Symmetric spaces of the non-compact type: Lie groups
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1 Introduction

This note is meant to give an introduction to the subjects of Lie groups and equivariant connections on homogeneous spaces. The final goal is the study of the Levi-Civita connection on a symmetric space of the non-compact type. An introduction to the subject of ”symmetric space” from the point of view of differential geometry is given in the course of J. Maubon [5].

2 Lie groups and Lie algebras: an overview

In this section, we review the basic notions concerning the Lie groups and the Lie algebras. For a more completed exposition, the reader is invited to consult standard textbook, for example [6], [1] and [3].
Definition 2.1 A Lie group $G$ is a differentiable manifold\(^1\) which is also endowed with a group structure such that the mappings
\[
G \times G \rightarrow G, \quad (x, y) \mapsto xy \quad \text{multiplication}
\]
\[
G \rightarrow G, \quad x \mapsto x^{-1} \quad \text{inversion}
\]
are smooth.

We can define in the same way the notion of a topological group: it is a topological space\(^2\) which is also endowed with a group structure such that ‘multiplication’ and ‘inversion’ mappings are continuous.

The most basic examples of Lie groups are $(\mathbb{R}, +)$, $(\mathbb{C} - \{0\}, \times)$, and the general linear group $\text{GL}(V)$ of a finite dimensional (real or complex) vector space $V$. The classical groups like
\[
\text{SL}(n, \mathbb{R}) = \{ g \in \text{GL}(\mathbb{R}^n), \det(g) = 1 \},
\]
\[
\text{O}(n, \mathbb{R}) = \{ g \in \text{GL}(\mathbb{R}^n), {\,}^tgg = \text{Id}_n \},
\]
\[
\text{U}(n) = \{ g \in \text{GL}(\mathbb{C}^n), {\,}^tgg = \text{Id}_n \},
\]
\[
\text{O}(p, q) = \{ g \in \text{GL}(\mathbb{R}^{p+q}), {\,}^tgI_{p,q}g = I_{p,q}, \text{ where } I_{p,q} = \left( \begin{array}{cc} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{array} \right) \}
\]
\[
\text{Sp}(\mathbb{R}^{2n}) = \{ g \in \text{GL}(\mathbb{R}^{2n}), {\,}^tgJg = J, \text{ where } J = \left( \begin{array}{cc} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{array} \right) \}
\]
are all Lie groups. It can be proved by hand, or one can use an old Theorem of E. Cartan.

Theorem 2.2 Let $G$ be a closed subgroup of $\text{GL}(V)$. Then $G$ is a embedded submanifold of $\text{GL}(V)$, and equipped with this differential structure it is a Lie group.

The identity element of any group $G$ will be denote by $e$. We denote the tangent space of the Lie groups $G, H, K$ at the identity element respectively by: $\mathfrak{g} = T_e G, \quad \mathfrak{h} = T_e H, \quad \mathfrak{f} = T_e K$.

Example : The tangent space at the identity element of the Lie groups $\text{GL}(\mathbb{R}^n), \text{SL}(n, \mathbb{R}), \text{O}(n, \mathbb{R})$ are respectively
\[
\mathfrak{gl}(\mathbb{R}^n) = \{ \text{endomorphism of } \mathbb{R}^n \},
\]
\[
\mathfrak{sl}(n, \mathbb{R}) = \{ X \in \mathfrak{gl}(\mathbb{R}^n), \text{ Tr}(X) = 0 \},
\]
\[
\mathfrak{o}(n, \mathbb{R}) = \{ X \in \mathfrak{gl}(\mathbb{R}^n), {\,}^tgX + X = 0 \},
\]
\[
\mathfrak{o}(p, q) = \{ X \in \mathfrak{gl}(\mathbb{R}^n), {\,}^tg\text{Id}_{p,q} + \text{Id}_{p,q}X = 0 \}, \text{ where } p + q = n.
\]

\(^1\)All manifolds are second countable.

\(^2\)Here “topological space” means Hausdorff and locally compact.
2.1 Group action

A morphism $\phi : G \to H$ of groups is by definition a map that preserves the product: $\Phi(g_1g_2) = \Phi(g_1)\phi(g_2)$.

**Exercise 2.3** Show that $\phi(e) = e$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

**Definition 2.4** An (left) action of a group $G$ on a set $M$ is a mapping

$$\alpha : G \times M \to M$$

such that $\alpha(e, m) = m, \forall m \in M, \text{ and } \alpha(g, \alpha(h, m)) = \alpha(gh, m)$ for all $m \in M$ and $g, h \in G$.

Let $\text{Bij}(M)$ be the group of all bijective maps from $M$ onto $M$. The conditions on $\alpha$ are equivalent to saying that the map $G \to \text{Bij}(M), g \to \alpha_g$ defined by $\alpha_g(m) = \alpha(g, m)$ is a group morphism.

If $G$ is a Lie (resp. topological) group and $M$ is a manifold (resp. topological space), the action of $G$ on $M$ is said to be smooth (resp. continuous) if the map (2.1) is smooth (resp. continuous). When the notations are understood we will write $g \cdot m$, or simply $gm$ for $\alpha(g, m)$.

A representation of a group $G$ on a real vector space (resp. complex) $V$ is a group morphism $\phi : G \to \text{GL}(V)$: the group $G$ acts on $V$ through linear endomorphism.

**Notation** : If $\phi : M \to N$ is a smooth map between differentiable manifolds, we denote by $T_m\phi : T_mM \to T_{\phi(m)}N$ the differential of $\phi$ at $m \in M$.

2.2 Adjoint representation

Let $G$ be a Lie group and let $\mathfrak{g}$ be the tangent space of $G$ at $e$. We consider the conjugation action of $G$ on itself defined by

$$c_g(h) = ghg^{-1}, \quad g, h \in G.$$ 

The mappings $c_g : G \to G$ are smooth and $c_g(e) = e$ for all $g \in G$, so one can consider the differential of $c_g$ at $e$

$$\text{Ad}(g) = T_ec_g : \mathfrak{g} \to \mathfrak{g}.$$ 

Since $c_{gh} = c_g \circ c_h$ we have $\text{Ad}(gh) = \text{Ad}(g) \circ \text{Ad}(h)$. That is, the mapping

$$\text{Ad} : G \to \text{GL}(\mathfrak{g})$$

(2.2)
is a smooth group morphism which is called the adjoint representation of $G$.

The next step is to consider the differential of the map $Ad$ at $e$:

$$\text{ad} = T_e\text{Ad} : g \rightarrow \mathfrak{gl}(g).$$

(2.3)

It is the adjoint representation of $g$. In (2.3), the vector space $\mathfrak{gl}(g)$ denotes the vector space of all linear endomorphisms of $g$, and is equal to the tangent space of $GL(g)$ at the identity.

**Lemma 2.5**. We have the fundamental relations

- $ad(Ad(g)X) = Ad(g) \circ ad(X) \circ Ad(g)^{-1}$ for $g \in G$, $X \in g$.
- $ad(Ad(Y)X) = ad(Y) \circ ad(X) - ad(X) \circ ad(Y)$ for $X, Y \in g$.
- $ad(X)Y = -ad(Y)X$ for $X, Y \in g$.

**Proof**: Since $Ad$ is a group morphism we have $Ad(ghg^{-1}) = Ad(g) \circ Ad(h) \circ Ad(g)^{-1}$. If we differentiate this relation at $h = e$ we get the first point, and if we differentiate it at $g = e$ we get the second one.

For the last point consider two smooth curves $a(t), b(s)$ on $G$ with $a(0) = b(0) = e$, $\frac{d}{dt}[a(t)]_{t=0} = X$, and $\frac{d}{ds}[b(t)]_{s=0} = Y$. We will now compute the second derivative $\frac{d^2 f}{dt ds}(0, 0)$ of the map $f(t, s) = a(t)b(s)a(t)^{-1}b(s)^{-1}$. Since $f(t, 0) = f(0, s) = e$, the term $\frac{d^2 f}{dt ds}(0, 0)$ is defined in an intrinsic manner as an element of $g$. For the first partial derivatives we get $\frac{df}{dt}(0, s) = X - Ad(b(s))X$ and $\frac{df}{ds}(t, 0) = Ad(a(t))Y - Y$. So $\frac{d^2 f}{dt ds}(0, 0) = ad(X)Y = -ad(Y)X$. \Box

**Definition 2.6** If $G$ is a Lie group, one defines a bilinear map, $[-,-]_g : g \times g \rightarrow g$ by $[X, Y]_g = ad(X)Y$. It is the Lie bracket of $g$. The vector space $g$ equipped with $[-,-]_g$ is called the Lie algebra of $G$. We have the fundamental relations

- anti symmetry : $[X, Y]_g = -[Y, X]_g$
- Jacobi identity : $ad([X, Y]_g) = ad(Y) \circ ad(X) - ad(X) \circ ad(Y)$.

On $\mathfrak{gl}(g)$, a direct computation shows that $[X, Y]_{\mathfrak{gl}(g)} = XY - YX$. So the Jacobi identity can be rewritten as $ad([X, Y]_g) = [ad(X), ad(Y)]_{\mathfrak{gl}(g)}$ or equivalently as

$$[X, [Y, Z]_g]_g + [Y, [Z, X]_g]_g + [Z, [X, Y]_g]_g = 0 \quad \text{for all} \ X, Y, Z \in g.$$  \hspace{1cm} (2.4)

**Definition 2.7** A Lie algebra $g$ is a real vector space equipped with the antisymmetric bilinear map $[-,-]_g : g \times g \rightarrow g$ satisfying the Jacobi identity.

- A linear map $\phi : g \rightarrow h$ between two Lie algebras is a morphism of Lie algebras if

$$\phi([X, Y]_g) = [\phi(X), \phi(Y)]_h.$$  \hspace{1cm} (2.5)
Remark 2.8 We have defined the notion of real Lie algebra. The definitions goes through on any field \( k \), in particular when \( k = \mathbb{C} \) we speak of complex Lie algebras. For example, if \( \mathfrak{g} \) is a real Lie algebra, the complexified vector space \( \mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C} \) inherits a canonical structure of complex Lie algebra.

The map \( \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) is the typical example of a morphism of Lie algebras. This example generalizes as follows.

Lemma 2.9 Consider a smooth morphism \( \Phi : G \to H \) between two Lie groups. Let \( \phi : \mathfrak{g} \to \mathfrak{h} \) be its differential at \( e \). Then:

- The map \( \phi \) is \( \Phi \)-equivariant: \( \phi \circ \text{Ad}(g) = \text{Ad}(\Phi(g)) \circ \phi \).
- \( \phi \) is a morphism of Lie algebras.

The proof works as in Lemma 2.5.

Example: If \( G \) is a closed subgroup of \( GL(V) \), the inclusion \( \mathfrak{g} \hookrightarrow \mathfrak{gl}(V) \) is a morphism of Lie algebra. In other words, if \( X, Y \in \mathfrak{g} \) then \([X, Y]_{\mathfrak{gl}(V)} = XY - YX\) belongs to \( \mathfrak{g} \) and corresponds to the Lie bracket \([X, Y]_\mathfrak{g}\).

### 2.3 Vectors fields and Lie bracket

Here we review the typical example of Lie bracket: those of vectors fields.

Let \( M \) be a smooth manifold. We denote by \( \text{Diff}(M) \) the group formed by the diffeomorphism of \( M \), and by \( \text{Vect}(M) \) the vector space of smooth vectors fields. Even if \( \text{Diff}(M) \) is not a Lie group (it’s not finite dimensional), many aspects discussed earlier apply here, with \( \text{Vect}(M) \) in the role of the Lie algebra of \( \text{Diff}(M) \). If \( a(t) \) is a smooth curve in \( \text{Diff}(M) \) passing through the identity at \( t = 0 \), the derivative \( V = \frac{d}{dt}[a]_{t=0} \) is a vector field on \( M \).

The "adjoint" action of \( \text{Diff}(M) \) on \( \text{Vect}(M) \) is defined as follows. If \( V = \frac{d}{dt}[a]_{t=0} \) one takes \( \text{Ad}(g)V = \frac{d}{dt}[g \circ a \circ g^{-1}]_{t=0} \) for every \( g \in \text{Diff}(M) \). The definition of \( \text{Ad} \) extends to any \( V \in \text{Vect}(M) \) through the following expression

\[
\text{Ad}(g)V|_m = T_{g^{-1}m}(g)(V_{g^{-1}m}), \quad m \in M. \tag{2.6}
\]

We can now define the adjoint action by differentiating (2.6) at the identity. If \( W = \frac{d}{dt}[b]_{t=0} \) and \( V \in \text{Vect}(M) \), we take

\[
\text{ad}(W)V|_m = \frac{d}{dt} \left[ T_{b(t)^{-1}m}(b(t))(V_{b(t)^{-1}m}) \right]_{t=0}, \quad m \in M. \tag{2.7}
\]

If we take any textbook on differential geometry we see that \( \text{ad}(W)V = -[W, V] \), where \([-,-]\) is the usual Lie bracket on \( \text{Vect}(M) \).
we get this minus sign, consider the group morphism

$$\Phi : \text{Diff}(M) \longrightarrow \text{Aut}(C^\infty(M))$$

$$g \longmapsto g$$

defined by $g \cdot f(m) = f(g^{-1}m)$ for $f \in C^\infty(M)$. Here $\text{Aut}(C^\infty(M))$ is the group of automorphism of the algebra $C^\infty(M)$. If $b(t)$ is a smooth curve in $\text{Aut}(C^\infty(M))$ passing through the identity at $t = 0$, the derivative $u = \frac{db}{dt}[t=0]$ belongs to the vector space $\text{Der}(C^\infty(M))$ of derivations of $C^\infty(M)$: $u : C^\infty(M) \rightarrow C^\infty(M)$ is a linear map and $u(fg) = u(f)g + fu(g)$. So the Lie algebra of $\text{Aut}(C^\infty(M))$ as a natural identification with $\text{Der}(C^\infty(M))$ equipped with the Lie bracket: $[u, v]_{\text{Der}} = u \circ v - v \circ u$, for $u, v \in \text{Der}(C^\infty(M))$.

Let $\text{Vect}(M) \sim \rightarrow \text{Der}(C^\infty(M)), V \mapsto \tilde{V}$ be the canonical identification defined by $\tilde{V}f(m) = \langle df_m, V_m \rangle$ for $f \in C^\infty(M)$ and $V \in \text{Vect}(M)$.

For the differential at the identity of $\Phi$ we get

$$d\Phi(V) = -\tilde{V}, \text{ for } V \in \text{Vect}(M).$$

Since $d\Phi$ is an algebra morphism we have $-\text{ad}(\tilde{V})W = [\tilde{V}, \tilde{V}]_{\text{Der}}$. Hence we see that $[V, W] = -\text{ad}(V)W$ is the traditional Lie bracket on $\text{Vect}(M)$ defined by posing $[\tilde{V}, \tilde{W}] = \tilde{V} \circ \tilde{W} - \tilde{W} \circ \tilde{V}$.

