Cobham's theorem(s) I

Fabien Durand

Université de Picardie Jules Verne

26 juin 2013

0-Motivations

Question

Given a set S, does there exists an algorithm (with finite memory) that recognizes the elements of S.

Examples

- Subsets of $\mathbb{N}$: $\mathbb{N}, 2\mathbb{N}, \mathbb{P}, \{2^n | n \in \mathbb{N}\}, \ldots;$

Examples

- Subsets of $\mathbb{N}$: $\mathbb{N}, 2\mathbb{N}, \mathbb{P}, \{2^n | n \in \mathbb{N}\}, \ldots$;
- Subsets of $\mathbb{N}^d$;

Examples

- Subsets of $\mathbb{N}$: $\mathbb{N}, 2\mathbb{N}, \mathbb{P}, \{2^n | n \in \mathbb{N}\}, \ldots;$
- Subsets of $\mathbb{N}^d$;
- Subsets of $\mathbb{R}^d$: intervals, balls, graph of curves, the set of rational numbers, ...

Examples

- Subsets of $\mathbb{N}$: $\mathbb{N}, 2\mathbb{N}, \mathbb{P}, \{2^n | n \in \mathbb{N}\}, \ldots;$
- Subsets of $\mathbb{N}^d$;
- Subsets of $\mathbb{R}^d$: intervals, balls, graph of curves, the set of rational numbers, ...
- Subsets of groups or rings, $\mathbb{F}_p[X]$, $\mathbb{Z} + i\mathbb{Z}$, ...

Some comments

Other questions

- How to represent the elements of the set S?

Local answers

Some comments

Other questions

- How to represent the elements of the set S?

Local answers

- We will use numeration systems

Some comments

Other questions

- How to represent the elements of the set S?
- What do we mean by algorithm?

Local answers

- We will use numeration systems

Some comments

Other questions

- How to represent the elements of the set S?
- What do we mean by algorithm?

Local answers

- We will use numeration systems
- and finite automata.

Two answers for $\mathbb{N}$

Two answers for $\mathbb{N}$

It strongly depends on the numeration base (Cobham, 1969)

Two answers for $\mathbb{N}$

It strongly depends on the numeration base (Cobham, 1969) and recognizable sets are not any subsets (Cobham, 1972).

An example

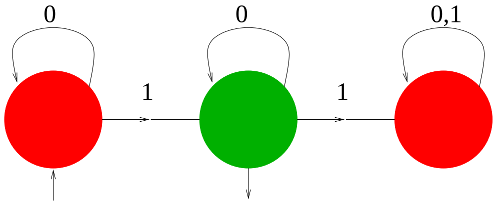
Let $E_{2^n} = \{2^n; n \in \mathbb{N}\}$.

An example

Let $E_{2^n} = \{2^n; n \in \mathbb{N}\}$.
Expansion of the elements of E_{2^n} in base 2 : $L_2(E_{2^n}) = 10^*$.

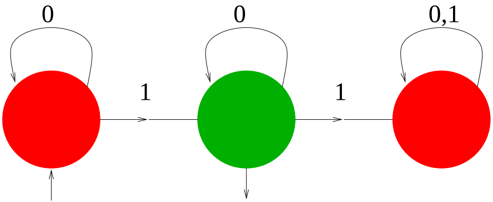
An example

Let $E_{2^n} = \{2^n; n \in \mathbb{N}\}$.
Expansion of the elements of E_{2^n} in base 2 : $L_2(E_{2^n}) = 10^*$.



An example

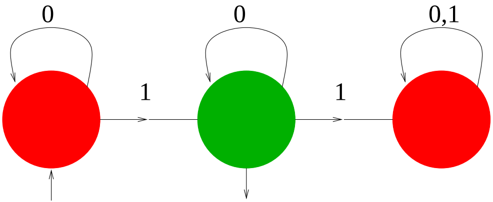
Let $E_{2^n} = \{2^n ; n \in \mathbb{N}\}$.
Expansion of the elements of E_{2^n} in base 2 : $L_2(E_{2^n}) = 10^*$.



E_{2^n} is 2-**recognizable**.

An example

Let $E_{2^n} = \{2^n; n \in \mathbb{N}\}$.
Expansion of the elements of E_{2^n} in base 2 : $L_2(E_{2^n}) = 10^*$.



