I. BASIC PERRON FROBENIUS THEORY AND INVERSE SPECTRAL PROBLEMS

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Contents

1. Introduction

By a nonnegative matrix we mean a matrix whose entries are nonnegative real numbers. By positive matrix we mean a matrix all of whose entries are strictly positive real numbers.

These notes (with appendices) give the core elements of the Perron-Frobenius theory of nonnegative matrices. This splits into three parts:

(1) the primitive case (due to Perron)
(2) the irreducible case (due to Frobenius)
(3) the general case (due to?)

2. The primitive case

Definition 2.1. A *primitive* matrix is a square nonnegative matrix some power of which is positive.

The primitive case is the heart of the Perron-Frobenius theory and its applications.

More definitions:

- The *spectral radius* of a square matrix is the maximum of the moduli of the roots of its characteristic polynomial.
- A number λ is a *simple* root of a polynomial $p(x)$ if it is a root of multiplicity one (i.e., $p(\lambda) = 0$ and $p'(\lambda) \neq 0$).
- For a matrix A or vector v, we define the norm ($||A||$ or $||v||$) to be the sum of the absolute values of its entries.

If $||\cdot||$ and $||\cdot||'$ are two norms on $\mathbb{R}^n$, then there are positive constants $C_1, C_2 > 0$ such that for all v in $\mathbb{R}^n$

$$||v|| \le C_1 ||v||' \quad \text{and} \quad ||v||' \le C_2 ||v|| \ .$$

So, our particular choice of norm isn't important.

In the sequel, inequalities of matrices or vectors are defined to hold entrywise.

Theorem 2.2 (Perron Theorem)**.** *Suppose A is a primitive matrix, with spectral radius λ. Then λ is a simple root of the characteristic polynomial which is strictly greater than the modulus of any other root, and λ has strictly positive eigenvectors.*

For example,

- $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ is primitive (eigenvalues are $2, -1$)
- $\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ is not primitive (eigenvalues are $2, -2$)
- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is not primitive (1 is a repeated root of char.polynomial)

3. Why the Perron Theorem is useful

The Perron theorem provides a very clear picture of the way large powers of a primitive matrix behave, with exponentially good estimates.

Theorem 3.1. *Suppose A is primitive. Let u be a positive left eigenvector and let v be a positive right eigenvector for the spectral radius λ, chosen such that $uv = (1)$. Then $((1/\lambda)A)^n$ converges to the positive matrix vu, exponentially fast.*

The theorem says that for large n, $A^n - \lambda^n vu$ has entries exponentially smaller than A^n; the dominant behavior of A^n is described by the positive rank one matrix $\lambda^n vu$.

For example, if A is a stochastic matrix defining a stationary Markov chain, then $\lambda = 1$ and $A^n(i,j)$ is the probability of being in state j after n steps from state i. Here the Perron Theorem makes a statement that for each j the probability of being in state j after n steps is positive and rapidly approaches independence of the initial state i.

Example 3.2. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$. Then A has spectral radius $\lambda = 4$, with left and right eigenvectors $(2, 3)$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Normalizing to achieve $uv = (1)$, we define

$$u = \begin{pmatrix} 2 & 3 \end{pmatrix} \quad \text{and} \quad v = (1/5) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ .$$

Then

$$vu = (1/5) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} = (1/5) \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2/5 & 3/5 \\ 2/5 & 3/5 \end{pmatrix} \ .$$

One can check that indeed

$$A^n \ = \ 4^n \begin{pmatrix} 2/5 & 3/5 \\ 2/5 & 3/5 \end{pmatrix} \ + \ (-1)^n \begin{pmatrix} 3/5 & -3/5 \\ -2/5 & 2/5 \end{pmatrix} \ .$$

Proof of Theorem. The matrix $(1/\lambda)A$ multiplies the eigenvectors u and v by 1 (i.e. leaves them unchanged).

Let W be the A-invariant codimension 1 subspace of column vectors complementary to $\mathbb{R}v$. Let β be the spectral radius of the restriction of A to V. Then there are $k > 0$ and $C > 0$ such that for all w in W, and for all positive integers n,

$$||A^n w|| \ \leq \ Cn^k \beta^n ||w|| \ .$$

By the Perron Theorem, $\beta < \lambda$, so

$$||\Big(\frac{1}{\lambda}A\Big)^n w|| \ \leq \ Cn^k \Big(\frac{\beta}{\lambda}\Big)^n ||w||$$

which goes to zero exponentially fast.

