MASTER 2 DEGREE IN FUNDAMENTAL MATHEMATICS 2018-19

GEOMETRY, TOPOLOGY AND APPLICATIONS TO ANALYSIS AND PROBABILITY

The Master 2 degree in fundamental mathematics (M2R) of the **University of Grenoble Alpes** offers each year a training in research in a selected topic, with possibilities to pursue in a **PhD thesis** on that subject, in the **Institut Fourier** (Grenoble), one of the French prominent institute in pure mathematics. Courses are **taught in English** if non-french-speaking students are registered.

This year, the selected topics lie at the crossroads of **geometry**, **topology** and **analysis**. The possible outlets and opportunities comprise a **large range of domains and applications**.

The courses are structured to provide the students with a **comprehensive background** on the subjects and preparing them to **several possible directions**, following their wishes.

Contents

Introduction

Geometry is a vast branch of mathematics concerned with questions of shape, size, and the properties of space. Differential geometry, in particular, uses techniques of calculus and linear algebra to study problems in geometry. It studies the structure of *differentiable manifolds* which, roughly speaking, formalize the concept of curve and surface. This M2R focuses on several contemporary sub-fields of geometry.

A first fundamental course is devoted to the classical topic of **Riemannian geometry**. This is the branch of differential geometry that studies manifolds endowed with local notions of angle, length of curves, surface area and volume. A particular emphasis is given to the interactions with the theory of **optimal transportation** and curvature bounds. Adding constraints on the admissible curves leads to **sub-Riemannian geometry**, the second fundamental course of the M2R. Its typical problems are tackled also with techniques coming from **optimal control theory**. The third fundamental course is devoted to **contact and symplectic geometry**, and in particular to their interaction with differential topology. Both symplectic and contact geometry are motivated by the mathematical formalism of classical mechanics, where one can consider either the even-dimensional phase space of a mechanical system or the odd-dimensional constant-energy hypersurface.

The second part of the M2R consists in advanced courses. The first one is devoted to **geometric analysis**, a mathematical discipline at the interface of differential geometry and differential equations. A particular emphasis is given to heat equation on Riemannian manifolds and Lie groups, and the interaction between the the analytic properties of its solutions and the geometrical properties of the manifold. The second one is a course in **random geometry**. This course has a probabilistic flavor, and applies techniques coming from topology and Riemannian geometry to study the *average* properties of the zero locus of random real polynomials (such as its size, shape and measure). A third possible advanced course is an introduction to **mathematical general relativity** and in particular to the Einstein equations on which the whole theory is based. Solutions of the Einstein equations are 4 dimensional Lorentzian manifolds. We will discuss the local theory of the Einstein equations as well as some particular solutions describing black holes.

Fundamental courses

1. Riemannian geometry and optimal transportation (H. Pajot)

Manifolds are generalizations of the classical notions of curves and surfaces which are fundamental concepts in geometry. Riemannian geometry is the study of "geometric structures" (in particular curvatures) on manifolds and has applications in a lot of fields in mathematics and physics, for instance analysis, group theory, differential and algebraic topology, and general relativity.

In the first part of the course, we will introduce fundamental notions and basic materials in Riemannian geometry which will be useful for all the other courses. The second part will concern geometric aspects of optimal transportation as developed very recently by Lott, Villani Sturm among others. The problem of optimal transportation was formalized by Gaspard Monge in 1781 and has many applications in probability, functional analysis and partial differential equations. We will mainly discuss the connections between Riemannian geometry and metric geometry: metric spaces with Ricci curvature bounded below (in the sense of optimal transportation) and metric spaces with positive sectional curvature (called Alexandrov spaces).

This course does not require any knowledge in differential geometry.

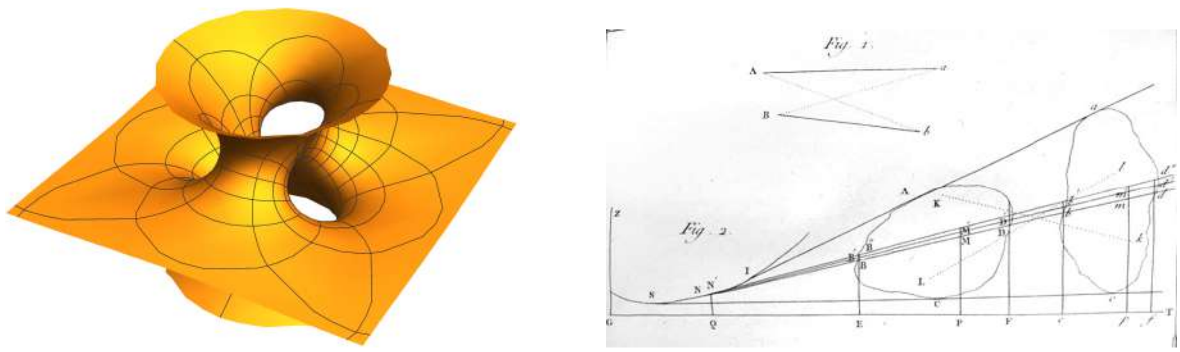

Figure 1. Left: De Costa surface. Right: optimal transport, by G. Monge

Program.