### 2.4 Group actions and Lie bracket

Let $M$ be a differentiable manifold equipped with a smooth action of a Lie group $G$. We can specialize (2.8) to a group morphism $G \rightarrow \text{Aut}(C^\infty(M))$. Its differential at the identity defines a map $g \rightarrow \tilde{g} : \text{Der}(C^\infty(M)) \rightarrow \text{Vect}(M)$, $X \mapsto X_g$ by $X_g|_m = \frac{d}{dt}[a(t)^{-1} \cdot m]_{t=0}$, $m \in M$. Here $a(t)$ is a smooth curve on $G$ such that $X = \frac{da}{dt}[t=0]$. This mapping is a morphism of Lie algebras:

$$[X, Y]_M = [X_g, Y_g].$$

**Example:** Consider the actions of translations $R, L$ of a Lie group $G$ on itself:

$$R(g)h = hg^{-1}, \quad L(g)h = gh \quad \text{for } g, h \in G.$$ 

These actions defines vectors field $X^L, X^R$ on $G$ for any $X \in \mathfrak{g}$, and (2.10) reads

$$[X, Y]^L = [X^L, Y^L], \quad [X, Y]^R = [X^R, Y^R].$$

These equations can be used to define the Lie bracket on $\mathfrak{g}$. Consider the subspaces $V^L = \{X^L, X \in \mathfrak{g}\}$ and $V^R = \{X^R, X \in \mathfrak{g}\}$ of $\text{Vect}(G)$. First
we see that $V^L$ (resp. $V^R$) coincides with the subspace of $\text{Vect}(G)^R$ (resp. $\text{Vect}(G)^L$) formed by the vectors fields invariant by the $R$-action of $G$ (resp. $L$-action of $G$). Second we see that the subspaces $\text{Vect}(G)^R$ and $\text{Vect}(G)^L$ are invariant under the Lie bracket of $\text{Vect}(G)$. Then for any $X, Y \in \mathfrak{g}$, the vectors field $[X^L, Y^L] \in \text{Vect}(G)^R$, so there exist a unique $[X, Y] \in \mathfrak{g}$ such that $[X, Y]^L = [X^L, Y^L]$.

2.5 Exponential map

Consider the usual exponential map $e : \mathfrak{gl}(V) \to \text{GL}(V)$: $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. We have the fundamental property

**Proposition 2.10** • For any $A \in \mathfrak{gl}(V)$, the map $\phi_A : \mathbb{R} \to \text{GL}(V)$, $t \mapsto e^{tA}$ is a smooth Lie group morphism with $\frac{d}{dt}[\phi_A]_{t=0} = A$.

• If $\phi : \mathbb{R} \to \text{GL}(V)$ is a smooth Lie group morphism we have $\phi = \phi_A$ for $A = \frac{d}{dt}[\phi]_{t=0}$.

Now, we will see that an exponential map together with Proposition 2.10 exists on all Lie group.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ we consider the vectors field $X^R \in \text{Vect}(G)$ defined by $X^R|_g = \frac{d}{dt}[ga(t)]_{t=0}$, $g \in G$. Here $a(t)$ is a smooth curve on $G$ such that $X = \frac{d}{dt}[a]_{t=0}$. The vectors fields $X^R$ are invariant under the left translations, that is

$$T_g(L(h))(X^R_g) = X^R_{hg}, \quad \text{for } g, h \in G. \quad (2.12)$$

We consider now the flow of the vectors field $X^R$. For any $X \in \mathfrak{g}$ we consider the differential equation

$$\frac{\partial}{\partial t} \phi(t, g) = X^R(\phi(t, g)) \quad (2.13)$$
$$\phi(0, g) = g.$$  

where $t \in \mathbb{R}$ belongs to an interval containing 0, and $g \in G$. Classical results assert that for any $g_0 \in G$ (2.13) admits a unique solution $\phi_X$ defined on $]-\varepsilon, \varepsilon[ \times \mathcal{U}$ where $\varepsilon > 0$ is small enough and $\mathcal{U}$ is a neighborhood of $g_0$. Since $X^R$ is invariant under the left translations we have

$$\phi_X(t, g) = g\phi_X(t, e). \quad (2.14)$$

The map $t \to \phi_X(t, -)$ is a 1-parameter subgroup of (local) diffeomorphisms of $M$: $\phi_X(t + s, m) = \phi_X(t, \phi_X(s, m))$ for $t, s$ small enough. Eq. (2.14) give then

$$\phi_X(t + s, e) = \phi_X(t, e)\phi_X(s, e) \quad \text{for } t, s \text{ small enough}. \quad (2.15)$$
The map \( t \mapsto \phi^X(t, e) \) initially defined on an interval \([-\varepsilon, \varepsilon[\) can be extended on \( \mathbb{R} \) thanks to (2.15). For any \( t \in \mathbb{R} \) take \( \Phi^X(t, e) = \phi^X \left( \frac{t}{\varepsilon}, e \right)^n \) where \( n \) is an integer large enough so that \( \left| \frac{t}{\varepsilon} \right| < \varepsilon \). It is not difficult to see that our definition make sense and that \( \mathbb{R} \to G, t \mapsto \Phi^X(t, e) \) is a Lie group morphism. Finally we have proved that the vectors field \( X^R \) is completed: its flow is defined on \( \mathbb{R} \times G \).

**Definition 2.11** For each \( X \in \mathfrak{g} \), the element \( \exp_G(X) \in G \) is defined as \( \Phi^X(1, e) \). The mapping \( \mathfrak{g} \to G, X \mapsto \exp_G(X) \) is called the exponential mapping from \( \mathfrak{g} \) into \( G \).

**Proposition 2.12**

1. \( \exp_G(tX) = \Phi^X(t, e) \) for each \( t \in \mathbb{R} \).
2. \( \exp_G : \mathfrak{g} \to G \) is \( C^\infty \) and \( T_e \exp_G \) is the identity map.

**Proof** : Let \( s \neq 0 \) in \( \mathbb{R} \). The maps \( t \mapsto \Phi^X(t, e) \) and \( t \mapsto \Phi^X(t \frac{s}{X}, e) \) are both solutions of the differential equation (2.13): so there are equal and \( a) \) is proved by taking \( t = s \). To proved \( b) \) consider the vectors field \( V \) on \( \mathfrak{g} \times G \) defined by \( V(X, g) = (X^R(g), 0) \). It is easy to see that the flow \( \Phi^V \) of the vectors field \( V \) satisfies \( \Phi^V(t, X, g) = (g \exp_G(tX), X) \), for \( (t, X, g) \in \mathbb{R} \times \mathfrak{g} \times G \). Since \( \Phi^V \) is smooth (a general property concerning the flows), the exponential map is smooth. □

Proposition 2.10 take now the following form.

**Proposition 2.13** If \( \phi : \mathbb{R} \to G \) is a \( (C^\infty) \) one parameter subgroup, we have \( \phi(t) = \exp_G(tX) \) with \( X = \frac{d}{dt}[\phi]_{t=0} \).

**Proof** : If we differentiate the relation \( \phi(t + s) = \phi(t)\phi(s) \) at \( s = 0 \), we see that \( \phi \) satisfies the differential equation (\( \star \)) \( \frac{d}{dt}[\phi]_t = X^R(\phi(t)) \), where \( X = \frac{d}{dt}[\phi]_{t=0} \). Since \( t \mapsto \Phi^X(t, e) \) is also solution of (\( \star \)), and \( \Phi^X(0, e) = \phi(0) = e \), we have \( \phi = \Phi^X(-e) \). □

We give now some easy consequences of Proposition 2.13.

**Proposition 2.14**

1. If \( \rho : G \to H \) is a morphism of Lie groups and \( d\rho : \mathfrak{g} \to \mathfrak{h} \) is the corresponding morphism of Lie algebras, we have \( \exp_H \circ d\rho = \rho \circ \exp_G \).
2. For \( \text{Ad} : G \to \text{GL}(\mathfrak{g}) \) we have \( \text{Ad}(\exp_G(X)) = e^{\text{ad}(X)} \).
3. \( \exp_G : \mathfrak{g} \to G \) is \( G \)-equivariant: \( \exp_G(\text{Ad}(g)X) = g \exp_G(X)g^{-1} \).
4. If \( [X, Y] = 0 \), then \( \exp_G(X) \exp_G(Y) = \exp_G(Y) \exp_G(X) = \exp_G(X + Y) \).

**Proof** : We use in each case the same type of proof. We consider two 1-parameters subgroup \( \Phi_1(t) \) and \( \Phi_2(t) \). After we verify that \( \frac{d}{dt}[\Phi_1]_{t=0} = \)
\[
\frac{d}{dt} [\Phi_2]_{t=0}, \text{ and from Proposition 2.13 we conclude that } \Phi_1(t) = \Phi_2(t), \forall t \in \mathbb{R}.
\]
The relation that we are looking for is \( \Phi_1(1) = \Phi_2(1) \).

For the first point, we take \( \Phi_1(t) = \exp_G(t \text{Ad}(g) X) \) and \( \Phi_2(t) = \rho \circ \exp_G(tX) \): for the second point we take \( \rho = \text{Ad} \), and for the third one we take \( \Phi_1(t) = \exp_G(t \text{Ad}(g) X) \) and \( \Phi_2(t) = g \exp_G(tX) g^{-1} \).

From the second and third point we have \( \exp_G(X) \exp_G(Y) \exp_G(-X) = \exp_G(e^{\text{ad}(X)Y}) \). Hence \( \exp_G(X) \exp_G(Y) \exp_G(-X) = \exp_G(Y) \) if \( \text{ad}(X)Y = 0 \). We consider after the 1-parameters subgroups \( \Phi_1(t) = \exp_G(tX) \) and \( \Phi_2(t) = \exp_G(t(X + Y)) \) to prove the second equality of the last point.

\[ \square \]

Exercise 2.15 We consider the Lie group \( \text{SL}(2, \mathbb{R}) \) with Lie algebra \( \text{sl}(2, \mathbb{R}) = \{ X \in \text{End}(\mathbb{R}^2), \text{Tr}(X) = 0 \} \). Show that the image of the exponential map \( \exp : \text{sl}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}) \) is equal to \( \{ g \in \text{SL}(2, \mathbb{R}), \text{Tr}(g) \geq -2 \} \).

Remark 2.16 The map \( \exp_G : g \to G \) is in general not surjective. Nevertheless the set \( U = \exp_G(g) \) is a neighborhood of the identity, and \( U = U^{-1} \). The subgroup of \( G \) generated by \( U \), which is equal to \( \bigcup_{n \geq 1} U^n \), is then a connected open subgroup of \( G \). Hence \( \bigcup_{n \geq 1} U^n \) is equal to the connected component of the identity, usually denoted \( G^o \).

Exercise 2.17 For any Lie group \( G \), show that \( \exp_G(X) \exp_G(Y) = \exp_G(X + Y + \frac{1}{2}[X,Y] + o(|X|^2 + |Y|^2)) \) in a neighborhood of \((0,0) \in g^2 \). Afterward show that

\[
\lim_{n \to \infty} (\exp_G(X/n) \exp_G(Y/n))^n = \exp_G(X + Y) \quad \text{and}
\]

\[
\lim_{n \to \infty} (\exp_G(X/n) \exp_G(Y/n) \exp_G(-X/n) \exp_G(-Y/n))^{n^2} = \exp([X,Y]).
\]

2.6 Lie subgroups and Lie subalgebras

Before giving the precise definition of a Lie subgroup, we look at the infinitesimal side. A Lie subalgebra of a Lie algebra \( g \) is a subspace \( h \subset g \) stable under the Lie bracket : \([X,Y]_g \in h \) whenever \( X, Y \in h \).

We have a natural extension of Theorem 2.2

Theorem 2.18 Let \( H \) be a closed subgroup of a Lie group \( G \). Then \( H \) is an imbedded submanifold of \( G \), and equipped with this differential structure it is a Lie group. The Lie algebra of \( H \), which is equal to \( h = \{ X \in g | \exp_G(tX) \in H \text{ for all } t \in \mathbb{R} \} \), is a subalgebra of \( g \).
Proof: The two limits given in the exercise 2.17 show that \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{g} \) (we use here that \( H \) is closed). Let \( \mathfrak{a} \) be any supplementary subspace of \( \mathfrak{h} \): one shows that \( (\exp(Y) \in H) \implies (Y = e) \) if \( Y \in \mathfrak{a} \) belongs to a small neighborhood of 0 in \( \mathfrak{a} \). Now we consider the map \( \phi: \mathfrak{h} \oplus \mathfrak{a} \to G \) given by \( \phi(X + Y) = \exp_G(X) \exp_G(Y) \). Since \( T_e\phi \) is the identity map, \( \phi \) defines a smooth diffeomorphism \( \phi|_\mathcal{V} \) from a neighborhood \( \mathcal{V} \) of 0 \( \in \mathfrak{g} \) to a neighborhood \( \mathcal{W} \) of \( e \) in \( G \). If \( \mathcal{V} \) is small enough we see that \( \phi \) map \( \mathcal{V} \cap \{Y = 0\} \) onto \( \mathcal{W} \cap H \), hence \( H \) is a submanifold near \( e \). Near any point \( h \in H \) we use the map \( \phi_h: \mathfrak{h} \oplus \mathfrak{a} \to G \) given by \( \phi_h(Z) = h \phi(Z) \): we prove in the same way that \( H \) is a submanifold near \( h \). Finally \( H \) is an imbedded submanifold of \( G \). We now look to the group operations \( m_G: G \times G \to G \) (multiplication), \( i_G: G \to G \) (inversion) and their restrictions \( m_G|_{H \times H}: H \times H \to G \) and \( i_G|_H: H \to G \) which are smooth maps. Here we are interested in the group operations \( m_H \) and \( i_H \) of \( H \). Since \( m_G|_{H \times H} \) and \( i_G|_H \) are smooth we have the equivalence:

\[
m_H \text{ and } i_H \text{ are smooth } \iff m_H \text{ and } i_H \text{ are continuous.}
\]

The fact that \( m_H \) and \( i_H \) are continuous follows easily from the fact that \( m_G|_{H \times H} \) and \( i_G|_H \) are continuous and that \( H \) is closed. \( \square \)

Theorem 2.18 has the following important corollary

**Corollary 2.19** If \( \phi: G \to H \) is a continuous group morphism between two Lie groups, then \( \phi \) is smooth.

**Proof:** Consider the graph \( L \subset G \times H \) of the map \( \phi: L = \{(g, h) \in G \times H \mid h = \phi(g)\} \). Since \( \phi \) is a continuous \( L \) is a closed subgroup of \( G \times H \). Following Theorem 2.18, \( L \) is an imbedded submanifold of \( G \times H \). Consider now the morphism \( p_1: L \to G \) (resp. \( p_2: L \to H \)) equals respectively to the composition of the inclusion \( L \hookrightarrow G \times H \) with the projection \( G \times H \to G \) (resp. \( G \times H \to H \)): \( p_1 \) and \( p_2 \) are smooth, \( p_1 \) is bijective, and \( \phi = p_2 \circ (p_1)^{-1} \). Since \( (p_1)^{-1} \) is smooth (see Exercise 2.24), the map \( \phi \) is smooth. \( \square \)

We have just seen the archetype of a Lie subgroup: a closed subgroup of a lie group. But this notion is too restrictive.

**Definition 2.20** \((H, \phi)\) is a Lie subgroup of a Lie group \( G \) if

- \( H \) is a Lie group,
- \( \phi: H \to G \) is a group morphism,
- \( \phi: H \to G \) is a one-to-one immersion.

In the next example we consider the 1-parameter Lie subgroups of \( S^1 \times S^1 \) : either they are closed or dense.
Example: Consider the group morphisms $\phi_\alpha : \mathbb{R} \to S^1 \times S^1$, $\phi_\alpha(t) = (e^{it}, e^{i\alpha t})$, defined for $\alpha \in \mathbb{R}$. Then:

- If $\alpha \notin \mathbb{Q}$, $\text{Ker}(\phi_\alpha) = 0$ and $(\mathbb{R}, \phi_\alpha)$ is a Lie subgroup of $S^1 \times S^1$ which is dense.
- If $\alpha \in \mathbb{Q}$, $\text{Ker}(\phi_\alpha) \neq 0$, and $\phi_\alpha$ factorizes in a smooth morphism $\tilde{\phi}_\alpha : S^1 \to S^1 \times S^1$. Here $\phi_\alpha(\mathbb{R})$ is a closed subgroup of $S^1 \times S^1$ diffeomorphic to the Lie subgroup $(S^1, \phi_\alpha)$.

