Threshold Saturation for Nonbinary Spatially-Coupled LDPC Codes on the Binary Erasure Channel

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Spatially-Coupled LDPC (SC-LDPC) codes are codes with a low-density parity-check matrix, the non-zero elements in which are located in the diagonal band [FZ99].

(Iterative) Belief Propagation (BP) algorithm is used for decoding.

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SC-LDPC codes show outstanding asymptotic performance for lots of channels and communication problems.

For the binary erasure channel (BEC), the BP threshold of a binary SC-LDPC ensemble achieves the optimal MAP threshold of the underlying LDPC ensemble (threshold saturation) [KRU11]. Extended to BMS channels [KMRU10].


Background

- An alternative proof technique based on potential functions for coupled scalar and vector recursions [YJNP12].


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- DE (BP decoding, vector recursion, BEC(ε)):

\[ x^{(\ell+1)} = f(g(x^{(\ell)}); \varepsilon), \]

where \( f(y) \) and \( g(x) \) are the variable and check node updates.

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Vector admissible system

- The functions \( f(y; \varepsilon) \) and \( g(x) \) non-decreasing in \( y \) and \( x \);
- \( f(y; \varepsilon) \) is differentiable in \( y \) and \( g(x) \) is twice differentiable in \( x \);
- \( f(0; \varepsilon) = f(y; 0) = g(0) = 0 \);
- \( f(y; \varepsilon) \) is strictly increasing with \( \varepsilon \).


Background

▶ Potential function [YJNP12b]:

\[ U(x; \varepsilon) = g(x)Dx^T - G(x) - F(g(x); \varepsilon), \]

where \( F \) and \( G \) are scalar functions that satisfy \( F'(y; \varepsilon) = f(y; \varepsilon)D \), and \( G'(x) = g(x)D \), with \( D \) a positive diagonal matrix.

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The BP threshold: $\epsilon^{\text{BP}} = \sup_{\epsilon} (\epsilon \in [0, 1] \mid U'(x; \epsilon) > 0, \forall x \in X).$
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- The BP threshold: \( \varepsilon^{BP} = \sup_{\varepsilon} (\varepsilon \in [0, 1] \mid U'(x; \varepsilon) > 0, \ \forall x \in X) \).

- The potential threshold: \( \varepsilon^* = \sup_{\varepsilon} (\varepsilon \in (\varepsilon^{BP}, 1] \mid \Delta E(\varepsilon) \geq 0, \ \forall x \in X) \),

where \( \Delta E(\varepsilon) = \inf_{x \in X \setminus U_0(\varepsilon)} U(x; \varepsilon) \) for some \( \varepsilon \), \( \varepsilon^{BP} \leq \varepsilon \leq \varepsilon^* \), and \( U_0(\varepsilon) = \{x \in X \mid x^\infty = 0\} \) is the basin of attraction for \( x^\infty = 0 \).

Proof of threshold saturation [YJNP12b]:

1. Define $U(x; \varepsilon)$ for the (uncoupled) vector recursion.
2. Derive the potential function for the coupled vector recursion.
3. Show that, below potential threshold $1$, the only fixed point of the DE is $x_\infty = 0$.

For several systems the MAP threshold and the potential threshold $\varepsilon^*$ are identical.

[Proved threshold saturation of irregular LDPC codes for a Slepian-Wolf problem with erasures, joint decoding of irregular LDPC codes on an erasure multiple-access channel, and protograph codes on the BEC.]

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Motivation

- Threshold saturation for binary SC-LDPC codes so far.

Figure: BP EXIT functions for nonbinary (3,6) SC-LDPC code ensembles over $\mathbb{F}_{2^m}$.


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- Nonbinary SC-LDPC codes:
  - Construction method for nonbinary SC-LDPC codes [UKS11].
  - The threshold saturation also occurs for nonbinary SC-LDPC codes. Contrary to uncoupled ensembles, the BP threshold of nonbinary SC-LDPC codes improves with field size and tends to the Shannon limit [PGiA13].

