ON RANDOM LINEAR FORMS

Chistyakov G.P. , Pergamentsev S.Yu.

B. Verkin Institute for Low Temperature Physics and
Engineering, National Academy of Sciences of Ukraine,
47 Lenin Ave., 310164, Kharkov, Ukraine

On étudie le problème de caractérisation des distributions grâce à l'indépendance des formes linéaires à coefficients aléatoires. On obtient une généralisation d'un théorème connu de Darmois-Skitovich.

1. Introduction. The Darmois–Skitovich theorem [1, 2] is one of the first results concerning characterization problems of the mathematical statistics. Consider independent random variables (i.r.v.'s) $X_1, \dots, X_n$, $n \geq 2$, and two linear statistics

$$L_1 = \alpha_1 X_1 + \dots + \alpha_n X_n, \qquad L_2 = \beta_1 X_1 + \dots + \beta_n X_n,$$

where α_i, β_i — are constant coefficients.

Theorem A (Darmois, Skitovich [1, 2]). *If L_1 and L_2 are independent, then those X_j which appear in the both forms L_1 and L_2, i.e., correspond to those j for which $\alpha_j \beta_j \neq 0$, are Gaussian.*

This theorem was extended by Linnik Yu.V. and Zinger A.A. [3] to linear forms with random coefficients. The studying of such random linear functionals was useful for the investigation of the independence of many non-linear statistics (see [4]). The Linnik–Zinger result is as follows. Let $\mathbf{X}^{(n)} = (X_1, \dots, X_n)$ and $\mathbf{U}^{(2n)} = (U_1, \dots, U_{2n})$ be n-dimensional and $2n$-dimensional random vectors respectively. Suppose that the random vector $\mathbf{U}^{(2n)}$ satisfies the following conditions:

1) its distribution has the bounded support in Euclidean space $\mathbb{R}^{2n}$,
2) there exists $\varepsilon > 0$ such that $\mathbf{P}\left(|U_j| > \varepsilon\right) > 0$ for $j = 1, \dots, n$,
3) $U_{n+j} = 1$ almost surely (a.s.) for $j = 1, \dots, n$,
4) the relation

$$Q_{m_1,\dots,m_n}(t) \not\equiv const$$

is valid for each collection of non-negative integers $(m_1, \dots, m_n)$ such that $\sum_{j=1}^{n} m_j \neq 0$, where

$$Q_{m_1,\dots,m_n}(t) = \mathbf{E}\left((1+U_1 t)^{m_1} \dots (1+U_n t)^{m_n}\right), \qquad t \in \mathbb{R}^1.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem B (Linnik–Zinger [3]). *Let a random vector* $\mathbf{X}^{(n)} = (X_1, \dots, X_n)$ *with independent components and a random vector* $\mathbf{U}^{(2n)} = (U_1, \dots, U_{2n})$ *be independent. Let the conditions* 1) – 4) *be satisfied. If the forms*

$$L_{r1} = U_1X_1 + \dots + U_nX_n, \qquad L_{r2} = U_{n+1}X_1 + \dots + U_{2n}X_n$$

are independent, then the vector $\mathbf{X}^{(n)}$ *is Gaussian.*

This theorem generalizes Theorem A in the case where all coefficients of the two forms L_1 and L_2 are not equal to zero.

The problem of the investigation of independent linear forms with random coefficients was raised in [4, p. 637]. One may consider the forms L_{r1}, L_{r2} as linear forms with random coefficients U_j, $j = 1, \dots, 2n$. In Theorem B the form L_{r2} has non-random coefficients. We will find conditions on the vectors $\mathbf{U}^{(2n)}$ and $\mathbf{X}^{(n)}$ such that Theorem B remains valid in the case where the both forms L_{r1} and L_{r2} have random coefficients.

Denote by P_j, $j = 1, \dots, n$, the probability distributions (pr.d.'s) of the r.v.'s X_j, $j = 1, \dots, n$, respectively and by Q_j, $j = 1, \dots, 2n$, the pr.d.'s of the r.v.'s U_j, $j = 1, \dots, 2n$, respectively.

Assume that the r.v.'s U_j satisfy the following conditions:

(i) the pr.d.'s Q_j, $j = 1, \dots, 2n$ have bounded supports,

(ii) there exists $\varepsilon > 0$ such that $\mathbf{P}(|U_j| > \varepsilon) > 0$ for $j = 1, \dots, 2n$,

(iii) there exist a constant $b \geq 1$ and a r.v. $U \geq 0$ such that

$$\frac{1}{b}\mathbf{E}\,U^k \leq \mathbf{E}\,|U_j|^k \leq b\,\mathbf{E}\,U^k$$

for all $k = 1, \dots$ and $j = n+1, \dots, 2n$.

Remark 1. If the r.v.'s U_j, $j = n+1, \dots, 2n$, are identically distributed, then (iii) is valid for $b = 1$ and $U = |U_{n+1}|$.

By the condition (i), the characteristic function (ch.f.) $\varphi(t; U_j)$ of the r.v. U_j is an entire function of order one and finite type for $j = 1, \dots, 2n$. Denote by $\{a_{k,j} : k = 1, 2, \dots\}$ the set of zeros of the function $\varphi(t; U_j)$.

We shall say that the r.v. X_j satisfies the condition (iv) if there exists $\varepsilon > 0$ such that $\mathbf{P}(|X_j| > \varepsilon) > 0$, a median μ_j of X_j is equal to zero, and the support of P_j is not contained in the sets

$$\mathbb{R}^1 \cap \{za_{k,j} : k = 1, 2, \dots\}, \qquad \mathbb{R}^1 \cap \{za_{k,n+j} : k = 1, 2, \dots\}$$

for any complex $z \in \mathbb{C} \setminus \{0\}$.

Here and in the sequel we denote by $\mathbb{C}$ the open complex plane.

Remark 2. Let the r.v. X_j be not equal to zero a.s. and its median $\mu_j = 0$. If X_j has a non-atomic pr.d., then it satisfies the condition (iv).

Let X_j have an atomic pr.d. and let $N_j(T)$ denote the number of its value in the interval $[-T, T]$, where $T > 0$. This number can be equal to $+\infty$. If

$$\limsup_{T \to \infty} \frac{N_j(T)}{T} = +\infty,$$

then X_j satisfies the condition (iv). Indeed, let $n_j(T)$ be the number of zeros of the entire function of order one and finite type $\varphi(t;U_j)$ in the circle $|t| < T$. The second assertion of the remark follows from the well-known fact (see 5, p.p. 14-16) that

$$\limsup_{T\to\infty} \frac{n_j(T)}{T} < +\infty.$$

Our main result is as follows.