E_{2^n} is 2-**recognizable**.
Does E_{2^n} be 3-recognizable ? : Does there exist a finite automaton that recognizes $L_3(E_{2^n})$?

Other examples

The integer Cantor set : $E_C = \{n = \sum \epsilon_i 3^i | \epsilon_i \in \{0, 2\}\}$.

The Morse set : $E_M = \{n = \sum \epsilon_i 2^i | \sum \epsilon_i = 0 \mod 2\}$.

Recognizability in $\mathbb{N}^d$

Exemple :

$$\begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

Recognizability in $\mathbb{N}^d$

Exemple :

$$\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 0011 \\ 1001 \end{pmatrix}$$

Recognizability in $\mathbb{N}^d$

Exemple :

$$\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 0011 \\ 1001 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

First answer (Cobham, 1969)

Cobham's theorem *Let E be a set of integers. Let $p, q \geq 2$ be two multiplicatively independent integers. Then,*

First answer (Cobham, 1969)

Cobham's theorem *Let E be a set of integers. Let $p, q \geq 2$ be two multiplicatively independent integers. Then,*

E is p-recognizable and q-recognizable

First answer (Cobham, 1969)

Cobham's theorem *Let E be a set of integers. Let $p, q \geq 2$ be two multiplicatively independent integers. Then,*

E is p-recognizable and q-recognizable

if and only if

E is a finite union of arithmetic progressions.

First answer (Cobham, 1969)

Cobham's theorem *Let E be a set of integers. Let $p, q \geq 2$ be two multiplicatively independent integers. Then,*

E is p-recognizable and q-recognizable

if and only if

E is a finite union of arithmetic progressions.

S. Eilenberg (*Automata, Languages, and Machines*, Acad. Press, 1972) : *The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem.*

Second answer (Cobham, 1972) : p-automatic sequences

Theorem (Cobham, 1972) *E is p-recognizable if and only if 1_E is p-automatic.*

.

Second answer (Cobham, 1972) : p-automatic sequences

Theorem (Cobham, 1972) *E is p-recognizable if and only if 1_E is p-automatic.*

Let $x \in \{a, b, c\}^{\mathbb{N}}$ be the fixed point starting with a of the substitution

$$a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$$

and ϕ the map defined by

$$a, c \mapsto 0, \quad b \mapsto 1,$$

.

Second answer (Cobham, 1972) : *p*-automatic sequences

Theorem (Cobham, 1972) *E is p-recognizable if and only if 1_E is p-automatic.*

Let $x \in \{a, b, c\}^{\mathbb{N}}$ be the fixed point starting with a of the substitution

$$a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$$

and ϕ the map defined by

$$a, c \mapsto 0, \quad b \mapsto 1,$$

then $1_{E_{2^n}} = \phi(x)$.

Second answer (Cobham, 1972) : p-automatic sequences

Theorem (Cobham, 1972) *E is p-recognizable if and only if 1_E is p-automatic.*

Let $x \in \{a, b, c\}^{\mathbb{N}}$ be the fixed point starting with a of the substitution

$$a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$$

and ϕ the map defined by

$$a, c \mapsto 0, \quad b \mapsto 1,$$

then $1_{E_{2^n}} = \phi(x)$.

We say it is a 2-**automatic sequence** (p-**automatic** in general).

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

Recall the lecture of Mark Pollicott : $\mathbb{P} \cap \{1, \ldots, n\} \sim \frac{n}{\log n}$.

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

Recall the lecture of Mark Pollicott : $\mathbb{P} \cap \{1, \ldots, n\} \sim \frac{n}{\log n}$.

Theorem (Cobham, 1972) Let $E \subset \mathbb{N}$ be p-recognizable and such that $\limsup \frac{\#E \cap \{1,\ldots,n\}}{n} = 0$ and $\#E = \infty$.

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

Recall the lecture of Mark Pollicott : $\mathbb{P} \cap \{1, \ldots, n\} \sim \frac{n}{\log n}$.

Theorem (Cobham, 1972) Let $E \subset \mathbb{N}$ be p-recognizable and such that $\limsup \frac{\#E \cap \{1,\ldots,n\}}{n} = 0$ and $\#E = \infty$.
Then E satisfies one of the two following properties.

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

Recall the lecture of Mark Pollicott : $\mathbb{P} \cap \{1, \ldots, n\} \sim \frac{n}{\log n}$.