Let $(H, \phi)$ be a Lie subgroup of $G$, and let $\mathfrak{h}, \mathfrak{g}$ be their respective Lie algebras. Since $\phi$ is an immersion, the differential at the identity, $d\phi : \mathfrak{h} \to \mathfrak{g}$, is an injective morphism of Lie algebras: $\mathfrak{h}$ is isomorphic with the subalgebra $d\phi(\mathfrak{h})$ of $\mathfrak{g}$. In practice we often “forget” $\phi$ in our notations, and speak of a Lie subgroup $H \subset G$ with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We have to be careful: when $H$ is not closed in $G$, the topology of $H$ is not the induced topology.

We state now the fundamental

**Theorem 2.21** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then there exists a unique connected Lie subgroup $H$ of $G$ with Lie algebra equal to $\mathfrak{h}$. Moreover $H$ is generated by $\exp_G(\mathfrak{h})$, where $\exp_G$ is the exponential map of $G$.

The proof uses Frobenius Theorem (see [6][Theorem 3.19]). This Theorem has an important corollary.

**Corollary 2.22** Let $G, H$ be two connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. If $G$ is simply connected there exists a (unique) Lie group morphism $\Phi : G \to H$ such that $d\Phi = \phi$.

**Proof:** Consider the graph $I \subset \mathfrak{g} \times \mathfrak{h}$ of the map $\phi : I := \{(X, Y) \in \mathfrak{g} \times \mathfrak{h} \mid \phi(X) = Y\}$. Since $\phi$ is morphism of Lie algebras $I$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Let $(L, \psi)$ be the connected Lie subgroup of $G \times H$ associated to $I$. Consider now the morphism $p_1 : L \to G$ (resp. $p_2 : L \to H$) equals respectively to the composition of $\phi : L \to G \times H$ with the projection $G \times H \to G$ (resp. $G \times H \to H$). The group morphism $p_2 : L \to G$ is onto with a discrete kernel since $G$ is connected and $dp_2 : I \to \mathfrak{g}$ is an isomorphism. Hence $p_2 : L \to G$ is a covering map (see Exercise 2.24). Since $G$ is simply connected, this covering map is a diffeomorphism. The group morphism $p_1 \circ (p_2)^{-1} : G \to H$ answers to the question. □

**Example:** The Lie group $\text{SU}(2)$ is composed by the $2 \times 2$ complex matrices of the form $\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$. Hence $\text{SU}(2)$ is simply connected since it is diffeomorphic to the 3-dimensional sphere. Since $\text{SU}(2)$
is a maximal compact subgroup of $\text{SL}(2, \mathbb{C})$, the Cartan decomposition (see Section 3.4) tells us that $\text{SL}(2, \mathbb{C})$ is also simply connected.

A subset $A$ of a topological space $M$ is path-connected if any points $a, b \in A$ can be joined by a continuous path $\gamma : [0, 1] \to M$ with $\gamma(t) \in A$ for all $t \in [0, 1]$. Any connected Lie subgroup of a Lie group is path-connected. We have the following characterization of the connected Lie subgroups.

**Theorem 2.23** Let $G$ be a Lie group, and let $H$ be a path-connected subgroup of $G$. Then $H$ is a Lie subgroup of $G$.

**Exercise 2.24** Let $\rho : G \to H$ be a smooth morphism of Lie groups, and let $d\rho : g \to h$ be the corresponding morphism of Lie algebras.

- Show that $\text{Ker}(\rho) := \{g \in G \mid \rho(g) = e\}$ is a closed (normal) subgroup with Lie algebra $\text{Ker}(d\rho) := \{X \in g \mid d\rho(X) = 0\}$.
- If $\text{Ker}(d\rho) = 0$, show that $\text{Ker}(\rho)$ is discrete in $G$. If furthermore $\rho$ is onto, then show that $\rho$ is a covering map.
- If $\rho : G \to H$ is bijective, then show that $\rho^{-1}$ is smooth.

### 2.7 Ideals

A subalgebra $\mathfrak{h}$ of a Lie algebra is called an ideal in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$; in other words $\mathfrak{h}$ is a stable subspace of $\mathfrak{g}$ under the endomorphism $\text{ad}(Y)$, $Y \in \mathfrak{g}$. A Lie subgroup $H$ of the Lie group $G$ is a normal subgroup if $gHg^{-1} \subset H$ for all $g \in G$.

**Proposition 2.25** Let $H$ be the connected Lie subgroup of $G$ associated to the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The following assertions are equivalent.

1) $H$ is a normal subgroup of $G^o$.

2) $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

**Proof:** 1) $\implies$ 2). Let $X \in \mathfrak{h}$ and $g \in G^o$. For every $t \in \mathbb{R}$, the element $g \exp_G(tX)g^{-1} = \exp_G(t\text{Ad}(g)X)$ belongs to $H$: if we take the derivative at $t = 0$ we get $\text{Ad}(g)X \in \mathfrak{h}$, $\forall g \in G^o$. If we take the differential of $\text{(1)}$ at $g = e$ we have $\text{ad}(Y)X \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

2) $\implies$ 1). If $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have $\exp_G(Y) \exp_G(X) \exp_G(Y)^{-1} = \exp_G(\text{ad}YX) \in H$. Since $H$ is generated by $\exp_G(\mathfrak{h})$, we have $\exp_G(Y)H \exp_G(Y)^{-1} \subset H$ for all $Y \in \mathfrak{g}$ (see Remark 2.16 and Proposition 2.21). Since $\exp_G(\mathfrak{g})$ generates $G^o$ we have finally that $gHg^{-1} \subset H$ for all $g \in G^o$. $\square$

**Examples of Ideals:** The center of $\mathfrak{g}$: $\mathcal{Z}_\mathfrak{g} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$. The commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. The kernel $\text{ker}(\phi)$ of a morphism of lie algebra $\phi : \mathfrak{g} \to \mathfrak{h}$.
We can associate to any Lie algebra \( g \) two sequences \( g_i, g^i \) of ideals of \( g \). The *commutator series* of \( g \) is the non increasing sequence of ideals \( g^i \) with
\[
g^0 = g \quad \text{and} \quad g^{i+1} = [g^i, g^i]. \tag{2.16}
\]
The *lower central series* of \( g \) is the non increasing sequence of ideals \( g_i \) with
\[
g_0 = g \quad \text{and} \quad g_{i+1} = [g, g_i]. \tag{2.17}
\]

**Exercise 2.26** Show that the \( g_i, g^i \) are ideals of \( g \).

**Definition 2.27** We say that \( g \) is
- solvable if \( g^i = 0 \) for \( i \) large enough,
- nilpotent if \( g_i = 0 \) for \( i \) large enough,
- abelian if \( [g, g] = 0 \).

**Exercise 2.28** Let \( V \) be a finite dimensional vector space, and let \( \{0\} = V_0 \subset V_1 \subset \cdots V_n = V \) be a strictly increasing sequence of subspaces. Let \( g \) be the Lie subalgebra of \( gl(V) \) defined by \( g = \{X \in gl(V) \mid X(V_{k+1}) \subset V_k\} \).
- Show that the Lie algebra \( g \) is nilpotent.
- Suppose now that \( \dim V_k = k \) for any \( k = 0, \ldots, n \). Show then that the Lie algebra of \( h = \{X \in gl(V) \mid X(V_k) \subset V_k\} \) is solvable.

**Exercise 2.29** For a group \( G \), the subgroup generated by the commutators \( \text{ghg}^{-1}h^{-1}, g, h \in G \) is the derived subgroup, and is denoted by \( G' \).
- Show that \( G' \) is a normal subgroup of \( G \).
- If \( G \) is a connected Lie group, show that \( G' \) is the connected Lie subgroup associated to the ideal \( [g, g] \).

**Exercise 2.30** For any Lie group \( G \), show that its center \( Z_G := \{g \in G \mid hg = hg \ \forall h \in G\} \) is a closed normal subgroup with Lie algebra \( Z_g := \{X \in g \mid [X, Y] = 0, \ \forall Y \in g\} \).
- Show that a lie algebra \( g \) is solvable if and only if \( [g, g] \) is solvable.
- Let \( h \) be the Lie algebra of the group \( H \) defined in Exercise 2.28. Show that \( [h, h] \) is nilpotent, and that \( h \) is not nilpotent.

### 2.8 Group actions and quotients

Let \( M \) be a set equipped with an action of group \( G \). For each \( m \in M \) the \( G \)-orbit through \( m \) is defined as the subset
\[
G \cdot m = \{g \cdot m \mid g \in G\}. \tag{2.18}
\]
For each \( m \in M \), the stabilizer group at \( m \) is
\[
G_m = \{ g \in G \mid g \cdot m = m \}.
\] (2.19)

The \( G \)-action is free if \( G_m = \{ e \} \) for all \( m \in M \). The \( G \)-action is transitive if \( G \cdot m = M \) for some \( m \in M \). The set-theoretic quotient \( M/G \) corresponds to the quotient of \( M \) by the equivalence relation \( m \sim n \iff G \cdot m = G \cdot n \).

Let \( \pi : M \to M/G \) be the canonical projection.

**Topological side:** Suppose now that \( M \) is a topological space equipped with a continuous action of a topological\(^3\) group \( G \). Note that in this situation the stabilizers \( G_m \) are closed in \( G \).

We define for any subsets \( A, B \) of \( M \) the set
\[
G_{A,B} = \{ g \in G \mid (g \cdot A) \cap B \neq \emptyset \}.
\]

**Exercise 2.31** Show that \( G_{A,B} \) is closed in \( G \) when \( A, B \) are compact in \( M \).

We take on \( M/G \) the quotient topology: \( V \subset M/G \) is open if \( \pi^{-1}(V) \) is open in \( M \). It is the smallest topology that makes \( \pi \) continuous. Note that \( \pi : M \to M/G \) is then an open map: if \( U \) is open in \( M \), \( \pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U \) is also open in \( M \), which means that \( \pi(U) \) is open in \( M/G \).

**Definition 2.32** The (topological) \( G \)-action on \( M \) is proper when the subsets \( G_{A,B} \) are compact in \( G \) whenever \( A, B \) are compact subsets of \( M \).

This definition of proper action is equivalent to the condition that the map
\[
\psi : G \times M \to M \times M, \ (g, m) \mapsto (g \cdot m, m)
\]
is proper, i.e. \( \psi^{-1}(\text{compact}) = \text{compact} \). Note that the action of a compact group is always proper.

**Proposition 2.33** If a topological space \( M \) is equipped with a proper continuous action of a topological group \( G \). The quotient topology is Hausdorff, locally compact.

The proof is left to the reader. The main result is the following,

**Theorem 2.34** Let \( M \) be a manifold equipped with a smooth, proper and free action of a Lie group. Then the quotient \( M/G \) equipped with the quotient topology carries the structure of a smooth manifold. Moreover the projection \( \pi : M \to M/G \) is smooth, and any \( n \in M/G \) has an open neighborhood \( U \) such that
\[
\pi^{-1}(U) \xrightarrow{\sim} U \times G, \ m \mapsto (\pi(m), \phi_U(m))
\]
is a \( G \)-equivariant diffeomorphism. Here \( \phi_U : \pi^{-1}(U) \to G \) is an equivariant map: \( \phi_U(g \cdot m) = g\phi_U(m) \).

\(^3\)Here the topological spaces are Hausdorff and locally compact.
For a proof see [1][Section 2.3].

**Remark 2.35** Suppose that $G$ is a discrete group. For a proper and free action of $G$ on $M$ we have: any $m \in M$ has an neighborhood $V$ such that $gV \cap V = \emptyset$ for every $g \in G$, $g \neq e$. Theorem 2.34 is true when $G$ is a discrete group. The quotient map $\pi : M \to M/G$ is then a covering map.

The typical example we are interested in is the action of translation of a closed subgroup $H$ of a Lie group $G$: the action of $h \in H$ is $G \to G, \, g \to gh^{-1}$. Its an easy exercise to see that this action is free and proper. The quotient space $G/H$ is a smooth manifold and the action of translation $g \to ag$ of $G$ on itself descend to a smooth action of $G$ on $G/H$. The manifolds $G/H$ are called homogeneous manifolds: these are the manifold with a transitive action of a Lie group $G$.

**Stiefel manifolds, Grassmanians:** Let $V$ be a (real) vector space of dimension $n$. for any integer $k \leq n$, let $\text{Hom}(\mathbb{R}^k, V)$ be the vector space of homomorphism equipped with the following (smooth) $\text{GL}(V) \times \text{GL}(\mathbb{R}^k)$-action: for $(g, h) \in \text{GL}(V) \times \text{GL}(\mathbb{R}^k)$ and $f \in \text{Hom}(\mathbb{R}^k, V)$, we take $(g, h) \cdot f(x) = g(f(h^{-1}x))$ for any $x \in \mathbb{R}^k$. Let $S_k(V)$ be the open subset of $\text{Hom}(\mathbb{R}^k, V)$ formed by the one-to-one linear map: we have a natural identification of $S_k(V)$ with the set of families $\{v_1, \ldots, v_k\}$ of linearly independent vectors of $V$. Moreover $S_k(V)$ is stable under the $\text{GL}(V) \times \text{GL}(\mathbb{R}^k)$-action: the $\text{GL}(V)$-action on $S_k(V)$ is transitive, and the $\text{GL}(\mathbb{R}^k)$-action on $S_k(V)$ is free and proper. The manifold $S_k(V)/\text{GL}(\mathbb{R}^k)$ admit a natural identification with the set $\{E \text{ subspace of } V \mid \dim E = k\}$: it is the grassmanian manifold $\text{Gr}_k(V)$. On the other hand the action of $\text{GL}(V)$ on $\text{Gr}_k(V)$ is transitive so that

$$
\text{Gr}_k(V) \cong \text{GL}(V)/H
$$

where $H$ is the closed Lie subgroup of $\text{GL}(V)$ that fixes a subspace $E \subset V$ of dimension $k$.

**2.9 Adjoint group**

Let $\mathfrak{g}$ be a (real) Lie algebra. The automorphism group of $\mathfrak{g}$ is

$$
\text{Aut}(\mathfrak{g}) := \{ \phi \in \text{GL}(\mathfrak{g}) \mid [\phi([X, Y])] = [\phi(X), \phi(Y)], \, \forall X, Y \in \mathfrak{g} \} \quad (2.20)
$$

It is a closed subgroup of $\text{GL}(\mathfrak{g})$ with Lie algebra equal to

$$
\text{Der}(\mathfrak{g}) := \{ D \in \text{gl}(\mathfrak{g}) \mid [D([X, Y])] = [D(X), Y] + [X, D(Y)], \, \forall X, Y \in \mathfrak{g} \} \quad (2.21)
$$

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The subspace $\text{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is called the set of derivations of $\mathfrak{g}$. Thanks to the Jacobi identity we know that $\text{ad}(X) \in \text{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$. So the image of the adjoint map $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, that we denote $\text{ad}(\mathfrak{g})$, is a Lie subalgebra of $\text{Der}(\mathfrak{g})$.

**Definition 2.36** The adjoint group $\text{Ad}(\mathfrak{g})$ is the connected Lie subgroup of $\text{Aut}(\mathfrak{g})$ associated to the Lie subalgebra of $\text{ad}(\mathfrak{g}) \subset \text{Der}(\mathfrak{g})$. As an abstract group, it is the subgroup of $\text{Aut}(\mathfrak{g})$ generated by the elements $e^{\text{ad}(X)}$, $X \in \mathfrak{g}$.

Consider now a connected Lie group $G$, with Lie algebra $\mathfrak{g}$, and the adjoint map $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. In this case, $e^{\text{ad}(X)} = \text{Ad}(\exp_G(X))$ for any $X \in \mathfrak{g}$, so the image of $G$ by $\text{Ad}$ is equal to the group $\text{Ad}(\mathfrak{g})$. If $g \in G$ belongs to the kernel of $\text{Ad}$, we have $g \exp_G(X) g^{-1} = \exp_G(\text{Ad}(g)X) = \exp_G(X)$, so $g$ commutes with all the element of $\exp_G(\mathfrak{g})$. But since $G$ is connected, $\exp_G(\mathfrak{g})$ generates $G$. Finally we have proved that the kernel of $\text{Ad}$ is equal to the center $Z_G$ of the Lie group $G$.

