MASTER 2 DEGREE
IN FUNDAMENTAL MATHEMATICS
2019–2020

Elements of algebraic geometry and topology

The Master 2 degree in fundamental mathematics of the **University of Grenoble Alpes** offers each year a training in research in a selected branch of mathematics, with possibilities to pursue in a PhD thesis on that subject, in the **Institut Fourier** (Grenoble), one of the French prominent institutes in pure mathematics. Courses are **taught in English** if non-French-speaking students are registered.

In 2019-2020, the proposed courses cover a wide range of subjects in **algebraic and analytic geometry**, with applications to **topology and representation theory**.

The courses are structured to provide the students with a **comprehensive background** on the subjects and preparing them to **several possible directions**. They are organized in two semesters, with more foundational material and tutorials being delivered during the first semester, and **more advanced subjects during the second semester**.

Students will have an opportunity to be coached in mathematical research through a **research internship** organized during the second semester. This internship is meant to be a **good preparation for subsequent PhD theses** in related domains.

Contents

Introduction

Algebraic geometry is one of the most central branches of mathematics. It has strong connections with commutative algebra, complex analysis and geometry, topology and number theory, at the very least. Its development has been pursued without interruption from Antiquity, e.g. with the theory of conics and plane curves by the ancien Greeks, to the modern period, which has seen a flourishing of new theories and new connections to other branches of science, and especially physics.

The goal of the present Master 2 degree is to give a wide overview of several central aspects of algebraic geometry, in connection with other related branches of mathematics such as differential geometry and topology.

Courses delivered during the first semester are meant to be introductory. The course by Philippe Eyssidieux is more algebraic in nature. It introduces the basic dictionary between algebraic varieties and their rings of functions, and then proceeds to discussing singular and regular points and the basic concepts of rational functions and divisors. Jean-Pierre Demailly's course has a more analytic focus: the objects under consideration can be similar to those of algebraic geometry, but the tools are borrowed from differential geometry and the theory of holomorphic functions of several variables. In the case of projective varieties, the two approaches have a strong interplay, as exemplified e.g. by Serre's famous GAGA theorem. The complex one dimensional case gives rise to theory of Riemann surfaces, initiated by Riemann in the XIX$^{\text{th}}$ century, which can also be thought of as real surfaces equipped with a complex structure; this is subject of the course of Louis Funar, Erwan Lanneau and Greg McShane, in which important topological concepts are also introduced (fundamental group, diffeotopies, mapping class group ...)

The second semester offers more advanced material. The course by Funar, Lanneau and McShane is a rather direct continuation of the first semester course on surfaces, covering classical topics in the geometry and topology of surfaces and their moduli – moduli spaces are spaces parametrizing possible deformations of a given structure. Applications to the study of geometric structures on low dimensional manifolds will be presented. The second advanced course by Jérémy Guéré deals with the enumerative theory of complex curves; to this aim, it also uses the theory of moduli spaces in a crucial manner, along with important theorems such as the Grothendieck-Riemann Roch formula. The theory has strong links with theoretical physics, namely string theory.

Additional advances courses might be given during the second semester if there is an interested audience.

1 Basic courses (first semester)

1.1 Intensive preparatory course on differential manifolds (S1P)

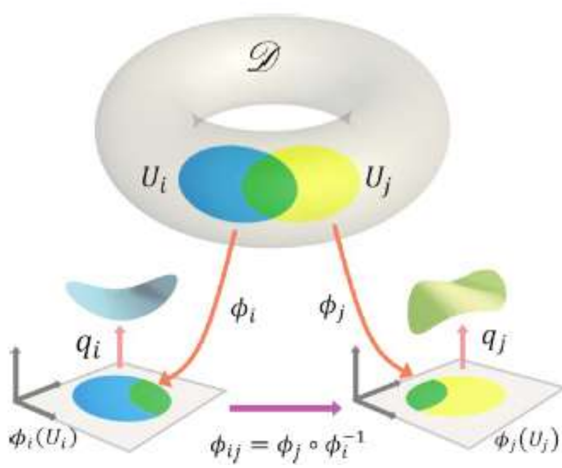
Lecturer: Catriona Maclean (18h course, 12h tutorials)

During the first 3 weeks of the first semester, an "intensive" preparatory course will be given on differential manifolds and De Rham cohomology. A differential manifold is a topological space that admits "coordinate charts" locally homeomorphic to open subsets of $\mathbb{R}^n$ in such a way that coordinate changes are diffeomorphisms. De Rham cohomology groups are defined in an algebraic manner by means of differential forms and their "exterior calculus". They describe very useful topological invariants of a manifolds, especially Betti numbers.