(1) Tensors, bundles, manifolds
(2) Riemannian metrics, connections and geodesics
(3) Curvatures and model spaces
(4) Laplace-Beltrami operator and introduction to analysis on manifolds

(5) The Monge-Kantorovitch problem, existence and regularity of optimal transports in the Riemannian setting
(6) Geometry of metric spaces with non-negative Ricci curvature
(7) Functional inequalities and analysis on metric spaces.

References.

- S. Gallot, D. Hulin, J. Lafontaine. *Riemannian geometry*, Springer-Verlag, Berlin, 2004
- J. Lee. *Riemannian manifolds. An introduction to curvature*, Springer-Verlag, New York, 1997
- J. Jost. *Riemannian geometry and geometric analysis*, Springer
- C. Villani. *Optimal transport. Old and new*, Springer-Verlag, Berlin, 2009
- L. Ambrosio, N. Gigli. *A user's guide to optimal transport*, Springer, Heidelberg, 2013

Prerequisites. A basic course in differential calculus and measure theory.

2. Sub-Riemannian geometry and optimal control theory (G. Charlot, L. Rizzi)

Sub-Riemannian geometry provides a mathematical model for many problems involving non-holonomic constraints. The paradigmatic example is the dynamics of parking a car. We can describe the configuration of a car by its position $(x, y) \in \mathbb{R}^2$ and the orientation of the steering wheel $\theta \in \mathbb{S}^1$. Indeed, not all movements are allowed: for example, the car cannot move sideways (a practical example of nonholonomic constraint). Nevertheless, through a combination of admissible movements, the car can attain the equivalent of a sideways displacement, making it possible to maneuver it into the parking spot. The total distance traveled is clearly

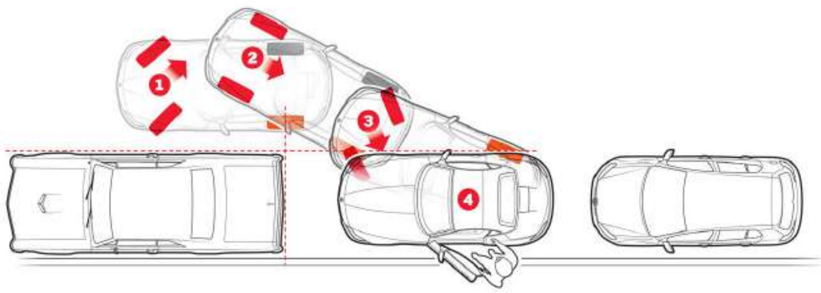

Figure 2. Parking a car in the optimal way, a typical sub-Riemannian problem.

longer than the straight line from the initial to the final position, and a non-trivial problem is to determine how to do it in an optimal way, i.e. minimizing the total distance. This gives rise to a metric space structure on the car's configuration manifold $\mathbb{R}^2 \times \mathbb{S}^1$, and constitute an example of sub-Riemannian structure.

The aim of this course is to provide an introduction to geometry and analysis on sub-Riemannian manifolds, and to illustrate some challenging open problems and research directions of this very active research domain. The first part of the course will be devoted to geometrical optimal control theory, a general framework extending the classical calculus of variations, which has many applications also outside sub-Riemannian geometry.

Program.

(1) Preliminaries of differential geometry (flows, Lie brackets, tangent and cotangent bundle, Poisson brackets)
(2) Control systems (controllability, orbit theorem, linear systems, Kalman condition)

(3) Optimal control (existence of minimizers, Filippov theorem, geometric Pontryagin maximum principle, sufficient conditions for optimality, examples)
(4) Sub-Riemannian structures (distributions, admissible curves, end-point map, singular and regular paths, Chow-Rashevskii theorem, examples)
(5) Sub-Riemannian geodesics (characterization of normal and abnormal extremals, local minimality of normal extremals, exponential map, second order conditions, examples)
(6) Sub-Riemannian tangent space (privileged coordinates, Ball-Box theorem, Mitchell's theorem, nilpotent approximation, Gromov's notion of tangent space, Carnot groups)
(7) Almost-Riemannian structures
(8) Sub-Laplacians (Hörmander type operators, hypoellipticity, sub-Riemannian heat equation, heat kernel)
(9) Some open problems (regularity of minimizers, Sard conjecture, topology of balls)

References.

