Representations of quivers

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Abstract. We give an introduction to the theory of quiver representations, in its algebraic and geometric aspects. The main result is Gabriel's theorem that characterizes quivers of finite representation type.

Résumé. Nous donnons une introduction à la théorie des représentations des carquois, sous ses aspects algébrique et géométrique. Le résultat principal est le théorème de Gabriel qui caractérise les carquois de type de représentation fini.

Introduction

Quivers are very simple mathematical objects: finite directed graphs. A representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow. Quiver representations were originally introduced to treat problems of linear algebra, for example, the classification of tuples of subspaces of a prescribed vector space. But it soon turned out that quivers and their representations play an important role in representation theory of finite-dimensional algebras; they also occur in less expected domains of mathematics including Kac-Moody Lie algebras, quantum groups, Coxeter groups, and geometric invariant theory.

These notes present some fundamental results and examples of quiver representations, in their algebraic and geometric aspects. Our main goal is to give an account of a theorem of Gabriel characterizing quivers of finite orbit type, that is, having only finitely many isomorphism classes of representations in any prescribed dimension: such quivers are exactly the disjoint unions of Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8, equipped with arbitrary orientations. Moreover, the isomorphism classes of indecomposable representations correspond bijectively to the positive roots of the associated root system.

This beautiful result has many applications to problems of linear algebra. For example, when applied to an appropriate quiver of type D_4, it yields a classification of triples of subspaces of a prescribed vector space, by finitely many combinatorial invariants. The

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corresponding classification for quadruples of subspaces involves one-parameter families (the so-called tame case); for r-tuples with $r \geq 5$, one obtains families depending on an arbitrary number of parameters (the wild case).

Gabriel's theorem holds over an arbitrary field; in these notes, we only consider algebraically closed fields, in order to keep the prerequisites at a minimum. Section 1 is devoted to the algebraic aspects of quiver representations; it requires very little background. The geometric aspects are considered in Section 2, where familiarity with some affine algebraic geometry is assumed. Section 3, on representations of finitely generated algebras, is a bit more advanced, as it uses (and illustrates) basic notions of affine schemes. The reader will find more detailed outlines, prerequisites, and suggestions for further reading, at the beginning of each section.

Many important developments of quiver representations fall beyond the limited scope of these notes; among them, we mention Kac's far-reaching generalization of Gabriel's theorem (exposed in [11]), and the construction and study of moduli spaces (surveyed in the notes of Ginzburg, see also [17]).

Conventions. Throughout these notes, we consider vector spaces, linear maps, algebras, over a fixed field k, assumed to be algebraically closed. All algebras are assumed to be associative, with unit; modules are understood to be left modules, unless otherwise stated.

1 Quiver representations: the algebraic approach

In this section, we present fundamental notions and results on representations of quivers and of finite-dimensional algebras.

Basic definitions concerning quivers and their representations are formulated in Subsection 1.1, and illustrated on three classes of examples. In particular, we define quivers of finite orbit type, and state their characterization in terms of Dynkin diagrams (Gabriel's theorem).

In Subsection 1.2, we define the quiver algebra, and identify its representations with those of the quiver. We also briefly consider quivers with relations.

The classes of simple, indecomposable, and projective representations are discussed in Subsection 1.3, in the general setting of representations of algebras. We illustrate these notions with results and examples from quiver algebras.

Subsection 1.4 is devoted to the standard resolutions of quiver representations, with applications to extensions and to the Euler and Tits forms.

The prerequisites are quite modest: basic material on rings and modules in Subsections 1.1-1.3; some homological algebra (projective resolutions, Ext groups, extensions) in Subsection 1.4.

We generally provide complete proofs, with the exception of some classical results for which we refer to [3]. Thereby, we make only the first steps in the representation theory of quivers and finite-dimensional algebras. The reader will find more complete expositions in the books [1, 2, 3] and in the notes [5]; the article [6] gives a nice overview of the subject.

1.1 Basic definitions and examples

DEFINITION 1.1.1. A *quiver* is a finite directed graph, possibly with multiple arrows and loops. More specifically, a quiver is a quadruple

$$Q = (Q_0, Q_1, s, t),$$

where Q_0, Q_1 are finite sets (the set of *vertices*, resp. *arrows*) and

$$s, t : Q_1 \longrightarrow Q_0$$

are maps assigning to each arrow its *source*, resp. *target*.