Program

- Basic concepts of differential geometry: manifolds, tangent and cotangent bundles
- Differential forms, exterior differential calculus, Stokes theorem
- Basic concepts of homological algebra: differential complexes and (co)homology
- De Rham cohomology, Poincaré duality

Figure 1: Charts of a manifold



References

- M. Berger, B. Gostiaux, *Differential Geometry: Manifolds, Curves, and Surfaces*, Springer, Graduate Texts in Mathematics, Volume 115, 1988
- R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, Springer, Graduate Texts in Mathematics, Volume 82, 1982

Prerequisites

- Basic knowledge of general topology
- Preliminary master course in differential calculus

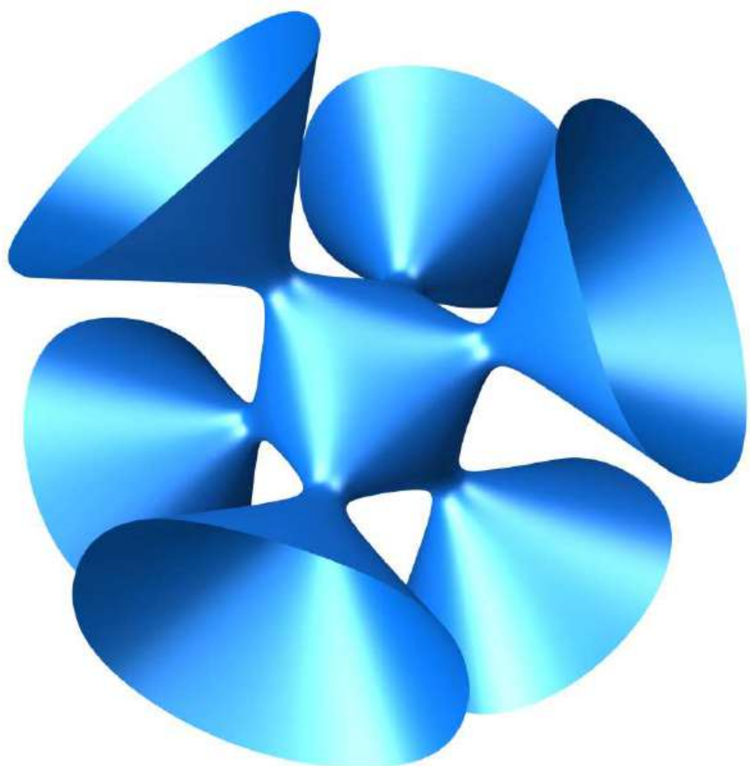
1.2 Introduction to algebraic geometry (S1A)

Lecturer: Philippe Eyssidieux (28h course, 18h tutorials)

Algebraic Geometry studies the sets defined by **polynomial equations in several variables** over a given field, the so-called **algebraic varieties**. The theory is characterized by the profusion of examples: line and conics have been known since the antiquity but the very first non-trivial example, the group law of the cubic curve, emerged in the 18th and 19th century from the study of the abelian integrals $\int \frac{dx}{y}$ where $y^2 = x^3 + ax + b$. Although the classification of algebraic curves is rather tame, surfaces and higher dimensional varieties exhibit wilder phenomena.

The first part of the course will introduce the basic dictionary between algebraic varieties and their rings of functions. The second part will be devoted to singular and regular points. The third part will introduce their main inhabitants. In the case of the fields of the real and the complex numbers, regular algebraic varieties (those having no singular points) define manifolds and we will use the intuition from differential geometry acquired in the preliminary course of the program to motivate the constructions.

Figure 2: A K3 surface (after Alessandra Sarti)



$$1 + x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2 + 1)^2 = 0, \quad a = -0.49$$

This introductory course on algebraic geometry will cover a selection of topics from chapters I-II-III-VIII of the book "Basic Algebraic Geometry" by I.R. Shafarevich. The tutorials will also borrow the exercises from Shafarevich's book.