- A. Agrachev, D. Barilari, U. Boscain. Lecture notes on sub-Riemannian geometry
- L. Rifford. *Sub-Riemannian Geometry and Optimal Transport*
- A. Bellaïche. *The tangent space in sub-Riemannian geometry*
- F. Jean. *Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning*
- A. Agrachev, Y. Sachkov. *Control Theory from the Geometric Viewpoint*
- R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*

Prerequisites. Basic concepts of differential geometry (smooth manifolds, tangent and cotangent space, flow of vector fields, Lie bracket), at the level of the book *Introduction to Smooth Manifolds* by J. Lee. All necessary notions will be introduced in the first part of the course, and during the course **Riemannian geometry and optimal transportation**.

3. Introduction to contact and symplectic geometry (S. Courte, S. Guillermou)

The objective of this course is to introduce some basic notions of symplectic and contact geometry. Contact geometry arises in the formalisation of classical mechanics (Hamilton equations), while contact geometry is linked to optics (wave propagation). Despite their classical roots, these are very active and modern subjects, with strong connection with low dimension topology and Riemannian geometry. The following is a tentative list of topics that will be discussed, not necessarily in this order, being symplectic and contact geometry deeply intertwined.

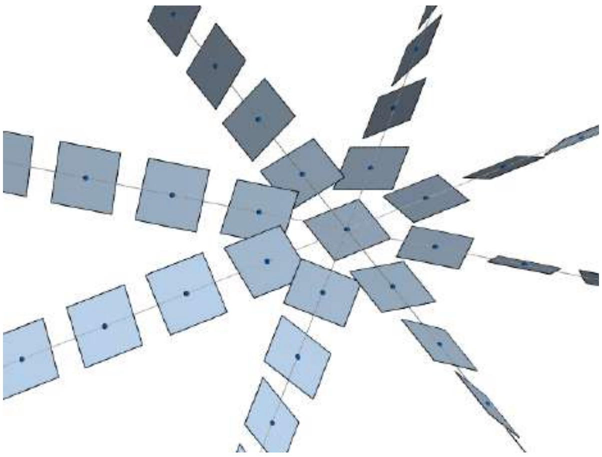

Figure 3. Standard contact structure on $\mathbb{R}^3$. (picture by P. Massot)

Program.

- Symplectic geometry
  1. Symplectic linear algebra, Lagrange Grassmannian, linear symplectic reduction, local Maslov index
  2. Symplectic manifolds, Darboux and Moser theorems
  3. Poincaré-Birkhoff theorem
  4. Co-isotropic submanifolds, isotropic foliations, symplectic reduction
  5. Construction of symplectic manifolds, blow-up, connected sum
  6. Lagrangian submanifolds, Weinstein tubular neighborhood theorem, generating functions, Arnold conjecture
- Contact geometry
  1. Darboux and Gray theorems
  2. Contact topology in dimension 3: Eliashberg overtwisted/tight dichotomy, Bennequin theorem

(3) Convex surfaces in dimension 3, classification of contact structures on some manifolds following Eliashberg, Giroux

References.

- McDuff-Salamon. *Introduction to symplectic topology*
- Geiges. *An introduction to contact topology*
- P. Massot. Topological methods in 3-dimensional contact geometry

Prerequisites. Basic concepts of differential geometry (smooth manifolds, tangent and cotangent space, flow of vector fields, Lie bracket), at the level of the book *Introduction to Smooth Manifolds* by J. Lee. All Necessary notions will be introduced in the first part of the course, and during the course **Riemannian geometry and optimal transportation**.

Advanced courses

4. Geometric analysis on Riemannian manifolds and Lie groups (E. Russ)

Real analysis in non Euclidean contexts, such as Riemannian manifolds and Lie groups (as well as on graphs for the discrete counterpart) developed significantly from the eighties up to now. Due to the lack of Fourier transform, an essential tool in this kind of analysis is the *heat kernel* h_t, that is the fundamental solution of the heat equation $\partial_t u = \Delta u$, where Δ stands for the Laplacian. In $\mathbb{R}^n$, one has

$$h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1}$$

In this course, we will see how the estimates of the heat kernel h_t in Riemannian manifolds and Lie groups (especially the Heisenberg group) are related to geometric assumptions (such as the growth of the volume of balls, Poincaré inequalities, Ricci curvature...). In particular, in some cases, the behaviour of h_t will be of Gaussian type, as in (1). We will also describe the links with isoperimetric inequalities. We will finally investigate some consequences about PDEs and real analysis problems.