Theorem 1. *Let* $\mathbf{X}^{(n)} = (X_1,\dots,X_n)$ *and* $\mathbf{U}^{(2n)} = (U_1,\dots,U_{2n})$ *be independent random vectors with independent components. Let the r.v.'s.* U_j, $j = 1,\dots,2n$, *satisfy the conditions* (i)–(iii) *and let the forms* L_{r1} *and* L_{r2} *be independent. Then, for every* j *such that* X_j *satisfies the condition* (iv), *the r.v.* X_j *is Gaussian and the r.v.'s* U_j *and* U_{n+j} *are a.s. constant.*

Assume in addition that the r.v.'s X_j have moments of order two and consider the condition

(v) $\sum\limits_{j=1}^{n} \mathbf{E}\,U_j\,\mathbf{E}\,U_{j+n}\ \mathrm{Var}\,X_j = 0.$

Theorem 1 easily implies

Theorem 2. *Let* $\mathbf{X}^{(n)} = (X_1,\dots,X_n)$ *and* $\mathbf{U}^{(2n)} = (U_1,\dots,U_{2n})$ *be independent random vectors with independent components. Let the r.v.'s* U_j, $j = 1,\dots,2n$, *satisfy the conditions* (i)–(iii) *and the r.v.'s* X_j, $j = 1,\dots,n$, *satisfy the condition* (iv). *The forms* L_{r1} *and* L_{r2} *are independent iff the r.v.'s* X_j, $j = 1,\dots,n$, *are Gaussian, the r.v.'s* U_l, $l = 1,\dots,2n$, *are a.s. constant, and the condition* (v) *is valid.*

Let us show that Theorem 1 is a generalization of Theorem A in the case where all coefficients of the two forms L_1 and L_2 are not equal to zero. We assume, without loss of generality, that the coefficients β_j, $j = 1,\dots,n$, of the form L_2 are equal to one and medians of all r.v.'s X_j, $j = 1,\dots,n$, are equal to zero. Indeed, in the opposite case we shall consider the r.v.'s $(X_j - \mu_j)/\beta_j$ instead of X_j for $j = 1,\dots,n$. It is easy to see that the r.v.'s $U_j = \alpha_j \neq 0$, $j = 1,\dots,n$, and $U_j = 1$, $j = n+1,\dots,2n$, satisfy the conditions (i)—(iii). Since in this case $\varphi(t;U_j) \neq 0$ for all $t \in \mathbb{C}$, we see that the condition (iv) for X_j, $j = 1,2,\dots,n$, is also valid.

Our nearest aim is to prove that the ch.f.'s of the r.v.'s X_j, satisfying the assumptions of Theorem 1, are entire functions of finite order.

Theorem 3. *Let* $\mathbf{X}^{(n)} = (X_1,\dots,X_n)$ *and* $\mathbf{U}^{(2n)} = (U_1,\dots,U_{2n})$ *be independent random vectors with independent components. Let the components* U_j, $j = 1,\dots,2n$, *satisfy the conditions* (i)–(iii). *If the random forms* L_{r1} *and* L_{r2} *are independent, then the ch.f.'s of all components of the random vector* $\mathbf{X}^{(n)}$ *can be continued to* $\mathbb{C}$ *as entire functions of finite order.*

2. Proof of Theorem 3. To prove Theorem 3 we use some ideas and the following result of the paper [3].

Theorem C (Linnik–Zinger [3]). *Let a random vector* $\mathbf{X}^{(n)} = (X_1, \dots, X_n)$ *with independent components and a random vector* $\mathbf{U}^{(2n)} = (U_1, \dots, U_{2n})$ *be independent. Let the distribution of* $\mathbf{U}^{(2n)}$ *satisfy the condition* 1) *and*

$$\mathbf{P}\left(|U_j| > \varepsilon,\ |U_{n+j}| > \varepsilon\right) > 0, \qquad j = 1, \dots, n,$$

for some $\varepsilon > 0$. *If the forms* L_{r1}, L_{r2} *are independent, then*

$$\mathbf{E}\,|X_j|^N < +\infty, \qquad j = 1, \dots, n, \tag{2.1}$$

for all positive integers N.

It is easy to see that the random vectors $\mathbf{X}^{(n)}$ and $\mathbf{U}^{(2n)}$ from Theorem 3 satisfy the assumptions of Theorem C. Therefore the inequalities (2.1) are true for the r.v.'s X_j, $j = 1, \dots, n$. Our next step is to show that the ch.f. $\varphi(t; X_j)$ of every r.v. X_j is regular in some horizontal strip of $\mathbb{C}$.

In the sequel we need the following notation

$$P^{(n)} = P_1 \times \dots \times P_n, \qquad Q^{(n)} = Q_1 \times \dots \times Q_n,$$

$$Q^{(2n)} = Q_1 \times \dots \times Q_{2n}, \qquad Q_1^{(n)} = Q_{n+1} \times \dots \times Q_{2n},$$

so that the masures $P^{(n)}$, $Q^{(n)}$, $Q^{(2n)}$, $Q_1^{(n)}$ are the product–measures of the corresponding pr.d.'s. We shall denote by $c_1, c_2, \dots$ positive constants depending on the r.v.'s $X_1, \dots, X_n,$, $U_1, \dots, U_{2n}$, U, and the parameter n only.

Since the r.v.'s $X_1, \dots, X_n$ satisfy (2.1) and L_{r1} and L_{r2} are independent, we have the relation

$$\mathbf{E}\left(|L_{r1}|^{2n}|L_{r2}|^N\right) = \mathbf{E}\,|L_{r1}|^{2n}\,\mathbf{E}\,|L_{r2}|^N \tag{2.2}$$

for all positive integers N. Consider the set

$$A = \Big\{(x_1, \dots, x_n, u_1, \dots, u_{2n}) \in \mathbb{R}^{3n} : \\ |x_1| > c_1^2, (x_2, \dots, x_n) \in G, |u_1| > \frac{1}{c_1}, |u_{n+1}| > \frac{1}{c_1}\Big\}.$$

Here G is a $(n-1)$– dimensional bounded set such that $(P_2 \times \dots \times P_n)(G) = c_2$. We select c_1 sufficiently large, so that

$$Q_1\big(\{|u_1| > \frac{1}{c_1}\big) \ge c_2, \qquad Q_{n+1}\big(\{|u_1| > \frac{1}{c_1}\big) \ge c_2,$$

and the inequality

$$|u_1x_1 + \dots + u_nx_n| \ge c_3|x_1| \tag{2.3}$$

is valid in the set A for sufficiently small c_3. Let us find a lower bound for the left-hand side of (2.2). With the help of (2.3) one obtains

$$\begin{aligned}
&\mathbf{E}\left(|L_{r1}|^{2n}|L_{r2}|^N\right) = \\
&\quad \iint\limits_{\mathbb{R}^n \times \mathbb{R}^{2n}} |u_1x_1 + \dots + u_nx_n|^{2n}|u_{n+1}x_1 + \dots + u_{2n}x_n|^N\, d\big(P^{(n)} \times Q^{(2n)}\big) \ge \\
&\quad \iint\limits_{A} (c_3|x_1|)^{2n}|u_{n+1}x_1 + \dots + u_{2n}x_n|^N\, d\big(P^{(n)} \times Q^{(2n)}\big) \ge \\
&c_3^{2n} \iint\limits_{A} |x_1|^{2n}\,\big||u_{n+1}||x_1| - |u_{n+2}x_2 + \dots + u_{2n}x_n|\big|^N\, d\big(P^{(n)} \times Q^{(2n)}\big).
\end{aligned} \tag{2.4}$$

