An Hilbert to Chow morphism for the non-commutative Hilbert scheme and moduli spaces of linear representations

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Let k be a commutative ring and let A, R be $k-$algebras with R commutative. Given a positive integer n there a three schemes representing functors of points related to A and n, namely

- $\mathcal{R}_A^n$ represents $B \rightarrow \hom_R(A, \mathrm{Mat}(n, B))$ where $\mathrm{Mat}(n, B)$ are the $n \times n$ matrices over B (the $n-$dimensional linear representations of A on B).
- the non-commutative Hilbert scheme $Hilb_n^A$ (see [2]) represents

  $$B \rightarrow \{\text{left ideals of } A \otimes_k B \,:\, A \otimes_k B/I \text{ is a projective } R\text{−module of rank } n\}$$

- Spec $\Gamma_R^n(A)^{ab}$ represents

  $$B \rightarrow \{\text{multiplicative polynomial laws homogeneous of degree } n\}$$

where B is commutative $R-$algebra. When A is commutative $Hilb_A^n$ is the usual Hilbert scheme of $n-$points of $X = \operatorname{Spec} A$. A polynomial law is a kind of map generalizing polynomial mapping and coinciding with it over flat $R-$modules. The typical example of multiplicative polynomial law homogeneous of degree n is the determinant of $n \times n$ matrices. The $R-$algebra $\Gamma_R^n(A)^{ab}$ is the quotient by the ideal generated by commutators of the $R-$algebra $\Gamma_R^n(A)$ of the divided powers of degree n on A. When A is flat as $R-$module this coincides with the symmetric tensors of order n that is $\Gamma_R^n(A) \cong (A^{\otimes_R n})^{S_n}$, where S_n is the symmetric group. Therefore when A is commutative and flat we have $\operatorname{Spec} \Gamma_R^n(A)^{ab} \cong X^{(n)}$, the $n-$th symmetric product of $X = \operatorname{Spec} A$.

We discuss the connections between the coarse moduli space $\mathcal{R}_A^n//GL_n$ of the $n-$dimensional representations of A, with $Hilb_n^A$ and the affine scheme Spec $\Gamma_R^n(A)^{ab}$. We build a norm map from $Hilb_n^A$ to $\Gamma_R^n(A)^{ab}$ which specializes to the Hilbert-Chow morphism on the geometric points when A is commutative and k is an algebraically closed field. This generalizes the construction done by Grothendieck, Deligne and others. By using faithfully flat descent we show that this norm map can be factored through $\mathcal{R}_A^n//GL_n$. When k is an infinite field and $A = k\{x_1, \ldots, x_m\}$ is the free $k-$associative algebra on m letters, we use the isomorphism Spec $\Gamma_k^n(A)^{ab} \cong \mathcal{R}_A^n//GL_n$ given in [1] to give a simple description of this norm map.

1. F. Vaccarino, *Generalized symmetric functions and invariants of matrices*, Math.Z. (to appear) doi: 10.1007/s00209-007-0285-2.
2. M. Van den Bergh , *The Brauer-Severi scheme of the trace ring of generic matrices.* in "Perspectives in Ring Theory" (Antwerp, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **233**, Kluwer Acad. Publ., Dordrecht (1988), 333-338,.