Program

- Plane algebraic curves, their regular and singular points
- Affine algebraic varieties, their ring of functions
- Projective and quasi projective varieties
- Morphisms
- Dimension
- Regular points
- Rational functions
- Differential forms
- Divisors

References

- I.R. Shafarevich, *Basic Algebraic Geometry 1, 2*. 3rd edition translated by Miles Reid. Springer 2013. Original Russian first edition published by Nauka, Moscow 1988

Prerequisites

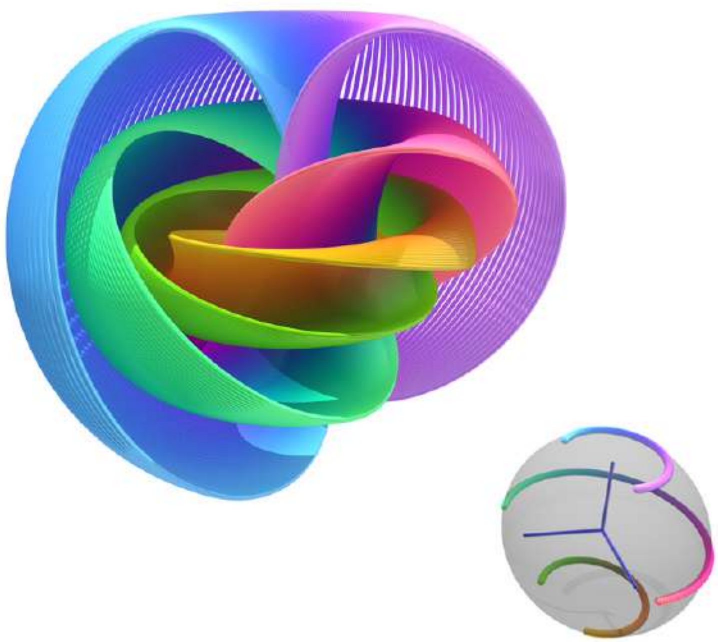
- Master course in Commutative Ring theory
- Preliminary course in Differential Geometry

1.3 Introduction to analytic geometry (S1B)

Lecturer: Jean-Pierre Demailly (28h course, 18h tutorials)

Algebraic varieties over the field of complex numbers can be studied through a transcendental point of view : such varieties, when they are non singular, are in fact complex analytic manifolds. Their study is then intimately related to the study of holomorphic functions of several complex variables. Just as in the case of one complex variable, holomorphic functions of several variables enjoy many deep properties that pertain to the fact that complex numbers form an algebraically closed field; also, holomorphic functions are in some sense a completion of the space of polynomials. Analytic properties can be linked to algebraic facts by some fundamental theorems such as Serre's GAGA theorem, which asserts that every holomorphic object living in a complex projective manifold is in fact algebraic. Such properties are reflected by strong "rigidity properties" of holomorphic functions, and can also be detected by computing relevant cohomology groups : these groups somehow describe the fundamental analytic "invariants" of complex manifolds. The basic Dolbeault-Grothendieck lemma asserts the local triviality of Dolbeault cohomology. The concepts of sheaf and locally free sheaf allow to establish certain canonical cohomology isomorphisms, and the fundamental Serre duality theorem generalizes Poincaré duality to complex geometry.

Figure 3: Complex projective space and Hopf fibration



$$\mathbb{P}^n_{\mathbb{C}} \simeq S^{2n+1}/S^1 \text{ and also } \mathbb{P}^n_{\mathbb{C}} \simeq \mathbb{C}^n \cup H_\infty$$

Program

- Holomorphic functions of several variables
- Complex manifolds, tangent and cotangent bundles, examples

- Real and complex structures, differential forms of type (p, q)
- ∂ and $\overline{\partial}$ operators
- Dolbeault-Grothendieck lemma and Dolbeault cohomology groups
- Locally free sheaves and holomorphic vector bundles, examples
- Finiteness theorems and Serre duality
- (if time permits) Connections and curvature, positivity concepts, first Chern class ...