Let us find an upper bound for the right-hand side of (2.2). Select c_4, so that

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u_1x_1+\cdots+u_nx_n|^{2n}\, d\big(P^{(n)}\times Q^{(n)}\big) \le c_4^{2n}.$$

Then we obtain the estimate

$$\begin{aligned}
&\mathbf{E}\,(|L_{r1}|^{2n})E(|L_{r2}|^N) = \\
&\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u_1x_1+\cdots+u_nx_n|^{2n}\, d\big(P^{(n)}\times Q^{(n)}\big) \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u_{n+1}x_1+\cdots+u_{2n}x_n|^N\, d\big(P^{(n)}\times Q_1^{(n)}\big) \\
&\qquad \le c_4^{2n} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} (|u_{n+1}||x_1|+\cdots+|u_{2n}||x_n|)^N\, d\big(P^{(n)}\times Q_1^{(n)}\big). \qquad (2.5)
\end{aligned}$$

Writing together the estimates (2.4) and (2.5) and dividing by c_3^{2n} both sides of the obtained inequality, we get

$$\begin{aligned}
&\iint\limits_{A} |x_1|^{2n}\, \big||u_{n+1}||x_1| - |u_{n+2}x_2+\cdots+u_{2n}x_n|\big|^N\, d\big(P^{(n)}\times Q^{(2n)}\big) \le \\
&\qquad \left(\frac{c_4}{c_3}\right)^{2n} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} (|u_{n+1}||x_1|+\cdots+|u_{2n}||x_n|)^N\, d\big(P^{(n)}\times Q_1^{(n)}\big).
\end{aligned}$$

Let us add the integral

$$I_1 = \iint\limits_{\widetilde{A}} |x_1|^{2n}\, \big||u_{n+1}||x_1| - |u_{n+2}x_2+\cdots+u_{2n}x_n|\big|^N\, d\big(P^{(n)}\times Q^{(2n)}\big),$$

where

$$\begin{aligned}
\widetilde{A} = \Big\{&(x_1,\dots,x_n,u_1,\dots,u_{2n}) \in \mathbb{R}^{3n} : \\
&\qquad |x_1| \le c_1^2,\ (x_2,\dots,x_n)\in G,\ |u_1| > \frac{1}{c_1},\ |u_{n+1}| > \frac{1}{c_1}\Big\},
\end{aligned}$$

to both sides of the preceding inequality. We obtain

$$\begin{aligned}
&\iint\limits_{A\cup\widetilde{A}} |x_1|^{2n}\, (|u_{n+1}||x_1| - |u_{n+2}x_2+\cdots+u_{2n}x_n|)^N\, d\big(P^{(n)}\times Q^{(2n)}\big) \\
&\qquad \le I_1 + \left(\frac{c_4}{c_3}\right)^{2n} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} (|u_{n+1}||x_1|+\cdots+|u_{2n}||x_n|)^N\, d\big(P^{(n)}\times Q_1^{(n)}\big).
\end{aligned}$$

We expand the N-th powers of corresponding expressions under the integral signs and get

$$\iint\limits_{A\cup\tilde{A}} \sum_{\nu=0}^{N}(-1)^\nu\binom{N}{\nu}|x_1|^{N+2n-\nu}|u_{n+1}|^{N-\nu}|u_{n+2}x_2+\cdots+u_{2n}x_n|^\nu d(P^{(n)}\times Q^{(2n)}) \le$$
$$I_1+\Big(\frac{c_4}{c_3}\Big)^{2n}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\sum_{i_1+\cdots+i_n=N}\frac{N!}{i_1!\dots i_n!}|u_{n+1}|^{i_1}|x_1|^{i_1}\dots|u_{2n}|^{i_n}|x_n|^{i_n}d(P^{(n)}\times Q_1^{(n)}).$$

We shall carry over all summands on the left-hand side of the preceding inequality, except of corresponding to $\nu=0$, to the right-hand side of one and conclude that

$$c_2Q_1(\{|u_1|>\frac{1}{c_1}\})\iint\limits_{\mathbb{R}^1\times\{|u_{n+1}|>1/c_1\}}|x_1|^{N+2n}|u_{n+1}|^N\,d\big(P_1\times Q_{n+1}\big)\le I_1+$$
$$\Big(\frac{c_4}{c_3}\Big)^{2n}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\sum_{i_1+\cdots+i_n=N}\frac{N!}{i_1!\dots i_n!}|u_{n+1}|^{i_1}|x_1|^{i_1}\dots|u_{2n}|^{i_n}|x_n|^{i_n}\,d\big(P^{(n)}\times Q_1^{(n)}\big)+$$
$$\iint\limits_{A\cup\tilde{A}}\sum_{\nu=1}^{N}\binom{N}{\nu}|x_1|^{N+2n-\nu}|u_{n+1}|^{N-\nu}|u_{n+2}x_2+\cdots+u_{2n}x_n|^\nu\,d\big(P^{(n)}\times Q^{(2n)}\big).\tag{2.6}$$

Since $Q_{n+1}(\{|u_{n+1}|>1/c_1\})\ge c_2$, we see that there exists $c_5\in(0,1)$ such that

$$c_5\,\mathbf{E}\,|U_{n+1}|^N\le\int\limits_{\{|u_{n+1}|>1/c_1\}}|u_{n+1}|^N\,dQ_{n+1}$$

for all positive integers N. Taking into account this estimate we deduce from (2.6) the inequality

$$\begin{aligned}&c_6\,\mathbf{E}\,|X_1|^{N+2n}\,\mathbf{E}\,|U_{n+1}|^N\\ &\le I_1+\Big(\frac{c_4}{c_3}\Big)^{2n}\sum_{i_1+\cdots+i_n=N}\frac{N!}{i_1!\dots i_n!}\,\mathbf{E}\,|U_{n+1}|^{i_1}\,\mathbf{E}\,|X_1|^{i_1}\dots\mathbf{E}\,|U_{2n}|^{i_n}\,\mathbf{E}\,|X_n|^{i_n}\\ &\qquad+\sum_{\nu=1}^{N}\binom{N}{\nu}c_7^\nu\,\mathbf{E}\,|U_{n+1}|^{N-\nu}\,\mathbf{E}\,|X_1|^{N+2n-\nu},\end{aligned}$$

where $c_6\in(0,1)$, and c_7 is chosen from the condition

$$\iint\limits_{G\times\mathbb{R}^{n-1}}|u_{n+2}x_2+\cdots+u_{2n}x_n|^\nu\,d\big(P_2\times\cdots\times P_n\times Q_{n+2}\times\cdots\times Q_{2n}\big)\le c_7^\nu,$$
$$\nu=1,2,\dots.$$