It is worth to keep in mind the exact sequence of Lie group

$$0 \to Z_G \to G \to \text{Ad}(\mathfrak{g}) \to 0 \quad (2.22)$$

### 2.10 The Killing form

We have already defined the notions of solvable and nilpotent Lie algebra (see Def. 2.27). We have the following “opposite” notion.

**Definition 2.37** Let $\mathfrak{g}$ be (real) Lie algebra.

- $\mathfrak{g}$ is simple if $\mathfrak{g}$ is not abelian and does not contains ideals different from $\{0\}$ and $\mathfrak{g}$.
- $\mathfrak{g}$ is semi-simple if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$ which are simple (as Lie algebras).

The following remarks follows directly from the definition and give a first idea of the difference between “solvable” and “semi-simple”.

**Exercise 2.38** Let $\mathfrak{g}$ be a (real) Lie algebra.

- Suppose that $\mathfrak{g}$ is solvable. Show that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, and that $\mathfrak{g}$ possess a non-zero abelian ideal.
- Suppose that $\mathfrak{g}$ is semi-simple. Show that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and show that $\mathfrak{g}$ does not possess non-zero abelian ideals : in particular the center $Z_{\mathfrak{g}}$ is reduced to $\{0\}$.  


In order to give the characterization of semi-simplicity we define the Killing form of a Lie algebra \( g \). It is the symmetric \( \mathbb{R} \)-bilinear map \( B_g : g \times g \to \mathbb{R} \) defined by
\[
B_g(X,Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)),
\]
where \( \text{Tr} : \mathfrak{gl}(g) \to \mathbb{R} \) is the canonical trace map.

**Proposition 2.39** For \( \phi \in \text{Aut}(g) \) and \( D \in \text{Der}(g) \) we have
- \( B_g(\phi(X),\phi(Y)) = B_g(X,Y) \), and
- \( B_g(DX,Y) + B_g(X,DY) = 0 \) for all \( X, Y \in g \).
- We have \( B_g([X,Z],Y) = B_g(X,[Z,Y]) \) for all \( X, Y, Z \in g \).

**Proof**: If \( \phi \) is an automorphism of \( g \), we have \( \text{ad}(\phi(X)) = \phi \circ \text{ad}(X) \circ \phi^{-1} \) for all \( X \in g \) (see (2.20)). Then a) follows and b) comes from the derivative of a) at \( \phi = e \). For c) take \( D = \text{ad}(Z) \) in b). □

We recall now the basic interaction between the Killing form and the ideals of \( g \). If \( h \) is an ideal of \( g \), then
- the restriction of the Killing form of \( g \) on \( h \times h \) is the Killing form of \( h \),
- the subspace \( h^\perp = \{ X \in g \mid B_g(X,h) = 0 \} \) is an ideal of \( g \).
- the intersection \( h \cap h^\perp \) is an ideal of \( g \) with a Killing form identically equal to 0.

It was shown by E. Cartan that the Killing form gives criterion for semi-simplicity and solvability.

**Theorem 2.40** (Cartan’s Criterion for Semisimplicity) Let \( g \) be a (real) Lie algebra. The following statements are equivalent
- \( g \) is semi-simple,
- the Killing form \( B_g \) is non degenerate,
- \( g \) does not have non-zero abelian ideals.

The proof of Theorem 2.40 need the following characterization of the solvable Lie algebra. The reader will find a proof of the following theorem in [3][Section I].

**Theorem 2.41** (Cartan’s Criterion for Solvability) Let \( g \) be a (real) Lie algebra. The following statements are equivalent
- \( g \) is solvable,
- \( B_g(g,[g,g]) = 0 \).

We will not prove Theorem 2.41, but only use the following easy corollary.
Corollary 2.42 If \( \mathfrak{g} \) is a (real) Lie algebra with \( B_\mathfrak{g} = 0 \), then \([\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}\).

Before giving a proof of Theorem 2.40 let us show how Corollary 2.42 gives the implication \( b) \Rightarrow a) \) in Theorem 2.41.

If \( \mathfrak{g} \) is a Lie algebra with \( B_\mathfrak{g} = 0 \), then Corollary 2.42 tell us that \( \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \) is an ideal of \( \mathfrak{g} \) different from \( \mathfrak{g} \) with \( B_{\mathfrak{g}^1} = 0 \). If \( \mathfrak{g}^1 \neq 0 \), we do it again: \( \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] \) is an ideal of \( \mathfrak{g}^1 \) with \( B_{\mathfrak{g}^2} = 0 \). This induction ends after finite steps: let \( i \geq 0 \) such that \( \mathfrak{g}^i \neq 0 \) and \( \mathfrak{g}^{i+1} = 0 \). Then \( \mathfrak{g}^i \) is an abelian ideal of \( \mathfrak{g} \), and \( \mathfrak{g} \) is solvable. In the situation \( b) \) of Theorem 2.41, we have then that \([\mathfrak{g}, \mathfrak{g}] \) is solvable, so \( \mathfrak{g} \) is also solvable.

Proof of Theorem 2.40 using Corollary 2.42:

\( c) \Rightarrow b) \). The ideal \( \mathfrak{g}^\perp = \{ X \in \mathfrak{g} \mid B_\mathfrak{g}(X, \mathfrak{g}) = 0 \} \) of \( \mathfrak{g} \) as a zero Killing form. If \( \mathfrak{g}^\perp \neq 0 \) we know from the preceding remark that there exists \( i \geq 0 \) such that \( (\mathfrak{g}^\perp)^i \neq 0 \) and \( (\mathfrak{g}^\perp)^{i+1} = 0 \). We see easily that \( (\mathfrak{g}^\perp)^i \) is also an ideal of \( \mathfrak{g} \) (which is abelian). It gives a contradiction, then \( \mathfrak{g}^\perp = 0 \): the Killing form \( B_\mathfrak{g} \) is non-degenerate.

\( b) \Rightarrow a) \). We suppose now that \( B_\mathfrak{g} \) is non-degenerate. It gives first that \( \mathfrak{g} \) is not abelian. After we use the following dichotomy:

\( i) \) either \( \mathfrak{g} \) does not have ideals different from \{0\} and \( \mathfrak{g} \), hence \( \mathfrak{g} \) is simple,

\( ii) \) either \( \mathfrak{g} \) have an ideal \( \mathfrak{h} \) different from \{0\} and \( \mathfrak{g} \).

In case \( i) \) we have finish. In case \( ii) \), let us show that \( \mathfrak{h} \cap \mathfrak{g}^\perp \neq 0 \): since \( B_\mathfrak{g} \) is non-degenerate, it will implies that \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \). If \( \mathfrak{a} := \mathfrak{h} \cap \mathfrak{g}^\perp \neq 0 \), the Killing form on \( \mathfrak{a} \) is equal to zero. Following Corollary 2.42 there exists \( i \geq 0 \) such that \( \mathfrak{a}^i \neq 0 \) and \( \mathfrak{a}^{i+1} = 0 \). Moreover since \( \mathfrak{a} \) is an ideal of \( \mathfrak{g} \), \( \mathfrak{a}^i \) is also an ideal of \( \mathfrak{g} \). By considering a supplementary \( F \) of \( \mathfrak{a}^i \) in \( \mathfrak{g} \), every endomorphism \( \text{ad}(X), X \in \mathfrak{g} \) as the following matricial expression

\[
\text{ad}(X) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},
\]

with \( A : \mathfrak{a}^i \to \mathfrak{a}^i, B : F \to \mathfrak{a}^i \), and \( D : F \to F \). The zero term is due to the fact that \( \mathfrak{a}^i \) is an ideal of \( \mathfrak{g} \). If \( X_o \in \mathfrak{a}^i \), then

\[
\text{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.
\]

because \( \mathfrak{a}^i \) is an abelian ideal. Finally for every \( X \in \mathfrak{g} \),

\[
\text{ad}(X)\text{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}
\]

and then \( B_\mathfrak{g}(X, X_o) = 0 \). It is a contradiction since \( B_\mathfrak{g} \) is non-degenerate.
So if $h$ is an ideal different from $\{0\}$ and $g$, we have the $B_g$-orthogonal decomposition $g = h \oplus h^\perp$. Since $B_g$ is non-degenerate we see that $B_h$ and $B_{h^\perp}$ are non-degenerate, and we apply the dichotomy to the Lie algebras $h$ and $h^\perp$. After finite steps we obtain a decomposition $g = g_1 \oplus \ldots \oplus g_r$ where the $g_k$ are simple ideals of $g$.

Exercise 2.43 • For the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ show that $B_{\mathfrak{sl}(n, \mathbb{R})}(X, Y) = 2n \text{Tr}(XY)$. Conclude that $\mathfrak{sl}(n, \mathbb{R})$ is a semi-simple Lie algebra.

• For the Lie algebra $\mathfrak{su}(n)$ show that $B_{\mathfrak{su}(n)}(X, Y) = 2n \text{Re}(\text{Tr}(XY))$. Conclude that $\mathfrak{su}(n)$ is a semi-simple Lie algebra.

Exercise 2.44 $\mathfrak{sl}(n, \mathbb{R})$ is a simple Lie algebra.

Let $(E_{i,j})_{1 \leq i, j \leq n}$ be the canonical basis of $\mathfrak{gl}(\mathbb{R}^n)$. Consider a non-zero ideal $a$ of $\mathfrak{sl}(n, \mathbb{R})$. Up to a change of $a$ in $a^\perp$ we can assume that $\dim(a) \geq \frac{n^2 - 1}{2}$.

• Show that $a$ possess an element $X$ which is not diagonal.

• Compute $[X, E_{i,j}], E_{i,j}$ and conclude that some $E_{i,j}$ with $i \neq j$ belongs to $a$.

• Show that $E_{k,l}, E_{k,k} - E_{l,l} \in a$ when $k \neq l$. Conclude.

2.11 Complex Lie algebras

We worked out the notions of solvable, nilpotent, simple and semi-simple real Lie algebras. The definitions go through for Lie algebras defined over any field $k$, and all the result of section 2.10 are true for $k = \mathbb{C}$.

Let $\mathfrak{h}$ be a complex Lie algebra. The Killing form is here a symmetric $\mathbb{C}$-bilinear map $B_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by (2.23), where $\text{Tr} : \mathfrak{gl}_\mathbb{C}(\mathfrak{h}) \to \mathbb{C}$ is the trace defined on the $\mathbb{C}$-linear endomorphism of $\mathfrak{h}$.

Theorem 2.40 is valid for the complex Lie algebras: a complex Lie algebra is direct sum of simple ideals if and only if its Killing form is non-degenerate.

A usefull toll is the complexification of real Lie algebras. If $\mathfrak{g}$ is a real Lie algebra, the complexified vector space $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$ carries a canonical structure of complex Lie algebras. We see easily that the Killing forms $B_{\mathfrak{g}}$ and $B_{\mathfrak{g}_\mathbb{C}}$ coincide on $\mathfrak{g}$:

$$B_{\mathfrak{g}_\mathbb{C}}(X, Y) = B_{\mathfrak{g}}(X, Y) \quad \text{for all} \ X, Y \in \mathfrak{g}. \quad (2.24)$$
With (2.24) we see that a real Lie algebra $\mathfrak{g}$ is semi-simple if and only if the complex Lie algebra $\mathfrak{g}_C$ is semi-simple.

## 3 Semi-simple Lie groups

**Definition 3.1** A connected Lie group $G$ is semi-simple (resp. simple) if its Lie algebra $\mathfrak{g}$ is semi-simple (resp. simple).

If we use Theorem 2.40 and Proposition 2.25 we have the following equivalent characterization of semi-simple Lie group that will be used in the lecture of J. Maubon (see Proposition 6.3).

**Proposition 3.2** A connected Lie group $G$ is semi-simple if and only if $G$ does not have non-trivial connected normal abelian Lie subgroup.

In particular the center $Z_G$ of a semi-simple Lie group is discrete. We have the following refinement for the simple Lie groups.

**Proposition 3.3** A normal subgroup $A$ of a (connected) simple Lie Group $G$ which is not equal to $G$ belongs to the center $Z$ of $G$.

**Proof** : Let $A_o$ be subset of $A$ defined as follow : $a \in A_o$ if there exits a continuous curve $c(t)$ in $A$ with $c(0) = e$ and $c(1) = a$. Obviously $A_o$ is a path-connected subgroup of $G$, so according to Theorem 2.23 $A_o$ is a Lie subgroup of $G$. If $c(t)$ is continuous curve in $A$, $gc(t)g^{-1}$ is also a continuous curve in $A$ for all $g \in G$, and then $A_o$ is a normal subgroup of $G$. From Proposition 2.25 we know that the Lie algebra of $A_o$ is an ideal of $\mathfrak{g}$, hence is equal to $\{0\}$ since $\mathfrak{g}$ is simple and $A \neq G$. We have proved that $A_o = \{e\}$, which means that every continuous curve in $A$ is constant. For every $a \in A$ and all continous curve $\gamma(t)$ in $G$, the continuous curve $\gamma(t)a\gamma(t)^{-1}$ in $A$ must be constant. It proves that $A$ belongs to the center of $G$. □

We come back to the exact sequence (2.22).

**Lemma 3.4** If $\mathfrak{g}$ is a semi-simple Lie algebra, the vector space of derivation Der($\mathfrak{g}$) is equal to ad($\mathfrak{g}$).

**Proof** : Let $D$ be a derivation of $\mathfrak{g}$. Since $B_\mathfrak{g}$ is non-degenerate there exist a unique $X_D \in \mathfrak{g}$ such that $\text{Tr}(D\text{ad}(Y)) = B_\mathfrak{g}(X_D, Y)$, for all $Y \in \mathfrak{g}$.
Now we compute

\[ B_g([X_D, Y], Z]) = B_g(X_D, [Y, Z]) = \text{Tr}(D\text{ad}([Y, Z])) \]
\[ = \text{Tr}(D[\text{ad}(Y), \text{ad}(Z)]) \]
\[ = \text{Tr}([D, \text{ad}(Y)]\text{ad}(Z)) \quad (1) \]
\[ = \text{Tr}(\text{ad}(DY)\text{ad}(Z)) \quad (2) \]
\[ = B_g(DY, Z). \]

(1) is a general fact about the trace: \( \text{Tr}(A[B, C]) = \text{Tr}([A, B]C) \) for any \( A, B, C \in \mathfrak{g}(\mathfrak{g}) \). (2) uses the definition of a derivation (see (2.21)). Using now the non-degeneracy of \( B_g \) we get \( D = \text{ad}(X_D) \). □

The equality of Lie algebras \( \text{ad}(\mathfrak{g}) = \text{Der}(\mathfrak{g}) \) tells us that the adjoint group is equal to identity component of the automorphism group: \( \text{Ad}(\mathfrak{g}) = \text{Aut}(\mathfrak{g}) \).

**Lemma 3.5** If \( G \) is a (connected) semi-simple Lie group, it’s center \( Z_G \) is discrete and the adjoint group as zero center.

