JEAN-PIERRE WINTENBERGER

Grenoble

Pierre Colmez

Jean-Pierre Wintenberger did his thèse de 3-ième cycle and his thèse d'État in Grenoble, under the supervision of Jean-Marc Fontaine. This is where I met him for the fist time. I spent two years (1985-87) in Grenoble for my PhD : my thesis problem had been given to me by John Coates who was then Professor at Orsay, but I had ended up in Grenoble, with Fontaine as an official adviser, by some bizarre twist. My first year there was rather miserable as my thesis (about complex L-functions) had a big gap. The second year was much more fun: Fontaine had just returned from his year at Minneapolis where he was collaborating with William Messing on their proof [6] of Fontaine's $C_{\rm cris}$ conjecture [3] on periods of p-adic algebraic varieties with good reduction, and everybody was speaking of p-adic periods (including me: I was fantasizing about a product formula for these numbers, analogous to the product formula for rational numbers, and most of what was being discussed found its way in the ouput [1] of my fantasies). Roland Gillard [8] had just proved, in the case of ordinary reduction, a p-adic analog of Shimura's multiplicative relations between periods of CM abelian varieties (a vast generalization of the celebrated Chowla-Selberg formula expressing periods of elliptic curves with complex multiplication in terms of values of the Γ-function at rational arguments – the simplest formula of this type being $\int_1^\infty \frac{dx}{\sqrt{x^3-x}} = \frac{\Gamma(1/4)\Gamma(1/2)}{2\,\Gamma(3/4)}$) using methods introduced by Gross [9] in his geometric proof of the Chowla-Selberg formula. Wintenberger had started to attack the general case which is more difficult as the periods do not live in $\mathbb{C}_p$ (the completion of the algebraic closure of $\mathbb{Q}_p$) anymore, but in the bigger (and much scarier) ring $\mathbb{B}_{\rm dR}$ of p-adic periods, constructed by Fontaine [3]. I remember him telling me one day that $\mathbb{B}_{\rm dR}$ lives in families (i.e., you have relative $\mathbb{B}_{\rm dR}$'s over bases) and p-adic periods of a family satisfy the same kind of differential equations as complex periods; his remark played a big role in my work with Fontaine [2] some 12 years later. He completed half of this project, namely the definition of the p-adic periods of a family of Abelian varieties [18], but part II announced in the introduction of that paper never materialised which is a pity.

At the time, Wintenberger had already a solid reputation in the field. He had developed, with Fontaine [7, 15], the Field of Norms theory which attaches to any "reasonable" infinite extension L of the field $\mathbb{Q}_p$ of p-adic numbers, a characteristic p field $X(L)$ isomorphic to $\mathbb{F}_q((T))$ for some $q = p^f$. The association $L \mapsto X(L)$ looks very strange at first (see the formula for the addition law below), but it is functorial, and provides a bridge between the absolute Galois groups of finite extensions of $\mathbb{Q}_p$ and those of finite extensions of $\mathbb{F}_p((T))$. The Field of Norms theory is the fundation upon which rests the powerful theory of (φ, Γ)-modules of Fontaine [4] which gives a description of all $\mathbb{Q}_p$-representations of these absolute Galois groups; it is also the 0-dimensional case of Scholze's tilting equivalence [13] between characteristics 0 and p.

Important examples of reasonable infinite extensions of $\mathbb{Q}_p$ are the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$, the Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$, or extensions fixed by the kernel of representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_d(\mathbb{Q}_p)$ "coming from geometry" (i.e., from the étale cohomology of algebraic varieties defined over $\mathbb{Q}_p$ or its finite extensions). The case of the cyclotomic extension gives a dévissage of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ whith the following shape: one has a natural exact sequence

$$1 \to G_{\mathbb{F}_p((T))} \to G_{\mathbb{Q}_p} \to \mathbb{Z}_p^* \to 1,$$

where $G_{\mathbb{F}_p((T))}$ is the absolute Galois group of $\mathbb{F}_p((T))$. A reasonable infinite extension of $\mathbb{Q}_p$ can be written as an increasing union of finite extensions L_n of $\mathbb{Q}_p$, and $X(L)$ is the set of sequences $(x_n)_{n\in\mathbb{N}}$, with $x_n \in L_n$ and $\mathrm{N}_{L_{n+1}/L_n}(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$. The set $X(L)$ is turned into a field of characteristic p, by setting $(x_n) + (y_n) = (s_n)$ and $(x_n)(y_n) = (t_n)$, with

$$t_n = x_n y_n \quad \text{and} \quad s_n = \lim_{k\to\infty} \mathrm{N}_{L_{n+k}/L_n}(x_{n+k} + y_{n+k})$$

(that the limit exists is the non trivial part of this construction and uses crucially the fact that the extension is reasonable).