We shall denote the vertices by letters $i, j, \ldots$. An arrow with source i and target j will be denoted by $\alpha : i \to j$, or by $i \xrightarrow{\alpha} j$ when depicting the quiver.

For example, the quiver with vertices i, j and arrows $\alpha : i \to j$ and $\beta_1, \beta_2 : j \to j$ is depicted as follows:

$$\begin{tikzcd} i \arrow[r, "\alpha"] & j \arrow[loop above, "\beta_1"] \arrow[loop below, "\beta_2"] \end{tikzcd}$$

DEFINITION 1.1.2. A *representation* M of a quiver Q consists of a family of vector spaces V_i indexed by the vertices $i \in Q_0$, together with a family of linear maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ indexed by the arrows $\alpha \in Q_1$.

For example, a representation of the preceding quiver is just a diagram

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[loop above, "g_1"] \arrow[loop below, "g_2"] \end{tikzcd}$$

where V, W are vector spaces, and f, g_1, g_2 are linear maps.

DEFINITION 1.1.3. Given two representations $M = \big((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1}\big)$, $N = (W_i, g_\alpha)$ of a quiver Q, a *morphism* $u : M \to N$ is a family of linear maps $(u_i : V_i \to W_i)_{i \in Q_0}$ such that the diagram

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{f_\alpha} & V_{t(\alpha)} \\ {\scriptstyle u_{s(\alpha)}}\big\downarrow & & {\scriptstyle u_{t(\alpha)}}\big\downarrow \\ W_{s(\alpha)} & \xrightarrow{g_\alpha} & W_{t(\alpha)} \end{array}$$

commutes for any $\alpha \in Q_1$.

For any two morphisms $u : M \to N$ and $v : N \to P$, the family of compositions $(v_i u_i)_{i \in Q_0}$ is a morphism $vu : M \to P$. This defines the composition of morphisms, which is clearly associative and has identity element $\mathrm{id}_M := (\mathrm{id}_{V_i})_{i \in Q_0}$. So we may consider the *category of representations of* Q, that we denote by $\mathrm{Rep}(Q)$.

Given two representations M, N as above, the set of all morphisms (of representations) from M to N is a subspace of $\prod_{i \in Q_0} \mathrm{Hom}(V_i, W_i)$; we denote that subspace by $\mathrm{Hom}_Q(M, N)$. If $M = N$, then

$$\mathrm{End}_Q(M) := \mathrm{Hom}_Q(M, M)$$

is a subalgebra of the product algebra $\prod_{i \in Q_0} \mathrm{End}(V_i)$.

Clearly, the composition of morphisms is bilinear; also, we may define direct sums and exact sequences of representations in an obvious way. In fact, one may check that $\mathrm{Rep}(Q)$ *is a k-linear abelian category*; this will also follow from the equivalence of $\mathrm{Rep}(Q)$ with the category of modules over the quiver algebra kQ, see Proposition 1.2.2 below.

Definition 1.1.4. A representation $M = (V_i, f_\alpha)$ of Q is *finite-dimensional* if so are all the vector spaces V_i. Under that assumption, the family

$$\underline{\dim}\, M := (\dim V_i)_{i \in Q_0}$$

is the *dimension vector* of M; it lies in the additive group $\mathbb{Z}^{Q_0}$ consisting of all tuples of integers $\underline{n} = (n_i)_{i \in Q_0}$.

We denote by $(\varepsilon_i)_{i \in Q_0}$ the canonical basis of $\mathbb{Z}^{Q_0}$, so that $\underline{n} = \sum_{i \in Q_0} n_i\, \varepsilon_i$.

Note that every exact sequence of finite-dimensional representations

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

satisfies

$$\underline{\dim}\, M = \underline{\dim}\, M' + \underline{\dim}\, M''.$$

Also, any two isomorphic finite-dimensional representations have the same dimension vector. A central problem of quiver theory is *to describe the isomorphism classes of finite-dimensional representations of a prescribed quiver, having a prescribed dimension vector.*

EXAMPLES 1.1.5. 1) The *loop* is the quiver L having a unique vertex i and a unique arrow α (then $s(\alpha) = t(\alpha) = i$). Thus, a representation of L is a pair (V, f), where V is a vector space and f an endomorphism of V; the dimension vector is just the dimension of V.

