NOTES ON COSECTION LOCALIZED VIRTUAL CYCLES

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Abstract. We briefly describe the construction of cosection localized virtual cycles; we present two recent applications of this construction: the GW-invariants of stable morphisms with fields.

Recently, aimed at understanding Lee-Parker's work on Gromov-Witten invariants of surfaces [LP], jointly with Kiem we constructed a new class of virtual cycles: the *cosection localized virtual cycles* [KL2]. Given a Deligne-Mumford stack $\mathcal{M}$, suppose it has a perfect obstruction theory and its obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ admits a cosection (a homomorphism to $\mathscr{O}_{\mathcal{M}}$)

$$\sigma : \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}};$$

let $\mathcal{M}(\sigma)$ be the loci where σ is non-surjective; then we constructed a localized virtual cycle of $\mathcal{M}$ supported on $\mathcal{M}(\sigma)$:

$$[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\mathcal{M}(\sigma).$$

This construction has two significant applications. Given a cosection σ, in case the degeneracy loci $\mathcal{M}(\sigma)$ is proper, the resulting localized virtual cycle $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ is properly supported, and its degree defines a Gromov-Witten type invariants of non-proper moduli space $\mathcal{M}$.

As an application, in a joint work with Chang [CL], we use cosection localized virtual cycle to construct the Gromov-Witten invariants of the moduli of stable morphisms to $\mathbb{P}^4$ with fields. The significance of these invariants are that they coincide up to signs with the Gromov-Witten invariants of quintic Calabi-Yau threefolds [CL]. Our work generalizes the Guffin-Sharpe-Witten model to all genus.

The cosection localization technique can be used to construct reduced virtual cycles by modifying the obstruction theory. The first example is the reduced Gromov-Witten invariants of polarized K3 surfaces, introduced by Okunkov-Pandharipande [OP], also see [MPT].

We expect that this will have mANY applications to studying virtual cycles of moduli spaces.

1. Cosection localized virtual cycles

The cosection localized virtual cycles take the following form. Let $p : \mathcal{M} \to \mathcal{S}$ be a representable morphism from a DM-stack $\mathcal{M}$ to a smooth Artin stack $\mathcal{S}$, both locally of finite type, that has a perfect relative obstruction theory

$$E \longrightarrow L_{\mathcal{M}/\mathcal{S}}.$$

Here E is a derive object on $\mathcal{M}$ locally quasi-isomorphic to a two-term complex of locally free sheaves placed at $[-1, 0]$, and $L_{\mathcal{M}/\mathcal{S}}$ is the cotangent complex of $\mathcal{M} \to \mathcal{S}$

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(cf. [BF]). We define the relative obstruction sheaf of $\mathcal{M}\to\mathcal{S}$ be the cohomology sheaf

$$\mathcal{O}b_{\mathcal{M}/\mathcal{S}} = H^1(E^\vee); \tag{1.1}$$

we define its (absolute) obstruction sheaf be the cokernel

$$\mathcal{O}b_{\mathcal{M}} = \operatorname{coker}\{p^*\Omega_{\mathcal{S}}^\vee \longrightarrow \mathcal{O}b_{\mathcal{M}/\mathcal{S}}\},$$

where the arrow in the bracket is the composite

$$p^*\Omega_{\mathcal{S}}^\vee \longrightarrow p^*L_{\mathcal{S}}^\vee \longrightarrow L_{\mathcal{M}/\mathcal{S}}^\vee[1] \longrightarrow E^\vee[1] \longrightarrow H^1(E^\vee) = \mathcal{O}b_{\mathcal{M}/\mathcal{S}},$$

where the first arrow uses that $\mathcal{S}$ is smooth; the second arrow is from the distinguished triangle $p^*L_{\mathcal{S}} \to L_{\mathcal{M}} \to L_{\mathcal{M}/\mathcal{S}} \to$.

Definition 1.1. We define a meromorphic cosection of $\mathcal{O}b_{\mathcal{M}}$ be a homomophism $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathscr{O}_U$ defined on an open subset $U \subset \mathcal{M}$. We call σ a cosection if $U = \mathcal{M}$.