Theorem (Cobham, 1972) Let $E \subset \mathbb{N}$ be p-recognizable and such that $\limsup \frac{\#E \cap \{1,\ldots,n\}}{n} = 0$ and $\#E = \infty$.
Then E satisfies one of the two following properties.

1. $\exists p \geq 1, s \in]0,1[$,

$$0 < \liminf_n \frac{\#E \cap \{1, \ldots, n\}}{n^s (\log n)^{p-1}} < \limsup_n \frac{\#E \cap \{1, \ldots, n\}}{n^s (\log n)^{p-1}} < \infty,$$

The prime numbers?

Is $\mathbb{P}$ p-recognizable for some p ?

Recall the lecture of Mark Pollicott : $\mathbb{P} \cap \{1, \ldots, n\} \sim \frac{n}{\log n}$.

Theorem (Cobham, 1972) Let $E \subset \mathbb{N}$ be p-recognizable and such that $\limsup \frac{\#E \cap \{1,\ldots,n\}}{n} = 0$ and $\#E = \infty$.
Then E satisfies one of the two following properties.

1. $\exists p \geq 1, s \in]0, 1[$,

$$0 < \liminf_n \frac{\#E \cap \{1, \ldots, n\}}{n^s (\log n)^{p-1}} < \limsup_n \frac{\#E \cap \{1, \ldots, n\}}{n^s (\log n)^{p-1}} < \infty,$$

2. $\exists p \geq 1, m \geq 2, c \in \mathbb{Q}^+, \#E \cap \{1, \ldots, n\} \sim c \left(\frac{\log n}{\log m}\right)^{p-1}$.

Summary

TODAY
I-Survey of Cobham's type results (logic, algebraic (transcendance), geometric (tilings), combinatorics on words, languages, automata, ...)

FRIDAY

II-Proof of Cobham's theorem (1969)
(using dynamical systems)

III-Open problems

I-Survey of Cobham's type results

Equivalent definition

$E \subset \mathbb{N}^d$ is p-recognizable if and only if

Equivalent definition

$E \subset \mathbb{N}^d$ is p-recognizable if and only if

- (Buchi, 1960) it is **definable** (by a first order formula) in $< \mathbb{N}, +, V_p >$.

Equivalent definition

$E \subset \mathbb{N}^d$ is p-recognizable if and only if

- (Buchi, 1960) it is **definable** (by a first order formula) in $< \mathbb{N}, +, V_p >$.
- (Christol, 1979) ($d = 1$, p prime) $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is **algebraic** over $\mathbb{F}_p(X)$.

Equivalent definition

$E \subset \mathbb{N}^d$ is p-recognizable if and only if

- (Buchi, 1960) it is **definable** (by a first order formula) in $< \mathbb{N}, +, V_p >$.
- (Christol, 1979) ($d = 1$, p prime) $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is **algebraic** over $\mathbb{F}_p(X)$.
- (Eilenberg, 1972) the p-**kernel** $\#\{(1_E(a + p^k n))_{n \in \mathbb{N}} \mid a \leq p^k - 1, k \geq 1\}$ is finite.

"Logical" extension

Theorem (Semenov, 1977) *p and q multiplicatively independent.* $E \subset \mathbb{N}^d$ *is both p and q-recognizable (or p and q-definable) if and only if E is definable in* $< \mathbb{N}, + >$.

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if* E *is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \ldots$

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \ldots$
- equality : $=$

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \ldots$
- equality : $=$
- fonctions : $+$, (resp. V_p)

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \ldots$
- equality : $=$
- fonctions : $+$, (resp. V_p)
- connectors : $\Leftarrow, \Rightarrow, \Leftrightarrow, \vee, \wedge, \neg$

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \dots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \dots$
- equality : $=$
- fonctions : $+$, (resp. V_p)
- connectors : $\Leftarrow, \Rightarrow, \Leftrightarrow, \vee, \wedge, \neg$
- quantifiers : $\exists, \forall$.