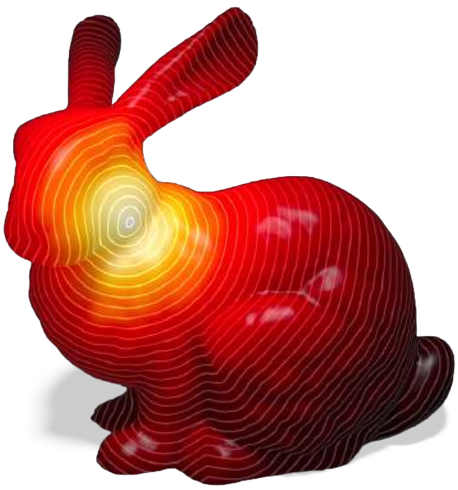

Figure 4. Reconstruction of the distance from a point on a given domain via heat methods. Picture by K. Crane *et al.*

Program.

(1) Estimates of the heat kernel on Lie groups and Riemannian manifolds: the case of Gaussian estimates, the role of geometric assumptions. Harnack inequalities.

(2) The link between heat kernel estimates and isoperimetric inequalities. The isodiametric problem in the Heisenberg group.
(3) Applications: the L^p-boundedness of the Riesz transform $\nabla\Delta^{-1/2}$, properties of Sobolev spaces on Lie groups and Riemannian manifolds.

References.

- A. Grigor'yan. *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, vol 47, 2009.
- P. Auscher, T. Coulhon, X. T. Duong, S. Hofmann. *Riesz transforms on manifolds and heat kernel regularity*, *Ann. Sci. Ecole Norm. Sup.*, 37, 6, 911-957, 2004.
- F. Bernicot, T. Coulhon, D. Frey. *Gaussian heat kernel bounds through elliptic Moser iteration*, *J. Math. Pures Appl.* 106, 6, 995-1037, 2016.
- T. Coulhon, E. Russ, V. Tardivel-Nachef. *Sobolev algebras on Lie groups and Riemannian manifolds*, *Amer. J. Math.* 123, 2, 283-342, 2001.
- N. Th. Varopoulos, L. Saloff-Coste, T. Coulhon. *Analysis and Geometry on groups*, Cambridge Tracts in Mathematics 100, Cambridge University Press, 1992.

Prerequisites. The basic notions in Riemannian and sub-Riemannian geometry seen in Courses **Riemannian geometry and optimal transportation** and **Sub-Riemannian geometry and optimal control theory**.

5. Random Geometry (D. Gayet)

If we choose at random a real polynomial in one variable and with fixed degree, how many real roots does one expect, statistically speaking? For instance, what is the mean number of these real roots? If the polynomial is in two variables, the vanishing locus is generically a certain number of connected plane curves (the nodal curves). Can we say something about this number, again statistically? Can we estimate the mean total length of this locus in a fixed bounded open set? If the polynomial is in three variables, can we say something about the topology of its vanishing locus, which is now a sum of real connected surfaces? On the real plane, what is the probability that there exists a long connected nodal line? This course is an introduction to this kind questions of *random geometry*, which is a very active research field. The methods needed come from **topology**, **Riemannian geometry** and **probability**.

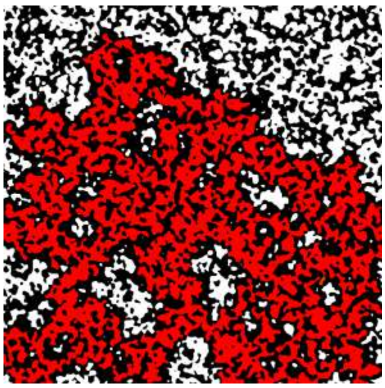

Figure 5. Left: nodal lines of a random polynomial. Right: a percolating nodal domain (in red) of a random analytic function. Pictures by V. Beffara.

Program.

(1) Gaussian fields
(2) Kac-Rice formula
(3) Mean length of random nodal lines
(4) Mean number of connected components
(5) Nodal lines and percolation

References.