In view of Lyapunov's moment inequalities we have

$$\mathbf{E}\,U^l\le\big(\mathbf{E}\,U^N\big)^{l/N},\qquad l=1,\dots N.$$

Then using (ii) and the right-hand side of the inequality (iii) we obtain

$$0 < \mathbf{E}\,|U_{n+j}|^l \le b\left(\mathbf{E}\,U^N\right)^{l/N}, \qquad l = 1,\ldots,N,\ j = 1,\ldots n.$$

By the left-hand side of (iii), we get

$$\mathbf{E}\,|U_{n+1}|^N \ge \frac{1}{b}\,\mathbf{E}\,U^N.$$

With the help of the two last inequalities we conclude that

$$\frac{c_6}{b}\,\mathbf{E}\,U^N\,\mathbf{E}\,|X_1|^{N+2n} \le$$
$$I_1 + \left(\frac{c_4}{c_3}\right)^{2n} \sum_{i_1+\cdots+i_n=N} \frac{N!}{i_1!\ldots i_n!} b^n\,\mathbf{E}\,U^N\,\mathbf{E}\,|X_1|^{i_1}\ldots\mathbf{E}\,|X_n|^{i_n} +$$
$$+\sum_{\nu=1}^{N} \binom{N}{\nu} c_7^\nu b\left(\mathbf{E}\,U^N\right)^{(N-\nu)/N}\,\mathbf{E}\,|X_1|^{N+2n-\nu}.$$

Taking into account that $\mathbf{E}\,U^N > 0$ we obtain from this estimate the main relation

$$\mathbf{E}\,|X_1|^{N+2n} \le \frac{b}{c_6}\left\{\frac{1}{\mathbf{E}\,U^N} I_1 + \right.$$
$$\left(\frac{c_4}{c_3}\right)^{2n} b^n \sum_{i_1+\cdots+i_n=N} \frac{N!}{i_1!\ldots i_n!}\,\mathbf{E}\,|X_1|^{i_1}\ldots\mathbf{E}\,|X_n|^{i_n} +$$
$$\left. b\sum_{\nu=1}^{N} \binom{N}{\nu} c_7^\nu \left(\mathbf{E}\,U^N\right)^{-\nu/N}\,\mathbf{E}\,|X_1|^{N+2n-\nu}\right\}. \qquad (2.7)$$

We shall show that there exists a positive number M such that

$$\mathbf{E}\,|X_j|^k \le M^k k!, \qquad k = 1,2,\ldots,\ j = 1,\ldots,n. \qquad (2.8)$$

Select M such that (2.8) is true for all $k = 1,\ldots,2n$ and $j = 1,\ldots,n$. Let us prove by induction on k that (2.8) is valid for $k > 2n$. Let (2.8) be valid for $k \le N+2n-1$ and $j = 1$. Verify it for $k = N+2n$ and $j = 1$, using the estimate (2.7).

We first note that there exists c_8 such that

$$I_1 \le c_8^N.$$

Since the estimate $\mathbf{E}\,U^N \ge c_9^N$ is valid for some c_9 and for all positive integers N, we shall choose $M \ge 1 + (3bc_8)/(c_6c_9)$, so that

$$\frac{1}{\mathbf{E}\,U^N} I_1 \le \left(\frac{c_8}{c_9}\right)^N \le \frac{1}{3}\frac{c_6}{b} M^{N+2n}(N+2n)! \qquad (2.9)$$

for all positive integer N.

We shall estimate the second summand in braces on the right-hand side of (2.7), taking into account inductive hypothesis. Choosing M such that $M > (c_4/c_3)(3b^{n+1}/c_6)^{1/(2n)}$, we see that this summand does not exceed

$$\left(\frac{c_4}{c_3}\right)^{2n} b^n M^N N! \sum_{i_1+\dots+i_n=N} 1 \le \left(\frac{c_4}{c_3}\right)^{2n} b^n M^N N!(N+1)^{n-1}$$
$$\le \frac{1}{3}\frac{c_6}{b} M^{N+2n}(N+2n)! \,. \quad (2.10)$$

Let us estimate the third summand in braces on the right-hand side of (2.7). We obtain with the help of inductive hypothesis

$$b\sum_{\nu=1}^{N}\binom{N}{\nu} c_7^\nu \left(\mathbf{E}\, U^N\right)^{-\nu/N} \mathbf{E}\,|X_1|^{N+2n-\nu}$$
$$\le b\sum_{\nu=1}^{N}\frac{N(N-1)\dots(N-\nu+1)}{\nu!} c_7^\nu \left(\mathbf{E}\, U^N\right)^{-\nu/N} M^{N+2n-\nu}(N+2n-\nu)!$$
$$\le bM^{N+2n}(N+2n)!\sum_{\nu=1}^{N}\frac{(c_7/M)^\nu}{\nu!}\left(\mathbf{E}\, U^N\right)^{-\nu/N} \le$$
$$\left(\exp\left(\frac{c_7}{M(\mathbf{E}\, U^N)^{1/N}}\right) - 1\right) bM^{N+2n}(N+2n)! \,.$$

Since $\mathbf{E}\, U^N \ge c_9^N$, we shall choose $M \ge (6b^2c_7)/(c_6c_9)$, so that the first factor on the right-hand side of the last inequality does not exceed $c_6/3b^2$. Then we conclude that the third summand in braces on the right-hand side of (2.7) does not exceed $\frac{c_6}{3b}M^{N+m}(N+m)!$. In view of this estimate and (2.9), (2.10), one finally obtains

$$\mathbf{E}\,|X_1|^{N+2n} \le M^{N+2n}(N+2n)!,$$

where the parameter M depend on the r.v.'s $X_1,\dots,X_n$ and n, b, and $c_1,\dots,c_9$ only. We prove the last estimate for the r.v.'s $X_2,\dots,X_n$ in the same way. Thus, we have proved that the ch.f.'s of the r.v.'s $X_1,\dots,X_n$ are regular at least in the strip $|\operatorname{Im} z| < 1/M$ for some $M > 0$. We write as usual $\varphi(z; X_j)$, $j = 1,\dots,n$, for the functions of the complex argument $z = t + iy$ $(t, y$ real) which agree with $\varphi(t; X_j)$, $j = 1,\dots,n$, on the real axis respectively.

It is clear we may assume, without loss of generality, that $|U_j| \le 1$ a.s. for all $j = 1,\dots,2n$ and one is a point of increase of the distribution function (d.f.) either of $|U_j|$ or $|U_{n+j}|$ for all $j = 1,\dots,n$.