A morphism from a pair (V, f) to another pair (W, g) is a linear map $u : V \to W$ such that $uf = gu$. In particular, the endomorphisms of the pair (V, f) are exactly the endomorphisms of V that commute with f.

Given a representation (V, f) having a prescribed dimension n, we may choose a basis $(v_1, \ldots, v_n)$ of V, and hence identify f with an $n \times n$ matrix A. Choosing another basis amounts to replacing A with a conjugate BAB^{-1}, where B is an invertible $n \times n$ matrix. It follows that the isomorphism classes of n-dimensional representations of L correspond bijectively to the conjugacy classes of $n \times n$ matrices. The latter are classified in terms of the Jordan canonical form.

In particular, there are infinitely many isomorphism classes of representations of the loop having a prescribed dimension.

More generally, for any integer $r \geq 1$, the *r-loop* is the quiver L_r having a unique vertex and r arrows $\alpha_1, \ldots, \alpha_r$.

$$L_2 : \qquad \begin{tikzcd} i \arrow[loop left, "\alpha_1"] \arrow[loop right, "\alpha_2"] \end{tikzcd}$$

The representations of L_r consist of a vector space V equipped with r endomorphisms $f_1, \ldots, f_r$. Thus, *the isomorphism classes of representations of L_r having a prescribed dimension (vector) n correspond bijectively to the r-tuples of $n \times n$ matrices up to simultaneous conjugation.*

2) The *r-arrow Kronecker quiver* is the quiver having two vertices i, j and r arrows $\alpha_1, \ldots, \alpha_r : i \to j$. The representations of K_r consist of two vector spaces V, W together with r linear maps $f_1, \ldots, f_r : V \to W$. The dimension vectors are pairs of non-negative integers.

$$K_2 : \qquad \begin{tikzcd} i \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & j \end{tikzcd}$$

As in the preceding example, the isomorphism classes of representations with dimension vector (m, n) correspond bijectively to the r-tuples of $n \times m$ matrices, up to simultaneous multiplication by invertible $n \times n$ matrices on the left, and by invertible $m \times m$ matrices on the right.

When $r = 1$, these representations are classified by the rank of the unique $n \times m$ matrix; in particular, they form only finitely many isomorphism classes.

In the case where $r = 2$, the classification is due (in essence) to Kronecker and is much more involved (see e.g. [3, Thm. 4.3.2]).

When $r \geq 2$, the classification of representations of K_r contains that of L_{r-1} in the following sense. Consider a representation of K_r with dimension vector (n, n), such that the map f_1 is invertible. Choosing appropriate bases of V and W, we may assume that f_1 is the identity of k^n; then $f_2, \ldots, f_r$ are $n \times n$ matrices, uniquely determined up to simultaneous conjugation. As a consequence, such representations of K_r form infinitely many isomorphism classes.

3) We denote by S_r the quiver having $r + 1$ vertices $i_1, \ldots, i_r, j$, and r arrows $\alpha_1, \ldots, \alpha_r$ with sources $i_1, \ldots, i_r$ and common target j.

$$S_4 : \qquad \begin{tikzcd} & i_2 \arrow[d, "\alpha_2"] & \\ i_1 \arrow[r, "\alpha_1"] & j & i_3 \arrow[l, "\alpha_3"'] \\ & i_4 \arrow[u, "\alpha_4"'] & \end{tikzcd}$$

A representation M of S_r consists of $r + 1$ vector spaces $V_1, \ldots, V_r, W$ together with r linear maps $f_i : V_i \to W$. By associating with M the images of the f_i, one obtains a bijection between the isomorphism classes of representations with dimension vector $(m_1, \ldots, m_r, n)$, and the orbits of the general linear group $\mathrm{GL}(n)$ acting on r-tuples $(E_1, \ldots, E_r)$ of subspaces of k^n such that $\dim(E_i) \leq m_i$ for all i, via $g \cdot (E_1, \ldots, E_r) := \big(g(E_1), \ldots, g(E_r)\big)$. In other words, *classifying representations of S_r is equivalent to classifying r-tuples of subspaces of a fixed vector space.*

When $r = 1$, one recovers the classification of representations of $K_1 \simeq S_1$.

When $r = 2$, one easily checks that the pairs of subspaces (E_1, E_2) of k^n are classified by the triples $\big(\dim(E_1), \dim(E_2), \dim(E_1 \cap E_2)\big)$, i.e., by those triples $(a, b, c) \in \mathbb{Z}^3$ such that $0 \leq c \leq \min(a, b)$. In particular, there are only finitely many isomorphism classes of representations having a prescribed dimension vector.