For a meromorphic cosection, we define its degeneracy loci be

$$\mathcal{M}(\sigma) = \{x \in U \mid \sigma|_x : \mathcal{O}b_{\mathcal{M}}|_x \to \mathbf{k}(x) \text{ is zero}\} \cup (\mathcal{M} - U).$$

Theorem 1.2 (Cosection localized virtual cycles [KL2])**.** *Let $p : \mathcal{M} \to \mathcal{S}$ with perfect relative obstruction theory be as stated. Suppose $\mathcal{O}b_{\mathcal{M}}$ admits a meromorphic cosection with degeneracy loci $\mathcal{M}(\sigma)$. Then $\mathcal{M}$ has a cosection localized virtual cycle in the Chow group of $\mathcal{M}(\sigma)$:*

$$[\mathcal{M}]^{\mathrm{vir}}_\sigma \in A_*\mathcal{M}(\sigma).$$

In case the cosection σ is understood, we often use the subscript "loc" to replace σ; i.e. we write $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ instead of $[\mathcal{M}]^{\mathrm{vir}}_\sigma$.

The cosection localized virtual cycle is a lift of the ordinary virtual cycle $[\mathcal{M}]^{\mathrm{vir}} \in A_*\mathcal{M}$.

Theorem 1.3 (Comparison [KL2])**.** *Let $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$ be the inclusion, then under push-froward,*

$$\iota_*[\mathcal{M}]^{\mathrm{vir}}_\sigma = [\mathcal{M}]^{\mathrm{vir}} \in A_*\mathcal{M}.$$

We comment that when $\mathcal{M}(\sigma)$ is proper but $\mathcal{M}$ is not, we can use $[\mathcal{M}]^{\mathrm{vir}}_\sigma$ to substitute $[\mathcal{M}]^{\mathrm{vir}}$ to define the Gromov-Witten type invariants of the stack $\mathcal{M}$.

Like the ordinary virtual cycle, the localized virtual cycles remain constant in family in naturally arisen situations. Let

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \tilde{\mathcal{M}} \\ \big\downarrow & & \big\downarrow \\ 0 & \xrightarrow{\ \tau\ } & T \end{array} \tag{1.2}$$

be a Cartesian square of DM stacks over $\mathcal{S}$, where $0 \in T$ is a pointed smooth curve; let $\tilde{p} : \tilde{\mathcal{M}} \to \mathcal{S}$ be the projection, representable, extending the $p : \mathcal{M} \to \mathcal{S}$ stated before. Suppose there is a perfect relative obstruction theory $F \to L_{\tilde{\mathcal{M}}/\mathcal{S}}$

compatible to $E \to L_{\mathcal{M}/\mathcal{S}}$, given by a homomorphism of distinguished triangles in $D(\mathcal{M})$:

$$
\begin{array}{ccccccc}
F|_{\mathcal{M}} & \xrightarrow{g} & E & \longrightarrow & \mathcal{O}_{\mathcal{M}}[1] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
L_{\tilde{\mathcal{M}}/\mathcal{S}}|_{\mathcal{M}} & \xrightarrow{h} & L_{\mathcal{M}/\mathcal{S}} & \longrightarrow & L_{\mathcal{M}/\tilde{\mathcal{M}}} & \xrightarrow{+1} & .
\end{array} \tag{1.3}
$$

As before, we denote by $\mathcal{O}b_{\tilde{\mathcal{M}}/\mathcal{S}} = h^1((F)^\vee)$ the relative obstruction sheaf and define its (absolute) obstruction sheaf like (1.1), thus we have the exact sequence

$$
\mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}b_{\tilde{\mathcal{M}}}|_{\mathcal{M}} \longrightarrow 0. \tag{1.4}
$$

We suppose there is an open $\tilde{U} \subset \tilde{\mathcal{M}}$ and a homomorphism

$$\tilde{\sigma} : \mathcal{O}b_{\tilde{\mathcal{M}}}|_{\tilde{U}} \longrightarrow \mathcal{O}_{\tilde{U}}.$$

We let $U = \tilde{U} \times_{\tilde{\mathcal{M}}} \mathcal{M}$; let $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathscr{O}_U$ be the composition of $\mathcal{O}b_{\mathcal{M}} \to \mathcal{O}b_{\tilde{\mathcal{M}}}|_{\mathcal{M}}$ with $\tilde{\sigma}|_U$, and we let $\tilde{\mathcal{M}}(\tilde{\sigma})$ be the loci where either $\tilde{\sigma}$ is undefined or not surjective. Note that $\mathcal{M}(\sigma) = \tilde{\mathcal{M}} \times_T 0$.