We shall show that the ch.f.'s of all r.v.'s X_j, $j = 1,\dots,n$, are entire functions. We give an indirect proof and suppose that there exist j_0 and $\tau_{j_0} \in (0,\infty)$ such that $\mathbf{E}\, e^{\tau|X_{j_0}|} < \infty$ for $\tau < \tau_{j_0}$ and $\mathbf{E}\, e^{\tau|X_{j_0}|} = +\infty$ for $\tau > \tau_{j_0}$. Define the parameter τ_j for the rest X_j in the same way. If $\mathbf{E}\, e^{\tau|X_j|} < \infty$ for all $\tau > 0$, we assume that $\tau_j = +\infty$. Let, for the definiteness, $\tau_1 = \min\{\tau_1,\dots,\tau_n\}$. Consider the event $A_1 = \{(X_2,\dots,X_n) \in G\}$, where G is the set of the $(n-1)$-dimensional space which was earlier defined. Let $I_{A_1}(\omega)$ be equal to unit or zero according as ω does or does not satisfy $\omega \in A_1$. The inequalities

$$|U_1||X_1| \le |L_{r1}| + c_{10}, \qquad |U_{n+1}||X_1| \le |L_{r2}| + c_{10}$$

hold in the event A_1. Taking into account these inequalities we shall write

$$\mathbf{E}\, I_{A_1}\, \mathbf{E}\, e^{\tau|X_1|(|U_1|+|U_{n+1}|)} = \mathbf{E}\,(I_{A_1} e^{\tau|X_1|(|U_1|+|U_{n+1}|)}) \le$$
$$e^{2c_{10}\tau}\, \mathbf{E}\,(I_{A_1} e^{\tau(|L_{r1}|+|L_{r2}|)}) \le e^{2c_{10}\tau}\, \mathbf{E}\, e^{\tau(|L_{r1}|+|L_{r2}|)} = e^{2c_{10}\tau}\, \mathbf{E}\, e^{\tau|L_{r1}|}\, \mathbf{E}\, e^{\tau|L_{r2}|}.$$

Since $|L_{rk}| \le |X_1| + \cdots + |X_n|$, we conclude according to the choice of the parameter τ_1 that

$$\mathbf{E}\, e^{\tau|L_{rk}|} \le \prod_{j=1}^{n} \mathbf{E}\, e^{\tau|X_j|} < +\infty$$

for all $\tau < \tau_1$ and $k = 1, 2$. Therefore we have

$$\mathbf{E}\, e^{\tau|X_1|(|U_1|+|U_{n+1}|)} \le c_{11} e^{2c_{10}\tau} \left(\prod_{j=1}^{n} \mathbf{E}\, e^{\tau|X_j|}\right)^2 < +\infty, \qquad \tau < \tau_1. \tag{2.11}$$

Denote by $Q_{1,n+1}$ the pr.d. of the r.v. $|U_1| + |U_{n+1}|$. The following formula

$$\mathbf{E}\, e^{\tau|X_1|(|U_1|+|U_{n+1}|)} = \int_0^\infty \mathbf{E}\, e^{\tau y|X_1|}\, dQ_{1,n+1}$$

is true. Since one is a point of increase of the d.f. either of $|U_1|$ or $|U_{n+1}|$, the d.f. of the r.v. $|U_1| + |U_{n+1}|$ has a point of increase $\delta_1 > 1$. Therefore the inequality

$$c_{12}\, \mathbf{E}\, e^{(1+\delta_1)\tau|X_1|/2} \le \int_0^\infty \mathbf{E}\, e^{\tau y|X_1|}\, dQ_{1,n+1} < +\infty \tag{2.12}$$

holds for $\tau < \tau_1$. Choosing $\tau < \tau_1$ sufficiently close to τ_1, we arrive at the contradiction.

It remains to show that the ch.f.'s of the r.v.'s X_j, $j = 1, 2, \ldots, n$, are entire functions of finite order. We note that (2.11) holds for all $\tau > 0$. In addition the same inequality is true for the r.v.'s $X_2, \ldots, X_n$. The inequality (2.12) is also true for all $\tau > 0$ and for $X_2, \ldots, X_n$ with some constant $\delta \in (1, \delta_1]$ instead of δ_1. We obtain from these inequalities

$$c_{13}\, \mathbf{E}\, e^{(1+\delta)\tau|X_j|/2} \le \mathbf{E}\, e^{\tau|X_j|(|U_j|+|U_{n+j}|)} \le c_{14} e^{2c_{15}\tau} \left(\prod_{l=1}^{n} \mathbf{E}\, e^{\tau|X_l|}\right)^2, \ j = 1, \ldots, n.$$

The last estimate yields

$$c_{13}^n \prod_{j=1}^{n} \mathbf{E}\, e^{(1+\delta)\tau|X_j|/2} \le c_{14}^n e^{2c_{15}n\tau} \left(\prod_{l=1}^{n} \mathbf{E}\, e^{\tau|X_l|}\right)^{2n}.$$

Denote by $\psi(\tau)$ the sum $\sum\limits_{j=1}^{n} \log \mathbf{E}\, e^{\tau|X_j|}$ and rewrite the last estimate in the form

$$n\log c_{13} + \psi\left((1+\delta)\tau/2\right) \le n\log c_{14} + 2c_{15}n\tau + 2n\psi(\tau), \qquad \tau \ge 0.$$

It is not difficult to see from this inequality that $\psi(\tau)$ has the polynomial growth as $\tau \to +\infty$. Thus, we have proved that the ch.f. $\prod\limits_{j=1}^{n} \varphi(t;|X_j|)$ is an entire function of finite order. Then, by Raikov's theorem [6, p. 58], the ch.f.'s $\varphi(t;|X_j|)$, $j=1,\dots,n$, are also entire functions of finite order. It is clear that the same assertion holds for the ch.f's $\varphi(t;X_j)$, $j=1,\dots,n$, so that the theorem is proved.

3. Proof of Theorem 1. Since, by Theorem 3, the ch.f.'s of the r.v.'s X_j, $j=1,\dots,n$, are entire functions of finite order and the r.v.'s U_j, $j=1,\dots,2n$, are a.s. bounded, we conclude that the ch.f.'s of the random vectors $(U_jX_j, U_{n+j}X_j)$, $j=1,\dots,n$, are entire functions of finite order. (About entire functions of several variables of finite order see [7]). It follows from this fact that ch.f.'s of the r.v.'s U_jX_j and $U_{n+j}X_j$ are also entire functions of finite order for all $j=1,\dots,n$. We obtain from the independence of the forms L_{r1}, L_{r2} the following functional equation

$$\prod_{j=1}^{n} \mathbf{E}\, e^{itU_jX_j+isU_{n+j}X_j} = \prod_{j=1}^{n} \mathbf{E}\, e^{itU_jX_j} \prod_{j=1}^{n} \mathbf{E}\, e^{isU_{n+j}X_j}, \quad (t,s)\in\mathbb{R}^2. \tag{3.1}$$

It is obvious that this relation remains true for all complex t,s. We now need one simple result from the theory of several complex variables.

Lemma A. *Let $\varphi(z,w)$ be an entire function of finite order and let the set of its zeros counting the multiciplicity has the form*

$$\left(\bigcup_{k=1}^{\infty}\{(z,w): z=z_k\}\right)\bigcup\left(\bigcup_{m=1}^{\infty}\{(z,w): w=w_m\}\right). \tag{3.2}$$

. Then, for all complex z,w,

$$\varphi(z,w) = e^{D(z,w)}\varphi_1(z)\varphi_2(w),$$

where $D(z,w)$ is a polynomial such that $D(0,w)=D(z,0)\equiv 0$ and $\varphi_1(z)$ and $\varphi_2(w)$ are entire functions of finite order.