Another striking contribution was his construction [16] of a natural splitting of the Hodge filtration for varieties over a p-adic field. If X is a smooth projective algebraic variety of dimension d defined over a characteristic 0 field K, Grothendieck has defined its algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X/K)$ by means of algebraic differential forms. The $H^i_{\mathrm{dR}}(X/K)$'s are finite dimensional K-vector spaces which vanish for $i > 2d$, and are endowed with a decreasing filtration – the Hodge filtration – by sub-K-vector spaces. If K is a subfield of $\mathbb{C}$, then $\mathbb{C} \otimes_K H^\bullet_{\mathrm{dR}}(X/K)$ is isomorphic to the de Rham cohomology of the $2d$-dimensional differentiable manifold $X(\mathbb{C})$, and Hodge theory provides a description of $\mathbb{C} \otimes_K H^\bullet_{\mathrm{dR}}(X/K)$ in terms of harmonic forms which, in turn, induces a canonical splitting of the Hodge filtration on $\mathbb{C} \otimes_K H^\bullet_{\mathrm{dR}}(X/K)$ (but not on $H^\bullet_{\mathrm{dR}}(X/K)$ itself as this splitting usually involves complex numbers which are transcendental over K).

Now, if K is a finite extension of $\mathbb{Q}_p$, there is nothing like harmonic forms at our disposal (at least, up to now). But Wintenberger managed to define a natural splitting of the Hodge filtration in the case where X has "good reduction modulo p" and $K/\mathbb{Q}_p$ is unramified. In that case the cohomology of X is controled by that of its reduction and the morphism $x \mapsto x^p$ that exists in characteristic p – the Frobenius morphism – induces a morphism φ on the $H^i_{\mathrm{dR}}(X/K)$'s. Hence $H^i_{\mathrm{dR}}(X/K)$ is what Fontaine calls a filtered φ-module (i.e., a K vector space with a φ and a filtration). Now, p-adic Hodge theory (nothing to do with harmonic forms) implies that this filtered φ-module has special properties: there exists an $\mathcal{O}_K$-lattice M (with $\mathcal{O}_K$ the ring of integers of K) such that φ is divisible by p^i on $M \cap \mathrm{Fil}^i$ and $M = \sum_i p^{-i}\varphi(M \cap \mathrm{Fil}^i)$ (such a lattice is said to be strongly divisible). Wintenberger's result is a linear algebra result concerning these filtered φ-modules admitting a strongly divisible lattice, and there is no geometry involved. This result has remained a mystery: is there a theory of p-adic harmonic forms that would explain the existence of this natural splitting? Does this splitting exist without assuming $K/\mathbb{Q}_p$ to be unramified or X to have good reduction?

Wintenberger was interested in this splitting for the construction [17] of special representations $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ of the absolute Galois group G_K of K with $\rho(G_K)$ open in a given algebraic subgroup of $\mathrm{GL}_d(\mathbb{Q}_p)$ (some kind of inverse Galois problem for finite extensions of $\mathbb{Q}_p$): Fontaine-Laffaille theory [5] allows to translate the problem in terms of φ-modules admitting a strongly divisible lattice. This was not the last time that Wintenberger used his splitting for questions related to representations of Galois groups (see e.g. [19]).

I did not really follow very closely what he was doing later on, after he took a position in Strasbourg, and I was amazed to discover, at a conference that he organised in Strasbourg about Serre's conjecture [14] on the modularity of mod p representations of the absolute Galois group of $\mathbb{Q}$, that he was actually proving, in collaboration with Chandrashekhar Khare [10, 11, 12], this very conjecture (a dream of quite a few number theorists at the time)! He had been thinking about a strategy to attack it for a long time...

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