Definability and Presburger arithmetic (1929)

Definition $E \subset \mathbb{N}^d$ *is* **definable** *(resp. p-definable) if E is defined by a formula from* $\langle \mathbb{N}, + \rangle$ *(resp.* $\langle \mathbb{N}, +, V_p \rangle$*)*

- $V_p(n) = p^k$ if $n = p^k m$ with p not dividing m.
- First order formula $\phi(x_1, x_2, \ldots, x_n)$ in $\langle \mathbb{N}, + \rangle$ (resp. $\langle \mathbb{N}, +, V_p \rangle$) :
- integer variables : $x_1, x_2, \ldots$
- equality : $=$
- fonctions : $+$, (resp. V_p)
- connectors : $\Leftarrow, \Rightarrow, \Leftrightarrow, \vee, \wedge, \neg$
- quantifiers : $\exists, \forall$.
- A priori : no constant ... or you should defined them by a formula ...

Examples

Examples

- E_{2^n} is 2-definable and 4-definable :

Examples

- E_{2^n} is 2-definable and 4-definable :
- $\phi_2(x) := (V_2(x) = x)$ et

Examples

- E_{2^n} is 2-definable and 4-definable :
- $\phi_2(x) := (V_2(x) = x)$ et
- $\phi_4(x) := (V_4(x) = x) \vee (V_4(x + x) = x + x)$

Examples

- E_{2^n} is 2-definable and 4-definable :
- $\phi_2(x) := (V_2(x) = x)$ et
- $\phi_4(x) := (V_4(x) = x) \vee (V_4(x + x) = x + x)$
- Other example : $X = \{(x, y, z) \in \mathbb{N}^3; x + y = z\}$ is p-definable for all $p \geq 2$.

Examples

- E_{2^n} is 2-definable and 4-definable :
- $\phi_2(x) := (V_2(x) = x)$ et
- $\phi_4(x) := (V_4(x) = x) \vee (V_4(x + x) = x + x)$
- Other example : $X = \{(x, y, z) \in \mathbb{N}^3; x + y = z\}$ is p-definable for all $p \geq 2$.
- **Theorem.** *$E \subset \mathbb{N}$ is ultimately periodic iff E is definable in $\langle \mathbb{N}, + \rangle$.*

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is *p*-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is p-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

$f_{E_{2^n}}(X)$ is a solution of $Y^2 - Y + X = 0$ in $\mathbb{F}_2[[X]]$.

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is *p*-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

$f_{E_{2^n}}(X)$ is a solution of $Y^2 - Y + X = 0$ in $\mathbb{F}_2[[X]]$.

$f_{E_M}(X)$ is a solution of $(X+1)^3 Y^2 + (1+X)^2 Y + X = 0$ in $\mathbb{F}_2[[X]]$.

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is p-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

$f_{E_{2^n}}(X)$ is a solution of $Y^2 - Y + X = 0$ in $\mathbb{F}_2[[X]]$.

$f_{E_M}(X)$ is a solution of $(X+1)^3 Y^2 + (1+X)^2 Y + X = 0$ in $\mathbb{F}_2[[X]]$.
Hint : $u_{2n} = u_n, u_{2n+1} = u_n + 1$

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is p-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

$f_{E_{2^n}}(X)$ is a solution of $Y^2 - Y + X = 0$ in $\mathbb{F}_2[[X]]$.

$f_{E_M}(X)$ is a solution of $(X+1)^3 Y^2 + (1+X)^2 Y + X = 0$ in $\mathbb{F}_2[[X]]$.
Hint : $u_{2n} = u_n, u_{2n+1} = u_n + 1$

$f_{E_C}(X)$ is a solution of ... ? in $\mathbb{F}_3[[X]]$.

Algebraic extension: Examples

Recall (Christol, 1979) : $E \subset \mathbb{N}$ is p-recognizable if and only if $f_E(X) = \sum_{n \in E} X^n \in \mathbb{F}_p[[X]]$ is algebraic over $\mathbb{F}_p(X)$.

$f_{E_{2^n}}(X)$ is a solution of $Y^2 - Y + X = 0$ in $\mathbb{F}_2[[X]]$.