Let

$$\tau^! : A_*\tilde{\mathcal{M}}(\tilde{\sigma}) \longrightarrow A_*\mathcal{M}(\sigma)$$

be the Gysin map associated to the square (1.2).

Theorem 1.4 (Defromation invariance [KL2])**.** *Let the notation be as stated; let*

$$[\tilde{\mathcal{M}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\tilde{\mathcal{M}}(\tilde{\sigma}) \quad \textit{and} \quad [\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\mathcal{M}(\sigma)$$

be the cosection localized virtual cycles. Then $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} = \tau^![\tilde{\mathcal{M}}]^{\mathrm{vir}}_{\mathrm{loc}}$.

The construction of the cosection localized virtual cycles stems from a reduction of virtual normal cone of $\mathcal{M}$. Following [BF], given the perfect relative obstruction theory $E \to L_{\mathcal{M}/\mathcal{S}}$, we obtain a virtual normal cone in the bundle stack

$$\mathbf{C} \subset h^1/h^0(E^\vee)$$

on $\mathcal{M}$; letting s be the zero section of $h^1/h^0(E^\vee)$, and letting $s^!$ be the Gysin map by intersecting with the zero section [Kre], we obtain the virtual cycle

$$[\mathcal{M}]^{\mathrm{vir}} = s^![\mathbf{C}] \in A_*\mathcal{M}.$$

Now assume we have a cosection $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_U$. It produces a cosection reduced bundle stack

$$\jmath : h^1/h^0(E^\vee))(\sigma) \subset h^1/h^0(E^\vee)$$

defined as follows. Let $U_0 = \mathcal{M} - \mathcal{M}(\sigma)$ be the locus where σ is defined and surjective; let $E_\sigma^\vee \in D(U_0)$ be defined by the distinguished triangle

$$E_\sigma^\vee \longrightarrow E^\vee|_{U_0} \xrightarrow{\bar{\sigma}} \mathscr{O}_{U_0}[-1] \longrightarrow,$$

where $\bar{\sigma}$ is the composite $E^\vee|_{U_0} \to H^1(E^\vee)[-1]|_{U_0} \to \mathscr{O}_{U_0}[-1]$. We define

$$h^1/h^0(E^\vee)(\sigma) = \Big(h^1/h^0(E_\sigma^\vee) \bigcup h^1/h^0(E^\vee) \times_{\mathcal{M}} \mathcal{M}(\sigma)\Big)_{\mathrm{red}}.$$

Here the subscript "red" stands for taking the reduced stack structure.

Proposition 1.5 (Cone reduction [KL2])**.** *Let the situation be as in Theorem 1.2. Then there is a cycle* $[\mathbf{C}_\sigma] \in Z_* h^1/h^0(E^\vee)(\sigma)$ *so that*

$$\jmath_*[\mathbf{C}_\sigma] = [\mathbf{C}] \in Z_* h^1/h^0(E^\vee).$$

The other ingredient for constructing the cosection localized virtual cycles is the localized Gysin map, constructed by Kiem and the author [KL2].

The localized Gysin map of a vector bundle with a meromorphic cosection is constructed as follows. Let

$$\pi : V \longrightarrow \mathcal{M}$$

be a rank r vector bundle over a DM stack $\mathcal{M}$, and let $\sigma : V|_U \to \mathcal{O}_U$, $U \subset \mathcal{M}$ open, be a meromorphic cosection. We let $\mathcal{M}(\sigma)$ be the degeneracy loci of σ, which is where either σ is undefined or not surjective. Let

$$V(\sigma) = \Bigl(V|_{\mathcal{M}(\sigma)} \bigcup \ker\bigl\{\sigma : V|_{\mathcal{M}-\mathcal{M}(\sigma)} \to \mathcal{O}_{\mathcal{M}-\mathcal{M}(\sigma)}\bigr\}\Bigr)_{\mathrm{red}},$$

which is closed in V. Let s_V be the zero section of V. The localized Gysin map to be constructed is a homomorphism

$$s^!_{V,\sigma} : A_d V(\sigma) \to A_{d-r}\mathcal{M}(\sigma)$$

that has the usual properties of the Gysin map, and coincides with $s^!_V$ when composed with the tautological $A_{d-r}\mathcal{M}(\sigma) \to A_{d-r}\mathcal{M}$.

The localized Gysin map is defined at the cycle level, after picking their proper representatives.