Therefore $((1/\lambda)A)^n$ converges to the (unique) rank one matrix M which annihilates W and satisfies $Mv = v$. For any w in W,

$$\begin{aligned} uw \ &= \ \Big(u\Big(\frac{1}{\lambda}A\Big)^n\Big)w \\ &= \ u\Big(\Big(\frac{1}{\lambda}A\Big)^n w\Big) \ \longrightarrow \ 0 \quad \text{as } n \to \infty \end{aligned}$$

and therefore $uw = (0)$. Now to check $vu = M$, we check that (vu) is a rank one matrix fixing v and annihilating W:

$$(vu)w = v(uw) = 0 \qquad \text{for all } w \in W$$
$$(vu)v = v(uv) = v(1) = v \ .$$

□

4. A proof of the Perron Theorem

We'll give a proof of the Perron Theorem. There are others.

Theorem 4.1 (Perron Theorem)**.** *Suppose A is a primitive matrix, with spectral radius λ. Then λ is a simple root of the characteristic polynomial which is strictly greater than the modulus of any other root, and λ has strictly positive eigenvectors.*

Note that the "simple root" condition is stronger than the condition that λ have a one dimensional eigenspace, because a one-dimensional eigenspace may be part of a larger-dimensional generalized eigenspace. For example, consider

$$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \end{pmatrix} \ .$$

We begin with a geometrically compelling lemma.

Lemma 4.2. *Suppose T is a linear transformation of a finite dimensional real vector space, S' is a polyhedron containing the origin in its interior, and a positive power of T maps S' into its interior. Then the spectral radius of T is less than 1.*

Proof of the lemma. Without loss of generality, we may suppose T maps S' into its interior. Clearly, there is no root of the characteristic polynomial of modulus greater than 1.

The image of S' is a closed set which does not intersect the boundary of S'. Because $T^n(S') \subset T(S')$ if $n \geq 1$, no point on the boundary of S' can be an image of a power of T, or an accumulation point of points which are images of powers of T. But this is contradicted if T has an eigenvalue of modulus 1, as follows:

CASE I: a root of unity is an eigenvalue of a T.

In this case, 1 is an eigenvalue of a power of T, and a power of T has a fixed point on the boundary of S'. Thus the image of S' under a power of T intersects the boundary of S', a contradiction.

CASE II: there is an eigenvalue of modulus 1 which is not a root of unity.

In this case, let V be a 2-dimensional subspace on which T acts as an irrational rotation. Let p be a point on the boundary of S' which is in V. Then p is a limit point of $\{T^n(p) : n > 1\}$, so p is in the image of T, a contradiction.

This completes the proof. □

Proof of the Perron Theorem. There are three steps.

STEP 1: get the positive eigenvector.

The unit simplex S is the set of nonnegative vectors v such that $||v|| := \sum_i v_i$ equals 1. The map

$$\begin{aligned} S &\to S \\ v &\mapsto \frac{1}{||vA||} vA \end{aligned}$$

is well defined ($vA \neq 0$ because no row of A is zero) and continuous. By Brouwer's Fixed Point Theorem, this map has a fixed point, which must be a nonnegative eigenvector of A for some positive eigenvalue, λ. Because a power of A is positive, the eigenvector must be positive.

STEP 2: stochasticize A.

Let r be a positive right eigenvector. Let R be the diagonal matrix whose diagonal entries come from r, i.e. $R(i, i) = r_i$. Define the matrix $P = (1/\lambda)R^{-1}AR$. P is still primitive. The column vector with every entry equal to 1 is an eigenvector of P with eigenvalue 1. Therefore every row sum of P is 1, and P is stochastic. It now suffices to do Step 3.

STEP 3: show 1 is a simple root of the characteristic polynomial of P dominating the modulus of any other root.

