M2R Lecture Course "Diophantine approximation and values of special functions"

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Provisional program.

Generalities on Diophantine approximation

Dirichlet's Theorem on the approximation of irrational numbers by rational numbers. Almost sure converse; some results in metric theory.

Liouville's Theorem on the approximation of algebraic numbers by rational number. Liouville's minoration of non-zero polynomial values at algebraic points.

First constructions of transcendental numbers.

Continued fractions.

Transcendance of values of exp and log

Generalities of Hermite-Padé approximants, i.e. of analogues of continued fractions for power series.

Hermite proof of the transcendance of e

Generalisations : Hermite-Lindemann and Lindemann-Weierstrass Theorem on transcendence and algebraic independence of values of exp.

Application to the transcendence of π and values of log. Some geometrical applications, like the Mazurkiewicz-Sierpinski Theorem (paradoxical decomposition of the plane using isometries).

Values of polylogarithms $\sum_{n=1}^{\infty} \frac{z^n}{n^s}$

Nesterenko's linear independence criterion; the saddle point method.

Irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$: Beukers' proof and Gutnik-Nesterenko's proof.

Irrationality of infinitely values $\zeta(2k+1) = \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}}$ $(k \in \mathbb{N}_{\geq 1})$ and another proof of the transcendance of π.

Explicit irrationality measures of numbers like $\log(2)$, $\zeta(3)$, π. For instance, for all $p, q \in \mathbb{Z}$, $q \geq 1$.

$$\left| \pi - \frac{p}{q} \right| > \frac{1}{q^{42}}.$$

The Gel'fond-Schneider Theorem

Sketch of the proof the transcendence of e^π (Gel'fond)

Siegel's lemma and *inexplicit* transcendence construction

Proof of Hilbert 7th problem: Transcendence of α^β with α, β algebraic, $\alpha \neq 0, 1$ and β irrational: Gel'fond's proof and Schneider's proof.

Generalization: the Schneider-Lang criterion.

Application to elliptic functions and modular forms.

E-functions and G-functions

• E-functions: generalization of $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and Bessel's functions

Siegel-Shidlovskii Theorem on values of E-function. Shidlovskii's lemma.

• G-functions: generalization of $\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ and polylogarithms.

Chudnovskii's Theorem of values of G-functions. André zero lemma.

Chudnovskii's Theorem: G-operators are fuchsian. Katz Theorem: exponents of G-operators are rational.

André's Theorem on the structure of E-operators. Applications to values of E-functions.

Prerequisites :

A basic course in Complex Analysis

Basic properties of number fields, Galois theory.

Linear differential equations with polynomials coefficients.