mathcal{D}^{\mathrm{b}}(A) \to \mathcal{D}^{\mathrm{b}}(\bar{A}) \quad \text{and} \quad \mathcal{D}^{\mathrm{b}}(\bar{A}) \to \mathcal{D}^{\mathrm{b}}(A),$$

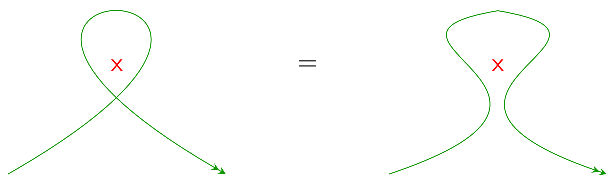
that induce a bijection between the isomorphism classes of indecomposable objects in $\mathcal{D}^{\mathrm{b}}(\bar{A})$ and the set

$$\{\sigma\text{-invariant indec. in } \mathcal{D}^{\mathrm{b}}(A)\} \times \mathbb{Z}_2 \cup \{\sigma\text{-orbits of non } \sigma\text{-invariant indec. in } \mathcal{D}^{\mathrm{b}}(A)\},$$

exactly as in Theorem 1.8 of Chapter 4 for cluster categories.

The objects in the derived category of $\mathcal{D}^{\mathrm{b}}(A)$ are described using a line field η_D defined from the dissection D as in 1.1. This line field η_D is clearly σ-invariant, thus it induces a line field $\bar{\eta}_{\bar{D}}$ on the surface $\bar{\mathcal{S}} \setminus X_{\times}$.

We define the *graded groupoid*[2] $\pi_1^{\mathrm{gr}}(\bar{\mathcal{S}} \setminus X_{\times}, \bar{M}_{\circ})$ as the set of graded curves $(\gamma, \mathbf{n})$ with endpoints in $\bar{M}_{\circ}$ up to regular homotopy, and where the grading is defined with respect to $\bar{D}$ and $\bar{\eta}_{\bar{D}}$. We define then $\pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_{\circ})$ as the quotient of the graded groupoid $\pi_1^{\mathrm{gr}}(\bar{\mathcal{S}} \setminus X_{\times}, \bar{M}_{\circ})$ by the relation



We refer to the appendix of this memoir for a precise definition. We also denote by $\pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}})$ the set of non contractible gradable closed loops up to free homotopy (see Appendix for precise definitions). Note that as in the previous section, if $([\gamma], \mathbf{n})$ is in $\pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}})$, then the winding number $w_{\bar{\eta}}(\gamma)$ vanishes.

Then one can show the following.

Theorem 2.4. *Let $\bar{A}$ be a skew-gentle algebra, and let $(\bar{\mathcal{S}}, \bar{M}_{\bullet}, \bar{P}_{\bullet}, X_{\times}, \bar{\eta})$ the corresponding graded orbifold surface. Then the indecomposable objects of $\mathcal{D}^{\mathrm{b}}(\bar{A})$ are in bijection with the following sets:*

1. $\left\{(\gamma, \mathbf{n}) \in \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_{\circ}) \mid \gamma^2 \neq 0\right\} / \sim$ *where* $(\gamma, \mathbf{n}) \sim (\gamma^{-1}, \mathbf{n})$;
2. $\left\{(\gamma, \mathbf{n}, \epsilon) \in \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_{\circ}) \times \{\pm 1\} \mid \gamma = \gamma^{-1}\right\}$;
3. $\left\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}}) \times k^* \mid [\gamma] \neq [\gamma^{-1}]\right\} / \sim$ *where* $([\gamma], \mathbf{n}, \lambda) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1})$;
4. $\left\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}}) \times k^* \setminus \{\pm 1\} \mid [\gamma] = [\gamma^{-1}], \gamma^2 \neq 0\right\} / \sim$;
5. $\left\{([\gamma], \mathbf{n}, \epsilon, \epsilon') \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}}) \times \{\pm 1\}^2 \mid [\gamma] = [\gamma^{-1}], \gamma^2 \neq 0\right\}$.

This result is proved in the appendix of this memoir. The method is very similar to the one used in [AP].

Note that a similar result has also been shown in [LSV]. Their proof is however completely different : it is based on the combinatorial description of the indecomposable objects of the derived category of a skew-gentle algebra in [BMM03], via [CB89].

[2]Note that it is not clear that the graded groupoid has a structure of groupoid (at least we don't prove it here), so the name is maybe not very well-chosen.

2.3 Derived equivalence

General results on skew-group algebras

We are interested in finding derived invariants for the skew-gentle algebras in topological terms. In order to do that, we need to investigate more generally the derived equivalences between algebras with a G-action, where G is a finite abelian group whose cardinal is invertible in k, and the derived equivalence between the corresponding skew-group algebras.

We define the notion of G-invariant object, and noticing that the endomorphism algebra of a G-invariant object has a natural G-action, we set the following

Definition 2.5. Let A and A' be algebras with G-action. Then A and A' are G*-derived equivalent* if and only if there exists a G-invariant tilting object $T \in \mathcal{D}^{\mathrm{b}}(A)$ together with an isomorphism $\mathrm{End}_{\mathcal{D}^{\mathrm{b}}(A)}(T) \simeq A'$ commuting with the action of G.

Given a G-algebra A, one can define a $\widehat{G}$-action on the algebra AG by $\chi(a \otimes g) = \chi(g)a \otimes g$, where $\widehat{G}$ is the group $\mathrm{Hom}(G, k^*)$.

Theorem 2.6. *[AB, Theorems 2.10 and 2.13]*

1. *Let A be a G-algebra. Then there is a bijection*

$$\{G\text{-tilting subcategories of } \mathcal{D}^b(A)\} \leftrightarrow \{\widehat{G}\text{-tilting subcategories of } \mathcal{D}^b(AG)\}.$$

2. *If moreover A' is a G-algebra, such that A and A' are G-derived equivalent, then AG and $A'G$ are $\widehat{G}$-derived equivalent.*

Applying this to the setup of skew-gentle algebras, we manage to prove the following

Theorem 2.7. *[AB, Theorem 5.6] Let k be a field of characteristic $\neq 2$. Let $\bar{A}$ and $\bar{A}'$ be two skew-gentle algebras together with their natural $\mathbb{Z}_2$-action. Denote by $(\mathcal{S}, M_{\bullet}, P_{\bullet}, \sigma, D)$ and $(\mathcal{S}', M'_{\bullet}, P'_{\bullet}, \sigma', D')$ their corresponding $\mathbb{Z}_2$-dissected surfaces, and A and A' be the corresponding $\mathbb{Z}_2$-gentle algebras. Then the following are equivalent*

1. *$\bar{A}$ and $\bar{A}'$ are $\mathbb{Z}_2$-derived equivalent;*

2. *A and A' are $\mathbb{Z}_2$-derived equivalent;*

3. *there exists an orientation preserving diffeomorphism $\Phi : \mathcal{S} \to \mathcal{S}'$ sending marked points to marked points, such that $\Phi \circ \sigma = \sigma' \circ \Phi$, and such that the line fields η_D and $\Phi^*(\eta_{D'})$ are homotopic.*