References

- J.-P. Demailly, *Complex analytic and differential geometry*, online book [B2] on `https://www-fourier.ujf-grenoble.fr/~demailly/documents.html`
- R.O. Wells, *Differential analysis on complex manifolds*, Graduate Texts in Math. **65**, 2nd edition, Springer-Verlag, Berlin (1980)

Prerequisites

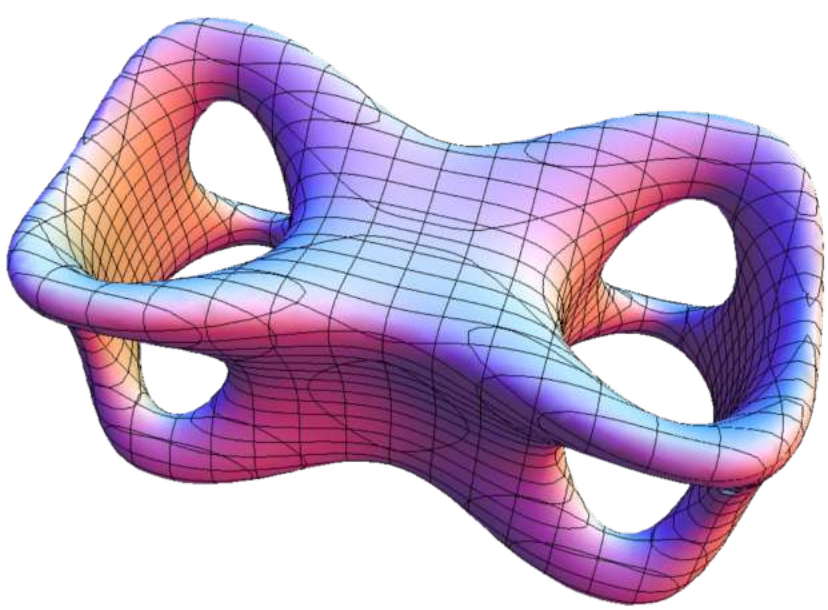
- Preliminary master course in differential calculus
- Preliminary course S1P (manifolds and De Rham cohomology)
- Holomorphic functions of one complex variable (Cauchy condition, Cauchy formula, residues, uniformly convergent sequences)
- Basic knowledge in functional analysis

1.4 Surfaces (S1C)

Lecturers: Louis Funar, Erwan Lanneau, Greg Mc Shane (28h course)
Tutorials: Catriona Maclean (18h tutorials)

This fundamental course is an introduction to the geometry and topology of surfaces and their moduli. This subject grew as the fruitful interaction of several domains of mathematics including algebra, geometry, topology and complex analysis. The methods developed here will provide firm ground and useful skills for several more advanced mathematical theories. We will first deal with surfaces and their topological classification which amounts to decide whether two surfaces are homeomorphic, i.e. they could be continuously deformed one into the other. Surfaces can be endowed with complex structures, and therefore called Riemann surfaces, by looking at them as the result of piecing together open subsets of the complex plane so that the transition maps between overlapping pieces are holomorphic. Next, to every manifold one associates a group whose role is essential in the study of their topology: the fundamental group is introduced by giving a group structure on the set of loops living in the manifold up to deformation and, in particular, groups of surfaces will be considered. We will consider elements of hyperbolic geometry in real dimension 2 and one complex variable theory, so as to investigate Fuchsian groups. The uniformization theorem will be discussed in connection with basic properties of algebraic curves. This will be followed by the study of surface automorphisms, diffeomorphism and diffeotopy groups, as well as automorphism groups of Riemann surface. Algebraic properties of these groups, along with a description of their natural action on the Teichmüller space, will be presented. If time permits, fiber bundles on algebraic curves will be presented, leading to a study of flat structures, namely flat surfaces, abelian and quadratic differentials.

Figure 4: A Riemann surface



Program

- Surfaces as topological objects. Classification
- Smooth and complex structures

- Fundamental groups
- Hyperbolic space and its isometries
- Fuchsian groups
- Uniformization, algebraic curves
- Surface diffeomorphisms, automorphisms
- Mapping class groups
- Flat structures

References

- S.Katok, *Fuchsian groups*, University of Chicago Press, 1992
- H. Paul de St. Gervais, *Uniformisation des surfaces de Riemann*, ENS Editions 2010
- B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Univ. Press 2012
- M. Hirsch, *Differential topology*, (2nd ed.), Springer 1994

Prerequisites

- Preliminary course in General Topology
- Preliminary course in Differential Geometry.