Consider the action of P on row vectors: P maps the unit simplex S into itself and a power of P maps S into its interior. From Step 1, we know there is a positive row vector v in S which is fixed by P. Therefore $S' = -v + S$ is a polyhedron, whose interior contains the origin. By the lemma the restriction of P to the subspace V spanned by S' has spectral radius less than 1. But V is P-invariant with codimension 1. □

Remarks 4.3 (Remarks on the proof above.).

(1) Any number of people have noticed that applicability of Brouwer's Theorem (Ky Fan in the 1950's.) It's a matter of taste as to whether to use it to get the eigenvector. There are other significant arguments for getting the existence of the positive eigenvector.
(2) The proof above, using the easy reduction to the geometrically clear and simple lemma, was found by Michael Brin in 1993. It is dangerous in this area to claim a proof is new. I haven't seen an earlier explicit use of this reduction.
(3) The utility of the stochasticization trick is by no means confined to this theorem.

We can now note that a primitive matrix has (up to scalar multiples) just one nonnegative eigenvector.

Corollary 4.4. *Suppose A is a primitive matrix and w is a nonnegative vector, with eigenvalue β. Then β must be the spectral radius of A.*

Proof. Because A is primitive, we can choose $k > 0$ such that $A^k w$ is positive. Thus, $w > 0$ (since $A^k w = \beta^k w$) and $\beta > 0$. By the Perron Theorem, there is a positive eigenvector v which has eigenvalue λ, the spectral radius, such that $v < w$. Then for all $n > 0$,

$$\lambda^n v = A^n v \le A^n w = \beta^n w \ .$$

This is impossible if $\beta < \lambda$, so $\beta = \lambda$. □

The following fact, whose proof doesn't need the Perron Theorem, can be quite useful.

Theorem 4.5. *Suppose A and B are square nonnegative matrices, with spectral radii λ_A and λ_B, such that A is primitive, $A \ge B$ and $A \ne B$.*
Then $\lambda_B < \lambda_A$.

Proof. For k any positive integer, $\lambda_B < \lambda_A$ is equivalent to $\lambda_{B^k} < \lambda_{A^k}$. Because A is primitive, after passing to a power we may assume A is positive. Then $A^2 > AB \ge B^2$, so after passing to another power we may assume $A > B$, and therefore $A > (1-\epsilon)B$ for some positive ϵ. By the Spectral Radius Theorem,

$$\begin{aligned} \lambda_B &= \lim_n ||B^n||^{1/n} \\ &\le \lim_n ||\Big((1-\epsilon)A\Big)^n||^{1/n} = (1-\epsilon)\lim_n ||A^n||^{1/n} = (1-\epsilon)\lambda_A \ . \end{aligned}$$

□

The theorem also holds with $|B|$ replacing B in the statement, by the same proof. After the next section, it will be easy to prove that the theorem still holds if also "primitive" is replaced by "irreducible" in the statement.

5. The irreducible case

Given a nonnegative $n \times n$ matrix A, we let its rows and columns be indexed in the usual way by $\{1, 2, \dots n\}$, and we define a directed graph $G(A)$ with vertex set $\{1, 2, \dots, n\}$ by declaring that there is an edge from i to j if and only if $A(i,j) \ne 0$. A loop of length k in $G(A)$ is a path of length k (a path of k successive edges) which begins and ends at the same vertex.

Definition 5.1. An irreducible matrix is a square nonnegative matrix such that for every i, j there exists $k > 0$ such that $A^k(i,j) > 0$.

Notice, for any positive integer k, $A^k(i,j) > 0$ if and only if there is a path of length k in $G(A)$ from i to j.

Definition 5.2. The *period* of an irreducible matrix A is the greatest common divisor of the lengths of loops in $G(A)$.

E.g., the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ has period 1 and the matrix $\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ has period 2.

Now suppose A is irreducible with period p. Pick some vertex v, and for $0 \leq i, p$ define a set of vertices

$$C_i = \{u : \text{there is a path of length } n \text{ from } v \text{ to } u \text{ such that } n \equiv i \mod p\} .$$

The sets $C(i)$ partition the vertex set. An arc from a vertex in $C(i)$ must lead to a vertex in $C(j)$ where $j = i + 1 \mod p$. If we reorder the indices for rows and columns of A, listing indices for C_0, then C_l, etc., and replace A with PAP^{-1} where P is the corresponding permutation matrix, then we get a matrix B with a block form which looks like a cyclic permutation matrix. For example, with $p = 4$, we have a block matrix

$$B = \begin{pmatrix} 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \\ A_4 & 0 & 0 & 0 \end{pmatrix} .$$

A specific example with $p = 3$ is

$$\begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Note the blocks of B are rectangular (not necessarily square). B and A agree on virtually all interesting properties, so we usually just assume A has the form given as B (i.e., we tacitly replace A with B, not bothering to rename). We call this a cyclic block form.