Definition 1.6. Let $\rho : X \to \mathcal{M}$ be a morphism from a quasi-projective variety X to $\mathcal{M}$ such that $\rho(X) \cap U \neq \emptyset$. We call ρ a σ-regularizing morphism if ρ is proper, its restriction to a dense open $X_0 \subset X$, $\rho|_{X_0} : X_0 \to \rho(X_0)$, is étale, and $\rho^*(\sigma)$ extends to a surjective homomorphism

$$\tilde\sigma : \tilde V := \rho^* V \to \mathcal{O}_X(D)$$

for a Cartier divisor $D \subset X$.

For the ρ given, we denote by $\tilde\rho : \tilde V \to V$ is the projection; $\tilde G := \ker\{\tilde\sigma\} \subset \tilde V$; $|D| \subset X$ is the support of D, and $\rho(\sigma) : |D| \to \mathcal{M}(\sigma)$ is the ρ restricted to $|D|$.

Basic Construction: *Let* $[B] \in Z_d V(\sigma)$ *be a cycle represented by a closed integral substack* $B \subset V(\sigma)$*. In case* $B \subset V|_{\mathcal{M}(\sigma)}$*, we define*

$$s^!_{V,\sigma}([B]) = s^!_{V|_{\mathcal{M}(\sigma)}}([B]) \in A_{d-r}\mathcal{M}(\sigma).$$

Otherwise, we pick a variety X *and a* σ*-regularizing* $\rho : X \to \mathcal{M}$ *such that* $\overline{\pi(B)} = \overline{\rho(X)}$*, there is a closed integral* $\tilde B \subset \tilde G$ *so that* $\tilde\rho_*([\tilde B]) = k \cdot [B] \in Z_d V$ *for some* $k \in \mathbb{Z}$*. We define*

$$s^!_{V,\sigma}([B])_{\rho,\tilde B} = k^{-1} \cdot \rho(\sigma)_*([D] \cdot s^!_{\tilde G}([\tilde B])) \in A_{d-r}\mathcal{M}(\sigma). \tag{1.5}$$

Here $[D]\cdot : A_* X \to A_{*-1}|D|$ *is the intersection with the divisor* D*.*

Definition-Proposition 1.7 (Localized Gysin map [KL2])**.** *Let the notation be as in the basic construction. Then for each closed integral* $B \subset V(\sigma)$ *not contained in* $V|_{\mathcal{M}(\sigma)}$*, we can find a pair* $(\rho, \tilde B)$ *so that* $s^!_{V,\sigma}([B])_{\rho,\tilde B}$ *is defined by the Basic construction. Furthermore the resulting cycle class* $s^!_{V,\sigma}([B])_{\rho,\tilde B} \in A_{d-r}\mathcal{M}(\sigma)$ *is independent of the choice of* $(\rho, \tilde B)$*.*

Construction of cosection localized virtual cycle. We prove Proposition 1.5. We let $\mathbf{E} = h^1/h^0(E^\vee)$; let $[A] \in Z_*\mathbf{E}$ be an irreducible cycle; let $\pi : \mathbf{E} \to \mathcal{M}$ be the projection, and let M_A be the closure of the image $\pi(A) \subset \mathcal{M}$. Applying Chow's Lemma [**?**, Cor. 16.6.1], we can find a quasi-projective variety X, a proper and surjective morphism $\rho : X \to M_A$ such that for a Zariski dense open $X_0 \subset X$, $\rho|_{X_0} : X_0 \to M_A$ is étale. Since X is quasi-projective, we can find a complex of locally free sheaves $F = [F_0 \to F_1]$ of $\mathcal{O}_X$-modules that is quasi-isomorphic to ρ^*E. By abuse of notation, we view F_1 as a vector bundle over X. The tautological morphism

$$\gamma : F_1 \longrightarrow h^1/h^0(F) = h^1/h^0(\rho^*E) = \rho^*\mathbf{E}$$

is flat. Since $\rho|_{X_0} : X_0 \to M_A$ is étale, the induced

$$\gamma|_{X_0} : F_1 \times_X X_0 \longrightarrow \mathbf{E}|_{M_A} = h^1/h^0(E) \times_{\mathcal{M}} M_A$$

is flat.