We then investigate the case where the skew-group algebras are derived equivalent, the derived equivalence given by a $\mathbb{Z}_2$-invariant tilting object T, but such that the isomorphism $\mathrm{End}_{\mathcal{D}^{\mathrm{b}}(\bar{A})}(T) \simeq \bar{A}'$ does not necessarily commutes with the action of $\mathbb{Z}_2$. In that case, we can still use 1. of Theorem 2.6 and interpret the $\mathbb{Z}_2$-tilting objects in the derived category of a G-gentle algebra, as the dual of a σ-invariant dissection. We obtain the following

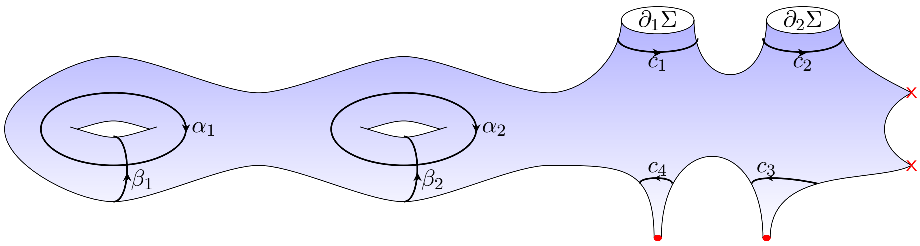
Theorem 2.8. *[AB, Thm 5.9] Let $\bar{A}$ and $\bar{A}'$ be skew-gentle algebras with their natural $\mathbb{Z}_2$-action. Denote by $(\bar{\mathcal{S}}, \bar{M}_{\bullet}, \bar{P}_{\bullet}, X_{\times}, \bar{D})$ and $(\bar{\mathcal{S}}', \bar{M}'_{\bullet}, \bar{P}'_{\bullet}, X'_{\times}, \bar{D}')$ their associated $\times$-dissected orbifolds. Then the following are equivalent:*

1. *the algebras $\bar{A}$ and $\bar{A}'$ are derived equivalent via a $\mathbb{Z}_2$-tilting object;*

2. *there exists an orientation preserving diffeomorphism $\bar{\Phi} : \bar{\mathcal{S}} \to \bar{\mathcal{S}}'$ sending marked points to marked points, and orbifold points to orbifold points such that the line fields $\bar{\Phi}^*(\eta_{\bar{D}'})$ and $\eta_{\bar{D}}$ are homotopic.*

Once again, one can use the numerical description of the mapping class group orbits of homotopy classes of line fields to deduce a numerical derived invariant.

For an orbifold $(\bar{\mathcal{S}}, X_{\times})$ with $x = |X_{\times}|$ orbifold points, such that $\bar{S} \setminus X$ is a surface of genus g, with b boundary components, $p + x$ punctures, we denote by $\mathcal{B} = \{c_1, \ldots, c_{b+p}\}$ and $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ sets of simple closed curves defined as before.



Corollary 2.9. *The derived equivalence via a $\mathbb{Z}_2$-tilting object class of a skew-gentle algebra $\bar{A}$ associated to a graded orbifold $(\bar{\mathcal{S}}, \bar{M}_{\bullet}, \bar{P}_{\bullet}, X_{\times}, \bar{\eta})$ is given by the numbers:*

- *g, b, p, x;*
- *$(w_{\bar{\eta}}(c_i), m_i)$, $i = 1, \ldots, b$;*
- *$w_{\bar{\eta}}(c_j)$; $j = b + 1, \ldots, b + p$;*
- *$w_{\bar{\eta}}(\gamma)$, $\gamma \in \mathcal{G}$.*

Note that here the winding number of the line field of a curve surrounding an orbifold point is always 1, so it is not needed in the theorem.

2.4 Further directions

These two results are both not completely satisfactory and lead to open related questions.

Numerical invariant

Using once again the numerical characterization of mapping class group orbits of homotopy classes of line fields, one can deduce from Theorem 2.7 a numerical $\mathbb{Z}_2$-derived invariant of skew-gentle algebras. But it is not clear that this numerical invariant is complete. Indeed, if two $\mathbb{Z}_2$-gentle algebras have the same numerical invariants, then we can deduce the following:

- the line fields η and η' are $\mathbb{Z}_2$-invariant (this is by construction)
- the surfaces $\mathcal{S}$ and $\mathcal{S}'$ are diffeomorphic and the diffeomorphism commutes with the action of σ and σ';
- there exists a diffeomorphism $\Phi : \mathcal{S} \to \mathcal{S}'$ such that $\Phi^*(\eta')$ is homotopic to η.

But it is not clear that this Φ commutes with σ and σ'.

The problem is in general difficult and related with some difficult question in topology such as Birman-Hilden's property. For example, it is not clear when two $\mathbb{Z}_2$-invariant line fields are homotopic, that one can find a homotopy preserving the $\mathbb{Z}_2$-invariance property.

Tilting objects

A clear desadvantage of Theorem 2.8 is this condition "given by a $\mathbb{Z}_2$-invariant tilting object". This condition is closely related to the fact that the class of skew-gentle algebras is not closed under derived equivalence. We need this hypothesis to apply Theorem 2.6 (1). But using the geometric model for skew-gentle algebras, one should be able to describe all tilting objects in terms of "generalized orbifold dissections". Then a natural problem would be to describe the endomorphism algebra of such a tilting object from the combinatorial data of the dissected surface. This would give a class of algebras closed under derived equivalence containing skew-gentle algebras. One would expect to find the Jacobian algebras associated to triangulated surfaces with punctures as defined in [LF09], and associated surface cut algebras. One would also expect an analogue of Theorem 2.11 and Corollary 2.9 to be true (note that in these two results, the same numbers are involved).

A_∞-structure and graded case

One also could want to enhance this geometric model describing the derived category of skew-gentle algebra in a A_∞-structure. Is there a notion of $\mathbb{Z}_2$-A_∞-structure, and do Reiten and Riedtmann's results carry over to this setting ? Can we describe higher multiplications for graded curves in this setting ? Is there a $\mathbb{Z}_2$-Fukaya category associated to a $\mathbb{Z}_2$-graded surface, or a Fukaya category associated with an orbifold ? More generally is there a notion of G-Fukaya category associated with an orbifold surface $\mathcal{S}/G$ where G is a finite sub-group of homeomorphisms of $\mathcal{S}$?

2.5 Geometric models for other triangulated categories

Other triangulated categories appearing in this memoir arise with a natural topological data. It would be of interest to try to interpret algebraic information of the category in terms of the topological data.