We omit the proof of this fact and refer the reader to [8]. The function on the right-hand side of (3.1) is an entire function of finite order and the set of its zeros has the form (3.2). Then the functions $\mathbf{E}\, e^{itU_jX_j+isU_{n+j}X_j}$, $j=1,\dots,n$, satisfy the assumptions of Lemma A and, by this lemma, admit the representation

$$\mathbf{E}\, e^{itU_jX_j+isU_{n+j}X_j} = e^{D_j(t,s)}\,\mathbf{E}\, e^{itU_jX_j}\,\mathbf{E}\, e^{isU_{n+j}X_j}, \quad t\in\mathbb{C}, s\in\mathbb{C}, \tag{3.3}$$

where $D_j(t,s)$ are polynomials and $D_j(0,s)=D_j(t,0)\equiv 0$.

In the sequel we shall consider only those indices j for which the r.v.'s X_j satisfy the condition (iv). Let us show that $\mathbf{E}\, e^{itU_jX_j}$ does not vanish in $\mathbb{C}$. Let the opposite be true. Then there exists $t_0 \in \mathbb{C}$ such that $\mathbf{E}\, e^{it_0U_jX_j} = 0$. Consider the function

$$I_j(s) = \mathbf{E}\, e^{it_0U_jX_j} e^{isU_{n+j}X_j} = \int_{-\infty}^{\infty} \varphi(t_0x; U_j)\varphi(sx; U_{n+j})\, dP_j = \int_{-\infty}^{\infty} \varphi(sx; U_{n+j})\, d\nu_j, \quad (3.4)$$

where ν_j is a finite complex-valued measure. Denote by $\varphi(s; \nu_j)$ the ch.f. of the measure ν_j. It is easy to see that the function $\varphi(s; \nu_j)$ is an entire function, its modulus is uniformly bounded for all $s \in \mathbb{R}^1$. We obtain from (3.3) that

$$I_j(s) = 0, \qquad s \in \mathbb{C}. \quad (3.5)$$

Then it follows from (3.4), where $s = 0$, that $\varphi(0; \nu_j) = 0$. We now note that from (3.4) and (3.5) it follows the relation

$$\int_{-\infty}^{\infty} \varphi(sx; \nu_j) dQ_{n+j} = 0, \qquad s \in \mathbb{C}$$

(recall that Q_{n+j} is the pr.d. of the r.v. U_{n+j}).

Apply Mellina's transform to both sides of the preceding equality. One gets the relations

$$q^+_{n+j}(z)R^+_j(z) - q^-_{n+j}(z)R^-_j(z) = 0, \qquad q^+_{n+j}(z)R^-_j(z) - q^-_{n+j}(z)R^+_j(z) = 0$$

for all complex z such that $-1 < \operatorname{Re} z < 0$, where

$$q^+_{n+j}(z) = \int_0^{\infty} x^{-z} dQ_{n+j}, \qquad q^-_{n+j}(z) = \int_{-\infty}^{0} x^{-z} dQ_{n+j}$$

and

$$R^+_j(z) = \int_0^{\infty} y^{z-1}\varphi(y; \nu_j) dy, \qquad R^-_j(z) = \int_{-\infty}^{0} y^{z-1}\varphi(y; \nu_j) dy$$

are regular functions in the strip $-1 < \operatorname{Re} z < 0$. From these relations it follows that

$$\Big(q^+_{n+j}(z) + q^-_{n+j}(z)\Big)\Big(R^+_j(z) - R^-_j(z)\Big) = 0 \quad (3.6)$$

and

$$\Big(q^+_{n+j}(z) - q^-_{n+j}(z)\Big)\Big(R^+_j(z) + R^-_j(z)\Big) = 0 \quad (3.7)$$

for all complex z such that $-1 < \operatorname{Re} z < 0$. Since the pr.d. Q_{n+j} is not degenerate at zero, the first factor on the left-hand side of (3.6) is a regular function in the strip $-1 < \operatorname{Re} z < 0$ which is not identically equal to zero. Therefore the second factor on the left-hand side of (3.6) is equal to zero for $-1 < \operatorname{Re} z < 0$. This implies that $R_j^+(z) = R_j^-(z)$ for $-1 < \operatorname{Re} z < 0$. Then in a similar way one obtains from (3.7) that $R_j^+(z) = 0$ for the same z. This easily implies that $\varphi(y;\nu_j) = 0$ for $y \ge 0$. Since $\varphi(y;\nu_j)$ is an entire function, we obtain then that $\varphi(y;\nu_j) = 0$ for all real y and conclude that $\nu_j = 0$. In other words we have

$$\int_S \varphi(t_0 x; U_j)\, dP_j = 0$$

for all Borelian's set S. It is possible only in the case where the support of the pr.d. P_j is situated in the set of real zeros of the function $\varphi(t_0 x; U_j)$. It is impossible in our case where the r.v. X_j satisfies the condition (iv), so that $\mathbf{E}\, e^{itU_jX_j}$ does not vanish in $\mathbb{C}$ for every j under consideration. In the same way we prove that the function $\mathbf{E}\, e^{itU_{n+j}X_j}$ does not vanish in $\mathbb{C}$. Then, by (3.3), the function $\mathbf{E}\, e^{i(tU_j+sU_{n+j})X_j}$ does not vanish for all complex t, s. This function is entire function of two variables of finite order. By many-dimensional Marcinkiewicz's theorem [6,p. 41], the random vector $(U_jX_j, U_{n+j}X_j)$ is Gaussian and therefore the r.v.'s U_jX_j, $U_{n+j}X_j$ are Gaussian. We then conclude that in (3.3) $D_j(t,s) = cts$, where c is a constant. Differentiating sequentially both sides of (3.3) by t and s and setting $t = s = 0$, we obtain $c = -\mathbf{E}\, U_j\, \mathbf{E}\, U_{n+j} \operatorname{Var} X_j$.

Let us show that $\mathbf{E}\, U_j \ne 0$ and $\mathbf{E}\, U_{n+j} \ne 0$. If at least one of them vanishes, then $c = 0$ and in (3.3) the polynomial $D_j(t,s) \equiv 0$. This implies the independence of the r.v.'s U_jX_j and $U_{n+j}X_j$ and we can write the relation

$$\mathbf{E}\, U_j^2\, \mathbf{E}\, U_{n+j}^2\, \mathbf{E}\, X_j^4 = \mathbf{E}\,(U_j^2X_j^2 \cdot U_{n+j}^2X_j^2) = \mathbf{E}\,(U_j^2X_j^2)\, \mathbf{E}\,(U_{n+j}^2X_j^2) = \mathbf{E}\, U_j^2\, \mathbf{E}\, U_{n+j}^2 (\mathbf{E}\, X_j^2)^2.$$

By condition (ii), $\mathbf{E}\, U_j^2 \ne 0$ and $\mathbf{E}\, U_{n+j}^2 \ne 0$, and we obtain from the preceding relation that $\mathbf{E}\, X_j^4 = (\mathbf{E}\, X_j^2)^2$. Hence $X_j^2 = \mathbf{E}\, X_j^2$ a.s., and since a median of the r.v. X_j is equal to zero, we obtain $\mathbf{E}\, X_j = 0$. We now note that the r.v.'s U_jX_j and $U_{n+j}X_j$ are a.s. bounded and, as it was shown above, Gaussian. Therefore their variances are equal to zero and we have the relations

$$\mathbf{E}\, U_j^2\, \mathbf{E}\, X_j^2 = (\mathbf{E}\, U_j)^2 (\mathbf{E}\, X_j)^2 = 0,$$

$$\mathbf{E}\, U_{n+j}^2\, \mathbf{E}\, X_j^2 = (\mathbf{E}\, U_{n+j})^2 (\mathbf{E}\, X_j)^2 = 0.$$

Since, by (iv), $\mathbf{E}\, X_j^2 \ne 0$, we get from the preceding relations that $\mathbf{E}\, U_j^2 = \mathbf{E}\, U_{n+j}^2 = 0$. By condition (ii), it is impossible.

