Introduction to Linear Algebraic Groups

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Presentation of the main theme of this year

- Classical Galois theory provides a correspondence

$$\{\textbf{finite field extensions}\} \longleftrightarrow \{\textbf{finite groups}\}$$

- Differential Galois theory (also called Picard-Vessiot theory) provides a similar correspondence

$$\left\{\begin{matrix}\text{extensions of "}\textbf{fields with a}\\ \textbf{given derivation}\text{"}\end{matrix}\right\} \longleftrightarrow \{\textbf{linear algebraic groups}\}$$

- Roughly speaking a linear algebraic group is a subgroup of a group of **matrices** which is defined by **Algebraic conditions**. In other words the group is an *Algebraic Variety* or more generally a *Scheme*.

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Let's begin with a more specific description ...

Introduction to Linear Aglebraic Groups

- Linear algebraic groups form a broad generalization of the classical groups.
- They show up in various domains of mathematics such as
  - algebra (in particular, differential Galois theory),
  - algebraic geometry (in particular, classification problems),
  - number theory (in particular, arithmetic groups),...
- Among the courses of the first semester, this is probably the one that may provide you with the widest spectra of specializations into different theories and possible applications.

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As mentioned Linear Algebraic Groups are sub-groups of the group of square matrices.

- Examples of linear algebraic groups are
  - $\mathbb{G}_a$;
  - $\mathbb{G}_m$;
  - GL_n;
  - Upper triangular matrices.

  How do they show up ?
- As GL_n acts naturally on the several structures (for instance vector spaces or affine spaces), usually linear algebraic groups arises as group of transformations of a sub-structure.

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- The first part of the course is an introduction to **Algebraic Geometry** illustrated by many examples.

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More specifically, the contents of the course are as follows:

1. Commutative algebra and affine algebraic geometry: Hilbert's Nullstellensatz, Zariski topology, affine varieties, morphisms, products.
2. Basic properties of algebraic groups: examples, actions, orbits, the neutral component. Linear algebraic groups: Jordan decomposition, representations.
3. Commutative algebraic groups: structure, diagonalizable groups, tori, unipotent groups.
4. Lie algebras: derivations, differentials, smooth algebraic varieties. Lie algebras of linear algebraic groups.
5. Homogeneous spaces under linear algebraic groups: open morphisms, smooth morphisms, normality. Homogeneous spaces and quotients.
6. Connected solvable groups: structure, Borel's fixed point theorem. Borel subgroups and maximal tori of linear algebraic groups.
7. Reductive groups, semi-simple groups: structure theory (if there is enough time left!)

Our main reference will be the book

- T. A. Springer, *Linear algebraic groups. Second edition*, Progress in Mathematics **9**, Birkhauser, 1998.

Indeed, we will follow its approach of introducing notions of algebraic geometry when they are needed for developing algebraic group theory.

Further relevant references are the books

- A. Borel, *Linear algebraic groups. Second enlarged edition*, Graduate Texts in Mathematics **126**, Springer-Verlag, 1991
- J. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer-Verlag, 1991
- A. Onishchik, E. B. Vinberg, *Lie groups and algebraic groups*, Springer-Verlag, 1990

and the notes (available on the author's web page)

- T. Szamuely, *Lectures on algebraic groups*.