Geometric model for the stable category of the trivial extension of a gentle algebra

For a general gentle algebra (of finite global dimension), one can construct its trivial extension $\mathbf{T}(\Lambda) := \Lambda \oplus D\Lambda$ as in Section 1.2 in Chapter 2. By Happel's result, there is a triangle equivalence

$$\mathcal{D}^{\mathrm{b}}(\Lambda) \simeq \underline{\mathrm{mod}}^{\mathbb{Z}}\mathbf{T}\Lambda.$$

Using Keller's results on orbit categories in [Kel05], one can show that this equivalence induces an equivalence between the stable category $\underline{\mathrm{mod}}\mathbf{T}\Lambda$ and the triangulated hull of the category $\mathcal{D}^{\mathrm{b}}(\Lambda)/\mathbb{S}[1]$. Using this observation, the idea would be to try to obtain a geometric model for the category $\underline{\mathrm{mod}}\mathbf{T}\Lambda$. The auto-equivalences are indeed well understood geometrically, so at least one should be able to provide a geometric model for the orbit category $\mathcal{D}^{\mathrm{b}}(\Lambda)/\mathbb{S}[1]$. As such, the triangulated category $\underline{\mathrm{mod}}\mathbf{T}\Lambda$ plays a role analogue to the cluster category $\mathcal{C}_2(\Lambda)$. Moreover, this trivial extension is a Brauer graph algebra (see [Sch18]), and their representation theory is already well-known.

One could also try to check analogue results as the ones obtained in [AO14] (see subsection 3.4 in Chapter 3). For example, if Λ_1 and Λ_2 are gentle algebras such that we have an equivalence of triangulated categories

$$\underline{\mathrm{mod}}\mathbf{T}\Lambda_1 \simeq \underline{\mathrm{mod}}\mathbf{T}\Lambda_2$$

can we construct a grading on Λ_1 and Λ_2 so that we have an equivalence

$$\mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\,\Lambda_1) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\,\Lambda_2)?$$

Note that this question also makes sense for algebras that are not gentle.

Geometric model for categories associated with dimer models

Another family of triangulated categories associated with topological data was described in subections 5.3 in Chapter 1. To a consistent dimer model with a perfect matching, one can associate a graded Jacobian algebra Π and its degree zero subalgebra Λ. If there exists an idempotent e such that $\Pi/\Pi e \Pi$ is finite dimensional, and such that $e\Lambda(1-e)$ vanishes, then we obtain equivalences

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}(\underline{\Lambda}) & \xrightarrow{\sim} & \underline{\mathrm{CM}}^{\mathbb{Z}}(e\Pi e) \\ \downarrow & & \downarrow \\ \mathcal{C}_2(\underline{\Lambda}) & \xrightarrow{\sim} & \underline{\mathrm{CM}}(e\Pi e) \end{array} .$$

One can then wonder whether certain indecomposable objects of the category $\underline{\mathrm{CM}}(e\Pi e)$ could be interpreted in terms of curves on the torus. Moreover, would it be possible to construct a line field on the torus and understand some objects of the category $\underline{\mathrm{CM}^{\mathbb{Z}}}(e\Pi e) \simeq \mathcal{D}^{\mathrm{b}}(\underline{\Lambda})$ in terms of graded curves ?

Chapter 6

Appendix: Proof of Theorem 2.4 (page 81 in Chapter 5)

1 Indecomposable in term of graded curves on $\mathcal{S}$

Let $\bar{A}$ be a skew-gentle algebra attached to the dissected surface $(\bar{\mathcal{S}}, \bar{M}_\bullet, \bar{P}_\bullet, X_\times, \bar{D})$, and A be the corresponding gentle algebra attached to the $\mathbb{Z}_2$-dissected surface $(\mathcal{S}, M_\bullet, P_\bullet, D, \sigma)$ (see Section 2 of Chapter 5 for definitions).

Since we have a bijection between the isomorphism classes of indecomposable objects in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\bar{A})$ with the set

$$\{\sigma\text{-invariant indec. in } \mathcal{K}^{\mathrm{b}}(\mathrm{proj}A)\} \times \mathbb{Z}_2 \cup \{\sigma\text{-orbits of non } \sigma\text{-invariant indec. in } \mathcal{K}^{\mathrm{b}}(\mathrm{proj}A)\},$$

the first thing to understand is the action of σ on the indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}A)$.

We first fix a piece of notation. For $(\gamma, \mathbf{n}) \in \pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$, we denote by $P_{(\gamma,\mathbf{n})}$ the string object in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}A)$ through the bijection given in Theorem 1.8 in Chapter 5. For $([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^*$ we denote by $B_{([\gamma],\mathbf{n},\lambda)}$ the corresponding band object in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}A)$.

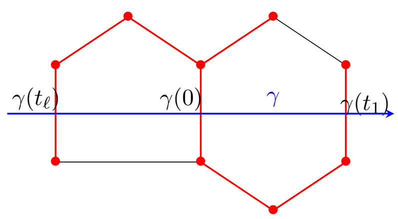
Lemma 1.1. *1. For $(\gamma, \mathbf{n}) \in \pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$, we have $(P_{(\gamma,\mathbf{n})})^\sigma \simeq P_{(\sigma\circ\gamma, \mathbf{n}\circ\sigma)}$.*

2. For $([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^*$, we have $(B_{([\gamma],\mathbf{n},\lambda)})^\sigma \simeq B_{([\sigma\circ\gamma],\mathbf{n}\circ\sigma,\lambda)}$.

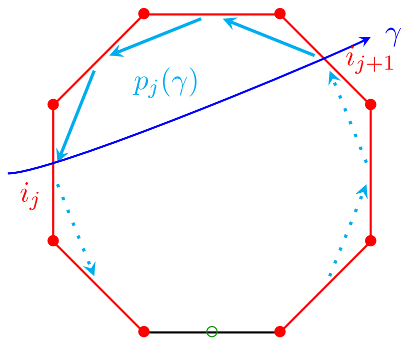
Proof. The first statement is proved in Lemma 5.4 in [AB].

For the second statement, let us define explicitely the bijection B. Let $([\gamma], \mathbf{n}, \lambda)$ be in $([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^*$. Assume first that γ is primitive. Let $\gamma : [0,1] \to \mathcal{S}$ be a regular representant of $[\gamma]$ intersecting transversally the arcs of D. Denote by $0 \le t_0 < \cdots < t_\ell < 1$ the elements in $[0,1]$ such that $\gamma(t_j)$ belongs to D, and denote by i_j the arc of D containing $\gamma(t_j)$.

Since $[\gamma]$ is defined up to free homotopy and since $w_\eta(\gamma) = 0$, we can assume that $t_0 = 0$, that $w_\eta(\gamma_{(0,t_1)}) = -1$ and that $w_\eta(\gamma_{(t_\ell,1)}) = +1$.



For $j = 0 \ldots, \ell$, one can associate a path $p_j(\gamma)$ of the quiver Q as in the following picture (where indices are taken modulo ℓ).



