Differential equations and their Galois theory

Julien Roques
Université Grenoble Alpes

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Galois's ideas

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The day before his fatal duel, Galois wrote a letter to his friend Auguste :

> *Tu sais, mon cher Auguste, que ces sujets ne sont pas les seuls que j'ai explorés. Mes principales méditations depuis quelque temps étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguïté. Il s'agissait de voir a priori dans une relation entre quantités ou fonctions transcendantes quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître tout de suite l'impossibilité de beaucoup d'expressions que l'on pouvait chercher. Mais je n'ai pas le temps et mes idées ne sont pas encore bien développées sur ce terrain qui est immense.*

Galois's ideas beyond algebraic equations

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The first important progress was made by Picard : he laid the foundations of the *differential* Galois theory.

Differential Galois theory : an historical motivation

Consider a linear differential equation

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  where

$$\mathfrak{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + zt\right) dt, \quad \mathfrak{Bi}(z) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3} + zt\right) + \sin\left(\frac{t^3}{3} + zt\right)\right] dt.$$

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An explicit computation of the differential Galois group attached to $y''(z) = zy(z)$ will prove that this equation cannot be solved by using "elementary functions".

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For instance, it has applications in the theory of dynamical systems, in number theory, *etc*.

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- Applications (including solvability by "elementary functions").
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- And also : hypergeometric equations, algebraic solutions of differential equations, differential equations in positive characteristic, applications to arithmetic, tannakian formalism, *etc*.