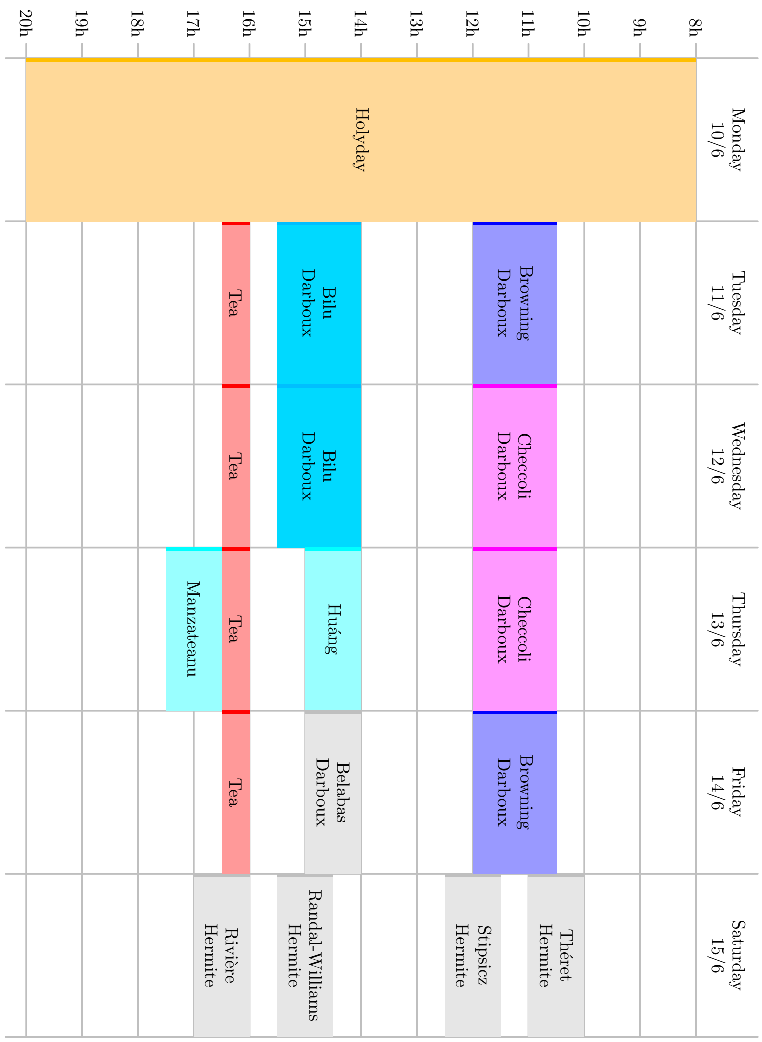
SCHEDULE WEEK 9



Yuri Bilu: *Effective André-Oort*

On every Shimura variety certain algebraic subvarieties are called "special"; in particular, "special points" are special subvarieties of dimension 0. Intuitively, if the Shimura variety is viewed as a moduli space of some objects (like Abelian varieties or so), then the special subvarities are moduli spaces of the same objects with some additional structure.

Date: Monday, June 10th–Sunday, June 16th.

The celebrated André-Oort Conjecture asserts, roughly speaking, that the Zariski closure of a set of "special points" is a "special subvariety"; equivalently, every algebraic subvariety of a Shimura variety may have at most finitely many maximal special subvarieties. This conjecture is proved by Klingler, Ullmo and Yafaev subject to GRH, and in many special cases unconditionally.

In particular, Pila (2011) proved unconditional André-Oort for Shimura varieties of modular type, that is, products of modular curves. Here every point is $(x_1, ..., x_n)$, where each x_i is an elliptic curve with some additional structure. Special subvarieties are (roughly) defined by conditions of the type "there is a cyclic isogeny of given degree between x_i and x_j" or "x_i is a given curve with Complex Multiplication".

While Pila's argument (based on the ground-breaking idea of a previous work of Pila and Zannier) is very clever and beautiful, it is non-effective, though the use of Siegel-Brauer lower estimate for the class number.

In the recent years, in the work of Masser, Zannier, Kühne, Binyamini and others effective proofs were given for various special cases of Pila's result. Most of them are based on the beautiful "Tatuzawa trick", discovered by Kühne, which allows one to achieve effectiveness, in many cases, by replacing the Siegel-Brauer by the Siegel-Tatuzawa Theorem.