As a graded A-module, $B_{([\gamma],\mathbf{n},\lambda)}$ is defined to be

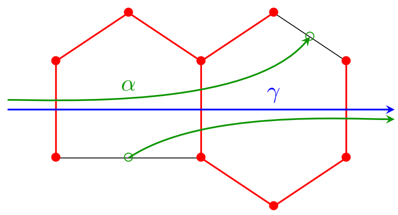
$$B_{([\gamma],\mathbf{n},\lambda)} := \bigoplus_{j=0}^{\ell} e_{i_j} A[\mathbf{n}(\gamma(t_j)].$$

The differential is given by the following $(\ell+1)\times(\ell+1)$ matrix $(d_{(j,k)})_{j,k}$

- if $w_\eta(\gamma_{|(t_j,t_{j+1})}) = +1$, then $d_{(j+1,j)} = p_j(\gamma)[\mathbf{n}(\gamma(t_j))]$
- if $w_\eta(\gamma_{|(t_j,t_{j+1})}) = -1$, then $d_{(j,j+1)} = p_j(\gamma)[\mathbf{n}(\gamma(t_{j+1}))]$
- $d_{(0,\ell)} = \lambda p_\ell(\gamma)[\mathbf{n}(\gamma(t_\ell))]$,
- all other values of $d_{(j,k)}$ are 0.

Note that in case $\ell = 1$, then we obtain $d_{(0,1)} = p_0[\mathbf{n}(\gamma(t_0))] + \lambda p_1[\mathbf{n}(\gamma(t_1))]$.

With the hypothesis on γ, we define an element $\alpha \in \pi_1(\mathcal{S}, M_\circ)$ as in the following picture,



and define a grading on it such that $\mathbf{n}((\alpha(t_j))) = \mathbf{n}((\gamma(t_j)))$ for $j = 1,\ldots,\ell$. Then one immediately sees that the map $(1,\lambda) : e_{i_0}A^2 \to e_{i_0}A$ induces a triangle

$$P_{(\alpha,\mathbf{n})} \longrightarrow B_{([\gamma],\mathbf{n},\lambda)} \longrightarrow e_{i_0}A[\mathbf{n}(\gamma(0))] \longrightarrow P_{(\alpha,\mathbf{n})}[1] \,.$$

Then we obtain statement 2. for primitive curves using statement 1..

For any curve, the proof is similar, since the band object can be seen as an iterated extension of a band object associated with a primitive curve.

□

We have $P_{(\gamma,\mathbf{n})} \simeq P_{(\gamma',\mathbf{n}')}$ if and only if $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$ and $\mathbf{n} = \mathbf{n}'$. We have $B_{([\gamma],\mathbf{n},\lambda)} \simeq B_{([\gamma'],\mathbf{n}',\lambda')}$ if and only if $([\gamma],\mathbf{n},\lambda) = ([\gamma'],\mathbf{n}',\lambda')$ or $([\gamma],\mathbf{n},\lambda) = ([\gamma'^{-1}],\mathbf{n}',\lambda^{-1})$. Therefore the indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\bar{A})$ are in bijection with the following sets:

1. $\{(\gamma,\mathbf{n}) \in \pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ) \mid \gamma^{-1} \neq \sigma\gamma\} / \sim$
   where $(\gamma,\mathbf{n}) \sim (\sigma\gamma, \mathbf{n}\circ\sigma) \sim (\gamma^{-1},\mathbf{n})$,

2. $\{(\gamma, \mathbf{n}, \epsilon) \in \pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ) \times \{\pm 1\}, \mid \sigma\gamma = \gamma^{-1}\} / \sim$
   where $(\gamma, \mathbf{n}, \epsilon) \sim (\gamma^{-1}, \mathbf{n}, \epsilon.$

3. $\Big\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^* \mid [\sigma\gamma] \neq [\gamma], [\gamma^{-1}]\Big\} / \sim$
   where $([\gamma], \mathbf{n}, \lambda) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1}) \sim ([\sigma\gamma], \mathbf{n} \circ \sigma, \lambda)$;

3'. $\Big\{([\gamma], \mathbf{n}, \lambda, \epsilon) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^* \times \{\pm 1\} \mid [\sigma\gamma] = [\gamma]\Big\} / \sim$
   where $([\gamma], \mathbf{n}, \lambda, \epsilon) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1}, \epsilon)$;

4. $\Big\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times k^* \setminus \{\pm 1\} \mid [\sigma\gamma] = [\gamma^{-1}]\Big\} / \sim$
   where $([\gamma], \mathbf{n}, \lambda) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1}) \sim ([\sigma\gamma], \mathbf{n} \circ \sigma, \lambda)$;

5. $\Big\{([\gamma], \mathbf{n}, \lambda, \epsilon) \in \pi_1^{\mathrm{free,gr}}(\mathcal{S}) \times \{\pm 1\} \times \{\pm 1\} \mid [\sigma\gamma] = [\gamma^{-1}]\Big\} / \sim$
   where $([\gamma], \mathbf{n}, \lambda, \epsilon) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1}, \epsilon)$

2 Indecomposables in term of graded curves on the orbifold

Now recall from [AP, Section 5] that there is a groupoid map

$$\Phi : \pi_1(\mathcal{S}, M_\circ) \to \pi_1^{\mathrm{orb}}(\bar{\mathcal{S}}, \bar{M}_\circ)$$

and a well defined map

$$\Psi : \pi_1^{\mathrm{free}}(\mathcal{S}) \to \pi_1^{\mathrm{orb,free}}(\bar{\mathcal{S}}).$$

The aim is to use these maps to translate the bijection above in term of graded curves on the orbifold $\bar{\mathcal{S}}$.

2.1 String objects

First, note that since $\bar{\eta}$ is the image of the line field η through the projection $p : \mathcal{S} \setminus X \to \bar{\mathcal{S}} \setminus X$, there is a natural map

$$\pi_1^{\mathrm{gr}}(\mathcal{S} \setminus X, M_\circ) \to \pi_1^{\mathrm{gr}}(\bar{\mathcal{S}} \setminus X, \bar{M}_\circ).$$

The first step consists of the definition of the set $\pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ)$ together with a map

$$\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ) \to \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ).$$

Definition 2.1. Let γ be in $\mathcal{C}^1((0,1), \bar{\mathcal{S}} \setminus X)$ such that its preimages $\widetilde{\gamma}$ and $\sigma\widetilde{\gamma}$ in $\mathcal{C}^1((0,1), \mathcal{S} \setminus X)$ do not contain any contractible loops and intersect transversally the dissection D.

Then, one defines a grading on γ as a map $\mathbf{n} : \gamma(0,1) \cap \bar{D} \to \mathbb{Z}$ such that

$$\mathbf{n}(\gamma(t_{i+1})) = \mathbf{n}(\gamma(t_i)) + w_{\bar{\eta}}(\gamma_{|_{[t_i, t_{i+1}]}}),$$

if $\gamma(t_i)$ and $\gamma(t_{i+1})$ are two consecutive intersections of γ with $\bar{D}$.

