**Workshop "Optimal Transport: Theory and Applications"**
**Fourier Institute (Grenoble I, France)**
**June 29th-July 3rd**
**Titles and abstracts of the talks**

*F. Bollet (Paris-Dauphine)*
*Phi-entropy inequalities and Fokker-Planck equations*

The convergence to equilibrium for Fokker-Planck type diffusion equations can be studied by means of functional inequalities such as the Poincaré and logarithmic Sobolev inequalities. We present this link and we derive and study a family of "Phi-entropy" inequalities: they provide an interpolation between these two functional inequalities and can make the convergence to equilibrium more precise. This is a joint work with I. Gentil.

*Y. Brenier (Nice)*
*Optimal transport, convection theory and semi-geostrophic equations*

There are several well known links between optimal transport and fluid mechanics. For instance, the polar factorization of maps is directly related to the time discretization of the Euler equations (in the Arnold style). Another link has been recently established with convection theory, the central theory of geophysical flows. The first step is an interpretation of the Angenent-Haker-Tannenbaum model for optimal transport as a convection model in porous media (described by the Darcy-Boussinesq equations). Thanks to this new connection with fluid mechanics, the first rigorous derivation of the semi-geostrophic equations from the Navier-Stokes equations has been established (in the 2 dimensional "x-z" configuration), jointly with Mike Cullen).

*V. Calvez (ENS Ulm)*
*The geometry of one-dimensional self-attracting particles*

Self-attracting particles can be modelled using a nonlinear Fokker-Planck equation, including a mean field term for the attracting potential. We focus here on the Keller-Segel model for biological cells (or the Smoluchowski-Poisson equation in astrophysics). This model has raised a lot of interest in the field of mathematical biology since it captures the critical mass phenomenon in a very simple manner: the number of cells determines whether some spatial structure emerges or not at the population level.
This model possesses the structure of a gradient flow for the Wasserstein metric on the space of probability densities. Interestingly enough the energy functional is the sum of convex and concave contributions. Despite this lack of convexity we shall

exhibit some underlying structure which makes this model somehow close to the gradient flow of a convex functional.
We present in this talk some analysis for a one-parameter family of Keller-Segel models in one dimension of space:

$$\partial_t \rho(t,x) = \partial_{xx}\rho^m(t,x) + \chi\partial_x \left(\rho(t,x)\partial_x \left(\frac{|x|^{1-m}}{1-m} * \rho(t,x)\right)\right), \quad t>0, \quad x \in \mathbb{R}.$$

We show how the system's behaviour (finite time blow-up, long-time asymptotics) can be simply re-interpreted in the gradient flow framework. A suitable numerical scheme and its consequences are also discussed.

This is joint work with José Antonio Carrillo (ICREA, Univ. Autonoma Barcelona).

*J. Delon (Telecom Paris)*
*Transport optimization on the circle and applications in image processing*

Distributions on the circle are common tools in computer vision and image processing. They can appear, for instance, as distributions of gradient orientation, as hue distributions in color images, or as distributions of displacement field orientations. The comparison of such distributions is an important issue for a large number of applications, such as image classification, image retrieval, image registration or color image transfer.

In the present work, we consider the problem of optimally matching two measures on the circle, or equivalently two periodic measures on R, and we suppose that the cost c(x, y) of matching two points x, y satisfies the Monge condition: $c(x_1, y_1) + c(x_2, y_2) < c(x_1, y_2) + c(x_2, y_1)$ whenever $x_1 < x_2$ and $y_1 < y_2$.
We introduce a notion of locally optimal transport plan and show that all locally optimal transport plans are conjugate to shifts. This theory is then applied to two sets of point masses with the same total mass. For the case of N real-valued point masses we present an O(N log epsilon) algorithm that approximates the optimal cost within epsilon; when all masses are integer multiples of $1/M$, the algorithm gives an exact solution in $O(NlogM)$ operations. Joint work with Andrei Sobolevskii and Julien Salomon.