Definition 1.8. A proper representative of an irreducible $[A] \in Z_*\mathbf{E}$ consists of (ρ, F_1) just constructed and a cycle $A_X = \sum_i m_i[A_i] \in Z_*F_1$, where A_i are closed and integral in F_1 and $m_i \in \mathbb{Z}$, such that $\sum_i m_i[A_i \times_X X_0] = (\gamma|_{X_0})^*(A)$, and $A_i = \overline{A_i \times_X X_0}$ for all i.

Let (ρ, X, F_1) with $\sum_i m_i A_i$ be a proper representative of $[A]$. Let $\sigma_X : F_1|_{\rho^{-1}U} \to \mathcal{O}_{\rho^{-1}U}$ be the composite of $F_1|_{\rho^{-1}U} \to \rho^*\mathcal{O}b_{\mathcal{M}/\mathcal{S}}|_{\rho^{-1}U}$ with $\rho^*\sigma$. Automatically $[A_i] \in Z_*F_1(\sigma_X)$. We let $\rho(\sigma) : X(\sigma_X) \to \mathcal{M}(\sigma)$ be the restricton of ρ to $X(\sigma_X) = X \setminus \rho^{-1}U$.

We define

$$s^!_{\mathbf{E},\sigma}([A]) = m_X^{-1}\rho(\sigma)_*(s^!_{F_1,\sigma_X}([A_X])),$$

where m_X is the degree of $\rho : X \to M_A$. We extend it to $s^!_{\mathbf{E},\sigma} : Z_*\mathbf{E} \to A_*\mathcal{M}(\sigma)$ by linearity.

It was verified in [KL2] that this construction is independent of the choice of proper-representatives of cycles, and it preserves rational equivalence; thus descends to

$$s^!_{\mathbf{E},\sigma} : A_*\mathbf{E}(\sigma) \longrightarrow A_*\mathcal{M}(\sigma),$$

called the localized Gysin map □

Remark. *Initially, the class $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ is constructed as a class in the homology group of $\mathcal{M}(\sigma)$. It is shown to lie in the Chow group after we constructed the localized Gysin map in algebraic geometry. Using the localized Gysin map, we show that all tools developed for studying virtual cycles remain valid for cosection localized virtual cycles.*

2. Applications of cosection localized virtual cycles

Applying cosection localized virtual cycle, in many situations one can reduce the virtual cycles to smaller subset $\mathcal{M}(\sigma) \subset \mathcal{M}$. An example is the GW-invariants of algebraic surfaces with non-trivial holomoprhic two-forms; it is an algebraic geometric analogue of Lee-Parker's reduction theorem of GW-invariants of surfaces in symplectic geometry [LP].

Let X be a smooth quasi-projective variety equipped with a holomorphic two-form $\theta \in \Gamma(\Omega^2_X)$. This form induces a cosection of the obstruction sheaf the moduli space $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable morphisms to X of class β. We denote by p :

$\mathcal{M} \to \mathcal{S}$ the forgetful morphism to the Artin stack of genus g connected nodal curves. We let $f:\mathcal{C} \to X$ and $\pi:\mathcal{C} \to \mathcal{M}$ be the universal family of $\mathcal{M}$; the relative obstruction theory of $\mathcal{M} \to \mathcal{S}$ is

$$(R^\bullet \pi_* f^* T_X)^\vee \longrightarrow L_{\mathcal{M}/\mathcal{S}}$$

(cf. [BF]), and its relative obstruction sheaf is $\mathcal{O}b_{\mathcal{M}/\mathcal{S}} = R^1\pi_* f^* T_X$.

By viewing the two-form θ as an anti-symmetric homomorphism

$$\hat\theta : T_X \longrightarrow \Omega_X, \quad (\hat\theta(v), v) = 0, \tag{2.1}$$

it defines the first arrow in the following sequence

$$R^1\pi_* f^* T_X \longrightarrow R^1\pi_* f^*\Omega_X \longrightarrow R^1\pi_*\Omega_{\mathcal{C}/\mathcal{M}} \longrightarrow R^1\pi_*\omega_{\mathcal{C}/\mathcal{M}}, \tag{2.2}$$

where the second is induced by $f^*\Omega_X \to \Omega_{\mathcal{C}/\mathcal{M}}$, and the last by the tautological $\Omega_{\mathcal{C}/\mathcal{M}} \to \omega_{\mathcal{C}/\mathcal{M}}$. Because $R^1\pi_*\omega_{\mathcal{C}/\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}$, the composite of this sequence provides

$$\sigma_\theta^{\mathrm{rel}} : R^1\pi_* f^* T_X = \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \longrightarrow \mathcal{O}_{\mathcal{M}}. \tag{2.3}$$