Proposition 5.3. *Let A be a square nonnegative matrix. Then A is primitive if and only if it is irreducible with period one.*

Proof. Exercise. □

Definition 5.4. We say two matrices have the same nonzero spectrum if their characteristic polynomials have the same nonzero roots, with the same multiplicities.

Proposition 5.5. *Let A be an irreducible matrix of period p in cyclic block form. Then A^p is a block diagonal matrix and each of its diagonal blocks is primitive. Moreover the diagonal blocks have the same nonzero spectrum.*

Proof. We'll give a proof in the special case $p = 3$ and A having block form

$$A = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix} .$$

(The proof of the general case involved no additional ideas and should be perfectly clear from this special case.) Note the diagonal blocks D_i of A^p:

$$A^p = \begin{pmatrix} A_1A_2A_3 & 0 & 0 \\ 0 & A_2A_3A_1 & 0 \\ 0 & 0 & A_3A_1A_2 \end{pmatrix} := \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix} .$$

These diagonal blocks must be irreducible of period 1, hence primitive. We have for example

$$\begin{aligned} D_1 &= (A_1)(A_2A_3) := RS \\ D_2 &= (A_2A_3)(A_1) := SR \,. \end{aligned}$$

Therefore D_1 and D_2 have equal trace, since

$$\begin{aligned} \text{trace}(RS) &= \sum_i (RS)(i,i) \\ &= \sum_i \sum_k R(i,k)S(k,i) \\ &= \sum_i \sum_k S(k,i)R(i,k) = \sum_k \sum_i S(k,i)R(i,k) \\ &= \text{trace}(SR) \,. \end{aligned}$$

For $n > 1$ likewise,

$$\begin{aligned} (D_1)^n &= \Big((D_1)^{n-1}R\Big)\, S \\ (D_2)^n &= S\, \Big((D_1)^{n-1}R\Big) \end{aligned}$$

and therefore $\text{trace}(D_1)^n = \text{trace}(D_2)^n$ for all $n > 0$. This forces D_1 and D_2 to have the same nonzero spectrum (we will see a formal proof of this in a later lecture). Likewise (applying the argument to the pair D_2, D_3), D_3 has this same nonzero spectrum. □

Proposition 5.6. *Let A be an irreducible matrix with period p and suppose that ξ is a primitive pth root of unity. Then the matrices A and ξA are similar.*

In particular, if c is root of the characteristic polynomial of A with multiplicity m, then ξc is also a root with multiplicity m.

Proof. The proof for the period 3 case already explains the general case:

$$\begin{pmatrix} \xi^{-1}I & 0 & 0 \\ 0 & \xi^{-2}I & 0 \\ 0 & 0 & \xi^{-3}I \end{pmatrix} \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^{1}I & 0 & 0 \\ 0 & \xi^{2}I & 0 \\ 0 & 0 & \xi^{3}I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \xi A_1 & 0 \\ 0 & 0 & \xi A_2 \\ \xi^{-2}A_3 & 0 & 0 \end{pmatrix} = \xi \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$$

since $\xi^{-2} = \xi$. □

Definition 5.7. If A is a matrix, then its characteristic polynomial away from zero is the polynomial $q_A(t)$ such that $q_A(0)$ is not 0 and the characteristic polynomial of A is a power of t times $q_A(t)$.

