Lecture III: Diophantine approximation on varieties.

Consider an algebraic variety $X=\{f_1=\ldots=f_s=0\}\subset \mathbb{C}^N$.
Suppose that $X(\mathbb{Q})$ is dense in $X(\mathbb{R})$.
Diophantine approximation?

Fix nonincreasing $\psi: \mathbb{R}^+ \to (0,1)$.

Def. $x\in X(\mathbb{R})$ is <u>ψ-approximable</u> if
$$\|x-r\| \leqslant \psi(\mathrm{den}(r))$$
has infinitely many solutions $r\in X(\mathbb{Q})$.

$$\mathcal{W}(X(\mathbb{Q}),\psi)=\{\psi\text{-approximable points in } X(\mathbb{R})\}.$$

By Borel-Cantelli Lemma, if $\forall$ compact $K\subset X(\mathbb{R})$:
$$\sum_{r\in X(\mathbb{Q})\cap K} \psi(\mathrm{den}(r))^{\dim(X)} < \infty,$$
then $\mathcal{W}(X(\mathbb{Q}),\psi)$ has measure 0.

Khinchin Thm?

$G < GL_N(\mathbb{C})$ – simple algebraic group defined over $\mathbb{Q}$
(ex. SL_N, Sp_{2N}, $SO(Q)$)

For simplicity, assume that G is simply connected.

$$\mathbb{Q}^d \rightsquigarrow \mathbb{Z}^{d+1} \underset{\text{lattice}}{<} \mathbb{R}^{d+1}$$

$$G(\mathbb{Q}) \underset{\text{lattice}}{<} (?) \to G(\mathbb{A})$$
$\mathbb{A}$ – adeles

$$\cup$$

$$G(\mathbb{Z}[\tfrac{1}{p}]) \underset{\text{lattice}}{<} (?) \to G(\mathbb{R}) \times G(\mathbb{Q}_p)$$

<u>p-adic numbers:</u> For $r \in \mathbb{Q}$, write $r = p^n \cdot \frac{\ell}{s}$ with ℓ, s coprime to p, and define p-adic norm $|r|_p = p^{-n}$.

p-adic numbers: $\mathbb{Q}_p$ = "completion of $\mathbb{Q}$ with respect to $|\cdot|_p$"

$$\cup$$

$\mathbb{Z}_p = \{x : |x|_p \leq 1\}$ – compact open ring

Then $\mathbb{Z}[\frac{1}{p}] \xhookrightarrow[\text{diag}]{} \mathbb{R} \times \mathbb{Q}_p$ – discrete cocompact subgroup,

and $\Gamma = G(\mathbb{Z}[\frac{1}{p}]) \xhookrightarrow[\text{diag}]{} G(\mathbb{R}) \times G(\mathbb{Q}_p)$
is a discrete subgroup of finite covolume.

Consider a "dynamical system":

$$G(\mathbb{Q}_p) \curvearrowright Y = (G(\mathbb{R}) \times G(\mathbb{Q}_p))/\Gamma.$$

Generalised Dani correspondence.

Diophantine approximation of $G(\mathbb{Z}[\tfrac{1}{p}]) \subset G(\mathbb{R})$ $\Longleftrightarrow$ Dynamics $G(\mathbb{Q}_p) \curvearrowright Y$

Notation: $\tilde{O}_\varepsilon = \{ g \in G(\mathbb{R}) : \|g - e\|_\infty \le \varepsilon \} \times G(\mathbb{Z}_p)$

$O_\varepsilon = \tilde{O}_\varepsilon \Gamma \subset Y$

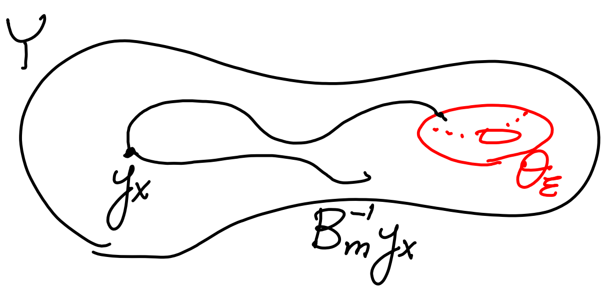
$B_m = \{ b \in G(\mathbb{Q}_p) : \|b\|_p = p^m \}$

(for $\gamma \in G(\mathbb{Z}[\tfrac{1}{p}])$, $\|\gamma\|_p = p^m \Leftrightarrow \mathrm{den}(\gamma) = p^m$).