Since the map $\Phi : \pi_1(\mathcal{S}, M_\circ) \to \pi_1^{\mathrm{orb}}(\bar{\mathcal{S}}, \bar{M}_\circ)$ is surjective, any element in $\pi_1^{\mathrm{orb}}(\bar{\mathcal{S}}, \bar{M}_\circ)$ has a representant that can be gradable.

We would like now to check that the grading is well-defined on the set $\pi_1^{\mathrm{orb}}(\bar{\mathcal{S}}, \bar{M}_\circ)$. This comes from the following two facts:

1. If $(\widetilde{\gamma}, \widetilde{\mathbf{n}})$ is a graded curve in $\mathcal{S}$, and $(\gamma, \mathbf{n})$ is a graded curve in $\bar{\mathcal{S}}$ such that $\Phi(\widetilde{\gamma}) = \gamma$ and $\mathbf{n}(\gamma(t_1)) = \widetilde{\mathbf{n}}(\widetilde{\gamma}(t_1))$, then for any i $\mathbf{n}(\gamma(t_i)) = \widetilde{\mathbf{n}}(\widetilde{\gamma}(t_i))$. This comes from the fact that $\bar{\eta}$ is the projection of η.

2. If $(\gamma, \mathbf{n})$ and $(\gamma', \mathbf{n}')$ be two graded curves on $\bar{\mathcal{S}}$ that have the same grading at their first intersection point with D, then they admit the same grading on any intersection point with D. Indeed their preimages starting at the same point are homotopic, so they admit the same grading in $\mathcal{S}$.

We denote by $\pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ)$, the set of graded curves up to homotopy, which is now well defined. It comes then with a natural surjective map

$$\pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ) \to \pi_1^{\mathrm{orb}}(\bar{\mathcal{S}}, \bar{M}_\circ)$$

whose fiber is in bijection with $\mathbb{Z}$.

Therefore the sets 1. and 2. described above are respectively in bijection with

1. $\Big\{(\gamma, \mathbf{n}) \in \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ) \mid \gamma^2 \neq 1\Big\} / \sim$ where $(\gamma, \mathbf{n}) \sim (\gamma^{-1}, \mathbf{n})$;

2. $\Big\{(\gamma, \mathbf{n}, \epsilon) \in \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ) \times \{\pm 1\} \mid \gamma^2 = 1\Big\}$

2.2 Band objects

Here again, we first define the notion of gradable closed curves on the orbifold $\bar{\mathcal{S}}$.

Let $[\gamma] \in \pi_1^{\mathrm{orb,free}}(\bar{\mathcal{S}})$ represented by a smooth curve γ without contractible loops and intersecting transversally $\bar{D}$. Denote by $x_0 = \gamma(0)$ its starting point, and by x_0^+, and x_0^- its preimages in $\mathcal{S}$. There exists a curve $\widetilde{\gamma} \in \mathcal{C}^1((0,1), \mathcal{S})$ satisfying :

- $\widetilde{\gamma}$ does not contain any contractible loops;
- $\widetilde{\gamma}(0) = x_0^+$, and $\widetilde{\gamma}(1) \in \{x_0^+, x_0^-\}$;
- $\dot{\widetilde{\gamma}}(0) = \dot{\widetilde{\gamma}}(1)$ if $\widetilde{\gamma}(1) = x_0^+$;
- $\sigma(\dot{\widetilde{\gamma}}(0)) = (T\sigma)\dot{\widetilde{\gamma}}(1)$ if $\widetilde{\gamma}(1) = x_0^-$, and where $T\sigma : T_{x_0^-}\mathcal{S} \to T_{x_0^+}\mathcal{S}$ is induced by the diffeomorphism σ.

Then the winding number of $\widetilde{\gamma}$ with respect to η is defined, and so is the winding number of $\gamma = p\widetilde{\gamma}$ with respect to $\bar{\eta}$. Moreover we have

$$w_{\bar{\eta}}(\gamma) = w_\eta(\widetilde{\gamma}).$$

Furthermore since the preimage $\tilde{\gamma}$ (starting in x_0^+) is unique up to homotopy, the winding number of $[\gamma]$ is well defined as a map

$$w_{\bar{\eta}} : \pi_1^{\mathrm{orb,free}}(\bar{\mathcal{S}}) \longrightarrow \mathbb{Z}.$$

Definition 2.2. Denote by $\mathbb{S}^1$ the segment $[0,1]$, where 0, and 1 are identified. Let $\gamma : \mathbb{S}^1 \to \bar{\mathcal{S}} \backslash X$ be a closed smooth map with $\gamma(0) = x_0$ and such that its preimage $\widetilde{\gamma} : [0,1] \to \mathcal{S}$ on $\mathcal{S}$ starting at x_0^+ is as above. A *grading* on γ is a map $\mathbf{n} : \gamma(\mathbb{S}^1) \cap \bar{D} \to \mathbb{Z}$ satisfying:

$$\mathbf{n}(\gamma(t_{i+1})) = \mathbf{n}(\gamma(t_i)) + w_{\bar{\eta}}(\gamma_{|[t_i, t_{i+1}]}),$$

if $\gamma(t_i)$ and $\gamma(t_{i+1})$ are two consecutive intersections of γ with $\bar{D}$.

Note that if γ has a grading and if $\widetilde{\gamma}$ is not closed (that is $\widetilde{\gamma}(1) = x_0^-$), then one can consider the closed curve $\beta := \sigma\widetilde{\gamma}.\widetilde{\gamma} : [0,2] \to \mathcal{S}$. The grading $\mathbf{n}$ defines a grading $\widetilde{\mathbf{n}}$ on β (and on $\widetilde{\gamma}$) with for any i $\widetilde{\mathbf{n}}(\widetilde{\gamma}(t_{i+1})) = \widetilde{\mathbf{n}}(\widetilde{\gamma}(t_i)) + w_\eta(\widetilde{\gamma}_{|_{[t_i,t_{i+1}]}})$ and such that

$$\widetilde{\mathbf{n}}((\widetilde{\gamma})(t_\ell)) = \widetilde{\mathbf{n}}(\beta(1+t_1) = \widetilde{\mathbf{n}}(\beta(1+t_1)) + w_\eta(\beta_{|_{[t_\ell,1+t_1]}}) = \widetilde{\mathbf{n}}(\widetilde{\gamma}(t_1))) + w_\eta(\beta|_{[t_\ell,1+t_1]}). \qquad (2.1)$$

Then, with the same argument as before, we see that if the gradings two graded closed curves that are equal when viewed in $\pi_1^{\text{orb,free}}(\bar{\mathcal{S}})$ coincide at their first point, then they coincide at every intersection point with the dissection. Therefore, the set $\pi_1^{\text{orb,free,gr}}(\bar{\mathcal{S}})$ is well defined.

Moreover we have the following

Proposition 2.3. *Let* $[\gamma] \in \pi_1^{\text{orb,free}}(\bar{\mathcal{S}})$. *Then* $[\gamma]$ *is gradable if and only if* $w_{\bar{\eta}}([\gamma]) = 0$.