This finiteness property may still be proved in the case where $r = 3$, but fails whenever $r \geq 4$. Consider indeed the representations with dimension vector $(1, 1, \ldots, 1, 2)$, such that the maps $f_1, \ldots, f_r$ are all non-zero. The isomorphism classes of these representations are in bijection with the orbits of the projective linear group $\mathrm{PGL}(2)$ acting on the product $\mathbb{P}^1(k) \times \cdots \times \mathbb{P}^1(k)$ of r copies of the projective line. Since $r \geq 4$, there are infinitely many orbits; for $r = 4$, an explicit infinite family is provided by the representations

$$\begin{tikzcd} & k \arrow[d, "(0,1)"] & \\ k \arrow[r, "(1,0)"] & k^2 & k \arrow[l, "(1,1)"'] \\ & k \arrow[u, "(1,\lambda)"] & \end{tikzcd}$$

where $\lambda \in k$.

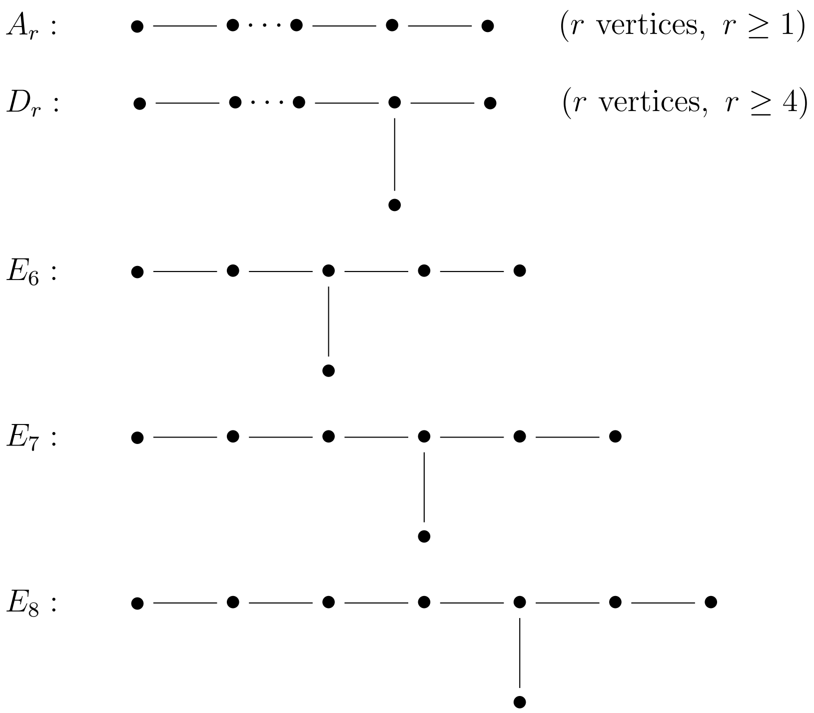
These examples motivate the following:

Definition 1.1.6. A quiver Q is *of finite orbit type* if Q has only finitely many isomorphism classes of representations of any prescribed dimension vector.

A remarkable theorem of Gabriel yields a complete description of these quivers:

Theorem 1.1.7. *A quiver is of finite orbit type if and only if each connected component of its underlying undirected graph is a simply-laced Dynkin diagram.*

Here the simply-laced Dynkin diagrams are those of the following list:



For example, $K_1 = S_1$ has type A_2, whereas S_2 has type A_3, and S_3 has type D_4.

We shall prove the "only if" part of Gabriel's theorem in Subsection 2.1, and the "if" part in Subsection 2.4. For a generalization of that theorem to arbitrary fields (possibly not algebraically closed), see [3, Sec. 4.7].