$f_{E_M}(X)$ is a solution of $(X+1)^3 Y^2 + (1+X)^2 Y + X = 0$ in $\mathbb{F}_2[[X]]$.
Hint : $u_{2n} = u_n, u_{2n+1} = u_n + 1$

$f_{E_C}(X)$ is a solution of ... ? in $\mathbb{F}_3[[X]]$.
Hint : ... not difficult

Algebraic extension

Theorem (Christol, Kamae, Mendès-France, Rauzy, 1980) *Let A be a finite alphabet, $x \in \mathcal{A}^{\mathbb{N}}$, and, p and q two different prime numbers. Let $\alpha_p : A \to \mathbb{F}_p$ and $\alpha_q : A \to \mathbb{F}_q$ be one-to-one maps. Then,*

Algebraic extension

Theorem (Christol, Kamae, Mendès-France, Rauzy, 1980) *Let A be a finite alphabet, $x \in \mathcal{A}^{\mathbb{N}}$, and, p and q two different prime numbers. Let $\alpha_p : A \to \mathbb{F}_p$ and $\alpha_q : A \to \mathbb{F}_q$ be one-to-one maps. Then,*

$\sum_{n\in\mathbb{N}} \alpha_p(x_n)X^n \in \mathbb{F}_p[[X]]$ *is algebraic over* $\mathbb{F}_p(X)$ *and*
$\sum_{n\in\mathbb{N}} \alpha_q(x_n)X^n \in \mathbb{F}_q[[X]]$ *is algebraic over* $\mathbb{F}_q(X)$
if, and only if,
they are rational.

Algebraic extension

Theorem (Christol, Kamae, Mendès-France, Rauzy, 1980) *Let A be a finite alphabet, $x \in \mathcal{A}^{\mathbb{N}}$, and, p and q two different prime numbers. Let $\alpha_p : A \to \mathbb{F}_p$ and $\alpha_q : A \to \mathbb{F}_q$ be one-to-one maps. Then,*

$$\sum_{n\in\mathbb{N}} \alpha_p(x_n)X^n \in \mathbb{F}_p[[X]] \text{ is algebraic over } \mathbb{F}_p(X) \text{ and}$$
$$\sum_{n\in\mathbb{N}} \alpha_q(x_n)X^n \in \mathbb{F}_q[[X]] \text{ is algebraic over } \mathbb{F}_q(X)$$

if, and only if,

they are rational.

Conjecture ? : If one of these numbers is irrational, $\sum_n \epsilon_n 3^{-n}$ and $\sum_n \epsilon_n 2^{-n}$, then one of them is transcendental.

Fatou's result (1906)

All integer series with uniformly bounded integer coefficients is transcendental or rational.

Fatou's result (1906)

All integer series with uniformly bounded integer coefficients is transcendental or rational.
Proof of Allouche, 1999

Digression on transcendancy

Question : If (u_k) is p-automatic then $\sum_k u_k p^{-n}$ is transcendental or rational ?

Digression on transcendancy

Question : If (u_k) is p-automatic then $\sum_k u_k p^{-n}$ is transcendental or rational ?

Theorem (Adamczewski-Bugeaud-Lucas, 2004) If $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ is algebraic then

$$\lim_{n \to \infty} \frac{p(n, b, \zeta)}{n} = +\infty,$$

where $p(n, b, \zeta)$ is the number of words of length n in the base b expansion of ζ.

Digression on transcendancy

Question : If (u_k) is p-automatic then $\sum_k u_k p^{-n}$ is transcendental or rational ?

Theorem (Adamczewski-Bugeaud-Lucas, 2004) If $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ is algebraic then

$$\lim_{n\to\infty} \frac{p(n,b,\zeta)}{n} = +\infty,$$

where $p(n,b,\zeta)$ is the number of words of length n in the base b expansion of ζ.

Consequence : "All non ultimately periodic automatic sequences are transcendental".

Digression on transcendancy

Question : If (u_k) is p-automatic then $\sum_k u_k p^{-n}$ is transcendental or rational ?

Theorem (Adamczewski-Bugeaud-Lucas, 2004) If $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ is algebraic then

$$\lim_{n\to\infty} \frac{p(n, b, \zeta)}{n} = +\infty,$$

where $p(n, b, \zeta)$ is the number of words of length n in the base b expansion of ζ.

Consequence : "All non ultimately periodic automatic sequences are transcendental".

Conjecture : If ζ is an irrational algebraic number then for all n and b, $p(n, b, \zeta) = b^n$.

α-substitutive sequences

- Recall $1_{E_{2^n}} = \phi(x)$ where x is the fixed point of $\tau : a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$ starting with a and ϕ the map defined by $a, c \mapsto 0, \quad b \mapsto 1,$

α-substitutive sequences

- Recall $1_{E_{2^n}} = \phi(x)$ where x is the fixed point of $\tau : a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$ starting with a and ϕ the map defined by $a, c \mapsto 0, \quad b \mapsto 1$,
- Remark : the dominant eigenvalue of (the incidence matrix of) τ is 2. We say $1_{E_{2^n}}$ is 2-substitutive.