**Proof**: The center \( Z(G) \) is discrete because the semi-simple Lie algebra \( \mathfrak{g} \) as zero center. Let \( \text{Ad}(g) \) be an element of the center of \( \text{Ad}(\mathfrak{g}) \): we have

\[ \text{Ad}(\exp_G(X)) = \text{Ad}(g)\text{Ad}(\exp_G(X))\text{Ad}(g)^{-1} = \text{Ad}(g\exp_G(X)g^{-1}) = \text{Ad}(\exp_G(\text{Ad}(g)X)) \]

for any \( X \in \mathfrak{g} \). So \( \exp_G(-X)\exp_G(\text{Ad}(g)X) \in Z(G), \forall X \in \mathfrak{g} \). But since \( Z(G) \) is discrete it implies that \( \exp_G(X) = \exp_G(\text{Ad}(g)X), \forall X \in \mathfrak{g} : g \) commutes with any element of \( \exp_G(\mathfrak{g}) \). Since \( \exp_G(\mathfrak{g}) \) generates \( G \), we have finally that \( g \in Z(G) \) and so \( \text{Ad}(g) = 1 \). □

The important point here is that a (connected) semi-simple Lie group is a central extension by a discrete subgroup of a quasi-algebraic group. The Lie group \( \text{Aut}(\mathfrak{g}) \) is defined by finite polynomial identities in \( \text{GL}(\mathfrak{g}) \) : it is an algebraic group. And \( \text{Ad}(\mathfrak{g}) \) is a connected component of \( \text{Aut}(\mathfrak{g}) \) : it is a quasi-algebraic group. There is an important case where the Lie algebra structure impose some restriction on the center.

**Theorem 3.6 (Weyl)** Let \( G \) be a connected Lie group such that \( B_g \) is negative definite. Then \( G \) is a compact semi-simple Lie group and the center \( Z_G \) is finite.

There are many proofs, for example [2][Section II.6], [1][Section 3.9]. Here we only stress that the condition “\( B_{\mathfrak{g}} \) is negative definite” imposes that \( \text{Aut}(\mathfrak{g}) \) is a compact subgroup of \( \text{GL}(\mathfrak{g}) \), hence \( \text{Ad}(\mathfrak{g}) \) is compact. Now if we consider the exact sequence \( 0 \to Z_G \to G \to \text{Ad}(\mathfrak{g}) \to 0 \) we see that \( G \) is compact if and only if \( Z_G \) is finite.

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Definition 3.7 A real Lie algebra is compact if its Killing form is negative definite.

3.1 Cartan decomposition on subgroups of $GL(\mathbb{R}^n)$

Let $\text{Sym}_n$ be the vector subspace of $\mathfrak{gl}(\mathbb{R}^n)$ formed by the symmetric endomorphisms, and let $\text{Sym}_n^+$ be the open subspace of $\text{Sym}_n$ formed by the positive definite symmetric endomorphisms. Consider the exponential $e : \mathfrak{gl}(\mathbb{R}^n) \to GL(\mathbb{R}^n)$. We compute its differential.

Lemma 3.8 For any $X \in \mathfrak{gl}(\mathbb{R}^n)$, the tangent map $T_X e : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ is equal to $e^X \left( \frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)} \right)$. In particular, $T_X e$ is a singular map if and only if the adjoint map $\text{ad}(X) : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ has a non-zero eigenvalue belonging to $2i\pi \mathbb{Z}$.

Proof: Consider the smooth functions $F(s, t) = e^{s(X + tY)}$, and $f(s) = \frac{\partial F}{\partial t}(s, 0)$: we have $f(0) = 0$ and $f(1) = T_X e(Y)$. If we differentiate $F$ first with respect to $t$, and after with respect to $s$, we find that $f$ satisfies the differential equation $f'(s) = Y e^{sX} + X f(s)$ which equivalent to

$$(e^{-sX} f)' = e^{-sX} Y e^{-sX} = e^{-s\text{ad}(X)} Y.$$

Finally we find $f(1) = e^{X} \left( \int_0^1 e^{-s \text{ad}(X)} ds \right) Y$. □

It is easy exercise to show that exponential map realize a one-to-one map from $\text{Sym}_n$ onto $\text{Sym}_n^+$. The last Lemma tells us that $T_X e$ is not singular for every $X \in \text{Sym}_n$. So we have prove the

Lemma 3.9 The exponential map $A \mapsto e^A$ realizes a smooth diffeomorphism from $\text{Sym}_n$ onto $\text{Sym}_n^+$.

Let $O(\mathbb{R}^n)$ the orthogonal group : $k \in O(\mathbb{R}^n) \iff t^k k = I d$. Every $g \in GL(\mathbb{R}^n)$ decomposes in a unique manner as $g = kp$ where $k \in O(\mathbb{R}^n)$ and $p \in \text{Sym}_n^+$ is the square root of $t^g g$. The map $(k, p) \mapsto kp$ defines a smooth diffeomorphism from $O(\mathbb{R}^n) \times \text{Sym}_n^+$ onto $GL(\mathbb{R}^n)$. If we use Lemma 3.9, we get the following

Proposition 3.10 (Cartan decomposition) The map

$$O(\mathbb{R}^n) \times \text{Sym}_n \longrightarrow GL(\mathbb{R}^n)$$

$$(k, X) \longmapsto ke^X$$

is a smooth diffeomorphism.
We will now extend the Cartan decomposition to an algebraic$^4$ subgroup $G$ of $\mathrm{GL}(\mathbb{R}^n)$ which is stable under the transpose map. In other term $G$ is stable under the automorphism $\Theta_o : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ defined by

\[ \Theta_o(g) = t^* g^{-1}. \]  

(3.26)

The classical groups like $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $\mathrm{Sp}(\mathbb{R}^{2n})$ fall into this category.

The Lie algebra $g \subset \mathfrak{gl}(\mathbb{R}^n)$ of $G$ is stable under the transpose map, so we have $g = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = g \cap \mathfrak{o}(n, \mathbb{R})$ and $\mathfrak{p} = g \cap \mathrm{Sym}_n$.

**Lemma 3.11** Let $X \in \mathrm{Sym}_n$ such that $e^X \in G$. Then $e^{tX} \in G$ for every $t \in \mathbb{R}$ : in other word $X \in \mathfrak{p}$.

**Proof**: The element $e^X$ can be diagonalized : there exist $g \in \mathrm{GL}(\mathbb{R}^n)$ and a sequence of real number $\lambda_1 \ldots \lambda_n$ such that $e^{tX} = g \mathrm{Diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_n}) g^{-1}$ for all $t \in \mathbb{R}$ (here $\mathrm{Diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_n})$ is a diagonal matrix). From the hypothesis we have that $\mathrm{Diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_n})$ belongs to the algebraic group $g^{-1} G g$ when $t \in \mathbb{Z}$. Now it an easy fact that for any polynomial in $n$-variables $P$, if $\phi(t) = P(e^{t\lambda_1}, \ldots, e^{t\lambda_n}) = 0$ for all $t \in \mathbb{Z}$, then $\phi$ is identically equal to 0. So we have prove that $e^{tX} \in G$ for every $t \in \mathbb{R}$ whenever $e^X \in G$. \(\square\)

Consider the Cartan decomposition $g = k e^X$ of an element $g \in G$. Since $G$ is stable under the transpose map $e^{2X} = t g g^{-1} \in G$. From Lemma 3.11 we get that $X \in \mathfrak{p}$ and $k \in G \cap \mathfrak{O}(\mathbb{R}^n)$. Finally, if we restrict the diffeomorphism 3.25 to the submanifold $(G \cap \mathfrak{O}(\mathbb{R}^n)) \times \mathfrak{p} \subset \mathfrak{O}(\mathbb{R}^n) \times \mathrm{Sym}_n$ we get a diffeomorphism

\[ (G \cap \mathfrak{O}(\mathbb{R}^n)) \times \mathfrak{p} \sim \rightarrow G. \]

(3.27)

Let $K$ be the connected Lie subgroup of $G$ associated to the subalgebra $\mathfrak{k}$: $K$ is equal to the identity component of the compact Lie group $G \cap \mathfrak{O}(\mathbb{R}^n)$ hence $K$ is compact. If we restrict the diffeomorphism (3.27) to the identity component $G_o$ of $G$ we get the diffeomorphism

\[ K \times \mathfrak{p} \sim \rightarrow G_o. \]

(3.28)

### 3.2 Cartan involutions

We start again with the situation of a closed subgroup $G$ of $\mathrm{GL}(\mathbb{R}^n)$ stable under the transpose map $A \mapsto t^* A$. Then the lie algebra $g \subset \mathfrak{gl}(\mathbb{R}^n)$ of $G$ is also stable under the transpose map.

$^4$ i.e. defined by a finite number of polynomial equalities.
Proposition 3.12 If the Lie algebra $\mathfrak{g}$ as a center reduced to 0, then $\mathfrak{g}$ is semi-simple. In particular, the bilinear map $(X,Y) \mapsto B_{\mathfrak{g}}(X,^tY)$ defines a scalar product on $\mathfrak{g}$. Moreover if we consider the transpose map $D \mapsto ^tD$ on $\mathfrak{gl}(\mathfrak{g})$ defined by this scalar product, we have $\text{ad}(^tX) = ^t\text{ad}(X)$ for all $X \in \mathfrak{g}$.

Proof : Consider the scalar product on $\mathfrak{g}$ defined by $(X,Y)_g := \text{Tr}(^tXY)$ where $\text{Tr}$ is the canonical trace on $\text{gl}(\mathbb{R}^n)$. With the help of $(-, -)_g$, we have a transpose map $D \mapsto ^tD$ on $\mathfrak{gl}(\mathfrak{g})$: $(D(X), Y)_g = (X, ^tD(Y))_g$ for all $X, Y \in \mathfrak{g}$ and $D \in \mathfrak{gl}(\mathfrak{g})$. A small computation shows that $^t\text{ad}(X) = \text{ad}(^tX)$, and then $B_{\mathfrak{g}}(X, ^tY) = \text{Tr}^\prime(\text{ad}(X)^t\text{ad}(Y))$ is a symmetric bilinear map on $\mathfrak{g} \times \mathfrak{g}$ (here $\text{Tr}^\prime$ is the trace map on $\text{gl}(\mathfrak{g})$). If $\mathfrak{g}$ as zero center then $B_{\mathfrak{g}}(X, ^tX) > 0$ if $X \neq 0$. Let $D \mapsto ^tD$ be the transpose map on $\mathfrak{gl}(\mathfrak{g})$ defined by this scalar product. We have

$$B_{\mathfrak{g}}(\text{ad}(X)Y, ^tZ) = -B_{\mathfrak{g}}(Y, [X, ^tZ]) = B_{\mathfrak{g}}(Y, ^t[X, Z]),$$

for all $X, Y, Z \in \mathfrak{g}$: in other terms $\text{ad}(^tX) = ^t\text{ad}(X)$. □

Definition 3.13 A linear map $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ on a Lie algebra is an involution if $\tau$ is an automorphism of the Lie algebra $\mathfrak{g}$ and $\tau^2 = 1$.

When $\tau$ is an involution of $\mathfrak{g}$, we define the bilinear map

$$B^\tau(X, Y) := -B_{\mathfrak{g}}(X, \tau(Y))$$

(3.29)

which is symmetric. We have the decomposition

$$\mathfrak{g} = \mathfrak{g}^\tau _1 \oplus \mathfrak{g}^\tau _{-1}$$

(3.30)

where $\mathfrak{g}^\tau _{\pm 1} = \{X \in \mathfrak{g} \mid \tau(X) = \pm X\}$. Since $\tau \in \text{Aut}(\mathfrak{g})$ we have

$$[\mathfrak{g}^\tau _{\varepsilon}, \mathfrak{g}^\tau _{\varepsilon'}] \subset \mathfrak{g}^\tau _{\varepsilon \varepsilon'} \quad \text{for all} \quad \varepsilon, \varepsilon' \in \{1, -1\},$$

(3.31)

and

$$B_{\mathfrak{g}}(X, Y) = 0 \quad \text{for all} \quad X \in \mathfrak{g}^\tau _1, \ Y \in \mathfrak{g}^\tau _{-1}. \quad (3.32)$$

The subspace $^5\mathfrak{g}^\tau$ is a sub-algebra of $\mathfrak{g}$, $\mathfrak{g}^\tau _{-1}$ is a module for $\mathfrak{g}^\tau$ through the adjoint action, and the subspace $\mathfrak{g}^\tau$ and $\mathfrak{g}^\tau _{-1}$ are orthogonal with respect to $B^\tau$.

Definition 3.14 An involution $\theta$ on a Lie algebra $\mathfrak{g}$ is a Cartan involution if the symmetric bilinear map $B^\theta$ defines a scalar product on $\mathfrak{g}$.

---

$^5$We will just denote by $\mathfrak{g}^\tau$ the subalgebra $\mathfrak{g}^\tau _1$.  

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Note that the existence of a Cartan involution implies the semi-simplicity of the Lie algebra.

**Example**: \( \theta_o(X) = -tX \) is an involution on the Lie algebra \( \mathfrak{gl}(\mathbb{R}^n) \). We prove in Proposition 3.12 that if a Lie sub-algebra \( \mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n) \) is stable under the transpose map and has zero center, then the linear \( \theta_o \) restricted to \( \mathfrak{g} \) is a Cartan involution. It is the case, for example, of the subalgebras \( \mathfrak{sl}(n, \mathbb{R}) \) and \( \mathfrak{o}(p, q) \).

In the other direction, if a semi-simple Lie algebra \( \mathfrak{g} \) is equipped with a Cartan involution \( \theta \), a small computation shows that

\[
\,^t\text{ad}(X) = -\text{ad}(\theta(X)), \quad X \in \mathfrak{g},
\]

where \( A \mapsto \,^tA \) is the transpose map on \( \mathfrak{gl}(\mathfrak{g}) \) defined by the scalar product \( B^\theta \). So the subalgebra \( \text{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g}) \), which is isomorphic to \( \mathfrak{g} \), is stable under the transpose map. Conclusion: for a real Lie algebra \( \mathfrak{g} \) with zero center, the following statements are equivalent:

- \( \mathfrak{g} \) can be realized as a subalgebra of matrices stable under the transpose map,
- \( \mathfrak{g} \) is a semi-simple Lie algebra equipped with a Cartan involution.

In the next section, we will see that any real semi-simple Lie algebra has a Cartan involution.

### 3.3 Compact real forms

We have seen the notion of *complexification* of a real Lie algebra. In the other direction, a complex Lie algebra \( \mathfrak{h} \) can be considered as a real Lie algebra and we denote it by \( \mathfrak{h}^\mathbb{R} \). The behavior of the Killing form with respect to this operation is

\[
B_{\mathfrak{h}^\mathbb{R}}(X, Y) = 2\Re(B_{\mathfrak{h}}(X, Y)) \quad \text{for all } X, Y \in \mathfrak{h}.
\]

(3.33)

For a complex Lie algebra \( \mathfrak{h} \), we speak of *anti-linear involutions*: it is the involutions of \( \mathfrak{h}^\mathbb{R} \) which anti-commute with the complex multiplication. If \( \tau \) is an anti-linear involution of \( \mathfrak{h} \) then \( \mathfrak{h}^\tau = i\mathfrak{h}^\tau \), i.e.

\[
\mathfrak{h} = \mathfrak{h}^\tau \oplus i\mathfrak{h}^\tau.
\]

(3.34)

**Definition 3.15** A real form of a complex Lie algebra \( \mathfrak{h} \) is a real subalgebra \( \mathfrak{a} \subset \mathfrak{h}^\mathbb{R} \) such that \( \mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a} \), i.e. \( \mathfrak{a}^\mathbb{C} \simeq \mathfrak{h} \). A compact real form of a complex Lie algebra is a real form which is a compact Lie algebra (see Def. 3.7).
For any real form $a$ of $h$, there exist a unique anti-linear involution $\tau$ such that $h^\tau = a$. Equation (3.34) tells us that $\tau \mapsto h^\tau$ is a one-to-one correspondence between the anti-linear involutions of $h$ and the real forms of $h$. If $a$ is a real form of a complex Lie algebra $h$, we have like in (2.24) that

$$B_a(X, Y) = B_h(X, Y) \quad \text{for all } X, Y \in a$$

(3.35)

In particular $B_h$ take real values on $a \times a$.

**Lemma 3.16** Let $\theta$ an anti-linear involution of a complex Lie algebra $h$. $\theta$ is a Cartan involution of the real Lie algebra $h^\mathbb{R}$ if and only if $h^\theta$ is a compact real form of $h$.