For $x \in G(\mathbb{R})$, $y_x = (x^{-1}, e)\Gamma \in Y$.

Prop. Fix compact $\Omega \subset G(\mathbb{R})$. Let $c = \sup_{x \in \Omega} \|x\|_\infty$. Then

$$B_m^{-1} y_x \cap O_\varepsilon \neq \emptyset \Rightarrow \exists \gamma \in G[\mathbb{Z}[\tfrac{1}{p}]] : \begin{cases} \|\gamma - x\|_\infty \le \varepsilon \cdot c \\ \mathrm{den}(\gamma) \le p^m. \end{cases}$$



Suppose that for $b \in B_m$ and $\gamma \in \Gamma$,

$$(e, b^{-1}) \cdot (x^{-1}, e) \cdot (\gamma, \gamma) \in \tilde{O}_\varepsilon \Rightarrow \|x^{-1}\gamma - e\| \le \varepsilon \Rightarrow \|\gamma - x\| \le \varepsilon \cdot c$$

$$\Rightarrow b\gamma^{-1} \in G(\mathbb{Z}_p) \Rightarrow \|\gamma\|_p \le p^m$$

Thm (property τ / Selberg, ... Burger–Sarnak, Clozel)

$\exists\, q < \infty$: $\forall f_1, f_2 \in L^2(Y)$ with $\int_Y f_i = 0$,

$$\int_Y f_1(yg) f_2(y)\, dy \in L^q(G(\mathbb{Q}_p)).$$

ex. 1) $G = SL_2$: Ramanujan Conj: $q = 2 + \varepsilon$, $\varepsilon > 0$

Kim–Sarnak : $q \le \frac{64}{25}$.

2) $G =$ anisotropic form of SL_2 : $q = 2 + \varepsilon$, $\varepsilon > 0$.

(Deligne)

Averaging operators: $A_m : L^2(Y) \longrightarrow L^2(Y)$

$$f \longmapsto \frac{1}{|B_m|} \int_{B_m} f(g^{-1}y)\, dg.$$

Thm (Mean Ergodic Thm / G.–Nevo)

$\forall f \in L^2(Y)$: $\|A_m(f) - \int_Y f\|_2 \ll |B_m|^{-\frac{1}{q}} \cdot \|f\|_2$.

Thm ("Khinchin Thm" / Ghosh – G.–Nevo)

Assume that for some bounded $K \subset G(\mathbb{R})$ and $\alpha > \frac{q}{2}\dim(G)$, $\sum_{r \in G(\mathbb{Z}[\frac{1}{p}]) \cap K} \psi(\mathrm{den}(r))^{\alpha} = \infty$.

Then $\mathcal{W}(X(\mathbb{Z}[\frac{1}{p}]), \psi)$ has full measure in $X(\mathbb{R})$.

Let $d = \dim(G)$. Fix compact $\Omega \supset K$ such that

$$a_m \overset{\mathrm{def}}{=} |\Gamma \cap (B_m \times \Omega)| \asymp |B_m|.$$

The series $\sum_{m \geq 1} a_m \psi(p^m)^{\alpha}$ $\begin{cases} \text{converges for } \alpha > \alpha_0 \\ \text{diverges for } \alpha < \alpha_0 \end{cases}$

In particular, $\alpha_0 > \frac{qd}{2}$.

For simplicity, assume that $\alpha_0 < \infty$.