![BP EXIT functions for nonbinary (3,6) SC-LDPC code ensembles over GF$_m^2$.](image)

**Figure**: BP EXIT functions for nonbinary (3,6) SC-LDPC code ensembles over GF$_m^2$.


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In this talk

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- Existence and computation of \(U(x; \varepsilon)\): Can be obtained by finding the functions \(F\) and \(G\) as the solution of a system of linear equations.
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- A necessary condition on the existence of \(F\) and \(G\) (thus \(U(x; \varepsilon)\)), assuming diagonal \(D \implies \text{Not verified for nonbinary codes!}\)
- A symmetric \(D\) is sufficient so that \(F\) and \(G\) exist.
- Proof of threshold saturation.
DE for \((d_v, d_c, m)\) LDPC code ensembles on the BEC

- Code symbols \(\alpha_0, \ldots, \alpha_{2^m-1} \in S = \text{GF}_2^m\)
Code symbols $\alpha_0, \ldots, \alpha_{2^m-1} \in S = \text{GF}_2^m$ \implies The messages exchanged in the BP decoding are probability vectors of length $2^m$, $(p_0, \ldots, p_{2^m-1})$, where $p_i$ is the probability that the symbol is $\alpha_i$. 
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- A message $(p_0, \ldots, p_{2^m-1})$ has dimension $k$ if it has $2^k$ non-zero entries, i.e., the symbol is known to be one out of $2^k$ possible symbols.
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- For the BEC the non-zero entries of a message are all equal, and the message is equivalent to a subspace of \(\mathbb{GF}_2^m\) of dimension \(k\).
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- DE simplifies to exchange of probability vectors of length \(m + 1\), \((\tilde{p}_0, \ldots, \tilde{p}_m)\), where \(\tilde{p}_i\) is the probability that the message has dimension \(i\).
DE for \((d_v, d_c, m)\) LDPC code ensembles on the BEC

- \(x^\ell_\circ = (x^\ell_\circ 0, \ldots, x^\ell_\circ m)\); \(x^\ell_\circ i\) is the probability that a message from variable nodes at iteration \(\ell\) has dimension \(i\).
- \(y^\ell_\circ = (y^\ell_\circ 0, \ldots, y^\ell_\circ m)\); \(y^\ell_\circ i\) is the probability that a message from check nodes at iteration \(\ell\) has dimension \(i\).
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- At check nodes, the BP decoder computes the check node updates (sum of the subspaces corresponding to the incoming messages),

\[
y_\circ^\ell = g_\circ(x_\circ^{\ell-1}).
\]
DE for \((d_v, d_c, m)\) LDPC code ensembles on the BEC

- \(x^\ell_o = (x^\ell_{o0}, \ldots, x^\ell_{om})\); \(x^\ell_{oi}\) is the probability that a message from variable nodes at iteration \(\ell\) has dimension \(i\).

- \(y^\ell_o = (y^\ell_{o0}, \ldots, y^\ell_{om})\); \(y^\ell_{oi}\) is the probability that a message from check nodes at iteration \(\ell\) has dimension dimension \(i\).

- At check nodes, the BP decoder computes the check node updates (sum of the subspaces corresponding to the incoming messages),
  \[y^\ell_o = g_{o}(x^\ell_{o}^{-1}).\]

- At variable nodes, the decoder computes the variable node updates (intersection of the subspaces of the incoming messages), corresponding to the incoming messages,
  \[x^\ell_o = f_{o}(y^\ell_o; \varepsilon).\]
The fixed-point DE equation for $x_\circ = x_\infty$ is

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- The fixed-point DE equation for \(x_0 = x_\infty\) is

\[ x_0 = f_0(g_0(x_0); \varepsilon). \]

Decoding is successful when it converges to \(x_\infty = (1, 0, \ldots, 0)\).

- \(f_0\) and \(g_0\) are not monotone! \(\implies\) Not an admissible system.
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- \(f_\circ\) and \(g_\circ\) are not monotone! \(\implies\) Not an admissible system.
- Does DE for nonbinary LDPC codes converge to a fixed point?
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- Does DE for nonbinary LDPC codes converge to a fixed point?