Theorem 5.8. *Let A be an irreducible matrix of period p. Let D be a diagonal block of A^p (so, D is primitive). Then*

$$q_A(t) = q_D(t^p) \,.$$

Equivalently, if ξ is a primitive pth root of unity and we choose complex numbers $\lambda_1, \dots, \lambda_j$ such that $q_D(t) = \prod_{j=1}^{k}(t - (\lambda_j^p))$, then

$$q_A(t) = \prod_{i=0}^{p-1}\prod_{j=1}^{k}(t - \xi^i \lambda_j) \ .$$

Proof. From the last proposition, a nonzero root c of q_{A^p} has multiplicity kp, where k is the number such that every pth root of c is a root of multiplicity k of q_A. Each c which is a root of multiplicity k for q_D is a root of multiplicity kp for q_{A^p} (since the diagonal blocks of A^p have the same nonzero spectrum. □

Theorem 5.9 (Perron-Frobenius Theorem)**.** *Let A be an irreducible matrix of period p.*

(1) *A has a nonnegative right eigenvector r. This eigenvector is strictly positive, its eigenvalue λ is the spectral radius of A, and any nonnegative eigenvector of A is a scalar multiple of r.*
(2) *The roots of the characteristic polynomial of A of modulus λ are all simple roots, and these roots are precisely the p numbers $\lambda, \xi\lambda, \dots, \xi^{p-1}\lambda$ where ξ is a primitive pth root of unity.*
(3) *The nonzero spectrum of A is invariant under multiplication by ξ.*

Proof. Everything is easy from what has gone before except the construction of the eigenvector. The general idea is already clear for $p = 3$. Then we can consider A in the block form

$$A = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix} \ .$$

Now $A_1A_2A_3$ is a diagonal block of A, primitive with spectral radius λ^3. Let r be a positive right eigenvector for $A_1A_2A_3$. Compute:

$$\begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^2 r \\ A_2A_3r \\ \lambda A_3 r \end{pmatrix} = \begin{pmatrix} A_1A_2A_3r \\ \lambda A_2A_3r \\ \lambda^2 A_3 r \end{pmatrix} = \lambda \begin{pmatrix} \lambda^2 r \\ A_2A_3r \\ \lambda A_3 r \end{pmatrix}$$

□

6. General nonnegative matrices

Theorem 6.1. *If A is a square nonnegative matrix, then there is a permutation matrix P such that $P^{-1}AP$ is block triangular, with each diagonal block either an irreducible matrix or a zero matrix.*

Proof. Suppose A is $m \times m$. Recall the directed graph $G(A)$: the vertex set is $\{1, \dots, m\}$ and there is an edge from i to j iff there exists $k > 0$ such that $A^k(i, j) > 0$. Partition $\{1, \dots, m\}$ into classes $\mathcal{C}_i$: two indices i, j are in the same class if in $G(A)$ there is a path from i to j and a path from j to i. Draw a new directed graph G' with vertex set the set of classes, with an edge from $\mathcal{C}$ to $\mathcal{C}'$ iff there is an edge in $G(A)$ from an index in $\mathcal{C}$ to an index in $\mathcal{C}'$. There is no cycle in G'. Thus by induction we may order the classes as $C_1, \dots, C_j$ such that $i < j$ implies there is no path in G' from C_j to C_i. Then define P to reorder $\{1, \dots, m\}$ compatible with the ordering $C_1, \dots, C_j$. □

Here is a simple example following the proof notation above:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Set $\mathcal{C}_1, C_2, C_3, C_4 = \{4\}, \{1,3\}, \{5\}, \{2\}$. Define P (acting on rows) to effect the permutation $4 \to 1$, $1 \to 2$, $3 \to 3$, $5 \to 4$, $2 \to 5$. Then

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P^{-1}AP = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Remark 6.2. A square nonnegative matrix will always have at least one nonnegative (not necessarily positive) eigenvector for eigenvalue the spectral radius.

Remark 6.3. The characteristic polynomial $\chi_A(t)$ of a square nonnegative matrix A will be the same as that for $P^{-1}AP$ above, which will be a product of those of characteristic polynomials of irreducible diagonal blocks and some power of t.

So, the basic picture: understanding the spectra of primitive matrices, we understand the spectra of irreducible matrices; understanding the spectra of irreducible matrices, we understand the spectra of general nonnegative matrices.

7. Perron numbers and Mahler measures

Here is our first example of a nontrivial inverse spectral problem for nonnegative matrices.

Question: What real numbers can be the spectral radius of a primitive matrix with integer entries?