Take $\alpha < \alpha_0$, $\alpha \approx \alpha_0$.

Let $\Phi_m = \mathcal{O}_{\psi(p^m)/c}$, $\varphi_m = c_m \chi_{\Phi_m}$ with $c_m = a_m \psi(p^m)^{\alpha - d}$.

We claim that:

1) $\sum_{m\geq 1} \int_Y \varphi_m = \infty$

2) $F_k = \sum_{m\geq k} \left| A_m(\varphi_m) - \int_Y \varphi_m \right| \in L^2(Y).$

1): $\sum_{m\geq 1} \int_Y \varphi_m = \sum_{m\geq 1} c_m \cdot |\Phi_m| \asymp \sum_{m\geq 1} a_m \psi(p^m)^{\alpha} = \infty$ since $\alpha < \alpha_0$.

2): By the Mean Ergodic Thm,

$$\|F_k\|_2 << \sum_{m\geq k} |B_m|^{-1/q} \|\varphi_m\|_2 = \sum_{m\geq k} |B_m|^{-1/q} c_m |\Phi_m|^{1/2}$$

$$<< \sum_{m\geq k} |B_m|^{1-1/q} \cdot \psi(p^m)^{\alpha - \frac{d}{2}}$$

$$= \sum_{m\geq k} |B_m|^{1-1/q+\varepsilon} \psi(p^m)^{\alpha-\frac{d}{2}} \cdot |B_m|^{-\varepsilon}, \quad \varepsilon > 0$$

$$\leq \left(\sum_{m\geq k} |B_m| \cdot \psi(p^m)^{r(\alpha - \frac{d}{2})} \right)^{1/r} \cdot \underbrace{\left(\sum_{m\geq k} |B_m|^{-\bar{r}\varepsilon} \right)^{1/\bar{r}}}_{<\infty}$$

Hölder inequality with $r = (1-1/q+\varepsilon)^{-1}$, $\bar{r} = (1/q - \varepsilon)^{-1}$.

We note that $\frac{\alpha - \frac{d}{2}}{1-1/q+\varepsilon} \approx \frac{\alpha_0 - \frac{d}{2}}{1-1/q} > \alpha_0 \Leftrightarrow \alpha_0 > \frac{q}{2} d.$

Hence, $F_k \in L^2(Y)$.

Let $Y_m = \{y : B_m^{-1} y \cap \mathcal{P}_m = \emptyset\}$.

On $\bigcap_{m \geq k} Y_m$, $F_k = \sum_{m \geq k} \int_Y \varphi_m = \infty$.

Since $F_k \in L^2(Y)$, $\left| \bigcap_{m \geq k} Y_m \right| = 0$.

Hence, $Y_\infty \overset{\underline{\text{def}}}{=} \underline{\lim} Y_m$ has measure 0.

Let $\Omega' = \{x \in \Omega : y_x \in Y_\infty\}$

Since $y_x = (x^{-1}, e)\Gamma$ and Y_∞ is $G(\mathbb{Z}_p)$-invariant,

$((\Omega \setminus \Omega')^{-1} \times G(\mathbb{Z}_p))\Gamma \subset Y_\infty$, so that

Ω' has full measure in Ω.

For all $x \in \Omega'$, $y_x \in Y_m$ infinitely often

$$\Updownarrow$$

$$B_m^{-1} y_x \cap \mathcal{O}_{\psi(p^m)/c} \neq \emptyset$$

$$\Downarrow \text{Prop.}$$

$$\exists \gamma \in G(\mathbb{Z}[\tfrac{1}{p}]) : \begin{cases} \|\gamma - x\| \leq \psi(p^m), \\ \operatorname{den}(\gamma) \leq p^m. \end{cases}$$

Hence, $\Omega' \subset \mathcal{W}(G(\mathbb{Z}[\tfrac{1}{p}]), \psi)$.

This proves that $\mathcal{W}(G(\mathbb{Z}[\tfrac{1}{p}]), \psi)$ has full measure.