The obstruction sheaf of $\mathcal{M}$ is the cokernel of $p^*\Omega_{\mathcal{S}}^\vee \to \mathcal{O}b_{\mathcal{M}/\mathcal{S}}$. Using the universal family f and that $R^1\pi_* f^* T_X = \mathcal{E}xt^1_\pi(f^*\Omega_X, \mathcal{O}_{\mathcal{C}})$, we have the exact sequence

$$p^*\Omega_{\mathcal{S}}^\vee = \mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/\mathcal{M}}, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt^1_\pi(f^*\Omega_X, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{O}b_{\mathcal{M}} \longrightarrow 0, \tag{2.4}$$

where the first arrow is induced by $f^*\Omega_X \to \Omega_{\mathcal{C}/\mathcal{M}}$. In [KL2], it is verified that the composition

$$\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/\mathcal{M}}, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt^1_\pi(f^*\Omega_X, \mathcal{O}_{\mathcal{C}}) \xrightarrow{\sigma_\theta^{\mathrm{rel}}} \mathcal{O}_{\mathcal{M}}$$

is trivial, which implies that $\sigma_\theta^{\mathrm{rel}}$ lifts to a cosection

$$\sigma_\theta : \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathscr{O}_{\mathcal{M}}.$$

The degeneracy (non-surjective) loci $\mathcal{M}(\sigma)$ of σ_θ can easily be described.

Definition 2.1. A stable maps $u:C \to X$ is called *θ-null* if the composite

$$u^*(\hat\theta) \circ du : T_{C_{\mathrm{reg}}} \longrightarrow u^*T_X|_{C_{\mathrm{reg}}} \longrightarrow u^*\Omega_X|_{C_{\mathrm{reg}}}$$

is trivial over the regular locus C_{reg} of C.

We have

Proposition 2.2. *Any holomorphic two-form $\theta \in H^0(\Omega_X^2)$ on a smooth quasi-projective variety X induces a homomorphism $\sigma_\theta:\mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}}$ of the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ of the moduli of stable morphisms $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X,\beta)$. The homomorphism σ_θ is surjective away from the set of θ-null stable maps in $\mathcal{M}$.*

When X is proper, the comparison Theorem (Thm. 1.3) implies that the ordinary virtual cycle $[\mathcal{M}]^{\mathrm{vir}} = [\mathcal{M}]^{\mathrm{vir}}_\sigma \in A_*\mathcal{M}$. In particular, if the class β can not be represented by combinations of curves in $\theta^{-1}(0)$, then $\mathcal{M}(\sigma_\theta) = \emptyset$, thus $[\mathcal{M}]^{\mathrm{vir}} = 0$. This recovers the vanishing results of Lee-Parker [LP] for GW-invariants of compact algebraic surfaces with non-trivial holomorphic two-forms.

One can also defined local GW-invariants of the surface S that is the total space of a theta characteristic L over a smooth curve. The complete understanding of this class of GW-invariants is conjectured to solve all GW-invariants of surfaces X with positive $h^0(K_X)$.

We now come to the examples of reduced invariants when the obstruction sheaf has a surjective homomoprhism to a locally free sheaf. A typical example is when X

is a K3 surface. In this case, the holomorphic two-form θ is nowhere vanishing; its induced cosection (2.3) is surjective, and the reduced bundle stack $h^1/h^0(E^\vee)(\sigma_\theta)$ is a codimension one sub-bundle stack of $h^1/h^0(E^\vee)$.

In this case, instead of obtaining $[\mathcal{M}]^{\rm vir}=0$, which gives no new information to the GW-invariants of surfaces, we can use the (lifted) cycle $[\mathbf{C}_\sigma]\in Z_*h^1/h^0(E^\vee)(\sigma_\theta)$ (cf. Prop. 1.5), and the zero section s_σ of $h^1/h^0(E^\vee)(\sigma_\theta)$ to define reduced virtual cycle

$$[\mathcal{M}]^{\rm vir}_{\rm red}=s_\sigma^![\mathbf{C}_\sigma]\in A_*\mathcal{M}.$$

This gives the reduced GW-invariants of K3 surfaces, which was first introduced by Okunkov-Pandharipande [OP], and also used extensively by Maulik-Pandharipande-Thomas [MPT].