**Proof**: Consider the decomposition $h = h^\theta \oplus i h^\theta$ and $X = a + ib$ with $a, b \in h^\theta$. We have

$$B_{h^\theta}(X, \theta(X)) = 2(B_h(a, a) + B_h(b, b)) \quad (1)$$

$$= 2(B_{h^\theta}(a, a) + B_{h^\theta}(b, b)) \quad (2).$$

(1) and (2) are consequence of (3.33) and (3.35). So we see that $-B_{h^\theta}$ is positive definite on $h^\mathbb{R}$ if and only if the Killing form $B_{h^\theta}$ is negative definite.

□

**Example**: the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is a real form of $\mathfrak{sl}(n, \mathbb{C})$. The complex Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ as other real forms like

- $\mathfrak{su}(n) = \{ X \in \mathfrak{sl}(n, \mathbb{C}) \mid \mathcal{I}X + X = 0 \}$,
- $\mathfrak{su}(p, q) = \{ X \in \mathfrak{sl}(n, \mathbb{C}) \mid \mathcal{I}p,qX + I_{p,q}X = 0 \}$, where $I_{p,q} = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}$.

Here the anti-linear involutions are respectively $\sigma(X) = \mathcal{I}X$, $\sigma_a(X) = -\mathcal{I}X$, and $\sigma_b(X) = -I_{p,q}X$. Among the real forms $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(n)$, $\mathfrak{su}(p, q)$ of $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{su}(n)$ is the only one which is compact.

Let $g$ be a real Lie algebra, and let $\sigma$ be the anti-linear involution of $g^\mathbb{C}$ associated to the real form $g$. We have a one-to-one correspondence

$$\tau \mapsto u(\tau) := (g^\mathbb{C})^{\tau \circ \sigma}$$

(3.36)

between the set of involution of $g$ and the set of real forms of $g^\mathbb{C}$ which are $\sigma$-stable. If $\tau$ is an involution of $g$, we consider its $\mathbb{C}$-linear extension to $g^\mathbb{C}$ (that we still denote by $\tau$). The composite $\tau \circ \sigma = \sigma \circ \tau$ is then an anti-linear involution of $g^\mathbb{C}$ which commutes with $\sigma$: hence the real form $u(\tau) := (g^\mathbb{C})^{\tau \circ \sigma}$ is stable under $\sigma$. If $a$ is a real form on $g^\mathbb{C}$ defined by a anti-linear involution $\rho$ which commutes with $\sigma$, then $\sigma \circ \rho$ is a $\mathbb{C}$-linear involution on $g^\mathbb{C}$ which commutes with $\sigma$: then it is the complexification of an involution $\tau$ on $g$, and we have $a = u(\tau)$.
Proposition 3.17 Let \( \mathfrak{g} \) be a real semi-simple Lie algebra. Let \( \tau \) be an involution of \( \mathfrak{g} \) and let \( \mathfrak{u}(\tau) \) be the real form of \( \mathfrak{g}_C \) defined by (3.36). The following statements are equivalents

- \( \tau \) is a Cartan involution of \( \mathfrak{g} \),
- \( \mathfrak{u}(\tau) \) is compact real form of \( \mathfrak{g}_C \) (which is \( \sigma \)-stable).

**Proof**: If \( \mathfrak{g} = \mathfrak{g}^\tau \oplus \mathfrak{g}^\tau_{-1} \) is the decomposition related to the eigen-spaces of \( \tau \) then \( \mathfrak{u}(\tau) = \mathfrak{g}^\tau \oplus i \mathfrak{g}^\tau_{-1} \). Take \( X = a + ib \in \mathfrak{u}(\tau) \) with \( a \in \mathfrak{g}^\tau \) and \( b \in \mathfrak{g}^\tau_{-1} \). We have

\[
B_{\mathfrak{u}(\tau)}(X, X) = B_{\mathfrak{g}_C}(X, X) \quad (1)
\]

\[
= B_{\mathfrak{g}}(a, a) - B_{\mathfrak{g}}(b, b) \quad (2)
\]

\[
= -B_{\mathfrak{g}}(\tilde{X}, \tilde{X}),
\]

where \( \tilde{X} = a + b \in \mathfrak{g} \). (1) is due to (3.35). In (2) we use (2.24) and the fact that \( \mathfrak{g}^\tau \) and \( \mathfrak{g}^\tau_{-1} \) are \( B_{\mathfrak{g}} \)-orthogonal. Then we see that \( B_{\mathfrak{u}(\tau)} \) is negative definite if and only if \( B_{\mathfrak{g}} \) is positive definite. \( \square \)

Now we give the way we can prove that a real semi-simple Lie algebra \( \mathfrak{g} \) has a Cartan involution. Let \( \mathfrak{g}_C \) be the complexification of \( \mathfrak{g} \) and let \( \sigma \) the anti-linear involution of \( \mathfrak{g}_C \) corresponding to the real form \( \mathfrak{g} \). We now from Proposition 3.17 that it is equivalent to look to the \( \sigma \)-stable compact real forms of \( \mathfrak{g}_C \). We use first the following fundamental fact.

Theorem 3.18 Any complex semi-simple Lie algebra has a compact real form.

A proof can be found in [3][Section 7.1]. The existence of a \( \sigma \)-stable compact real form is given by the following

Lemma 3.19 Let \( \tau : \mathfrak{g}_C \to \mathfrak{g}_C \) be anti-linear involution corresponding to a compact real form of \( \mathfrak{g}_C \). There exists \( \phi \in \text{Aut}(\mathfrak{g}_C) \) such that the anti-linear involution \( \phi \tau \phi^{-1} \) commutes with \( \sigma \). Hence \( \phi \tau \phi^{-1} \big|_{\mathfrak{g}} \) is a Cartan involution of \( \mathfrak{g} \).

**Proof**: The complex vector space \( \mathfrak{g}_C \) is equipped with the hermitian metric : \( (X, Y) \to B_{\mathfrak{g}_C}(X, \tau(Y)) \). It easy to check that \( \tau \sigma \) belongs to the intersection

\[
\text{Aut}(\mathfrak{g}_C) \cap \{ \text{hermitian endomorphism} \} = \{ \phi \in \text{Aut}(\mathfrak{g}_C) \mid \tau \phi \tau = \phi^{-1} \}
\]

(3.37)

\( \rho = (\tau \sigma)^2 \) is positive definite. Following Lemma 3.11, the one parameter subgroup \( r \in \mathbb{R} \mapsto \rho^r \) belongs to the identity component \( \text{Aut}(\mathfrak{g}_C)_o \) (since \( \text{Aut}(\mathfrak{g}_C) \) is an algebraic subgroup of \( GL((\mathfrak{g}_C)^\mathbb{R}) \)). We leave as an exercise to check that \( \rho^r \) commutes with \( \tau \sigma \) for all \( r \in \mathbb{R} \). Since \( \tau \rho^r \tau = \rho^{-r} \) (see (3.37)) it is easy to see that \( \rho^r \tau \rho^{-r} \) commutes with \( \sigma \) if \( r = \frac{1}{4} \). \( \square \)
3.4 Cartan decomposition at the group level

Let $G$ be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$. So we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}^{\theta,1}$ is a $\mathfrak{k}$-module. Let $K$ be the connected Lie subgroup of $G$ associated to $\mathfrak{k}$. This section is devoted to the proof of the following

**Theorem 3.20**

(a) $K$ is a closed subgroup of $G$

(b) the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp_G(X)$ is a diffeomorphism onto

(c) $K$ contains the center $Z$ of $G$

(d) $K$ is compact if and only if $Z$ is finite

(e) there exists a Lie group automorphism $\Theta$ of $G$, with $\Theta^2 = 1$ and with differential $\theta$

(f) the subgroup of $G$ fixed by $\Theta$ is $K$.

**Proof** : The Lie group $\hat{G} = \text{Ad}(\mathfrak{g})$ which is equal to the image of $G$ by the adjoint action is the identity component of Aut($\mathfrak{g}$). The Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ which is equal to the subspace of derivations Der($\mathfrak{g}$) $\subset \mathfrak{gl}(\mathfrak{g})$ is stable under the transpose map $A \mapsto tA$ on $\mathfrak{gl}(\mathfrak{g})$ associated to the scalar product $B_\theta$ on $\mathfrak{g}$ (since $-t\text{ad}(X) = \text{ad}(\theta(X))$). Since $\hat{G}$ is generated by $e^{\text{ad}(X)}$, $X \in \mathfrak{g}$, $\hat{G}$ is stable under the group morphism $A \mapsto tA^{-1}$. We have $\hat{\mathfrak{k}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{p}}$ where $\hat{\mathfrak{k}} = \{ A \in \hat{\mathfrak{g}} \mid tA = -A \}$ and $\hat{\mathfrak{p}} = \{ A \in \hat{\mathfrak{g}} \mid tA = A \}$. We have of course $\hat{\mathfrak{g}} = \text{ad}(\mathfrak{g})$, $\hat{\mathfrak{k}} = \text{ad}(\mathfrak{k})$ and $\hat{\mathfrak{p}} = \text{ad}(\mathfrak{p})$. Let $\hat{K}$ be the compact Lie group equal to $\hat{G} \cap O(\mathfrak{g})$ : its Lie algebra is $\hat{\mathfrak{k}}$. Since Aut($\mathfrak{g}$) is an algebraic subgroup of GL($\mathfrak{g}$), (3.28) applies here and gives the diffeomorphism

$$\hat{K} \times \hat{\mathfrak{p}} \longrightarrow \hat{G} \quad (3.38)$$

$$(k, A) \longmapsto ke^A.$$

We consider the *closed* Lie subgroup

$$K := \text{Ad}^{-1}(\hat{K})$$

of $G$ : its Lie algebra is $\mathfrak{k}$. By definition $K$ contains the center $Z = \text{Ad}^{-1}(\text{Id})$ of $G$. If we take the pull-back of (3.38) through Ad : $G \to \hat{G}$ we get the diffeomorphism

$$K \times \mathfrak{p} \longrightarrow G \quad (3.39)$$

$$(k, X) \longmapsto k \exp_G(X),$$

which proves that $K$ is connected since $G$ is connected : hence $K$ is the connected Lie subgroup of $G$ associated to the Lie subalgebra $\mathfrak{k}$. Finally $Z$
belongs to $K$ and $K/Z \simeq \hat{K}$ is compact: the point (a), (b), (c) and (d) are proved.

Let $\Theta : G \to G$ defined by $\Theta(k \exp(X)) = k \exp(-X)$ for $k \in K$ and $X \in \mathfrak{p}$. We have obviously $\Theta^2 = 1$ and $\text{Ad}(\Theta(g)) = t \text{Ad}(g)^{-1}$. If we take $g_1, g_2$ in $G$ we see that

$$\text{Ad}(\Theta(g_1 g_2) \Theta(g_2)^{-1} \Theta(g_1)^{-1}) = (t \text{Ad}(g_1) \text{Ad}(g_2)^{-1}) (t \text{Ad}(g_2)^{-1}) (t \text{Ad}(g_1)^{-1})$$

$$= 1.$$  

So $\Theta(g_1 g_2) \Theta(g_2)^{-1} \Theta(g_1)^{-1} \in Z$ for every $g_1, g_2$ in $G$. Since $G$ is connected and $Z$ is discrete it gives $\Theta(g_1 g_2) \Theta(g_2)^{-1} \Theta(g_1)^{-1} = 1$: (e) and (f) are proved.

\[\square\]

## 4 Invariant connections

A connection $\nabla$ on the tangent bundle $\mathbf{T}M$ of a manifold $M$ is a differential linear operator

$$\nabla : \Gamma(\mathbf{T}M) \to \Gamma(\mathbf{T}^*M \otimes \mathbf{T}M)$$

(4.40)
satisfying the Leibniz’s rule: $\nabla(fs) = df \otimes s + f \nabla s$ for every $f \in C^\infty(M)$ and $s \in \Gamma(\mathbf{T}M)$. Here $\Gamma(-)$ denotes the space of sections of the corresponding bundle. The contraction of $\nabla s$ by $v \in \Gamma(\mathbf{T}M)$ is a vectors field on $M$ denoted $\nabla_v s$.

The torsion of a connection $\nabla$ on $\mathbf{T}M$ is the $(2, 1)$-tensor $T^\nabla$ defined by

$$T^\nabla(u, v) = \nabla_u v - \nabla_v u - [u, v],$$

(4.41)

for all vectors fields $u, v$ on $M$. The curvature of a connection $\nabla$ on $\mathbf{T}M$ is the $(3, 1)$-tensor $R^\nabla$ defined by

$$R^\nabla(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u, v]},$$

(4.42)

for all vectors fields $u, v$ on $M$. Here $R^\nabla(u, v)$ is a differential operator acting on $\Gamma(\mathbf{T}M)$ which commutes with the multiplication by functions on $M$; so it is defined by the action of an element of $\Gamma(\text{End}(\mathbf{T}M))$. For convenience we denote $R^\nabla(u, v) \in \Gamma(\text{End}(\mathbf{T}M))$ this element. We can specialize the curvature tensor $R^\nabla$ at each $m \in M$: $R^\nabla_m(U, V) \in \text{End}(\mathbf{T}_m M)$ for each $U, V \in \mathbf{T}_m M$.

### 4.1 Connections invariant under a group action

Suppose now that a Lie group $G$ acts smoothly on a manifold $M$. The corresponding action of $G$ on the vectors spaces $C^\infty(M)$, $\Gamma(\mathbf{T}M)$ and $\Gamma(\mathbf{T}^*M)$
is

\[ g \cdot f(m) = f(g^{-1}m), \quad m \in M, \]

\[ g \cdot s(m) = T_{g^{-1}m}g(s(g^{-1}m)), \quad m \in M, \]

and

\[ g \cdot \xi(m) = \xi(g^{-1}m) \circ T_m g^{-1}, \quad m \in M, \]

for every \( f \in C^\infty(M) \), \( s \in \Gamma(TM) \), \( \xi \in \Gamma(T^*M) \) and \( g \in G \). Here we denote \( T_n g \) the differential at \( n \in M \) of the smooth map \( m \mapsto gm \).

Note that the G-action is compatible with the canonical bracket \( \langle - , - \rangle : \Gamma(T^*M) \times \Gamma(TM) \to C^\infty(M) : \langle g \cdot \xi, g \cdot s \rangle = g \cdot \langle \xi, s \rangle \). We still denote \( g \) the action of \( g \in G \) on \( \Gamma(T^*M \otimes TM) \).

**Definition 4.1** A connection \( \nabla \) on the tangent bundle \( TM \) is G-invariant if

\[ g \nabla g^{-1} = \nabla, \quad \text{for every } g \in G. \quad (4.43) \]

This condition is equivalent to asking that \( \nabla_{g \cdot v}(g \cdot s) = g \cdot (\nabla_v s) \) for every vectors fields \( s, v \) on \( M \) and \( g \in G \).