For all j under consideration, medians of the r.v.'s X_j are equal to zero. Since the r.v.'s U_j, U_{n+j}, and X_j are independent, we see that medians of the r.v.'s U_jX_j and $U_{n+j}X_j$ are also equal to zero. Hence U_jX_j and $U_{n+j}X_j$ are Gaussian r.v.'s with the mathematical expectations are equal to zero. Then there exists c_{16} such that U_jX_j and $c_{16}U_{n+j}X_j$ are identically distributed. Therefore we obtain

$$\mathbf{E}\, U_j^k\, \mathbf{E}\, X_j^k = \mathbf{E}\,(U_jX_j)^k = (c_{16})^k\, \mathbf{E}\,(U_{n+j}X_j)^k = (c_{16})^k\, \mathbf{E}\, U_{n+j}^k\, \mathbf{E}\, X_j^k$$

for all positive integers k. We see from these equalities that, for all even positive integers k,

$$\mathbf{E}\, U_j^k = (c_{16})^k\, \mathbf{E}\, U_{n+j}^k. \tag{3.8}$$

Since the random vector $(U_jX_j, U_{n+j}X_j)$ is Gaussian, then $Z_j = (\alpha U_j + \beta U_{n+j})X_j$ are Gaussian r.v.'s for all real values of the parameters α and β. We assume, without loss of generality, that $\mathbf{E}\, U_j^k = \mathbf{E}\, U_{n+j}^k$ for all even k. Indeed, by (3.8), this is true for U_j and $c_{16}U_{n+j}$. Choosing instead of β the value $c_{16}\beta$, we obtain the required assumption. It is easy to see that a median of the r.v. Z_j is equal to zero, therefore $\mathbf{E}\, Z_j = 0$. On the other hand,

$$\operatorname{Var} Z_j = (\alpha^2\, \mathbf{E}\, U_j^2 + 2\alpha\beta\, \mathbf{E}\, U_j\, \mathbf{E}\, U_{n+j} + \beta^2\, \mathbf{E}\, U_{n+j}^2)\, \mathbf{E}\, X_j^2.$$

Therefore the value of event moments of the r.v. Z_j is calculated by the formulas

$$\begin{aligned}
\mathbf{E}\, Z_j^{2k} &= \mathbf{E}\, (\alpha U_j + \beta U_{n+j})^{2k}\, \mathbf{E}\, X_j^{2k},\\
\mathbf{E}\, Z_j^{2k} &= (\alpha^2\, \mathbf{E}\, U_j^2 + 2\alpha\beta\, \mathbf{E}\, U_j\, \mathbf{E}\, U_{n+j} + \beta^2\, \mathbf{E}\, U_{n+j}^2)^k (\,\mathbf{E}\, X_j^2)^k (2k-1)!!,\\
k &= 1, 2, \ldots .
\end{aligned}$$

We now note that polynomials of the variables α and β stand on the right-hand sides of the last equalities. Comparing coefficients of the powers α^{2k}, $\alpha^{2k-1}\beta$, $\alpha^{2k-2}\beta^2$, and $\alpha^{2k-3}\beta^3$ in these polynomials we obtain the relations

$$\mathbf{E}\, U_j^{2k}\, \mathbf{E}\, X_j^{2k} = (\,\mathbf{E}\, U_j^2)^k (\,\mathbf{E}\, X_j^2)^k (2k-1)!!\ , \tag{3.9}$$

$$\mathbf{E}\, U_j^{2k-1}\, \mathbf{E}\, U_{n+j}\, \mathbf{E}\, X_j^{2k} = (\,\mathbf{E}\, U_j^2)^{k-1}\, \mathbf{E}\, U_j\, \mathbf{E}\, U_{n+j} (\,\mathbf{E}\, X_j^2)^k (2k-1)!!\ , \tag{3.10}$$

$$\begin{aligned}
\binom{2k}{2} \mathbf{E}\, U_j^{2k-2}\, \mathbf{E}\, U_{n+j}^2\, \mathbf{E}\, X_j^{2k} = \Big\{ \binom{k}{1} (\,\mathbf{E}\, U_j^2)^{k-1}\, \mathbf{E}\, U_{n+j}^2 + \\
+ 4\binom{k}{2} (\,\mathbf{E}\, U_j^2)^{k-2} (\,\mathbf{E}\, U_j)^2 (\,\mathbf{E}\, U_{n+j})^2 \Big\} (\,\mathbf{E}\, X_j^2)^k (2k-1)!!\ ,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\binom{2k}{3} \mathbf{E}\, U_j^{2k-3}\, \mathbf{E}\, U_{n+j}^3\, \mathbf{E}\, X_j^{2k} = \Big\{ 4\binom{k}{2} (\,\mathbf{E}\, U_j^2)^{k-2}\, \mathbf{E}\, U_j\, \mathbf{E}\, U_{n+j}\, \mathbf{E}\, U_{n+j}^2 + \\
+ 8\binom{k}{3} (\,\mathbf{E}\, U_j^2)^{k-3} (\,\mathbf{E}\, U_j)^3 (\,\mathbf{E}\, U_{n+j})^3 \Big\} (\,\mathbf{E}\, X_j^2)^k (2k-1)!!
\end{aligned} \tag{3.12}$$

for $k = 2, 3, \ldots$. Since $\mathbf{E}\, U_j \neq 0$ and $\mathbf{E}\, U_{n+j} \neq 0$, and the r.v. X_j is not a.s. zero, we see that the left-hand sides of (3.9) – (3.11) do not vanish. Dividing (3.9) by (3.10) we arrive at the relation

$$\mathbf{E}\, U_j^{2k} = \frac{\mathbf{E}\, U_j^2}{\mathbf{E}\, U_j}\, \mathbf{E}\, U_j^{2k-1}. \tag{3.13}$$