Proof. Let γ representing $[\gamma]$ be such that its pre-image $\widetilde{\gamma}$. If $\widetilde{\gamma}$ is closed on $\mathcal{S}$, this is clear since we have

$$\gamma \text{ gradable } \Leftrightarrow \widetilde{\gamma} \text{ gradable } \Leftrightarrow w_\eta(\widetilde{\gamma}) = 0 \Leftrightarrow w_{\bar{\eta}}(\gamma) = 0.$$

If $\widetilde{\gamma}$ is not closed, then we have

$$\gamma \text{ gradable } \Leftrightarrow \widetilde{\gamma} \text{ is gradable with condition (2.1)} \Leftrightarrow w_\eta(\sigma\widetilde{\gamma}.\widetilde{\gamma}) = 0 \Leftrightarrow w_{\bar{\eta}}(\gamma) = 0,$$

since $w_\eta(\sigma\widetilde{\gamma}.\widetilde{\gamma}) = 2w_\eta(\widetilde{\gamma}) = 2w_{\bar{\eta}}(\gamma)$.

□

Therefore we obtain a map

$$\pi_1^{\text{orb,free,gr}}(\bar{\mathcal{S}}) \longrightarrow \pi_1^{\text{orb,free}}(\bar{\mathcal{S}}),$$

whose image consists of curves with winding number 0, and whose fiber is in bijection with $\mathbb{Z}$.

Definition 2.4. We call an element $\gamma \in \pi_1(\bar{\mathcal{S}}, x_0)$ *primitive* if it is torsionfree, and if it is a generator of the maximal cyclic group containing it.

Hence if $\gamma \in \pi_1^{\text{orb}}(\bar{\mathcal{S}}, x_0)$ satisfies $\gamma^2 \neq 1$ then γ is torsionfree, and so can be written in a unique way as a positive power of a primitive element.

Now, a small adaptation of Corollary 5.18 in [AP] yields the following.

Proposition 2.5. *Let* $\Psi : \pi_1^{\text{free}}(\mathcal{S}) \to \pi_1^{\text{orb,free}}(\bar{\mathcal{S}})$ *be the map induced by the projection* $p : \mathcal{S} \to \bar{\mathcal{S}}$.

1. *We have a bijection between the following sets:*
   (a) $\Big\{\{[\widetilde{\gamma}], [\sigma\widetilde{\gamma}]\} \mid [\widetilde{\gamma}] \in \pi_1^{\text{free}}(\mathcal{S})$ *primitive with* $w_\eta(\widetilde{\gamma}) = 0$ *and such that* $[\sigma\widetilde{\gamma}] \neq [\widetilde{\gamma}], [\widetilde{\gamma}^{-1}]\Big\}$;
   (b) $\Big\{[\gamma] \in \pi_1^{\text{orb,free}}(\bar{\mathcal{S}}) \mid [\gamma] \in \text{Im}\Psi$ *primitive with* $w_{\bar{\eta}}(\gamma) = 0$, $[\gamma] \neq [\gamma^{-1}]\Big\}$.
2. *We have a bijection between the sets*
   (a) $\Big\{\{[\widetilde{\gamma}], [\sigma\widetilde{\gamma}]\} \mid [\widetilde{\gamma}] \in \pi_1^{\text{free}}(\mathcal{S})$ *primitive with* $w_\eta(\widetilde{\gamma}) = 0$ *and such that* $[\sigma\widetilde{\gamma}] = [\widetilde{\gamma}^{-1}]\Big\}$;
   (b) $\Big\{[\gamma] \in \pi_1^{\text{orb,free}}(\bar{\mathcal{S}}) \mid [\gamma]$ *primitive with* $w_{\bar{\eta}}(\gamma) = 0$ *and* $[\gamma] = [\gamma^{-1}]\Big\}$.
3. *We have a bijection between the sets*
   (a) $\Big\{[\widetilde{\gamma}] \mid [\widetilde{\gamma}] \in \pi_1^{\text{free}}(\mathcal{S})$ *primitive with* $w_\eta(\widetilde{\gamma}) = 0$ *and such that* $[\sigma\widetilde{\gamma}] = [\widetilde{\gamma}]\Big\} \times k^* \times \mathbb{Z}_2$;
   (b) $\Big\{[\alpha] \in \pi_1^{\text{orb,free}}(\bar{\mathcal{S}}) \mid [\alpha] \notin \text{Im}\Psi$ *primitive with* $w_{\bar{\eta}}(\alpha) = 0\Big\} \times k^*$.

Proof. The bijections in items 1. and 2. are induced by Ψ, and we always have $w_\eta([\widetilde{\gamma}]) = w_{\bar\eta}(\Psi[\widetilde{\gamma}])$, so the proof here follows from 1. and 2. of Corollary 5.18 in [AP].

Bijection 3. is constructed as follows (see proof of Corollary 5.18 in [AP]) : for any $[\widetilde{\gamma}] \in \pi_1^{\text{free}}(\mathcal{S})$ primitive such that $[\sigma\widetilde{\gamma}] = [\widetilde{\gamma}]$ there exists a primitive element $[\alpha] \in \pi_1^{\text{orb,free}}(\bar{\mathcal{S}})$ such that $\Psi([\widetilde{\gamma}]) = [\alpha^2]$. If $w_\eta(\widetilde{\gamma}) = 0$, then $w_{\bar\eta}(\alpha^2) = 0 = 2w_{\bar\eta}(\alpha)$. Thus α has winding number zero if and only if so does $\widetilde{\gamma}$. We associate to $([\widetilde{\gamma}], \lambda, \pm 1)$ the element $([\alpha], \pm\lambda')$ where λ' is a square root of λ in k.

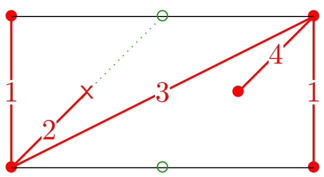
□

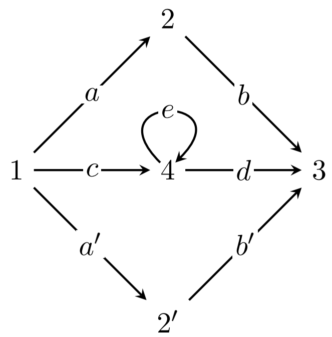
Then combining Propositions 2.3 and 2.5, one easily gets that the sets 3. and 3'. (resp 4., resp 5.) are in bijection respectively with

3. $\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\text{orb,free,gr}}(\bar{\mathcal{S}}) \times k^* \mid [\gamma] \neq [\gamma^{-1}]\}/\sim$ where $([\gamma], \mathbf{n}, \lambda) \sim ([\gamma^{-1}], \mathbf{n}, \lambda^{-1})$;
4. $\{([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\text{orb,free,gr}}(\bar{\mathcal{S}}) \times k^* \setminus \{\pm 1\} \mid [\gamma] = [\gamma^{-1}], \gamma^2 \neq 0\}/\sim$;
5. $\{([\gamma], \mathbf{n}, \epsilon, \epsilon') \in \pi_1^{\text{orb,free,gr}}(\bar{\mathcal{S}}) \times \{\pm 1\}^2 \mid [\gamma] = [\gamma^{-1}], \gamma^2 \neq 0\}$.