1.2 The quiver algebra

In this subsection, we fix a quiver $Q = (Q_0, Q_1, s, t)$. To any representation $M = (V_i, f_\alpha)$ of Q, we associate the vector space

$$V := \bigoplus_{i \in Q_0} V_i \tag{1.2.1}$$

equipped with two families of linear self-maps: the projections

$$f_i : V \longrightarrow V \quad (i \in Q_0)$$

(the compositions $V \to V_i \hookrightarrow V$ of the projections with the inclusions), and the maps

$$f_\alpha : V \longrightarrow V \quad (\alpha \in Q_1)$$

obtained similary from the defining maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$. Clearly, these maps satisfy the relations

$$f_i^2 = f_i, \quad f_i f_j = 0 \ (i \neq j), \quad f_{t(\alpha)} f_\alpha = f_\alpha f_{s(\alpha)} = f_\alpha$$

and all other products are 0. This motivates the following:

Definition 1.2.1. The *algebra of the quiver* Q is the (associative) algebra kQ determined by the generators e_i, where $i \in Q_0$, and α, where $\alpha \in Q_1$, and the relations

$$e_i^2 = e_i, \quad e_i e_j = 0 \ (i \neq j), \quad e_{t(\alpha)} \alpha = \alpha e_{s(\alpha)} = \alpha. \tag{1.2.2}$$

In particular, $e_i e_j = 0$ unless $i = j$, so that the e_i are *orthogonal idempotents* of kQ. Also, $\sum_{i \in Q_0} e_i = 1$, since this equality holds after multiplication by any generator. Likewise, $e_i \alpha = 0$ unless $i = t(\alpha)$, and $\alpha e_j = 0$ unless $j = s(\alpha)$.

Proposition 1.2.2. *The category of representations of any quiver Q is equivalent to the category of left kQ-modules.*

Indeed, we have seen that any representation M of Q defines a representation V of kQ. Conversely, any kQ-module V yields a family of vector spaces $(V_i := e_i V)_{i \in Q_0}$, and the decomposition (1.2.1) holds in view of the relations (1.2.2). Moreover, we have a linear map $f_\alpha : V_i \to V_j$ for any arrow $\alpha : i \to j$ (since the image of the multiplication by α in V is contained in V_j, by the relation $\alpha = e_j \alpha$). One may check that these constructions extend to functors, and yield the desired equivalence of categories; see the proof of [2, Thm. II.1.5] for details.

In what follows, we shall freely identify representations of Q with left modules over kQ, and the category $\mathrm{Rep}(Q)$ with the (abelian) category of kQ-modules.

For any arrows α, β, the product $\beta\alpha = \beta e_{s(\beta)}\alpha$ is zero unless $s(\beta) = t(\alpha)$. Thus, a product of arrows $\alpha_\ell \cdots \alpha_1$ is zero unless the sequence $\pi := (\alpha_1, \ldots, \alpha_\ell)$ is a *path*, i.e., $s(\alpha_j) = t(\alpha_{j+1})$ for $j = 1, \ldots, \ell - 1$. We then put $s(\pi) := s(\alpha_1)$ (the *source* of the path π), $t(\pi) := t(\alpha_\ell)$ (the *target* of π), and $\ell(\pi) := \ell$ (the *length*). For any vertex i, we also view e_i as the *path of length* 0 *at the vertex* i.

Clearly, the paths generate the vector space kQ. They also are linearly independent: consider indeed the *path algebra* with basis the set of all paths, and multiplication given by the concatenation of paths. This algebra is generated by the paths of length 0 (the vertices) and of length 1 (the arrows), and satisfies the relations of kQ. Thus, the path algebra is a quotient of kQ, which implies the desired linear independence, and shows that the quiver algebra and the path algebra are in fact the same.

Examples 1.2.3. We describe the path algebras of the quivers considered in Examples 1.1.5, and of an additional class of examples.

1) The algebra of the loop L has basis the monomials α^n, where $n \in \mathbb{N}$. In other words, the algebra kL is freely generated by α.

More generally, the algebra of the r-loop L_r is the free algebra $k\langle X_1, \ldots, X_r\rangle$ on the r arrows. The paths are just the words (or non-commutative monomials) in $X_1, \ldots, X_r$.

2) The algebra of the r-arrow Kronecker quiver K_r has basis $e_i, e_j, \alpha_1, \ldots, \alpha_r$. Thus, kK_r is the direct sum of $k\alpha_1 \oplus \cdots \oplus k\alpha_r$ (a two-sided ideal of square 0), with $ke_i \oplus ke_j$ (a subalgebra isomorphic to $k \times k$).

3) Likewise, kS_r is the direct sum of the two-sided ideal $k\alpha_1 \oplus \cdots \oplus k\alpha_r$ of square 0, with the subalgebra $ke_{i_1} \oplus \cdots \oplus ke_{i_r} \oplus ke_j \simeq k \times \cdots \times k$ ($r + 1$ copies).