Definition 7.1. A Perron number is an algebraic integer which is strictly greater than the modulus of any of its algebraic conjugates over $\mathbb{Q}$.

Theorem 7.2 (Lind, Bulletin AMS 1983)**.** *For a real number λ, the following conditions are equivalent.*

1. *λ is the spectral radius of a primitive matrix with integer entries.*
2. *λ is a Perron number.*

That (1) implies (2) follows from the Perron Theorem. Lind's proof that (2) implies (1) is a pleasant geometric construction.

The set $\mathbb{P}$ of Perron numbers has some algebraic properties, which (as in [Lind 1983]) are left as excercises for the interested:

1. $\mathbb{P}$ is closed under addition and multiplication.
2. If α, β and λ are in $\mathbb{P}$ and $\lambda = \alpha\beta$, then $\{\alpha, \beta\} \subset \mathbb{Q}(\lambda)$.
3. $\mathbb{Q}(\lambda) \cap \mathbb{P}$ is a discrete subset of $\mathbb{R}$.
4. A Perron number has only finitely many factorizations as a product of Perron numbers greater than 1.

By (4), any Perron number greater than 1 is a product of "irreducible" Perron numbers: those Perron numbers greater than 1 which are not a product of other

Perron numbers which are greater than 1. Factorization into irreducibles in $\mathbb{P}$ is not unique. For example, letting $\alpha = \frac{1+\sqrt{5}}{2}$, we have $(\alpha+2)^2 = 4\alpha^2$.

Definition 7.3. The Mahler measure of a polynomial $p(z) = a(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$ is $|a(\alpha_1\alpha_2\cdots\alpha_n)|$.

The Mahler measure can be defined by an integral, and there is a definition by an integral for polynomials in several variables. Mahler measure is a rich and multifaceted topic that we won't go into here except to remark on some relations and parallels to Perron numbers:

(1) If $p(z)$ is a monic polynomial with coefficients in $\mathbb{Z}$, then its Mahler measure is a Perron number.
(2) (Dixon and Dubickas, Mathematika 2004) Given λ the Mahler measure of a degree d polynomial over $\mathbb{Z}$, there is $K = K(d)$ such that λ has degree at most K.
(3) Given a polynomial p with integer coefficients and a root λ of p, there is an algorithm to determine whether λ is a Mahler measure.

The description of Perron numbers gives a satisfying characterization of the numbers which are spectral radii of primitive matrices. Whether there is a satisfying characterization of the numbers which are Mahler measures, I leave as an open question.

8. The Spectral Conjecture

In this section we complement the basic Perron-Frobenius theory above, by considering the inverse spectral problem for nonnegative matrices. This is a step to more algebraic relations in later lectures.

There is a long history of looking for sufficient conditions for a list of n complex numbers (possibly repeated) to be the spectrum of a nonnegative $n \times n$ matrix. The literature contains ingenious special results and also complete characterizations for some small n. Overall, the problem is quite complicated. However, by focusing on the nonzero spectrum one can recover simple conditions.

As we've seen, if you understand the possible spectra of primitive matrices, then you understand the possible spectra of general nonnegative matrices. Also, for applications you need to understand the primitive case. So the nonzero spectra of primitive matrices are our focus, and it is here that we find clear conditions.

Spectral Conjecture (Boyle-Handelman, Annals of Math. 1991)
Let $\Lambda = (\lambda_1, \dots, \lambda_k)$ be a list of nonzero complex numbers. Let $\mathcal{S}$ be a unital subring of $\mathbb{R}$. Then the following are equivalent.

(1) There exists primitive matrix A of size n whose characteristic polynomial is $t^{n-k}\prod_{i=1}^{k}(t-\lambda_i)$ (i.e., Λ is the nonzero spectrum of A).
(2) The list Λ satisfies the following conditions:
  (a) (Perron Condition)
  There exists a unique index i such that λ_i is a positive real number and $\lambda_i > |\lambda_j|$ whenever $j \neq i$.
  (b) (Coefficents Condition)
  The polynomial $\prod_{i=1}^{k}(t-\lambda_i)$ has coefficients in $\mathcal{S}$.
  (c) (Trace Conditions)

(i) (In the case $\mathcal{S} \neq \mathbb{Z}$.)
(Let $\mathrm{tr}(\Lambda^n)$ denote $\sum_{i=1}^{k}(\lambda_i)^n$.)
For all positive integers n, k the following hold:
(A) For all n, $\mathrm{tr}(\Lambda^n) \geq 0$.
(B) If $\mathrm{tr}(\Lambda^n) > 0$, then $\mathrm{tr}(\Lambda^{nk}) > 0$.