3 Example

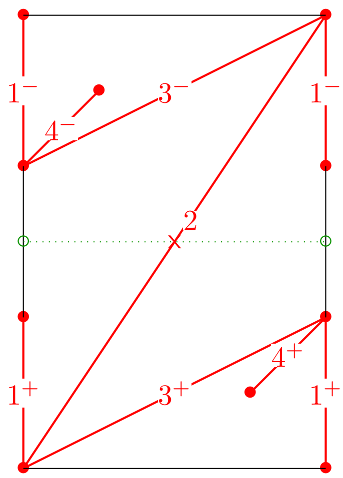
Let us consider a cylinder with one puncture and one orbifold point with the following x-dissection, and its corresponding skew-gentle algebra.

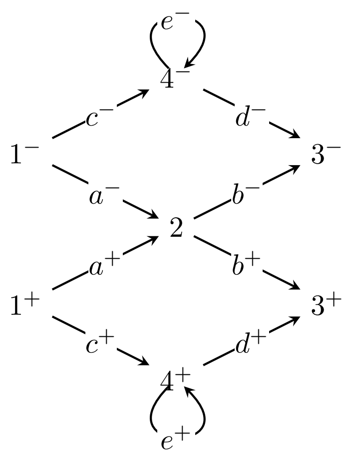




$$\bar{I} = (ba + b'a', dc, e^2)$$

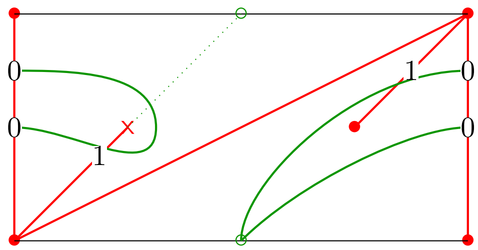
The dissected surface $\mathcal{S}$, and the $\mathbb{Z}_2$-gentle algebra associated to the skew-gentle algebra is as follows.



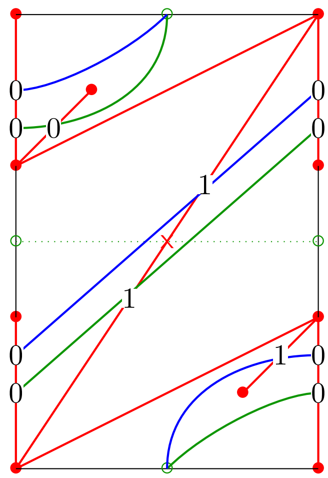


$$I = (b^-a^-, b^+a^+, d^-c^-, d^+c^+, (e^-)^2, (e^+)^2)$$

Let $(\gamma, \mathbf{n}) \in \pi_1^{\mathrm{orb,gr}}(\bar{\mathcal{S}}, \bar{M}_\circ)$ be as follows.



The element γ satisfies $\gamma^2 \neq 1$, therefore $(\gamma, \mathbf{n})$ is in the set 1. and has two preimages in $\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$.



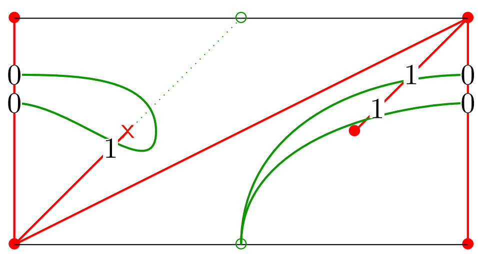
These two graded curves correspond to the following complexes in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} A)$:

$$P_{1^+} \oplus P_{1^-} \xrightarrow{\begin{pmatrix} a^+ & a^- \\ 0 & c^- \end{pmatrix}} P_2 \oplus P_{4^-} \quad \text{and} \quad P_{1^-} \oplus P_{1^+} \xrightarrow{\begin{pmatrix} a^- & a^+ \\ 0 & c^+ \end{pmatrix}} P_2 \oplus P_{4^+} \ .$$

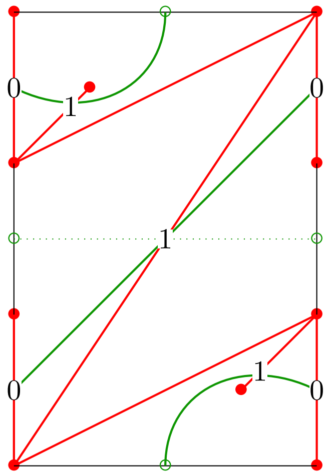
Their image through the functor $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} A) \to \mathcal{K}^{\mathrm{b}}(\operatorname{proj} \bar{A})$ gives the following complexes which are isomorphic:

$$P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} a & a \\ a' & -a' \\ 0 & c \end{pmatrix}} P_2 \oplus P_{2'} \oplus P_4 \ , \quad \text{and} \quad P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} a & a \\ -a' & a' \\ 0 & c \end{pmatrix}} P_2 \oplus P_{2'} \oplus P_4 \ .$$

Take now a $(\gamma, \mathbf{n}) \in \pi_1(\bar{\mathcal{S}}, \bar{M}_\circ)$ such that $\gamma^2 = 1$ as follows:



The graded curve $(\gamma, \mathbf{n})$ is in the set 2. and has a unique preimage in $\pi_1^{\mathrm{gr}}(\mathcal{S}, M_\circ)$.



The corresponding object in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,A)$ is

$$P_{1^+} \oplus P_{1^-} \xrightarrow{\begin{pmatrix} c^+ & 0 \\ a^+ & a^- \\ 0 & c^- \end{pmatrix}} P_{4^+} \oplus P_2 \oplus P_{4^-}$$

Its image in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\bar{A})$ is the following complex

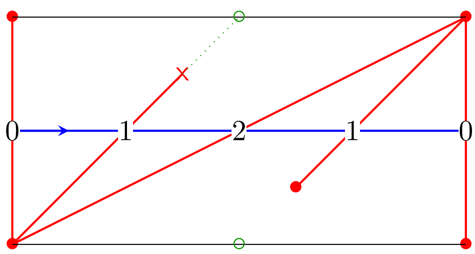
$$P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} c & 0 \\ a & a \\ a' & -a' \\ 0 & c \end{pmatrix}} P_4 \oplus P_2 \oplus P_{2'} \oplus P_4 \,,$$

which is isomorphic to the complex

$$P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} c & 0 \\ a & 0 \\ 0 & a' \\ 0 & c \end{pmatrix}} P_4 \oplus P_2 \oplus P_{2'} \oplus P_4$$

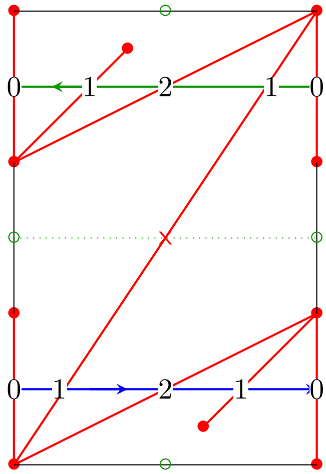
which is clearly decomposable.