4) Let H_r denote the quiver having two vertices i, j, an arrow $\alpha : i \to j$, and r loops $\beta_1, \ldots, \beta_r$ at j (so that H_2 is our very first example). Then kH_r is the direct sum of $k\langle\beta_1, \ldots, \beta_r\rangle\alpha$ (a two-sided ideal of square 0) with $ke_i \oplus k\langle\beta_1, \ldots, \beta_r\rangle$ (a subalgebra isomorphic to $k \times k\langle X_1, \ldots, X_r\rangle$).

Returning to an arbitrary quiver Q, let $kQ_{\geq 1}$ be the linear span in kQ of all paths of positive length. Then $kQ_{\geq 1}$ is the two-sided ideal of kQ generated by all arrows, and we have the decomposition

$$kQ = kQ_{\geq 1} \oplus \bigoplus_{i \in Q_0} ke_i \ , \tag{1.2.3}$$

where $\bigoplus_{i \in Q_0} ke_i$ is a subalgebra isomorphic to the product algebra $\prod_{i \in Q_0} k$. Moreover, for any positive integer n, the ideal $(kQ_{\geq 1})^n$ is the linear span of all paths of length $\geq n$; we shall also denote that ideal by $kQ_{\geq n}$.

Clearly, the vector space kQ is finite-dimensional if and only if Q does not contain any *oriented cycle*, that is, a path π of positive length such that $s(\pi) = t(\pi)$ (the oriented cycles of length 1 are just the loops). Under that assumption, all paths in Q have length at most the number r of vertices. Thus, $(kQ_{\geq 1})^r = \{0\}$. In particular, the ideal $kQ_{\geq 1}$ is nilpotent.

To obtain a more general class of algebras, it is convenient to introduce quivers with relations:

DEFINITION 1.2.4. A *relation* of a quiver Q is a subspace of kQ spanned by linear combinations of paths having a common source and a common target, and of length at least 2.

A *quiver with relations* is a pair (Q, I), where Q is a quiver, and I is a two-sided ideal of kQ generated by relations. The quotient algebra kQ/I is the *path algebra* of (Q, I).

For instance, if Q is the r-loop, then a relation is a subspace of $kQ = k\langle X_1, \ldots, X_r\rangle$ spanned by linear combinations of words of length at least 2. As an example, take the linear span of all the commutators $X_iX_j - X_jX_i$, then the path algebra is just the polynomial algebra $k[X_1, \ldots, X_r]$.

The representations of arbitrary finite-dimensional algebras may be described in terms of quivers with relations. Namely, to any such algebra A, one can associate a quiver with relations (Q, I) such that I contains a power of the ideal generated by the arrows (hence the path algebra kQ/I is finite-dimensional), and $\mathrm{Rep}(A)$ is equivalent to the category $\mathrm{Rep}(Q, I)$ defined in an obvious way. This follows from the results of [3, Sec. 4.1], especially Prop. 4.1.7.

In contrast, finite-dimensional quiver algebras (without relations) satisfy very special properties among all finite-dimensional algebras, as we shall see in Subsection 1.4.

1.3 Structure of representations

In this subsection, we fix an (associative) algebra A and consider (left) A-modules, assumed to be finitely generated. We begin by discussing the *simple* A-modules, also called *irreducible*, i.e., those non-zero modules that have no non-zero proper submodule.

Let M, N be two simple A-modules; then every non-zero A-morphism $f : M \to N$ is an isomorphism by Schur's lemma. As a consequence, $\mathrm{Hom}_A(M, N) = \{0\}$ unless $M \simeq N$; moreover, $\mathrm{End}_A(M)$ is a division algebra. If M is finite-dimensional, then so is $\mathrm{End}_A(M)$; in particular, each $f \in \mathrm{End}_A(M)$ generates a finite-dimensional subfield. Since k is algebraically closed, it follows that $\mathrm{End}_A(M) = k\,\mathrm{id}_M$.