3. Gromov-Witten invariants of stable morphisms with fields

The most recent application of the cosection localized virtual cycle is the mathematical treatment and generalization of Witten's Gauged-Linear-Sigma model for all genus: the GW-invariants of stable morphisms with fields.

We begin withe the moduli spaces. Given non-negative integer g and positive d, we form the moduli $\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p$ of genus g degree d stable morphisms to $\mathbb{P}^4$ with p-fields:

$$\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p=\{[u,C,p]\mid [u,C]\in\overline{\mathcal{M}}_g(\mathbb{P}^4,d),\ p\in\Gamma(C,u^*\mathscr{O}_{\mathbb{P}^4}(-5)\otimes\omega_C)\}/\sim.$$

A standard argument shows that this is a Deligne-Mumford stack; forgetting the fields, the induced morphism

$$\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p\longrightarrow\overline{\mathcal{M}}_g(\mathbb{P}^4,d)$$

has fiber $H^0(u^*\mathscr{O}_{\mathbb{P}^4}(-5)\otimes\omega_C)$ over $[u,C]\in\overline{\mathcal{M}}_g(\mathbb{P}^4,d)$. When g is positive, $\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p$ is not proper.

The moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p$ has a perfect obstruction theory, thus has a virtual class. However, its usual GW type invariant is ill defined since $\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p$ is not proper. To overcome its non-properness, we construct a cosection of its obstruction sheaf:

$$\sigma_{\mathbf{w}}:\mathcal{O}b_{\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p}\longrightarrow\mathscr{O}_{\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p};$$

the choice of $\sigma_{\mathbf{w}}$ depends on the choice of a degree five homogeneous polynomial, like $\mathbf{w}=x_1^5+\ldots+x_5^5$. The non-surjective loci (called the degeneracy loci) of the cosection $\sigma_{\mathbf{w}}$ is

$$\overline{\mathcal{M}}_g(Q,d)\subset\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p,$$

where ,$Q=(x_1^5+\ldots+x_5^5=0)\subset\mathbb{P}^4$. Note that $\overline{\mathcal{M}}_g(Q,d)$ is proper.

Applying cosection localized virtual class construction (Theorem 1.2), we obtain a localized virtual cycle

$$[\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p]^{\rm vir}_\sigma\in A_0\overline{\mathcal{M}}_g(Q,d).$$

We define the Gromov-Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p$ be

$$N_g(d)^p_{\mathbb{P}^4}=\deg[\overline{\mathcal{M}}_g(\mathbb{P}^4,d)^p]^{\rm vir}_\sigma.$$

The miracle of this invariant is the following equivalence result:

Theorem 3.1 (Chang-Li [CL]). *For $g \geq 0$ and $d > 0$, the Gromov-Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ coincides with the Gromov-Witten invariant $N_g(d)_Q$ of the quintic Q up to a sign:*

$$N_g(d)^p_{\mathbb{P}^4} = (-1)^{5d+1-g} N_g(d)_Q.$$

When $g = 0$, this is derived in Guffin-Sharpe [GS] using path-integral. This identity also is the Kontsevich's formula on $g = 0$ Gromov-Witten invariants of quintics. If one views the localized virtual cycle of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ as "Euler class of bundles", this theorem is a substitute of the "hyperplane property" of the Gromov-Witten invariants of quintics in high genus.

This construction is an algebro-geometric construction of Guffin-Sharpe-Witten model for all genus. The moduli of stable morphisms with p-fields is the algebro-geometric substitute of the phase space of all smooth maps with smooth fields. The cosection localized virtual cycle is the analogue of Witten's perturbed equation. Theorem 3.1 shows that the Gromov-Witten invariants of the algebro-geometric Guffin-Sharpe-Witten model of all genus coincide up to signs with the Gromov-Witten invariants of quintic threefolds.

This construction applies to global complete intersection Calabi-Yau threefolds of toric varieties. This techniques can be applied to the moduli of stable quotients (cf. [MOP]) to obtain all genus invariants of massive theory of $(K_{\mathbb{P}^4}, \mathbf{w})$; one can also apply it to the linear Landau-Gingzberg model to obtain an alternative algebro-geometric construction of Fan-Jarvis-Ruan-Witten invariants. In the later case, using the analytic interpretation of resulting invariants are equal to those defined using perturbed the Witten equations [FJRW].

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