*B. De Meyer (Paris I)*
*Continuous Martingales of Maximal Variation and Price Dynamics on the Stock Market*

The appearance of a Brownian term in the price dynamics on a stock market was interpreted in [De Meyer, Moussa-Saley (2003)[1]] as a consequence of the informational asymmetries between agents. To take benefit of their private information without

[1] De Meyer B. and H. Moussa-Saley (2003): On the strategic origin of Brownian Motion in Finance, *Internationnal Journal of game theory* ,31, 285-319.

revealing it to fast, the informed agents have to introduce a noise on their actions, and all these noises introduced in the day after day transactions for strategic reasons will aggregate in a Brownian Motion. We prove in the present paper that this kind of argument leads not only to the appearance of the Brownian motion, but it also narrows the class of the price dynamics: the price process will be, as defined in this paper, a continuous martingale of maximal variation. This class of dynamics contains in particular Black and Scholes' as well as Bachelier's dynamics. The main result in this paper is that this class is quite universal and independent of a particular model: the informed agent can choose the speed of revelation of his private information. He determines in this way the posterior martingale $L$, where $L_q$ is the expected value of an asset at stage $q$ given the information of the uninformed agents. The payoff of the informed agent at stage $q$ can typically be expressed as a 1-homogeneous function $M$ of $L_{q+1} - L_q$. In a game with $n$ stages, the informed agent will therefore chose the martingale $L^n$ that maximizes the $M$-variation. Under a mere continuity hypothesis on $M$, we prove in this paper that $L^n$ will converge to a continuous martingale of maximal variation. This limit is independent of $M$.

*U. Frisch (Nice)*
*Optimal transport and the reconstruction of the early Universe*

We show that the deterministic past history of the Universe can be uniquely reconstructed from the knowledge of the present mass density field, the latter being inferred from the 3D distribution of luminous matter, assumed to be tracing the distribution of dark matter up to a known bias. Reconstruction ceases to be unique below those scales – a few Mpc – where multi-streaming becomes significant. Above $6\,h^{-1}$ Mpc we propose and implement an effective Monge–Ampère–Kantorovich method of unique reconstruction. At such scales the Zel'dovich approximation is well satisfied and reconstruction becomes an instance of optimal mass transportation. After discretization into $N$ point masses one obtains an assignment problem that can be handled by effective algorithms with not more than $O(N^3)$ time complexity and reasonable CPU time requirements. Testing against $N$-body cosmological simulations gives over 60% of exactly reconstructed points.

We apply several interrelated tools from optimization theory that were not used in cosmological reconstruction before, such as the Monge–Ampère equation, its relation to the mass transportation problem, the Kantorovich duality and the auction algorithm for optimal assignment. Self-contained discussion of relevant notions and techniques is provided.

*N. Gozlan (Marne la Vallée)*
*A refined version of Otto and Villani's Theorem*

This talk is devoted to Otto and Villani's Theorem stating that the Logarithmic Sobolev inequality implies Talagrand's $T_2$ transportation cost inequality.
In the first part of the talk, we will give a new and very robust proof of this result. This proof is based on the characterization of dimension free concentration in terms of transportation cost inequalities recently obtained by the author using a large deviations approach.
The second part of the talk is devoted to a recent improvement of Otto and Villani's Theorem.
We will show that Talagrand's inequality is *equivalent* to a Logarithmic Sobolev inequality restricted to a subclass of functions. This new result enables us to show that Talagrand's inequality and dimension free Gaussian concentration are stable under bounded perturbations. Joint work with P-M. Samson.

*A. Guillin (Clermont-Ferrand)*
*Transportation, Poincaré and other inequalities via Lyapunov conditions*

We will show here how a simple integration by parts formula associated with local inequalities and Lyapunov condition lead to various functional inequalities such as transportation inequalities (wrt Kullback or Fisher information), Poincare inequalities or logarithmic Sobolev inequalities.