(ii) (In the case $\mathcal{S} = \mathbb{Z}$.)
(Let $\mathrm{tr}_n(\Lambda)$ denote $\sum_{k|n} \mu(n/k)\mathrm{tr}(\Lambda^n)$.)
For all positive integers n, $\mathrm{tr}_n(\Lambda) \geq 0$

The three conditions are necessary conditions for existence of the primitive matrix with nonzero spectrum Λ; this is explained below. Also, if a nonzero spectrum can be realized at matrix size $n \times n$, then it can be realized at all larger sizes. So the inverse spectral problem for primitive matrices given the Spectral Conjecture reduces to finding the minimum dimension allowing a given nonzero spectrum.

Results on the Spectral Conjecture

- (BH, Annals of Math 1991)
  True whenever the large entry of Λ is in $\mathcal{S}$. In particular:
  True for $\mathcal{S} = \mathbb{R}$.
  (Very complicated proof using symbolic dyanmics.)

- (Kim-Ormes-Roush, JAMS 2000)
  True for $\mathcal{S} = \mathbb{Z}$.
  (Complicated proof using polynomial matrix presentations, formal power series, etc.)

- (Laffey, Linear Algebra Appl. 2012)
  For the special (but central) case that $\mathrm{tr}(\Lambda) > 0$ and also
  $\mathcal{S} = \mathbb{R}$ or $\mathcal{S}$ is any subfield of $\mathbb{R}$:
  A short, practical and constructive proof is given, via an elegantly structured family of matrices, with meaningful bounds on the size matrix required in terms of the spectral gap (the difference between the spectral radius and the next largest modulus of an element of the spectrum).

Just how large a primitive matrix must be to accommodate a given nonzero spectrum is in general still poorly understood. The proofs above are not geometric. The problem seems geometric, but evidently nobody has understood this geometry well enough to say much.

Why the conditions of the Spectral Conjecture are necessary.

The first condition of course follows from the Perron Theorem.

The second condition is obvious.

The trace conditions for $\mathcal{S} \neq \mathbb{Z}$ follow easily from the following fact: if A has nonzero spectrum Λ, then $\mathrm{tr}(\Lambda^n) = \mathrm{tr}(A^n) \geq 0$.

To understand the trace conditions for $\mathcal{S} = \mathbb{Z}$, imagine the nonnegative matrix A as the adjacency matrix of a directed graph G. (The vertices of G are the indices of the rows/columns of A, and $A(i,j)$ is the number of edges from i to j in G.) A loop is a path of edges from i to i, for some i. A loop ℓ is simple if there is no shorter loop ℓ' such that ℓ is a concatenation of copies of ℓ'.

The trace conditions for $\mathcal{S} = \mathbb{Z}$ are stating that the number of simple loops of length n is nonnegative, for all n. To see this, let s_k denote the number of simple loops in G of length k, and recall $\mathrm{tr}(A^n) = \mathrm{tr}(\Lambda^n)$ is the number of loops in G of length n. Then

$$\mathrm{tr}(\Lambda^n) \;=\; \sum_{k|n} s_k$$

because for any loop ℓ of length n, there is unique simple loop ℓ' such that ℓ is the concatenation of copies of ℓ'. If ℓ' has length k and ℓ is a concatenation of copies of ℓ', then k divides n.

Consequently, by applying the combinatorial Mobius inversion formula we get

$$s_n \;=\; \sum_{k|n} \mu(n/k)\mathrm{tr}(\Lambda^k)\ .$$

Here $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Mobius function:

$$\begin{aligned} \mu(1) &= 1 \\ \mu(n) &= 0 \quad \text{if the square of a prime divides } n \\ &= 1 \quad \text{if } n \text{ is the product of an even number of distinct primes} \\ &= -1 \ \text{if } n \text{ is the product of an odd number of distinct primes}\ . \end{aligned}$$

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