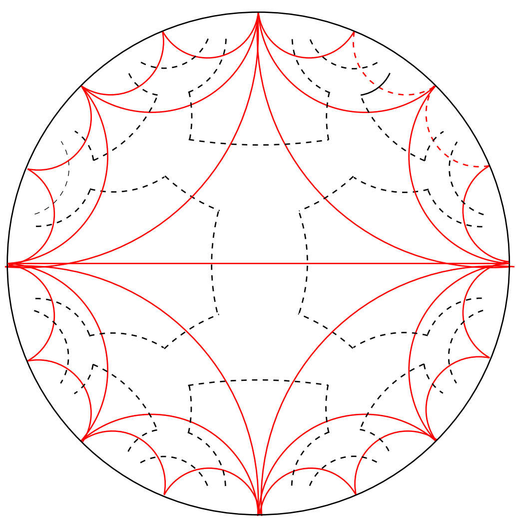
2 Advanced courses (second semester)

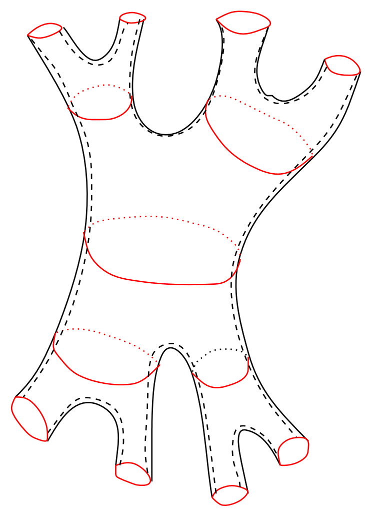
2.1 Geometry and topology of moduli spaces (S2A)

Lecturers: Louis Funar, Erwan Lanneau, Greg Mc Shane
(24h course)

The aim of this course is to cover some classical topics in the geometry and topology of surfaces and their moduli, in particular the study of the geometry and the topology of moduli spaces issued from varieties of representations of discrete groups into Lie groups. In particular we wish to study geometric structures on low dimensional manifolds, notably surfaces and 3-manifolds. A particularly interesting case is the Teichmüller theory, including mapping class groups, moduli spaces of curves and dynamics. Finally we would like to introduce the Chern-Simons theory in connection with the symplectic geometry of the varieties of representations. We would like to bring forth an approach that uses only a very little amount of advanced mathematics which lead us deep enough and yields information in a physicist-oriented manner.

Figure 5: Universal cover and rigid structure





Program:

- Mapping class groups: homotopy versus isotopy for curves and homeomorphisms of surfaces, generating mapping class group by Dehn twists, the Dehn-Nielsen-Kneser theorem, basic relations (braid-type, lantern, 2-chain), the symplectic group, the Torelli group and its generators, the Johnson homomorphism, complexes of curves and arcs, Harer-Hatcher theorem about the contractibility of the arc complex.
- Spaces of discrete subgroups: non-rigidity phenomena, Teichmüller space, Thurston-Bonahon-Penner-Fock alias shearing coordinates on the Teichmüller space.
- The interplay between mapping class groups and Teichmüller spaces: proper discontinuity, stabilizers, Ptolemy groupoids, explicit computations in terms of shearing coordinates, Belyi surfaces.

- Remarkable identities. Counting simple curves on surfaces.
- Moduli spaces of surface groups representations and their symplectic structure.
- Geometric structures on 3-manifolds.
- Chern-Simons invariants.

Prerequisites

Rudiments of algebraic topology (homology, fundamental group), surfaces, linear algebra, basic complex analysis. The fundamental course S1C on Surfaces is highly recommended.

References

- W. Thurston, *Three-Dimensional Geometry and Topology*, Volume 1, Princeton University Press, 1997.
- F. Labourie, *Lectures on representations of surfaces groups*, European Math. Society, 2013.

2.2 Enumerative theory of complex curves (S2B)

Lecturer: Jérémy Guéré (24h course)

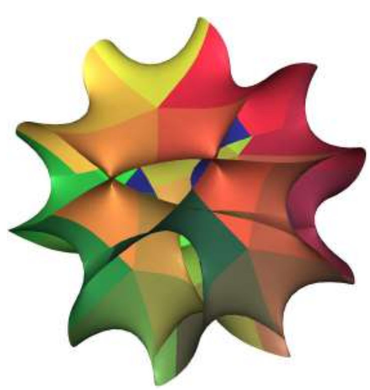
If we look for the space of lines through a point in an affine space, we end up with the projective space. It is our first example of a moduli space, i.e. a space of parameters for a given class of objects (lines in this example). If we want to classify all vector subspaces of dimension k inside a vector space of dimension $n > k$, then the corresponding moduli space is called the Grassmannian $\mathrm{Gr}(k, n)$. It is a variety of dimension $\binom{n}{k}$. The purpose of this course is to describe the moduli space of complex curves, i.e. Riemann surfaces. At a given genus g (which is the number of holes in the Riemann surface), the dimension of the moduli space is $3g - 3$. Therefore, the complexity of that space increases a lot with the genus and that's why we will start with $g = 0, 1$, and 2.