**Idea**

Rewrite the DE equation using complementary cumulative distribution function (CCDF) vectors.
Definition

Given a probability vector \( x_o \), define the CCDF vector \( x = (x_1, \ldots, x_m) \), where
\[
x_i = \sum_{k=i}^{m} x_{ok}.
\]
We also define \( x_{m+1} = 0 \). Then, it follows that
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x_{oi} = x_i - x_{i+1}.
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Note also that \( x_0 = 1 \).
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- We define new vector functions \(f = (f_0, \ldots, f_m)\) and \(g = (g_0, \ldots, g_m)\),

\[
\begin{align*}
    f_i &= \sum_{k=i}^{m} f_{\circ k}(y_\circ; \varepsilon) = \sum_{k=i}^{m} f_{\circ k}(y^{-1} - y; \varepsilon), \\
g_i &= \sum_{k=i}^{m} g_{\circ k}(x_\circ) = \sum_{k=i}^{m} g_{\circ k}(x^{-1} - x),
\end{align*}
\]

where \(x^{-1} = (1, x_1, \ldots, x_{m-1})\).
Using CCDF vectors, the DE equation can be written in an equivalent form as

\[ x = f(g(x); \varepsilon). \]  \hspace{1cm} (1)
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**Theorem 1**
The functions \( f(y; \varepsilon) \) and \( g(x) \) are increasing in \( y \) and \( x \).
Convergence of the DE for nonbinary LDPC codes

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Corollary
The DE for regular nonbinary LDPC codes given by (1) converges to a fixed point.
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The DE for regular nonbinary LDPC codes given by (1) converges to a fixed point.

**Successful decoding**
Successful decoding corresponds to convergence of the DE equation to the fixed point \( x^\infty = 0 = (0, 0, \ldots, 0) \).
Potential Function

$x = f(g(x); \varepsilon)$ is a vector admissible system.
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$\mathbf{x} = f(\mathbf{g}(\mathbf{x}); \varepsilon)$ is a vector admissible system. Can define a potential function

$$U(\mathbf{x}; \varepsilon) = g(\mathbf{x}) \mathbf{D} \mathbf{x}^T - G(\mathbf{x}) - F(g(\mathbf{x}); \varepsilon).$$
\( x = f(g(x); \varepsilon) \) is a vector admissible system. Can define a potential function

\[
U(x; \varepsilon) = g(x)Dx^T - G(x) - F(g(x); \varepsilon).
\]

However...

\( U(x; \varepsilon) \) with diagonal \( D \) does not exist! \( \implies \) Conditions on the existence of \( U(x; \varepsilon) \).
Properties of $D$ and calculation of $U(x; \varepsilon)$

\[ U(x; \varepsilon) = g(x)Dx^\top - G(x) - F(g(x); \varepsilon) \]
Properties of $D$ and calculation of $U(x; \varepsilon)$

$$U(x; \varepsilon) = g(x) D x^\top - G(x) - F(g(x); \varepsilon)$$

Theorem 4 [for the $(d_v, d_c, m)$ ensemble]

$F(y; \varepsilon)$ and $G(x)$ exist (hence $U(x; \varepsilon)$ exists) if there exist sets of values $\{d_{js}\}$, $\{\varphi(i_1, \ldots, i_m)\}$ and $\{\mu(k_1, \ldots, k_m)\}$ that satisfy the following equations,

$$
\begin{align*}
\dot{i}_s \varphi(i_1, \ldots, i_s, \ldots, i_m) &= \sum_{j=1}^{m} d_{js} \varphi(j)(i_1, \ldots, i_{s-1}, \ldots, i_m)(\varepsilon) \\
\dot{k}_t \mu(k_1, \ldots, k_t, \ldots, k_m) &= \sum_{j=1}^{m} d_{jt} \gamma(j)(k_1, \ldots, k_t-1, \ldots, k_m)
\end{align*}
$$