The purpose of my course is to give an introduction into this topic. I will restrict to the "Shimura variety" C^n (viewed as the product of n j-lines). The special subvarieties are (roughly) defined by equations of the type $F_N(x_i, x_j) = 0$ or $x_i =$(singular modulus), where F_N is the modular polynomial of level N, and singular moduli are j-invariants of elliptic curves with CM. In particular, special points are those whose all coordinates are singular moduli.

No preliminary knowledge beyond university course of Algebra, Analysis and Number Theory is required. I will even define what Complex Multiplication is. I will also try to highlight ideas as clearly as possible, in many cases sacrificing generality to lucidity. Here is an approximate plan.

(1) Introductory material: Complex Multiplication, Class Field Theory etc.
(2) Definable sets and the (non-effective) Pila-Zannier argument.
(3) One-dimensional effective/explicit results on individual curves and families on curves.
(4) Multidimensional results: the "naive" approach and the Binyamini approach (using his effective estimates for definable sets).

Tim Browning: *Rational points via the circle method*

I will survey recent highlights around applications of the Hardy-Littlewood circle method to rational points, before focusing on what a version of this method over global fields of positive characteristic has to say about the geometry of rational curves on smooth hypersurfaces of low degree. Inspired by recent ideas of Peyre, moreover, I will also describe the parallel situation over the rational numbers, in which one counts points of bounded height satisfying the additional constraint that an associated tangent lattice is not too lopsided.

Sara Checcoli: *Small height in big fields: around the properties of Northcott and Bogomolov and Lehmer's conjecture*

In this course we will study certain problems concerning algebraic numbers of small height.

The (logarithmic Weil) height of an algebraic number α is a non-negative real number that measures the "arithmetic complexity" of α. By Kronecker's theorem the algebraic numbers of height zero are precisely zero and the roots of unity, but what about numbers of non-zero "small" height?

There are two important statements in this context:

- The first, Northcott's theorem, ensures that a set of algebraic numbers whose all elements have both their height and degree "small" (i.e. bounded) is finite.
- This makes the height (and its variants/generalisations) a very important tool in diophantine geometry: to show the finiteness of a certain set of points (e.g. the rationals points on a variety), one tries usually to bound their height and their degree.
- The second is a famous conjecture of Lehmer, which states that for every algebraic number the product of its height and its degree is either 0 or always bigger than an absolute positive constant. This conjecture was proved for many classes of algebraic numbers, but it is still open in general.

Now, one could ask: in which cases the above statements are still true if one "forgets the degree"? More precisely, following Bombieri and Zannier, we say that a set K of algebraic numbers has the Northcott property (N) if it contains only finitely many points of bounded height and we say that K has the Bogomolov property (B) if 0 is not an accumulation point for the values of the height of the elements in K.

It is easy to see that property (N) implies property (B) and that they both hold when K is a number field. However deciding the validity of these properties for an infinite algebraic extension K of $\mathbb{Q}$ is in general a difficult problem, which has been studied by many authors.

The goal of this mini-course is to give an overview of the known results on this subject and some open problems. If time permits, we will discuss some recent works of Breuillard and Varjù, which show the equivalence between the Lehmer's conjecture and the growth conjecture in geometric group theory.

Zhìzhōng Huáng: *Rational approximations on toric varieties*

Motivated by the local behaviour of rational points on varieties over number fields, McKinnon and M. Roth introduced the notion "approximation constant" attached to every rational point. Roughly speaking, for a given rational point Q and a place v, it measures how quickly the height must grow for any sequence of rational points approaching Q with respect to a v-adic distance. A conjecture of McKinnon predicts that rational curves should be the unique medias on which this constant is achieved. Building on Salberger's universal torsor method and Batyrev's toric "Bend-and-Break", we verify McKinnon's conjecture for certain split toric varieties and we give the precise locally accumulating subvariety.

Adelina Manzateanu: *Counting points on Hilbert schemes over function fields*

We consider the Hilbert scheme $\mathrm{Hilb}^2\mathbb{P}^2$ defined over a global field of positive characteristic K. We give an asymptotic formula for the number of K-points of bounded height on $\mathrm{Hilb}^2\mathbb{P}^2$ that supports the thin set version of Manin's conjecture and show that the leading constant agrees with the prediction of Peyre. Moreover, we extend the analogy between the integers and 0-cycles on a variety V over a finite field to 0-cycles on a variety V over K and establish a version of the prime number theorem in the case when $V = \mathbb{P}^2$.