Intersection theory is a way to focus on objects satisfying particular conditions. For instance, in the Grassmannian $\mathrm{Gr}(k, n)$, we can look at subvector spaces containing a given line, yielding a subvariety of $\mathrm{Gr}(k, n)$. When this subvariety is zero-dimensional, then it is just made out of points and we can count them, that is enumerative theory. For instance, the number of lines in the plane meeting at two given points is just 1.

In this course, we aim at counting complex curves with special properties. For this reason, we'll develop intersection theory on the moduli space of complex curves. We'll use nice geometric tools such as gluing of curves and crucial theorems such as Grothendieck–Riemann–Roch formula. At the very end of the course, we will survey some deep connections of this counting problem with fundamental physics (string theory).

Figure 6: Complex curves of genus 1, 2, and 3

Figure 7: Example of a complex variety used in string theory
(Calabi-Yau manifold)



Program:

- Projective space and Grassmannian
- Moduli space of complex curves
- Introduction to intersection theory
- The Grothendieck–Riemann–Roch formula

References:

- Zvonkine, *An introduction to moduli spaces of curves and their intersection theory*
- Arbarello–Cornalba–Griffiths, *Geometry of algebraic curves I & II*
- Katz, *Enumerative geometry and string theory*

Prerequisites:

The courses of the first semester, especially the introduction to algebraic geometry.

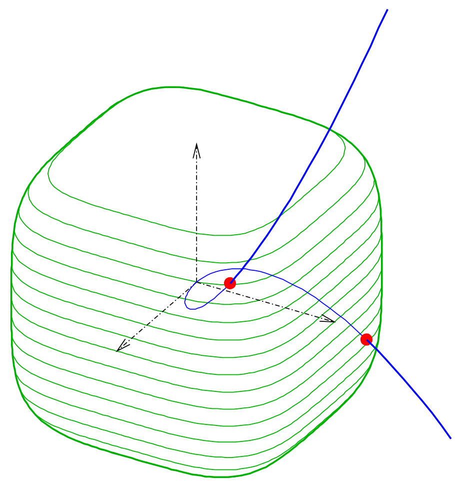
3 Supplementary possible course

If the audience is sufficient and there are interested students further courses on related subjects could be organized.
One such possible course could be :

Lecturer: Jean Fasel (24h course)

Title. Characteristic classes and intersection theory in algebraic geometry

Figure 8: Intersection of algebraic cycles



The purpose of the course is to give an introduction to Chow groups, after Fulton and MacPherson. We will start with the notion of cycles and rational equivalence necessary for the definition of Chow groups and then study the functorial properties of the latter. Then, we will pass to the definition of Segre and Chern classes of vector bundles, starting with Cartier divisors. Time permitting, we will consider the intersection product and make some basic calculations.

Reference

- W. Fulton, *Intersection theory*, Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, 2, 1998

4 Contacts and additional information

For more information, send an e-mail to Andrea Pulita, who is the responsible of the Master 2 degree, at the following address:
– andrea.pulita@univ-grenoble-alpes.fr

For more information or questions about the courses, it is possible to contact the reference persons at the addresses below:

- *Introduction to manifolds*
  – Catriona Maclean : catriona.maclean@univ-grenoble-alpes.fr
- *Algebraic geometry*
  – Philippe Eyssidieux : philippe.eyssidieux@univ-grenoble-alpes.fr
- *Analytic geometry*
  – Jean-Pierre Demailly : jean-pierre.demailly@univ-grenoble-alpes.fr
- *Theory of surfaces and moduli spaces*
  – Catriona Maclean : catriona.maclean@univ-grenoble-alpes.fr
  – Louis Funar : louis.funar@univ-grenoble-alpes.fr
  – Erwan Lanneau : erwan.lanneau@univ-grenoble-alpes.fr
  – Grag McShane : greg.mcshane@univ-grenoble-alpes.fr
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- *Characteristic classes and intersection theory in algebraic geometry*
  – Jean Fasel : jean.fasel@univ-grenoble-alpes.fr