- R. J. Adler and J. E. Taylor, *Random fields and geometry*, New York, NY: Springer, 2007.
- K. S. Alexander, *Boundedness of level lines for two-dimensional random fields*, Ann. Probab., 24 (1996), pp. 1653–1674.

- J.-M. Azaïs and M. Wschebor, *Level sets and extrema of random processes and fields*, Hoboken, NJ: John Wiley & Sons, 2009.
- E. Kostlan, *On the distribution of roots of random polynomials*, in From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990), Springer, New York, 1993, pp. 419–431.
- F. Nazarov and M. Sodin, *On the number of nodal domains of random spherical harmonics*, Amer. J. Math., 131 (2009), pp. 1337–1357.

Prerequisites. The course in **Riemannian geometry** and a course in probability (L3 level).

Supplementary possible courses

This advanced course will be activated only if a reasonable number of students is interested and under positive advise of the administration.

6. Introduction to Mathematical General Relativity (D. Hafner)

Although the Einstein equations describing General Relativity are now more than 100 years old, only little is known about the long time behavior of their solutions. Solutions are Lorentzian manifolds of dimension 4. Initial data are Riemannian manifolds of dimension 3 together with a symmetric 2-tensor which have to fulfill the so called constraint equations. Since the fundamental result of Choquet-Bruhat from 1952 we know that the Einstein equations have local solutions for sufficiently smooth data which fulfill the constraint equations. The study of the long time behavior of such solutions turns out to be very difficult. Of particular interest is the question of stability of black hole type solutions of the Einstein equations. Such a stability result has been shown in 2016 by Hintz and Vasy for positive cosmological constant, but the question is still open for zero cosmological constant. The lectures give an introduction to the mathematical theory of the Einstein equations.

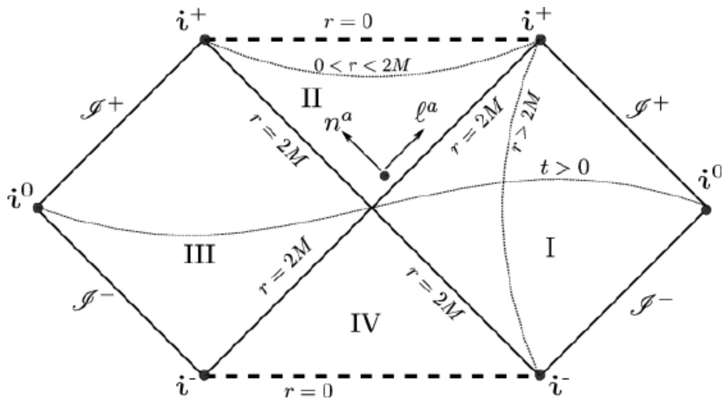

Figure 6. Penrose diagram for a Schwarzschild black hole.

Program.

(1) Introduction to the Einstein equations
(2) Constraint equations
(3) Local existence for the Einstein equations

(4) Explicit solutions of the Einstein equations : Minkowski, Schwarzschild and Kerr spacetimes
(5) Main conjectures in mathematical general relativity

References.

- S. Alinhac, *Hyperbolic partial differential equations*, Springer 2009.
- Yvonne Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford mathematical Monographs 2008.
- Alan Rendall, *Partial differential equations in General Relativity*, Oxford University Press 2008.
- H. Ringström, *The Cauchy problem in General Relativity*, European Mathematical Society 2009.

Prerequisites. The course in **Riemannian geometry**.

Contacts and more information

For more information, send an e-mail to Andrea Pulita, who is the responsible of the Master 2 degree, at the following address:

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For more information or questions about the courses, it is possible to contact the reference persons at the addresses below:

- *Sub-Riemannian geometry and optimal control theory*
  - Grégoire Charlot: gregoire.charlot@univ-grenoble-alpes.fr
  - Luca Rizzi: luca.rizzi@univ-grenoble-alpes.fr
- *Riemannian geometry and optimal transport*
  - Hervé Pajot: herve.pajot@univ-grenoble-alpes.fr
- *Introduction to contact and symplectic geometry*
  - Sylvain Courte: sylvain.courte@univ-grenoble-alpes.fr
  - Stéphane Guillermou: stephane.guillermou@univ-grenoble-alpes.fr
- *Geometric analysis on Riemannian manifolds and Lie groups*
  - Emmanuel Russ: emmanuel.russ@univ-grenoble-alpes.fr
- *Random geometry*
  - Damien Gayet: damien.gayet@univ-grenoble-alpes.fr
- *Introduction to Mathematical General Relativity*
  - D. Hafner: dietrich.hafner@univ-grenoble-alpes.fr