Now let $([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}}) \times k^*$ be the following graded curve



The element $([\gamma], \mathbf{n}, \lambda)$ is the set 3. and $[\gamma]$ is in the image of Ψ (indeed it intersects the green dotted lines an even number of times).

The graded curve $([\gamma], \mathbf{n})$ has two preimages in $\pi_1^{\mathrm{free,gr}}(\mathcal{S})$ (that are in the set 3.) which are as follows:



These graded curves correspond respectively to the following objects in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} A)$:

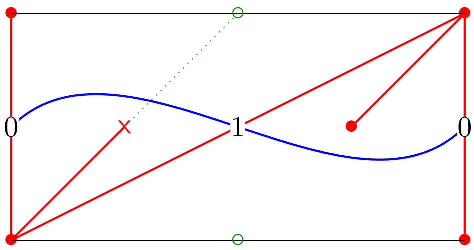
$$P_{1^+} \xrightarrow{\begin{pmatrix}\lambda a^+\\ d^+\end{pmatrix}} P_2 \oplus P_{4^+} \xrightarrow{\begin{pmatrix}b^+ & c^+\end{pmatrix}} P_{3^+} \quad \text{and} \quad P_{1^-} \xrightarrow{\begin{pmatrix}\lambda a^-\\ d^-\end{pmatrix}} P_2 \oplus P_{4^-} \xrightarrow{\begin{pmatrix}b^- & c^-\end{pmatrix}} P_{3^-}$$

The corresponding complexes in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} \bar{A})$ are

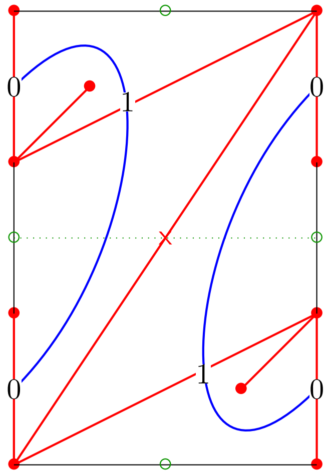
$$P_1 \xrightarrow{\begin{pmatrix}\lambda a\\ \lambda a'\\ d\end{pmatrix}} P_2 \oplus P_{2'} \oplus P_4 \xrightarrow{\begin{pmatrix}b & b' & c\end{pmatrix}} P_3 \quad \text{and} \quad P_1 \xrightarrow{\begin{pmatrix}\lambda a\\ -\lambda a'\\ d\end{pmatrix}} P_2 \oplus P_{2'} \oplus P_4 \xrightarrow{\begin{pmatrix}b & -b' & c\end{pmatrix}} P_3$$

which are isomorphic.

Now let $([\gamma], \mathbf{n}, \lambda) \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}}) \times k^*$ be the following graded curve. It is in the set 3. and $[\gamma]$ is not the image of Ψ since it intersects the green dotted lines an odd number of times.



However, the concatenation of its two preimages is in the set 3'. and is a primitive closed curve as follows:



The corresponding band object in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} A)$ is given by

$$P_{1^+} \oplus P_{1^-} \xrightarrow{\begin{pmatrix} \lambda b^- a^+ & d^- e^- c^- \\ d^+ e^+ c^+ & b^+ a^- \end{pmatrix}} P_{3^-} \oplus P_{3^+} \ ,$$

whose image in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} \bar{A})$ is

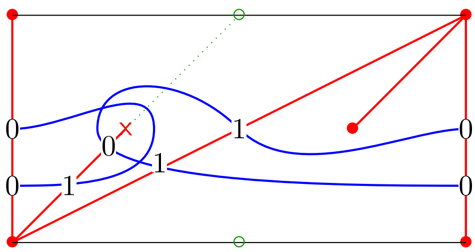
$$P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} \lambda(ba - b'a') & dec \\ dec & ba - b'a' \end{pmatrix}} P_3 \oplus P_3$$

which can be shown to be isomorphic to

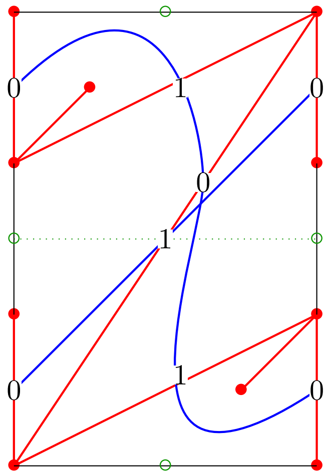
$$\left(P_1 \xrightarrow{\lambda'(ba - b'a') + dec} P_3 \right) \oplus \left(P_1 \xrightarrow{-\lambda'(ba - b'a') + dec} P_3 \right)$$

where $(\lambda')^2 = \lambda$.

Finally, let $([\gamma], \mathbf{n}) \in \pi_1^{\mathrm{orb,free,gr}}(\bar{\mathcal{S}})$ be such that $[\gamma] = [\gamma^{-1}]$ as follows



The closed curve $[\gamma]$ is in the image of Ψ and its preimage is unique as follows:



The corresponding complex in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} A)$ is the following

$$P_{1^+} \oplus P_{1^-} \oplus P_2 \xrightarrow{\begin{pmatrix} \lambda a^+ & a^- & 0 \\ 0 & d^- e^- c^- & b^- \\ d^+ e^+ c^+ & 0 & b^+ \end{pmatrix}} P_2 \oplus P_{3^-} \oplus P_{3^+} .$$

Its image in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} \bar{A})$ is the following complex

$$P_1^2 \oplus P_2 \oplus P_{2'} \xrightarrow{\begin{pmatrix} \lambda a & a & 0 & 0 \\ \lambda a' & -a' & 0 & 0 \\ 0 & dec & b & -b' \\ dec & 0 & b & b' \end{pmatrix}} P_2 \oplus P_{2'} \oplus P_3^2$$

For $\lambda \neq \pm 1$ (so in the case where $[\gamma], \mathbf{n}, \lambda)$ is in the set 4.), this complex is indecomposable. For $\lambda = 1$ this complex is isomorphic to

$$P_1^2 \oplus P_2 \oplus P_{2'} \xrightarrow{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & 0 \\ 0 & dec & 0 & b' \\ dec & 0 & b & 0 \end{pmatrix}} P_2 \oplus P_{2'} \oplus P_3^2$$

which decomposes.

For $\lambda = -1$ this complex is isomorphic to

$$P_1^2 \oplus P_2 \oplus P_{2'} \xrightarrow{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & 0 \\ 0 & dec & b & 0 \\ dec & 0 & 0 & b' \end{pmatrix}} P_2 \oplus P_{2'} \oplus P_3^2$$

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