*D. Jacobs (Maryland)*
*Applications of the Earth Mover's Distance in Computer Vision, with a new Approximate Algorithm*

The Earth Mover's Distance (EMD) has been widely used in Computer Vision to compare histograms of image descriptors. I will first review some of its applications in color, texture, and image matching. These applications have been somewhat limited, though, by the computational complexity of exact computation of the EMD. To deal with this problem, we have developed a novel linear time algorithm for approximating the EMD for low dimensional histograms using the sum of absolute values of the weighted wavelet coefficients of the difference histogram. EMD computation is a special case of the Kantorovich-Rubinstein transshipment problem, and we exploit the Holder continuity constraint in its dual form to convert it into a simple optimization problem with an explicit solution in the wavelet domain. We prove that the resulting wavelet EMD metric is equivalent to EMD, i.e. the ratio of the two is bounded.

We also provide estimates for the bounds. The weighted wavelet transform can be computed in time linear in the number of histogram bins, and the comparison is about as fast as for normal Euclidean distance or $\chi_2$ statistic. We experimentally show that wavelet EMD is a good approximation to EMD, has similar performance, but requires much less computation.
This is joint work with Sameer Sheorey.

*C. Jimenez (Brest)*
*Optimal transport with a convex obstacle*

We deal with the problem of solving the Monge problem with the squared geodesic distance $c(\cdot,\cdot)$ in a compact subset of $\mathbb{R}^n$ with a convex hole.
In case there is no obstacle, the solution is very well known (see [1]), the main idea is to derive the primal-dual optimality condition and get the direction and the length of the displacement. Unfortunately in our setting these informations are insufficient. To overcome this difficulty we introduce an equivalent optimization problem with a space-time cost replacing $c$. This new problem is similar to the Monge problem with the (non-squared) geodesic distance. We then apply similar techniques as in [2].
Joitn work with Pierre Cardaliaget.

References

[1] Y. Brenier, *Polar Factorization and monotone rearrangements of vector valued functions*, Comm. Pure and Applied Math., **44**, pp. 375–417 (1991).
[2] M. Feldman, R.J. McCann, *Mass transfer around convex obstacles*: Personnal communication.

*A. Joulin (Toulouse)*
*Curvature and concentration for empirical means of Markov chains*

Using a notion of Ricci curvature for Markov Chains on metric spaces developed recently by Y. Ollivier, we give in this talk new concentration inequalities for empirical means of Markov chains. The main ingredient in the proofs relies on a tensorization procedure of the Laplace transform and allows us to consider various examples such as Markov chains on the hypercube, the Ising model, queueing or diffusion processes. This is a joint work with Y. Ollivier.

*R. Mc Cann (Toronto)*
*Curvature and the continuity of optimal transportation maps*

I shall briefly review new and old results concerning optimal transportation of mass between two manifolds, in particular the regularity theory of Ma, Trudinger, Wang and Loeper, and counterexamples due to Loeper. I will describe an unexpected pseudo-Riemannian structure underlying these results, which yields the much-desired direct proof of a key result in this theory, and opens new connections to differential geometry.

*F. Maggi (Firenze)*
Stability problems for anisotropic surface tensions

The equilibrium shape of a crystal is determined by the minimization under a volume constraint of its free energy, consisting of an anisotropic interfacial surface energy plus a bulk potential energy. In the absence of the potential term, the equilibrium shape can be directly characterized in terms of the surface tension and turns out to be a convex set, the Wulff shape of the crystal.
Our first result is a sharp quantitative inequality implying that any shape with almost-optimal surface energy is close in the proper sense to the Wulff shape. This is a joint work with Aldo Pratelli (Pavia) and Alessio Figalli (Paris).
Under the action of a weak potential or, equivalently, if the total mass of the crystal is small enough, the surface energy of the equilibrium shape is actually close to that of the corresponding Wulff shape, and the previous result applies. However, stronger geometric properties are now expected, due to the fact that the considered shapes are minimizers. Indeed we can prove their convexity, as well as their proximity to the Wulff shape with respect to a stronger notion of distance. This is a joint work with Alessio Figalli (Paris).