For every \( X \in \mathfrak{g} \), the differential of \( t \to \exp_G(tX) \) at \( t = 0 \) defines linear operators on \( C^\infty(M) \), \( \Gamma(TM) \) and \( \Gamma(T^*M) \), all denoted \( \mathcal{L}(X) \). For \( f \in C^\infty(M) \) and \( s \in \Gamma(M) \) we have \( \mathcal{L}(X)f = X_M(f) \) and \( \mathcal{L}(X)s = [X_M, s] \) where \( X_M \) is the vectors field on \( M \) defined at Section 2.4. The map \( X \mapsto \mathcal{L}(X) \) is a Lie algebra morphism:

\[ [\mathcal{L}(X), \mathcal{L}(Y)] = \mathcal{L}([X, Y]), \quad \text{for all } X, Y \in \mathfrak{g}. \quad (4.44) \]

**Definition 4.2** The moment of a G-invariant connection \( \nabla \) on \( TM \) is the linear endomorphism of \( \Gamma(TM) \) defined by

\[ \Lambda(X) = \mathcal{L}(X) - \nabla_{X_M}, \quad X \in \mathfrak{g}. \quad (4.45) \]

Since the \( \Lambda(X), X \in \mathfrak{g} \) commute with the multiplication by functions on \( M \), we can and we will see the \( \Lambda(X) \) as element of \( \Gamma(\text{End}(TM)) \). The invariance condition (4.43) tells us that the map \( \Lambda : \mathfrak{g} \to \Gamma(\text{End}(TM)) \) is G-equivariant:

\[ \Lambda(\text{Ad}(g)Y) = g \Lambda(Y) g^{-1}, \quad \text{for every } (g, Y) \in G \times \mathfrak{g}. \quad (4.46) \]

If we differentiate (4.46) at \( g = 1 \), we get

\[ \Lambda([X, Y]) = [\mathcal{L}(X), \Lambda(Y)], \quad \text{for every } X, Y \in \mathfrak{g}. \quad (4.47) \]
We finish this section by computing the values of the torsion and curvature on vectors fields generated by the $G$-action. A direct computation gives

$$T^\nabla(X_M, Y_M) = [X, Y]_M - \Lambda(X)Y_M + \Lambda(Y)X_M.$$  (4.48)

for every $X, Y \in \mathfrak{g}$. Now using (4.44) and (4.47) we have for the curvature

$$R^\nabla(X_M, Y_M) = \Lambda([X, Y]) - \Lambda([X, Y]).$$  (4.49)

for every $X, Y \in \mathfrak{g}$.

### 4.2 Invariant Levi-Civita connections

Suppose now that the manifold $M$ carries a Riemannian structure invariant under the Lie group $G$. The scalar product of two vectors fields $u, v$ is just denote $(u, v)$. The invariance condition is that the equality

$$g \cdot (u, v) = (g \cdot u, g \cdot v) \quad (4.50)$$

holds in $C^\infty(M)$ for $u, v \in \Gamma(TM)$ and $g \in G$. If we differentiate (4.50) at $g = e$ we get

$$X_M(u, v) = ([X_M, u], v) + (u, [X_M, v]). \quad (4.51)$$

Let $\nabla^{LC}$ the Levi-Civita connection on $M$ relative to the Riemannian metric: it is the unique torsion free connection which preserve the Riemannian metric. Since the Riemannian metric is $G$-invariant, the connection $g\nabla^{LC}g^{-1}$ preserves also the Riemannian metric and is torsion free for every $g \in G$. Hence $\nabla^{LC}$ is a $G$-invariant connection. Recall that for $u, v \in \Gamma(TM)$ the vectors field $\nabla^ {LC} u v$ is defined by the relations

$$2(\nabla^ {LC} u v, w) = ([u, v], w) - ([v, w], u) + ([w, u], v) + u(v, w) + v(u, w) - w(u, v). \quad (4.52)$$

If we take $u = X_M$ and $v = Y_M$ in the former relation we find with the help of (4.51) that

$$2(\nabla^ {LC} X_M Y_M, w) = ([X, Y]_M, w) - w(X_M, Y_M). \quad (4.53)$$

So we have proved the

**Proposition 4.3** For any $X, Y \in \mathfrak{g}$ we have

$$\nabla^ {LC} X_M Y_M = \frac{1}{2} \left( [X, Y]_M - \nabla (X_M, Y_M) \right).$$

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5 Invariant connections on homogeneous spaces

The main references here are [2] and [4].

5.1 Existence of invariant connections

We work here with the homogeneous space $M = G/H$ where $H$ is a closed subgroup with Lie algebra $\mathfrak{h}$ of a Lie group $G$. We denote by $\pi : G \to M$ the quotient map. The quotient vector space $\mathfrak{g}/\mathfrak{h}$ is equipped with the $H$-action induced by the adjoint action. We consider the space $G \times \mathfrak{g}/\mathfrak{h}$ with the following $H$-action:

$$h \cdot (g, X) = (gh^{-1}, \text{Ad}(h)X).$$

This action is proper and free so the quotient $G \times_H \mathfrak{g}/\mathfrak{h}$ is a smooth manifold: the class of $(g, X)$ in $G \times_H \mathfrak{g}/\mathfrak{h}$ is denoted $[g, X]$. We use here the following $G$-equivariant isomorphism

$$G \times_H \mathfrak{g}/\mathfrak{h} \to TM$$

$$[g, X] \mapsto \frac{d}{dt} \pi(g \exp_G(tX))|_{t=0}.\tag{5.54}$$

Using the $G$-equivariant isomorphism (5.54) we have

$$\Gamma(TM) \xrightarrow{\sim} (C^\infty(G) \otimes \mathfrak{g}/\mathfrak{h})^H$$

$$s \mapsto \tilde{s}\tag{5.55}$$

and

$$\Gamma(\text{End}(TM)) \xrightarrow{\sim} (C^\infty(G) \otimes \text{End}(\mathfrak{g}/\mathfrak{h}))^H$$

$$A \mapsto \tilde{A}.\tag{5.56}$$

For example, the vectors field $X_M, X \in \mathfrak{g}$ give rise through the isomorphism (5.55) to the functions $\tilde{X}_M(g) = -\text{Ad}(g)^{-1}X \mod \mathfrak{g}/\mathfrak{h}$.

Let $\nabla$ be a $G$-invariant connection on the tangent bundle $TM$, and let $\Lambda : \mathfrak{g} \to \Gamma(\text{End}(TM))$ be the corresponding $G$-equivariant map defined by (4.45). Let $\tilde{\Lambda} : \mathfrak{g} \to (C^\infty(G) \otimes \text{End}(\mathfrak{g}/\mathfrak{h}))^H$ be the map $\Lambda$ through the identifications (5.56). The mapping $\tilde{\Lambda}$ is $G$-equivariant and each $\tilde{\Lambda}(X), X \in \mathfrak{g}$ is a $H$-equivariant map from $G$ to $\text{End}(\mathfrak{g}/\mathfrak{h})$:

$$\tilde{\Lambda}(\text{Ad}(g)X)(g') = \tilde{\Lambda}(X)(g'^{-1}g)$$

$$\tilde{\Lambda}(X)(gh^{-1}) = \text{Ad}(h) \circ \tilde{\Lambda}(X)(g) \circ \text{Ad}(h)^{-1}\tag{5.57}$$

for every $g, g' \in G, h \in H$ and $X \in \mathfrak{g}$. 

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Definition 5.1  Let $\lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h})$ the map defined by $\lambda(X) = \tilde{\Lambda}(X)(e)$.

From (5.57), we see that $\lambda$ is $H$-equivariant and determines completely $\Lambda$:

$$\tilde{\Lambda}(X)(g) = \lambda(\text{Ad}(g)^{-1}X).$$

(5.58)

So we have proved that the $G$-invariant connection $\nabla$ is uniquely determined by the mapping $\lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h})$.

Proposition 5.2  (a) The linear map $\lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h})$ is $H$-equivariant, and when restrict to $\mathfrak{h}$ is equal to the adjoint action.

(b) A linear map $\lambda$ satisfying the conditions of (a) determine a unique $G$-invariant connection on $T(G/H)$.

Proof: 

Let $\lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h})$ be a linear map satisfying the conditions (a), and let $\Lambda : \mathfrak{g} \to \Gamma(\text{End}(TM))$ be the corresponding $G$-equivariant map defined by $\lambda$: for $\bar{g} \in M$ and $X \in \mathfrak{g}$ the map $\Lambda(X)\bar{g}$ is

$$T_{\bar{g}}M \to T_{\bar{g}}M$$

$$[g,Y] \mapsto [g, \lambda(g^{-1}X)Y].$$

By definition we have $\Lambda(X)\bar{g} = \mathcal{L}(X)\bar{g} - \nabla_{X_M}$. So if $X_M(m) = 0$, we have $\Lambda(X)_m = \mathcal{L}(X)_m$ as endomorphism of $T_mM$. When $m = t \in M, X_M(t) = 0$ if and only if $X \in \mathfrak{h}$, and then the endomorphism $\mathcal{L}(X)_t$ of $T_tM = \mathfrak{g}/\mathfrak{h}$ is equal to $\text{ad}(X)$. So $\lambda(X) = \text{ad}(X)$ for all $X \in \mathfrak{h}$. The first point is proved.

Let $\lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h})$ be a linear map satisfying the conditions (a), and let $\Lambda : \mathfrak{g} \to \Gamma(\text{End}(TM))$ be the corresponding $G$-equivariant map defined by $\lambda$: for $\bar{g} \in M$ and $X \in \mathfrak{g}$ the map $\Lambda(X)\bar{g}$ is

$$T_{\bar{g}}M \to T_{\bar{g}}M$$

$$[g,Y] \mapsto [g, \lambda(g^{-1}X)Y].$$

By definition we have $\Lambda(X)\bar{g} = \mathcal{L}(X)\bar{g}$ when $X_M(\bar{g}) = 0$. Finally we define a $G$-invariant connection $\nabla$ on $TM$ by posing for any vectors field $v, s$ on $M$ and $m \in M$:

$$(\nabla_v s)|_m = (\mathcal{L}(X)s)|_m - \Lambda(X)_m(s|_m),$$

where $X \in \mathfrak{g}$ is chosen so that $X_M(m) = s|_m$. □

Counter example: Consider the homogeneous space $^7 M = \text{SL}(2, \mathbb{R})/H$ where

$H = \{ \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) \mid a, b \in \mathbb{R}, a \neq 0 \}.$

We are going to prove that the tangent bundle $TM$ does not carry a $G$-invariant connection. Consider the basis $(e, f, g)$ of $\mathfrak{sl}(2, \mathbb{R})$, where

$$e = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \quad f = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad g = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right).$$

$^6$X_M(m) = 0 if and only if m is fixed by the 1-parameter subgroup exp_G(\mathbb{R}X)

$^7$The manifold M is diffeomorphic to the circle
We have \([e, f] = 2e, [g, f] = -2g,\) and \([e, g] = -f.\) Since the Lie algebra of \(H\) is \(\mathfrak{h} := \mathbb{R}f \oplus \mathbb{R}g,\) we use the identifications \(\mathfrak{sl}(2, \mathbb{R})/\mathfrak{h} \cong \mathbb{R}e\) and \(\text{End}(\mathfrak{sl}(2, \mathbb{R})/\mathfrak{h}) \cong \mathbb{R}.\) For the induced adjoint action of \(\mathfrak{h}\) on \(\mathbb{R}e\) we have: \(\widehat{\text{ad}}(f) = -2\) and \(\widehat{\text{ad}}(g) = 0.\) We are interested in a map \(\lambda : \mathfrak{sl}(2, \mathbb{R}) \to \mathbb{R}\) satisfying

- \(\lambda\) is \(H\)-equivariant, i.e. \(\lambda([X, Y]) = 0\) whenever \(X \in \mathfrak{h}.\)
- \(\lambda(X) = \text{ad}(X)\) for \(X \in \mathfrak{h}.\)

These conditions can not be fulfilled since the first point gives \(\lambda(f) = \lambda([g, e]) = 0,\) and with the second point we have \(\lambda(f) = \widehat{\text{ad}}(f) = -2.\)

The previous example shows that some homogeneous spaces do not have invariant connection. For the remaining of Section 5 we work with the following

**Assumption 5.3** The subalgebra \(\mathfrak{h}\) has a \(H\)-invariant supplementary subspace \(\mathfrak{m}\) in \(\mathfrak{g}\).

In [4] the homogeneous spaces \(G/H\) are called of reductive type when the assumption 5.3 is satisfied. This hypothesis guarantees the existence of invariant connections as we will see now.

Let \(X \mapsto X_m\) denotes the \(H\)-equivariant projection onto \(\mathfrak{m}\) relatively to \(\mathfrak{h}.\) This projection induces an \(H\)-equivariant isomorphism \(\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}.\) Then a \(G\)-invariant connection on \(T(G/H)\) is determined uniquely by a linear \(H\)-equivariant mapping \(\lambda : \mathfrak{g} \to \text{End}(\mathfrak{m})\) which extends the adjoint action \(\text{ad} : \mathfrak{h} \to \text{End}(\mathfrak{m}).\) So \(\lambda\) is completely determined by its restriction

\[
\lambda|_{\mathfrak{m}} : \mathfrak{m} \to \text{End}(\mathfrak{m})
\]

The following definition defines a family \(\nabla^a, a \in \mathbb{R}\) of invariant connection when \(G/H\) is an homogeneous spaces of reductive type.

**Definition 5.4** Let \(G/H\) be an homogeneous spaces of reductive type. For any \(a \in \mathbb{R},\) we define a \(H\)-equivariant mapping \(\lambda^a : \mathfrak{g} \to \text{End}(\mathfrak{m})\) by \(\lambda^a(X) = \text{ad}(X)\) for \(X \in \mathfrak{h}\) and

\[
\lambda^a(X)Y = a[X, Y]_m \quad \text{for } X, Y \in \mathfrak{m}.
\]

We denote \(\nabla^a\) the \(G\)-invariant connection associated to \(\lambda^a.\)

The connection \(\nabla^0\) is called the canonical connection. Note that the connections \(\nabla^a, a \in \mathbb{R}\) are distincts except when the bracket \([-[,-]]_m = 0\) is identically equal to 0.

We finish this section by looking to the torsion free invariant connections.
Proposition 5.5 Let $\nabla$ be a $G$-invariant connection on $T(G/H)$ and let $\lambda : g \rightarrow \text{End}(m)$ be the associated $H$-equivariant map. The connection $\nabla$ is torsion free if and only if we have

$$[X, Y]_m = \lambda(X)Y - \lambda(Y)X \quad \text{for all } X, Y \in m. \quad (5.59)$$

Condition (5.59) is equivalent to asking that

$$\lambda(X)Y = \frac{1}{2}[X, Y]_m + b(X, Y), \quad (5.60)$$

where $b : m \times m \rightarrow m$ is a symmetric bilinear map.

**Proof**: The vectors fields $X_M, X \in g$ generates the tangent space of $M = G/H$, hence the connection is torsion free if and only if $T^\nabla(X_M, Y_M) = 0$ for every $X, Y \in g$. Following (4.48) the condition is

$$[X, Y]_M = \Lambda(X)Y_M - \Lambda(Y)X_M \quad \text{for all } X, Y \in g. \quad (5.61)$$

A small computations shows that the function $\overline{X}_M : G \rightarrow m$ associated to the vectors field $X_M$ via the isomorphism (5.55) is defined by $\overline{X}_M(g) = -[\text{Ad}(g)^{-1}X]_m$. For the function $\lambda\overline{X}_M : G \rightarrow m$ we have

$$\lambda\overline{X}_M(g) = -\lambda(\text{Ad}(g)^{-1}X)[\text{Ad}(g)^{-1}Y]_m, \quad \text{for all } X, Y \in g.$$

So condition (5.61) is equivalent to

$$[X, Y]_m = \lambda(X)Y_m - \lambda(Y)X_m \quad \text{for all } X, Y \in g. \quad (5.62)$$

It is now easy to see that (5.62) is equivalent to (5.59) and (5.60). $\square$

Corollary 5.6 Let $\nabla^a$ be the $G$-invariant connection introduced in Definition 5.4. After Proposition 5.5, we see that

- if the bracket $[-,-]_m$ is identically equal to 0: $\nabla^a = \nabla^0$ is torsion free.
- if the bracket $[-,-]_m$ is not equal to 0, $\nabla^a$ is torsion free if and only if $a = \frac{1}{2}$.