Also, recall that an A-module is *semi-simple* (or *completely reducible*) if it equals the sum of its simple submodules. Any finite-dimensional semi-simple module admits a decomposition of algebras

$$M \simeq \bigoplus_{i=1}^{r} m_i\, M_i\,, \tag{1.3.1}$$

where the M_i are pairwise non-isomorphic simple modules, and the m_i are positive integers. By Schur's lemma, the simple summands M_i and their multiplicities m_i are uniquely determined up to reordering. Moreover, we have a decomposition

$$\operatorname{End}_A(M) \simeq \prod_{i=1}^{r} \operatorname{End}_A(m_i\, M_i) \simeq \prod_{i=1}^{r} \operatorname{Mat}_{m_i \times m_i}\big(\operatorname{End}_A(M_i)\big)$$

and hence

$$\operatorname{End}_A(M) \simeq \prod_{i=1}^{r} \operatorname{Mat}_{m_i \times m_i}(k). \tag{1.3.2}$$

We may apply the decomposition (1.3.2) to an algebra A which is *semi-simple*, i.e., the (left) A-module A is semi-simple; equivalently, every A-module is semi-simple. Indeed, for an arbitrary algebra A, we have an isomorphism of algebras

$$\operatorname{End}_A(A) \xrightarrow{\sim} A^{\mathrm{op}}, \quad f \longmapsto f(1), \tag{1.3.3}$$

where A^{op} denotes the *opposite algebra*, with the order of multiplication being reversed. Moreover, each matrix algebra is isomorphic to its opposite algebra, via the transpose map. It follows that each finite-dimensional semi-simple algebra satisfies

$$A \simeq \prod_{i=1}^{r} \operatorname{Mat}_{m_i \times m_i}(k)\,,$$

where $m_1, \ldots, m_r$ are unique up to reordering; the simple A-modules are exactly the vector spaces k^{m_i}, where A acts via the ith factor.

It is easy to construct simple representations of a quiver $Q = (Q_0, Q_1, s, t)$: given $i \in Q_0$, consider the representation $S(i)$ defined by

$$S(i)_i = k, \qquad S(i)_j = 0 \quad (j \in Q_0,\ j \neq i), \qquad f_\alpha = 0 \quad (\alpha \in Q_1).$$

Clearly, $S(i)$ is simple with dimension vector ε_i (the ith basis vector of $\mathbb{Z}^{Q_0}$). This yields all the simple representations, if kQ is finite-dimensional:

PROPOSITION 1.3.1. *Assume that Q has no oriented cycle. Then any simple representation of Q is isomorphic to $S(i)$ for a unique $i \in Q_0$. Moreover, any finite-dimensional semi-simple representation is uniquely determined by its dimension vector, up to isomorphism.*

Proof. Consider a simple kQ-module M. Then $M \neq kQ_{\geq 1}M$ (otherwise, $M = (kQ_{\geq 1})^n M = kQ_{\geq n}M$ for any positive integer n, and hence $M = \{0\}$). Thus, $kQ_{\geq 1}M = \{0\}$, so that M may be viewed as a module over the algebra

$$kQ/kQ_{\geq 1} \simeq \bigoplus_{i \in Q_0} ke_i \simeq \prod_{i \in Q_0} k.$$

As a consequence, each subspace of $e_i M$ is a kQ-submodule of M. This readily implies the first assertion.

Next, let M be a finite-dimensional semi-simple kQ-module. Then, by the decomposition (1.3.1),

$$M \simeq \bigoplus_{i \in Q_0} m_i \, S(i)$$

for some non-negative integers m_i. Thus,

$$\underline{\dim}\, M = \sum_{i \in Q_0} m_i \, \underline{\dim}\, S(i) = \sum_{i \in Q_0} m_i \, \varepsilon_i.$$

$\square$

In the preceding statement, the assumption that Q has no oriented cycle cannot be omitted, as shown by the following:

Example 1.3.2. The irreducible representations of the loop L are exactly the spaces $S(\lambda) := k[X]/(X - \lambda)\, k[X]$, where $\lambda \in k$, viewed as modules over $kL = k[X]$. Each $S(\lambda)$ is just the vector space k, where the arrow α acts via multiplication by λ.

In contrast, the r-loop L_r, $r \geq 2$, has irreducible representations of arbitrary dimension n; for example, the vector space k^n with standard basis $(v_1, \ldots, v_n)$, where α_1 acts via the 'shift' $v_1 \mapsto v_2$, $v_2 \mapsto v_3$, $\ldots$, $v_n \mapsto v_1$; α_2 acts via $v_1 \mapsto v_2$, $v_j \mapsto 0$ for all $j \geq 2$, and α_i acts trivially for $i \geq 3$.