*B. Maury (Paris 11, Orsay)*
*Gradient flow formulation of crowd motion models*

We propose a macroscopic model for pedestrian motion in emergency situations. This model, which is a direct extension of a previous microscopic one (developped in collaboration with J. Venel), rests on very simple principles : people density is subject to remain below a maximal value, and it is advected by a velocity field which is the closest (in the least square sense) to the one individuals would like to have in the absence of others, among all those fields which do not violate the congestion constraint. In spite of its formal simplicity, this model does not fit into any standard framework, in particular because the regularity of the advecting velocity field cannot be controlled a priori. A minimizing movement formulation of this model can be introduced. We shall present how this formulation provides a natural framework for this kind of evolution problem, and how it throws light on the deep differences between micro and macro approaches. Joint work with A. Roudneff-Chupin and F. Santambrogio.

*Q. Mérigot (INRIA Nice)*
*Stability of Federer's Curvature Measures*

We study the boundary measures of compact subsets of the d-dimensional Euclidean space, which are closely related to Federer's curvature measures. Boundary measures can be computed efficiently for point clouds through Monte-Carlo sampling, and can be used for estimating some geometric properties of the underlying space.

The main contribution of this work is a quantitative stability theorem for boundary measures, which is proven using tools of convex analysis and geometric measure theory. As a corollary we obtain a stability result for Federer's curvature measures of a compact set under Hausdorff approximation, without any regularity assumption on the approximating compact set.

*W. Schachermayer (Vienna)*
*Duality for Borel measurble cost functions*

We consider the Monge-Kantorovich transport problem in an ab- stract measure theoretic setting. Our main result states that duality holds if $c : XxY \to [0, \infty)$ is an arbitrary Borel measurable cost function on the product of Polish spaces $X, Y$ . In the course of the proof we show how to relate a non - optimal transport plan to the optimal transport costs via a subsidy function and how to identify the dual optimizer. We also provide some examples showing the limitations of the duality relations.

*A. Sobolevskii (Moscow)*
*Transport in shock manifolds in Hamilton-Jacobi equations*

Characteristics of a solution to a Hamilton-Jacobi equation can be seen as trajectories of fluid particles, but usually this analogy is only pursued until trajectories hit a shock and cease to minimize the Lagrangian action. We show that in the limits of infinite and of vanishing Prandtl number, two uniquely defined, global in time, adhesive flows are obtained for any weak solution to a Hamilton-Jacobi equation with a general convex Hamiltonian. The two limits generally differ and coincide only for Hamiltonians quadratic in the momentum variable. Joint work with K. Khanin (U. Toronto).

*E. Stepanov (St Petersburg)*
*Optimal transportation with fractional costs*

The classical Monge-Kantorovich problem (with linear cost) admits a natural reformulation in terms of mass minimization of a Federer-Fleming currents with given boundary. We consider the analogous transportation costs for measures involving fractional masses. The local structure of optimal currents as well as the strcuture of measure space induced by the respective costs will be discussed. The setting is in generic metric spaces, so one uses Ambrosio-Kirchheim currents instead of classical euclidean flat chains.

*K.T. Sturm (Bonn)*
*Entropic measure on multidimensional spaces*

We consider stochastic perturbations of gradient flows for the entropy. The basic object to be considered here is the entropic measure, a random probability measure (= probability measure on the space of probability measures on M) with finite dimensional distributions determined by relative entropy. On higher dimensional spaces M, its construction relies on a new continuous and involutive map $C : P(M) \to P(M)$, the conjugation map, which might be of interest in its own right.

On one-dimensional spaces, a change-of-variable formula for the entropic measure admits the construction of a reversible stochastic process ("Wasserstein diffusion") on the space of probability measures which can be regarded as a stochastic perturbation of the heat flow.

*L.M. Wu (Clermont-Ferrand)*
*Lipschitzian norm estimate of Poisson equations and applications in transportation-information inequalities*

We establish several sharp estimates of Lipschitzian norm of the Poisson equation $-LG = g$ both on one-dimensional and multi-dimensional Riemannian manifolds. Those sharp estmates can be used to estimate the Cheeger's isoperimetric constant, the Gaussian constant for the diffusion generated by $L$, and sharp transportation-information inequalities.