α-substitutive sequences

- Recall $1_{E_{2^n}} = \phi(x)$ where x is the fixed point of $\tau : a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$ starting with a and ϕ the map defined by $a, c \mapsto 0, \quad b \mapsto 1,$
- Remark : the dominant eigenvalue of (the incidence matrix of) τ is 2. We say $1_{E_{2^n}}$ is 2-substitutive.
- In general : a sequence x is α-substitutive if it is the image under a letter-to-letter morphism of a fixed point of a (non-erasing) substitution $\sigma : A \to A^*$ whose dominant eigenvalue is α (all letters of A appearing in x).

α-substitutive sequences

- Recall $1_{E_{2^n}} = \phi(x)$ where x is the fixed point of $\tau : a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$ starting with a and ϕ the map defined by $a, c \mapsto 0, \quad b \mapsto 1$,
- Remark : the dominant eigenvalue of (the incidence matrix of) τ is 2. We say $1_{E_{2^n}}$ is 2-substitutive.
- In general : a sequence x is α-substitutive if it is the image under a letter-to-letter morphism of a fixed point of a (non-erasing) substitution $\sigma : A \to A^*$ whose dominant eigenvalue is α (all letters of A appearing in x).
- Fact : p-automatic sequences are p-substitutive. The converse is not true.

α-substitutive sequences

- Recall $1_{E_{2^n}} = \phi(x)$ where x is the fixed point of $\tau : a \mapsto ab, \quad b \mapsto bc, \quad c \mapsto cc$ starting with a and ϕ the map defined by $a, c \mapsto 0, \quad b \mapsto 1$,
- Remark : the dominant eigenvalue of (the incidence matrix of) τ is 2. We say $1_{E_{2^n}}$ is 2-substitutive.
- In general : a sequence x is α-substitutive if it is the image under a letter-to-letter morphism of a fixed point of a (non-erasing) substitution $\sigma : A \to A^*$ whose dominant eigenvalue is α (all letters of A appearing in x).
- Fact : p-automatic sequences are p-substitutive. The converse is not true.
- The Fibonacci sequence ($0 \mapsto 01$, $1 \mapsto 0$) is $\frac{1+\sqrt{5}}{2}$-substitutive.

Substitutive version of Cobham's theorem

Theorem (Cobham, 1969+1972) *Let* $p, q \geq 2$ *be two multiplicatively independent integers. Let* x *be a sequence on a finite alphabet.*

Substitutive version of Cobham's theorem

Theorem (Cobham, 1969+1972) *Let $p, q \geq 2$ be two multiplicatively independent integers. Let x be a sequence on a finite alphabet. Then, x is both p and q-automatic*

Substitutive version of Cobham's theorem

Theorem (Cobham, 1969+1972) *Let* $p, q \geq 2$ *be two multiplicatively independent integers. Let* x *be a sequence on a finite alphabet. Then,* x *is both* p *and* q*-automatic if and only if* $x = uvvvvv\ldots$*.*

Substitutive version of Cobham's theorem

Theorem (Cobham, 1969+1972) *Let* $p, q \geq 2$ *be two multiplicatively independent integers. Let* x *be a sequence on a finite alphabet. Then,* x *is both* p *and* q*-automatic if and only if* $x = uvvvvv\ldots$*.*

Theorem (Durand, 2011) *Let* $\alpha, \beta > 1$ *be two multiplicatively independent Perron numbers. Then,* x *is both* α *and* β*-substitutive if and only if* $x = uvvvvv\ldots$*.*

Non standard numeration systems

Question : What about numeration systems like Fibonacci ?

Non standard numeration systems

Question : What about numeration systems like Fibonacci ?

Definition. *A numeration system is an increasing sequence of integers* $U = (U_n; n \in \mathbb{N})$ *such that*

1. $U_0 = 1$,
2. $s = \sup\{\frac{U_{n+1}}{U_n}; n \in \mathbb{N}\} < \infty$.

Non standard numeration systems

Question : What about numeration systems like Fibonacci ?

