# A short presentation of the course *Algebraic number theory and Galois theory*

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'Number theory and Geometry'
UGA

# Topics

- Part I: Galois theory;

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- Part II: Introduction to $p$-adic fields.

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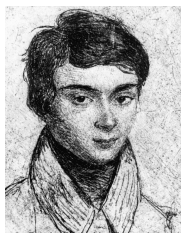

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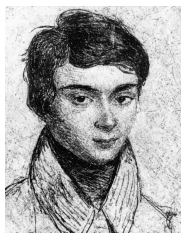

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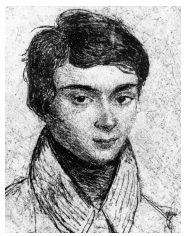

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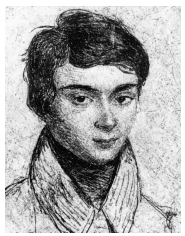

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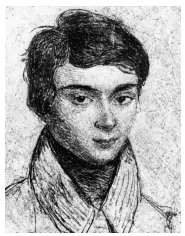

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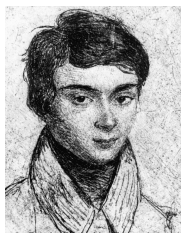

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- (If time) Some elements of Galois theory for infinite extensions

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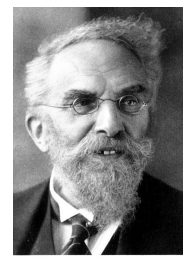

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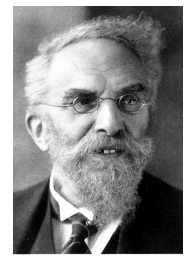


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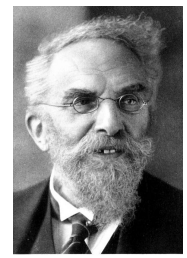


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# Part II: $p$-adic fields - Introduction

**$p$-adic fields:** Finite extensions of $\mathbb{Q}_p$.

- $K/\mathbb{Q}$ number field
- $\mathfrak{p}$ prime ideal of $K$ above $p$
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