(2)

for all possible $m$-tuples $(i_1, \ldots, i_m)$ and $(k_1, \ldots, k_m)$ and all $i_s$ and $k_t$ varying from 1 to $m$. The coefficients $\varphi$’s and $\gamma$’s in (2) are given by

$$
\begin{align*}
\varphi(j)(i_1, \ldots, i_m)(\varepsilon) &= \text{coeff}(f_j(x; \varepsilon), x_1^{i_1} \cdots x_m^{i_m}) \\
\gamma(j)(k_1, \ldots, k_m) &= \text{coeff}(g_j(x), x_1^{i_1} \cdots x_m^{i_m}).
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Properties of $D$ and calculation of $U(x; \varepsilon)$

$$U(x; \varepsilon) = g(x)Dx^T - G(x) - F(g(x); \varepsilon)$$

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\dot{i}_s \varphi(i_1, \ldots, i_s, \ldots, i_m) &= \sum_{j=1}^{m} d_{js} \varphi^{(j)}(i_1, \ldots, i_s-1, \ldots, i_m)(\varepsilon) \\
\dot{k}_t \mu(k_1, \ldots, k_t, \ldots, k_m) &= \sum_{j=1}^{m} d_{jt} \gamma^{(j)}(k_1, \ldots, k_t-1, \ldots k_m)
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\varphi^{(j)}(i_1, \ldots, i_m)(\varepsilon) = \text{coeff}(f_j(x; \varepsilon), x_1^{i_1} \cdots x_m^{i_m}),
$$

$$
\gamma^{(j)}(k_1, \ldots, k_m) = \text{coeff}(g_j(x), x_1^{i_1} \cdots x_m^{i_m}).
$$

Similar result for the coupled case. General result for coupled vector systems.
Properties of $D$ and calculation of $U(x; \varepsilon)$

Theorem 5 [Existence of $U(x; \varepsilon)$ with diagonal $D$]

A solution to (2) exists if, for all $i$ from 1 to $m$

$$
(i_1, \ldots, i_m) \in S_i^f
\Leftrightarrow (i_1, \ldots, i_i + 1, \ldots, i_j - 1, \ldots, i_m) \in S_j^f
$$

(3)

and

$$
(i_1, \ldots, i_m) \in S_i^g
\Leftrightarrow (i_1, \ldots, i_i + 1, \ldots, i_j - 1, \ldots, i_m) \in S_j^g
$$

(4)

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Not satisfied for nonbinary LDPC codes!

The potential function does not exist for diagonal $D$. 

Properties of $D$ and calculation of $U(x; \varepsilon)$

Theorem 5 [Existence of $U(x; \varepsilon)$ with diagonal $D$]

A solution to (2) exists if, for all $i$ from 1 to $m$

$$(i_1, \ldots, i_m) \in S^f_i$$

$\Leftrightarrow (i_1, \ldots, i_i + 1, \ldots, i_j - 1, \ldots, i_m) \in S^f_j$$

and

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1. Define $U(x; \varepsilon) = g(x)Dx^\top - G(x) - F(g(x); \varepsilon)$ with symmetric $D$ for the uncoupled system.
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Theorem 3 [Threshold saturation]
Given the spatially coupled $(d_v, d_c, m, L, w)$ LDPC code ensemble, for $\varepsilon < \varepsilon^*$ and $w > \frac{mK}{2\Delta E(\varepsilon)}$, the only fixed point of the system is $x^\infty = 0$. 
Conclusion

- **Threshold saturation** occurs for nonbinary SC-LDPC codes on the BEC!
- **Monotonicity of the variable node and check node updates** for nonbinary LDPC codes, and the existence of a fixed point in the DE.
- Existence and computation of $U(x; \varepsilon)$: Solution of a system of linear equations.
- A necessary condition on the existence of $F$ and $G$ (thus $U(x; \varepsilon)$), assuming diagonal $D$.
- A symmetric $D$ is sufficient for $U(x; \varepsilon)$ exist for nonbinary codes.