Definition. *A numeration system is an increasing sequence of integers* $U = (U_n; n \in \mathbb{N})$ *such that*

1. $U_0 = 1$,
2. $s = \sup\{\frac{U_{n+1}}{U_n}; n \in \mathbb{N}\} < \infty$.

$A_U = \{0, \cdots, S-1\}$ where $S = \lceil s \rceil - 1$

Non standard numeration systems

Question : What about numeration systems like Fibonacci ?

Definition. *A numeration system is an increasing sequence of integers* $U = (U_n; n \in \mathbb{N})$ *such that*

1. $U_0 = 1$,
2. $s = \sup\{\frac{U_{n+1}}{U_n}; n \in \mathbb{N}\} < \infty$.

$A_U = \{0, \cdots, S-1\}$ where $S = \lceil s \rceil - 1$

With the greedy algorithm, uniqueness of the expansion $\rho_U(n) = a_i \cdots a_0$

$$n = a_i U_i + a_{i-1} U_{i-1} + \cdots + a_0 U_0;$$

with $a_j \in A_U$

Non standard numeration systems

- We set $L_U(E) = \{0^n \rho_U(x); n \in \mathbb{N}, x \in E\}$.

Non standard numeration systems

- We set $L_U(E) = \{0^n \rho_U(x); n \in \mathbb{N}, x \in E\}$.
- $E \subset \mathbb{N}$ is *U-recognizable* if $L_U(E)$ is recognizable (by a finite automata)

Non standard numeration systems

- We set $L_U(E) = \{0^n \rho_U(x); n \in \mathbb{N}, x \in E\}$.
- $E \subset \mathbb{N}$ is *U-recognizable* if $L_U(E)$ is recognizable (by a finite automata)
- **Open question:** *Let U and V be two linear numeration systems with $U_{n+1}/U_n \to_n \alpha$ and $V_{n+1}/V_n \to_n \beta$ where α and β are multiplicatively independent real numbers. Then, $E \subset \mathbb{N}$ is U and V-recognizable iff $1_E = uvvv \dots$.*

Non standard numeration systems

- We set $L_U(E) = \{0^n \rho_U(x); n \in \mathbb{N}, x \in E\}$.
- $E \subset \mathbb{N}$ is *U-recognizable* if $L_U(E)$ is recognizable (by a finite automata)
- **Open question:** *Let U and V be two linear numeration systems with $U_{n+1}/U_n \to_n \alpha$ and $V_{n+1}/V_n \to_n \beta$ where α and β are multiplicatively independent real numbers. Then, $E \subset \mathbb{N}$ is U and V-recognizable iff $1_E = uvvv\ldots$.*
- **Answer** : (Durand, 1998) for Bertrand numeration systems, (Durand-Rigo, 2009) for abstract numeration systems.

p-kernel

- R commutative ring, $k \geq 2$,

p-kernel

- R commutative ring, $k \geq 2$,
- $x = (x_n)_{n \geq 0}$ taking values in some R-module,

p-kernel

- R commutative ring, $k \geq 2$,
- $x = (x_n)_{n \geq 0}$ taking values in some R-module,
- If the R-module generated by all sequences in the k-kernel $\{(x_{np^i+j})_n | i, j < p^i\}$ is finitely generated then the sequence x is said to be (R, k)-**regular**.

p-kernel

- R commutative ring, $k \geq 2$,
- $x = (x_n)_{n \geq 0}$ taking values in some R-module,
- If the R-module generated by all sequences in the k-kernel $\{(x_{np^i+j})_n | i, j < p^i\}$ is finitely generated then the sequence x is said to be (R, k)-**regular**.

Theorem (Bell, 2006) *Let k, l be two multiplicatively independent integers. If a sequence $x \in R^{\mathbb{N}}$ is both (R, k)-regular and (R, l)-regular, then it satisfies a linear recurrence over R.*

In $\mathbb{R}^d$

- **Weak automata** : automata such that each strongly connected component contains either only accepting or only non-accepting states.

In $\mathbb{R}^d$

- **Weak automata** : automata such that each strongly connected component contains either only accepting or only non-accepting states.
- **Weakly** r**-recognizable** : recognized by a weak automaton in base r.

In $\mathbb{R}^d$

- **Weak automata** : automata such that each strongly connected component contains either only accepting or only non-accepting states.
- **Weakly r-recognizable** : recognized by a weak automaton in base r.