An analogous relation is valid for the r.v. U_{n+j}. If we divide (3.10) by (3.11), then, taking into account that the even moments of the r.v.'s U_j and U_{n+j} are the same, we obtain

$$\mathbf{E}\,U_j^{2k-1} = \frac{(\mathbf{E}\,U_j^2)^2\,\mathbf{E}\,U_j}{(1-\frac{1}{2k-1})(\mathbf{E}\,U_j)^2(\mathbf{E}\,U_{n+j})^2 + \frac{1}{2k-1}(\mathbf{E}\,U_j^2)^2}\,\mathbf{E}\,U_j^{2k-2}. \tag{3.14}$$

We have from (3.13) and (3.14)

$$\mathbf{E}\,U_j^{2k} = \frac{(\mathbf{E}\,U_j^2)^3}{(1-\frac{1}{2k-1})(\mathbf{E}\,U_j)^2(\mathbf{E}\,U_{n+j})^2 + \frac{1}{2k-1}(\mathbf{E}\,U_j^2)^2}\,\mathbf{E}\,U_j^{2k-2}. \tag{3.15}$$

We see from the formula (3.13) for the r.v.'s U_j and U_{n+j} that $\mathbf{E}\,U_j^{2k-3} \neq 0$, $\mathbf{E}\,U_{n+j}^3 \neq 0$ and hence the left-hand side of (3.12) is not equal to zero. Dividing (3.9) by (3.12) we arrive at the relation

$$\mathbf{E}\,U_j^{2k} = \frac{\mathbf{E}\,U_{n+j}^3(\mathbf{E}\,U_j^2)^3}{8\binom{k}{3}(\mathbf{E}\,U_j)^3(\mathbf{E}\,U_{n+j})^3/\binom{2k}{3} + 4\binom{k}{2}(\mathbf{E}\,U_j^2)^2\,\mathbf{E}\,U_j\,\mathbf{E}\,U_{n+j}/\binom{2k}{3}}\,\mathbf{E}\,U_j^{2k-3} \tag{3.16}$$

for $k = 2, 3, \dots$. We get from (3.15) and (3.13) the formula

$$\mathbf{E}\,U_j^{2k} = \frac{(\mathbf{E}\,U_j^2)^4}{\mathbf{E}\,U_j}\,\frac{1}{(1-\frac{1}{(2k-1)})(\mathbf{E}\,U_j)^2(\mathbf{E}\,U_{n+j})^2 + \frac{1}{2k-1}(\mathbf{E}\,U_j^2)^2}\,\mathbf{E}\,U_j^{2k-3} \tag{3.17}$$

for $k = 2, 3, \dots$. We mentioned above that $\mathbf{E}\,U_j^{2k-1} \neq 0$ for all $k = 1, 2, \dots$. Comparing the right-hand sides of (3.16) and (3.17) and tending k to infinity, we deduce the equality

$$\mathbf{E}\,U_{n+j}^3 = \mathbf{E}\,U_j^2\,\mathbf{E}\,U_{n+j}. \tag{3.18}$$

On the other hand, we obtain from (3.14), where $k = 2$,

$$\mathbf{E}\,U_{n+j}^3 = \frac{3(\mathbf{E}\,U_j^2)^3\,\mathbf{E}\,U_{n+j}}{2(\mathbf{E}\,U_j)^2(\mathbf{E}\,U_{n+j})^2 + (\mathbf{E}\,U_j^2)^2}. \tag{3.19}$$

We see, comparing the right-hand sides of the formulas (3.18) and (3.19), that the equality

$$\mathbf{E}\,U_j^2 = |\mathbf{E}\,U_j||\mathbf{E}\,U_{n+j}| \tag{3.20}$$

is true. Taking into account (3.20) and (3.15) we obtain

$$\mathbf{E}\,U_j^{2k} = \mathbf{E}\,U_j^2\,\mathbf{E}\,U_j^{2k-2}$$

for $k = 1, 2 \dots$ and therefore

$$\mathbf{E}\,U_j^{2k} = (\mathbf{E}\,U_j^2)^k$$

for the same k. It is easily follows from these relations that $U_j^2 = \mathbf{E}\,U_j^2$ a.s. So that U_j takes the two values $-(\mathbf{E}\,U_j^2)^{1/2}$ and $(\mathbf{E}\,U_j^2)^{1/2}$ with probabilities p_1

and p_2 respectively. Since even moments of the r.v.'s U_j and U_{n+j} are the same, the r.v. U_{n+j} takes the values $-(E(U_j^2))^{1/2}$, $(E(U_j^2))^{1/2}$ with probabilities q_1 and q_2 respectively. Let us show that one of the numbers p_1 and p_2, and one of the numbers q_1 and q_2 are equal to one. Without loss of generality, we assume that $p_2 \geq \frac{1}{2}$ and $q_2 \geq \frac{1}{2}$. Then we rewrite (3.20) in the form $(2p_2-1)(2q_2-1) = 1$. One obtains then that p_2, q_2 satisfy the relation

$$\frac{1}{p_2} + \frac{1}{q_2} = 2.$$

It is possible only in the case where $p_2 = 1$ and $q_2 = 1$. Thus, it is proved that the r.v.'s U_j and U_{n+j} are a.s. constant .

Since, as we saw before, the r.v. $U_j X_j$ is Gaussian and $U_j = const$ a.s., the r.v. X_j is Gaussian. This completes the proof of the theorem.

References

1. *G.Darmois,* Analyse générale des liaisons stochastiques. Rev. Inst. Intern. Statist. (1953), v. 21, pp. 2–8.
2. *Skitovich V.P.* Linear forms of independent random variables and normal distribution function. Izv. AN USSR, ser. Matem. (1954), v. 18, No. 2, pp. 185–200 (Russian).
3. *Zinger A.A., Linnik Yu.V.* Non-linear statistics and random linear forms. Proceeding of the Steklov Institute of Mathematics, no. 111 (1970) (Russian).
4. *Kagan A.M., Linnik Yu.V, Rao S.R.* Characterization problems of mathematical statistic. Nauka, Moscow, 1972, 656 p. (Russian).
5. *Levin B.Ya.* Distribution of zeros of entire functions. Providence, Rhode Island, Amer. Math. Soc., 1964, 493 p.
6. *Linnik Yu.V., Ostrovskii I.V.* Decomposition of Random Variables and Vectors. Providence, Rhode Island, Amer. Math. Soc., 1977, 380 p.
7. *Ronkin L.I.* Introduction to the Theory of Entire Functions of Several Variables. Providence, Rhode Island, Amer. Math. Soc., 1974, 273 p.
8. *Papush D.E.* On growth of entire functions with "plane" zeros. Teor. Funkcii, Funkcional. Anal. i Prilozen., Vyp. 48, (1987), pp. 117–125 (Russian).

Gennadii Chistyakov

Institute for Low Temperature Physics and Engineering

National Academy of Sciences of Ukraine

47 Lenin Ave.

310164 Kharkov

Ukraine

Sergei Pergamentsev

Institute for Low Temperature Physics and Engineering

National Academy of Sciences of Ukraine

47 Lenin Ave.

310164 Kharkov

Ukraine

E-mail address: chistyakov@ilt.kharkov.ua

pergamentsev@ilt.kharkov.ua