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The Ising Model's Low Temperature Representation and Phase Transition

Spontaneous symmetry breaking at low temperatures

It is shown that for the nearest-neighbour Ising model on $\mathbb{Z}^d$, the critical inverse temperature $\beta_c(d)$ satisfies $\beta_c(d) < \infty$, for all dimensions $d \geq 2$, implying that the model undergoes a phase transition at low temperatures.

The Ising model

For a finite subset $\Lambda \subset \mathbb{Z}^d$, the set of *configurations* is $\Omega_\Lambda := \{-1, 1\}^\Lambda$. The *spin* σ_i at vertex $i \in \mathbb{Z}^d$ is the random variable giving the value of the configuration at vertex i.
The interactions among spins in the *nearest-neighbour* model are only between neighbouring spins and favour agreement of spin values.
The distribution of configurations, with $(+)$ boundary, is given by the *Gibbs measure* $\mu^+_{\Lambda;\beta}$.
One may extend $\mu^+_{\Lambda;\beta}$ on all of $\mathbb{Z}^d$ to the *infinite volume Gibbs state*, under which, expectation shall be denoted $\langle\cdot\rangle^+_\beta$.

Phase transition and the critical temperature

We say that a *phase transition* occurs at temperature β if at least two distinct Gibbs states can be constructed at β. The *spontaneous magnetisation* $m^*(\beta)$ is defined to be $m^*(\beta) := \langle\sigma_0\rangle^+_\beta$ and the *critical inverse temperature* $\beta_c(d) := \inf\{\beta \geq 0 \mid m^*(\beta) > 0\}$.

Low-temperature representation

A low temperature favors the alignment of nearest-neighbour spins. Therefore the contours, which are lines that seperate regions of $+$ and $-$ spins, should be sparse. This geometric observation is used, for the case $d = 2$, to derive an expression of $\mu^+_{\Lambda;\beta}$ called the *low-temperature representation*.

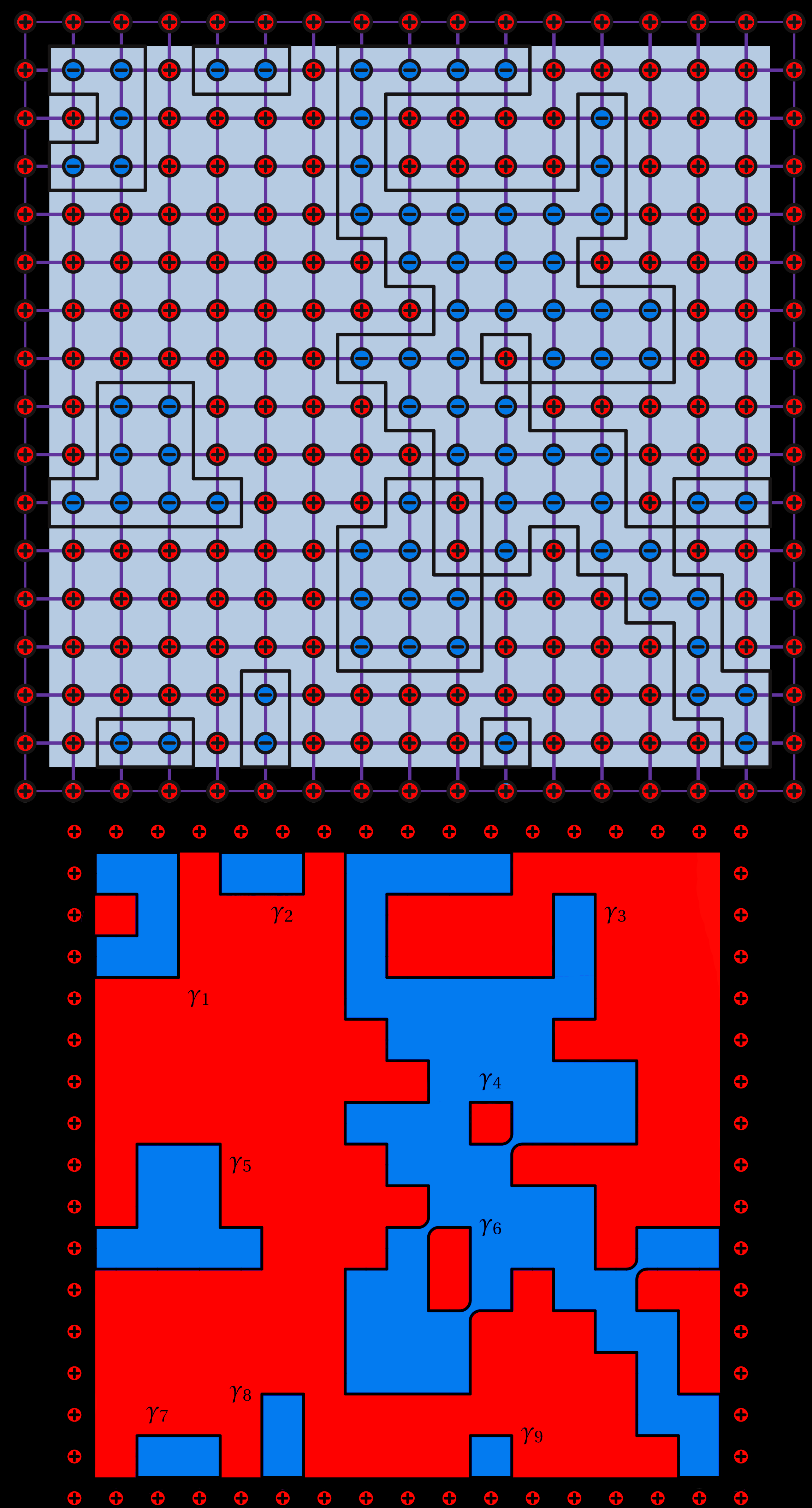

Fig. 1: Given the contrours of a configuration, which are paths on the dual lattice, it may be uniquely reconstructed.

Peierls' argument

Let $B_n = \{-n, ..., n\}^2$. For any $\omega \in \Omega^+_{B_n}$ let $\Gamma(\omega)$ be the set of contours of ω as in Fig.1.
For all $\beta > 0$ and any contour γ,

$$\mu^+_{B_n;\beta}(\Gamma \ni \gamma) \leq e^{-2\beta|\gamma|}.$$

Extension to higher dimensions

Using the two-dimensional low temperature representation and Peierls' argument, it is shown that $\beta_c(2) < \infty$. This analysis is extended for all $d \geq 3$ by embedding $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$ and using the GKS inequalities to show that $\beta_c(d)$ is non-increasing in d.

References

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