Theorem (Boigelot, Brusten, Bruyère, 2008) Let $k, l \geq 2$ be two multiplicatively independent integers. Let $X \subseteq \mathbb{R}$ be compact set. Then, X is both weakly k- and l-recognizable iff it is a finite union of intervals with rational extremities.

In $\mathbb{R}^d$

- **Weak automata** : automata such that each strongly connected component contains either only accepting or only non-accepting states.
- **Weakly r-recognizable** : recognized by a weak automaton in base r.

Theorem (Boigelot, Brusten, Bruyère, 2008) Let $k, l \geq 2$ be two multiplicatively independent integers. Let $X \subseteq \mathbb{R}$ be compact set. Then, X is both weakly k- and l-recognizable iff it is a finite union of intervals with rational extremities.

Theorem (Brusten PhD thesis, 2011) Let $k, l \geq 2$ be two multiplicatively independent integers. Let $X \subset \mathbb{R}^d$ be a compact set. Then, X is both weakly k- and l-recognizable iff it is definable in $\langle \mathbb{R}, +, <, 1 \rangle$.

Self-similar sets

Self-similar set K or attractor of the IFS (f_i) :
$K = f_1(K) \cup \ldots f_n(K)$ where the f_i are contractions.

Self-similar sets

Self-similar set K or attractor of the IFS (f_i) :
$K = f_1(K) \cup \ldots f_n(K)$ where the f_i are contractions.

The IFS is homogeneous if $f_i(x) = ax + b_i$

Self-similar sets

Self-similar set K or attractor of the IFS (f_i) :
$K = f_1(K) \cup \ldots f_n(K)$ where the f_i are contractions.

The IFS is homogeneous if $f_i(x) = ax + b_i$

Theorem (Feng-Wang, 2009) (Suppose K is not a finite union of intervals and (f_i) homogeneous)
If $\alpha K + \beta \subset K$ then $\log \alpha / \log a \in \mathbb{Q}$.

Self-similar sets

Self-similar set K or attractor of the IFS (f_i) :
$K = f_1(K) \cup \dots f_n(K)$ where the f_i are contractions.

The IFS is homogeneous if $f_i(x) = ax + b_i$

Theorem (Feng-Wang, 2009) (Suppose K is not a finite union of intervals and (f_i) homogeneous)
If $\alpha K + \beta \subset K$ then $\log \alpha / \log a \in \mathbb{Q}$.

Theorem (Elekes-Keleti-Mathé, 2010) (Suppose K is not a finite union of intervals)
If $\alpha K + \beta \subset K$ then $\log \alpha$ is a linear combination of the $\log a_i$.

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Graph Directed Iterated Function Systems (GDIFS)

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Graph Directed Iterated Function Systems (GDIFS)

Observation : K is p-recognizable iff it is an homogeneous GDIFS with contraction ratio $1/p$.

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Graph Directed Iterated Function Systems (GDIFS)

Observation : K is p-recognizable iff it is an homogeneous GDIFS with contraction ratio $1/p$.

Boigelot-Brusten theorem for GDIFS : *Let K be a compact which is not a finite union of rational polyhedrons (it can be expressed as a finite Boolean combination of linear constraints with rational coefficients).*

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Graph Directed Iterated Function Systems (GDIFS)

Observation : K is p-recognizable iff it is an homogeneous GDIFS with contraction ratio $1/p$.

Boigelot-Brusten theorem for GDIFS : *Let K be a compact which is not a finite union of rational polyhedrons (it can be expressed as a finite Boolean combination of linear constraints with rational coefficients). Suppose K is an attractor of an homogeneous GDIFS with contraction ratio a and of an other one with contraction ratio b.*

Relation with Cobham

Question : What is the relation with Cobham's theorem ?

The Cantor set ... but not only

Graph Directed Iterated Function Systems (GDIFS)

Observation : K is p-recognizable iff it is an homogeneous GDIFS with contraction ratio $1/p$.

Boigelot-Brusten theorem for GDIFS : *Let K be a compact which is not a finite union of rational polyhedrons (it can be expressed as a finite Boolean combination of linear constraints with rational coefficients). Suppose K is an attractor of an homogeneous GDIFS with contraction ratio a and of an other one with contraction ratio b. Then, $\log a / \log b \in \mathbb{Q}$.*