5.2 Geodesics on an homogeneous spaces

Let $\nabla$ be a $G$-invariant connection on $M = G/H$ associated to a $H$-equivariant map $\lambda : m \rightarrow \text{End}(m)$. A smooth curve $\gamma : I \rightarrow M$ is a geodesic relative to $\nabla$ if

$$\nabla_{\gamma'}(\gamma') = 0. \quad (5.63)$$
The last condition can be understood locally as follow. Let \( t_0 \in I \) and let \( \mathcal{U} \subset M \) be a neighborhood of \( \gamma(t_0) \) : if \( \mathcal{U} \) is small enough there exists a vectors field \( v \) on \( \mathcal{U} \) such that \( v(\gamma(t)) = \gamma'(t) \) for \( t \in I \) closed to \( t_0 \). Then for \( t \) near \( t_0 \), condition (5.63) is equivalent to

\[
\nabla_{x}v|_{\gamma(t)} = 0.
\]

(5.64)

**Proposition 5.7** For \( X \in \mathfrak{m} \), we consider the curve \( \gamma_{X}(t) = \pi(\exp_{G}(tX)) \) on \( G/H \), where \( \pi : G \to G/H \) denotes the canonical projection and \( \exp_{G} \) is the exponential map of the lie group \( G \). The curve \( \gamma_{X} \) is a geodesic for the connection \( \nabla \), if and only if \( \lambda(X)X = 0 \).

**Proof**: The vectors field \( X_{M} \), which is defined on \( M \), satisfies \( X_{M}(\gamma_{X}(t)) = \gamma'_{X}(t) \) for \( t \in \mathbb{R} \). Since \( \nabla_{X_{M}}X_{M} = \Lambda(X)X_{M} \) we get

\[
\nabla_{X_{M}}X_{M}|_{\gamma_{X}(t)} = [\gamma_{X}(t), \lambda(X)X] \quad \text{in} \quad TM \simeq G \times_{H} \mathfrak{m},
\]

so the conclusion follows. \( \square \)

**Corollary 5.8** Let \( \nabla^a \) be the connection on \( G/H \) defined in Def. (5.4). Then

- the maximal geodesic are the curves \( \gamma(t) = \pi(g\exp_{G}(tX)) \), where \( g \in G \) and \( X \in \mathfrak{m} \).
- the exponential mapping \( \exp_{\bar{e}} : \mathfrak{m} \to G/H \) is defined by \( \exp_{\bar{e}}(X) = \pi(\exp_{G}(X)) \).

### 5.3 Levi-civita connection on homogeneous spaces

We suppose now that one has a \( \text{Ad}(H) \)-invariant scalar product on the supplementary subspace \( \mathfrak{m} \) of \( \mathfrak{h} \), that we just denote \( (-,-) \).

We define a \( G \)-invariant Riemannian metric \( (-,-)_{M} \) on \( M = G/H \) as follows. Using the identification \( G \times_{H} \mathfrak{m} \simeq TM \), we take \( (v,w)_{M} = (X,Y) \) for the tangent vector \( v = [g,X] \) and \( w = [g,Y] \) of \( T_{g}M \). Let \( \nabla^{LC} \) the Levi-Civita connection on \( M \) relative to this Riemannian metric. Since the Riemannian metric is \( G \)-invariant, the connection \( \nabla^{LC} \) is \( G \)-invariant (see Section 4.2). Let \( \lambda^{LC} : \mathfrak{g} \to \text{End}(\mathfrak{m}) \) the \( H \)-equivariant map associated to the connection \( \nabla^{LC} \). Since \( \nabla^{LC} \) preserves the metric we have

\[
\lambda^{LC}(X) \in \text{so}(\mathfrak{m}) \quad \text{for every} \quad X \in \mathfrak{g}.
\]

(5.65)

Here \( \text{so}(\mathfrak{m}) \) denotes the Lie algebra of the orthogonal group \( \text{SO}(\mathfrak{m}) \).
Proposition 5.9 The map $\lambda^{LC}$ is determined by the following conditions: $\lambda^{LC}(X) = \text{ad}(X)$ for $X \in \mathfrak{h}$ and $\lambda^{LC}(X)Y = \frac{1}{2}[X,Y]_m + b^{LC}(X,Y)$ for $X,Y \in \mathfrak{m}$, where $b^{LC} : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ is the symmetric bilinear map defined by

$$2(b^{LC}(X,Y), Z) = ([X,Y]_m, Z) + ([X,Z]_m, Y).$$

(5.66)

Proof: We use the decomposition (5.60) together with the fact that $(\lambda^{LC}(X)Y, Z) = -(Y, \lambda^{LC}(X)Z)$ for $X, Y, Z \in \mathfrak{m}$. It gives

$$(b^{LC}(X,Y), Z) + (b^{LC}(Z,X), Y) = -\frac{1}{2}((X,Y)_m, Z) + ([X,Z]_m, Y)).$$

(5.67)

Now if we interchanging $X, Y, Z$ in $Z, X, Y$ and after in $Y, Z, X$, we get two other equalities. If we sum them with alternative sign we get on the LHS the term $2(b^{LC}(X,Y), Z)$ and on the RHS we get $-([X,Z]_m, Y) - ([Y,Z]_m, X)$. □

Example. Suppose that $G$ is a compact Lie group and $H$ is a closed subgroup. Let $(\cdot, \cdot)_g$ be a $G$-invariant scalar product on $\mathfrak{g}$. We take $\mathfrak{m}$ as the orthogonal subspace of $\mathfrak{h}$. We take on $G/H$ the $G$-invariant Riemannian metric coming from the scalar product $(\cdot, \cdot)_g$ restricted to $\mathfrak{m}$. In this situation we see that the bilinear map $b^{LC}$ vanishes. So, the Levi-Civita connection on $G/H$ is equal to the connection $\nabla^{1/2}$ (see Definition 5.4). Then we know after Corollary 5.8 that the geodesics on $G/H$ are of the form $\gamma(t) = \pi(g \exp_G(tX))$ with $X \in \mathfrak{m}$.

5.4 Levi-civita connection on symmetric spaces of the non-compact type.

We come back to the situation of section 3.4. Let $G$ be a connected semi-simple Lie group with algebra $\mathfrak{g}$. Let $\Theta : G \rightarrow G$ be an involution of $G$ such that $\theta = d\Theta$ is a Cartan involution of $\mathfrak{g}$. At the Lie algebra level we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the closed connected subgroup $K = G^\Theta$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. We denote by $X \mapsto X_\mathfrak{r}$ and $X \mapsto X_\mathfrak{p}$ the projections such that $X = X_\mathfrak{r} + X_\mathfrak{p}$ for $X \in \mathfrak{g}$.

We consider here the homogeneous space $M = G/K$. Since $\text{Ad}(K)$ is compact, the vector subspace $\mathfrak{p} \simeq T_eM$ carries $\text{Ad}(K)$-invariant scalar product that induces $G$-invariant Riemannian metric on $M$. One of them is of particular interest: the Killing form $B_g$.

Proposition 5.10 The Levi-Civita connection $\nabla^{LC}$ on $G/K$ associated to any $\text{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ coincides with the canonical connection $\nabla^0$ (see Definition 5.4).
Proof: Since \([p, p] \subset \mathfrak{k}\), we have \([X, Y]_p = 0\) when \(X, Y \in p\). After Proposition 5.9, we have then \(\lambda^{LC}(X) = \text{ad}(X_\mathfrak{k})\) for \(X \in p\), which means that \(\nabla^{LC} = \nabla^0\). □

In this setting Corollary 5.8 gives

**Corollary 5.11** • All the maximal geodesic on \(G/K\) are defined over \(\mathbb{R}\) : the Riemannian manifold \(G/K\) is completed.

• the exponential mapping \(\exp_e : \mathfrak{p} \rightarrow G/K\) is defined by \(\exp_e(X) = \pi(\exp_G(X))\).

We will now compute the curvature tensor \(R^{LC}\) of \(\nabla^{LC}\). By definition \(R^{LC}\) is a 2-form on \(M\) with values in \(\text{End}(TM)\). We take \(X, Y \in \mathfrak{g}\) and look at \(R^{LC}(X_M, Y_M) \in \Gamma(\text{End}(TM))\) or equivalently at the function \(R^{LC}(X_M, Y_M) : G \rightarrow \text{End}(\mathfrak{p})\) : (4.49) gives

\[
R^{LC}(X_M, Y_M)(g) = -[\lambda^{LC}(g^{-1}X), \lambda^{LC}(g^{-1}X)] + \lambda^{LC}([g^{-1}X, g^{-1}Y])
\]

\[
= -[\text{ad}((g^{-1}X)_\mathfrak{g}), \text{ad}((g^{-1}X)_\mathfrak{g})] + \text{ad}((g^{-1}X, g^{-1}Y)_\mathfrak{g})
\]

\[
= \text{ad}([((g^{-1}X)_\mathfrak{p}, (g^{-1}Y)_\mathfrak{p})].
\]

At the point \(\bar{e} \in M\), the curvature tensor \(R^{LC}\) specializes in a map \(R^{LC}_\bar{e} : \mathfrak{p} \times \mathfrak{p} \rightarrow \text{End}(\mathfrak{p})\).

**Proposition 5.12** For \(X, Y \in \mathfrak{p}\), we have

\[
R^{LC}_\bar{e}(X, Y) = \text{ad}([X, Y]).
\]

We will now compute the sectional curvature when the Riemannian metric on \(M = G/K\) is induced by the scalar product on \(\mathfrak{p}\) defined by the Killing form \(B_\mathfrak{g}\). The sectional curvature is a real function \(\kappa\) defined on the Grassmannian \(\text{Gr}_2(TM)\) of 2-dimensional vector subspaces of \(TM\) (see []). If \(S \subset T_{\bar{e}}M\) is generated by two orthogonal vectors \(X, Y \in \mathfrak{p}\) we have

\[
\kappa(S) = \frac{B_\mathfrak{g}(R^{LC}_\bar{e}(X, Y)X, Y)}{\|X\|^2\|Y\|^2} \quad [1]
\]

\[
= \frac{B_\mathfrak{g}([X, Y], X), Y)}{\|X\|^2\|Y\|^2} \quad [2]
\]

\[
= -\frac{\|X, Y\|^2}{\|X\|^2\|Y\|^2} \quad [3].
\]

[1] is the definition of the sectional curvature. [2] is due to Proposition 5.12, and [3] follows from the \(\mathfrak{g}\)-invariance of the Killing form and also to the fact that \(-B_\mathfrak{g}\) defines a scalar product on \(\mathfrak{k}\).
Conclusion: The homogeneous manifold $G/K$, when equipped with the Riemannian metric induced by the Killing form, is a completed Riemannian manifold with negative sectional curvature.

5.5 Flats on symmetric spaces of the non-compact type

Let $M$ be a Riemannian manifold and $N$ a connected submanifold of $M$. The submanifold $N$ is called totally geodesic if for each geodesic $\gamma : I \to M$ of $M$ we have for $t_0 \in I$

$$\left( \gamma(t_0) \in N \quad \text{and} \quad \gamma'(t_0) \in T_{\gamma(t_0)}N \right) \implies \gamma(t) \in N \quad \text{for all} \quad t \in I.$$

We consider now the case of the symmetric space $G/K$ equipped with the Levi-Civita connection $\nabla^0$.

Theorem 5.13 The set of totally geodesic submanifolds of $G/K$ containing $\bar{e}$ is in one to one correspondence with the subspaces $s \subset p$ satisfying $[s, [s, s]] \subset s$.

For a proof see [2][Section IV.7]. The correspondence works as follows. If $S$ is a totally geodesic submanifold of $G/K$, one has $R^n_{\gamma}(u,v)w \in T_nS$ for each $n \in S$ and $u, v, w \in T_nS$. Then when $\bar{e} \in S$ one takes $s := T_{\bar{e}}S$ : the last condition becomes $[[u, v], w] \in s$ for $u, v, w \in s$.

In the other direction, if $s$ is a Lie triple system one sees that $g_s := [s, s] \oplus s$ is a Lie subalgebra of $g$. Let $G_s$ be the connected Lie subgroup of $G$ associated to $g_s$. One can prove that the orbit $S := G_s \cdot \bar{e}$ is a closed submanifold of $G/K$ which is totally geodesic.

We are interested now in the "flats" of $G/K$. These are the totally geodesic submanifold with a curvature tensor that vanishes identically. If we use the last Theorem one sees that the set of flats in $G/K$ passing through $\bar{e}$ is in one to one correspondence with the set of abelian subspaces of $p$.

We will conclude this section with the

Lemma 5.14 Let $s, s'$ be two maximal abelian subspaces of $p$. Then there exists $k_o \in K$ such that $\text{Ad}(k_o)s = s'$. In particular the subspaces $s$ and $s'$ have the same dimension.

Proof: First step. Let us show that for any maximal abelian subspace $s$ there exists $X \in s$ such that the stabilizer $g^X := \{ Y \in g \mid [X, Y] = 0 \}$ satisfies $g^X \cap p = s$. We look at the commuting family $\text{ad}(X)$, $X \in s$ of

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Such subspace of $p$ are called Lie triple system.

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linear map on $\mathfrak{g}$. All these maps are *symmetric* relative to the scalar product $B^θ := -B_θ(\cdot, θ(\cdot))$, so they can be diagonalized all together: there exists a finite set $α_1, \cdots, α_r$ of non-zero linear maps from $\mathfrak{s}$ to $\mathbb{R}$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{i=1}^r \mathfrak{g}_{α_i},$$

with $\mathfrak{g}_{α_i} = \{ X ∈ \mathfrak{g} \mid [Z, X] = α_i(Z)X, \forall Z ∈ \mathfrak{s} \}$. Here the subspace $\mathfrak{s}$ belongs to $\mathfrak{g}_0 = \{ X ∈ \mathfrak{g} \mid [Z, X] = 0, \forall Z ∈ \mathfrak{s} \}$. Since we have assume that $\mathfrak{s}$ is maximal abelian in $\mathfrak{p}$ we have $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$. For any $X ∈ \mathfrak{s}$ we have obviously $\mathfrak{g}^X = \mathfrak{g}_0 \oplus \sum_{α_i(X) = 0} \mathfrak{g}_{α_i}$.

If we take $X ∈ \mathfrak{s}$ such that $α_i(X) \neq 0$ for all $i$, then $\mathfrak{g}^X = \mathfrak{g}_0$, hence $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$.

**SECOND STEP.** Take $X ∈ \mathfrak{s}$ and $X' ∈ \mathfrak{s}'$ such that $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{s}$ and $\mathfrak{g}^{X'} \cap \mathfrak{p} = \mathfrak{s}'$. We define the function $f(k) = B_θ(X', \text{Ad}(k)X), k ∈ K$. Let $k_0$ be a critical point of $f$ (which exits since $\text{Ad}(K)$ is compact). If we differentiate $f$ at $k_0$ we get $B_θ(X', [Y, \text{Ad}(k_o)X]) = 0, \forall Y ∈ \mathfrak{k}$. Since $B_θ$ is $\mathfrak{g}$-invariant we get $B_θ([X', \text{Ad}(k_o)X], Y) = 0, \forall Y ∈ \mathfrak{k}$, so $[X', \text{Ad}(k_o)X] = 0$. Since $\mathfrak{g}^{\text{Ad}(k_o)X} \cap \mathfrak{p} = \text{Ad}(k_o)(\mathfrak{g}^X \cap \mathfrak{p}) = \text{Ad}(k_o)\mathfrak{s}$, the last equality gives $X' ∈ \text{Ad}(k_o)\mathfrak{s}$. And since $\text{Ad}(k_o)\mathfrak{s}$ is an abelian subspace of $\mathfrak{p}$ we have then

$$\text{Ad}(k_o)\mathfrak{s} ⊂ \mathfrak{g}^{X'} \cap \mathfrak{p} ⊂ \mathfrak{s}'.\n$$

Finally since $\mathfrak{s}, \mathfrak{s}'$ are two maximal abelian subspaces, the last equality insures that $\text{Ad}(k_o)\mathfrak{s} = \mathfrak{s}'$. □

**References**