Next, we consider *indecomposable* modules over an algebra A, i.e., those non-zero modules that have no decomposition into a direct sum of non-zero submodules.

Clearly, an A-module M is indecomposable if and only if the algebra $\mathrm{End}_A(M)$ contains no non-trivial idempotent. Assuming that M is finite-dimensional, we obtain further criteria for indecomposability, analogous to Schur's lemma:

Lemma 1.3.3. *For a finite-dimensional module M over an algebra A, the following conditions are equivalent:*

(i) *M is indecomposable.*

(ii) *Any A-endomorphism of M is either nilpotent or invertible.*

(iii) $\mathrm{End}_A(M) = I \oplus k\, \mathrm{id}_M$*, where I is a nilpotent ideal.*

Proof. Some of the statements, and all the arguments of their proofs, may be found in [3, Sec. 1.4]; we provide details for completeness.

(i)⇒(ii) follows from the Fitting decomposition

$$M = \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n),$$

where $f \in \mathrm{End}_A(M)$ and $n \gg 0$ (see e.g. [3, Lem. 1.4.4]).

(ii)⇒(iii) Denote by I the set of all nilpotent elements of $\mathrm{End}_A(M)$. We first show that I is a two-sided ideal. Consider $x \in I$ and $y \in \mathrm{End}_A(M)$. Then xy is non-invertible in $\mathrm{End}_A(M)$, and hence is nilpotent: $xy \in I$ and likewise, $yx \in I$. If, in addition, $y \in I$, then $x + y \in I$: otherwise, $z := x + y$ is invertible, and hence $x = z - y = z(1 - z^{-1}y)$ is invertible as well, since $z^{-1}y$ is nilpotent.

Next, we show that the ideal I is nilpotent. Since the algebra $\mathrm{End}_A(M)$ is finite-dimensional, and $I^n \supset I^{n+1}$ for all n, there exists a positive integer n such that $I^n = I^{n+1}$. But $1 + x$ is invertible for all $x \in I$, and hence $I^n = \{0\}$ by Nakayama's lemma (see [3, Lem. 1.2.3]).

Finally, $\mathrm{End}_A(M)/I$ is a division algebra, since the complement of I in $\mathrm{End}_A(M)$ consists of invertible elements. On the other hand, the vector space $\mathrm{End}_A(M)/I$ is finite-dimensional; thus, $\mathrm{End}_A(M)/I = k$.

(iii)⇒(i) Consider an idempotent $e \in \mathrm{End}_A(M)$. Then the image of e in the quotient $\mathrm{End}_A(M)/I \simeq k$ is 1, and hence $e = 1 + x$ for some $x \in I$. Thus, e is invertible, and $e = 1$. □

We now obtain an important structure result for finite-dimensional modules and their endomorphism rings:

Theorem 1.3.4. *Let M be a finite-dimensional module over an algebra A. Then there is a decomposition of A-modules*

$$M \simeq \bigoplus_{i=1}^{r} m_i\, M_i, \tag{1.3.4}$$

where $M_1, \ldots, M_r$ are indecomposable and pairwise non-isomorphic, and $m_1, \ldots, m_r$ are positive integers. Moreover, the indecomposable summands M_i and their multiplicities m_i are uniquely determined up to reordering.

Finally, we have a decomposition of vector spaces

$$\mathrm{End}_A(M) = I \oplus B \tag{1.3.5}$$

where I is a nilpotent ideal, and B is a subalgebra isomorphic to $\prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$.

Proof. The first assertion is the classical Krull-Schmidt theorem, proved e.g. in [3, Sec. 1.4].

The second assertion follows from Lemma 1.3.3 (iii), in the case where M is indecomposable. In the general case, let $f \in \mathrm{End}_A(M)$ and consider the compositions

$$f_{ij} : m_j\, M_j \hookrightarrow M \xrightarrow{f} M \longrightarrow M/\bigoplus_{\ell \neq i} m_\ell\, M_\ell \xrightarrow{\sim} m_i\, M_i \quad (i,j = 1,\ldots,r).$$

Then we have the "block decomposition" $f = \sum_{i,j} f_{i,j}$, where

$$f_{i,j} \in \mathrm{Hom}_A(m_j\, M_j, m_i\, M_i) \simeq \mathrm{Mat}_{m_i \times m_j}\big(\mathrm{Hom}_A(M_j, M_i)\big).$$

In particular,

$$f_{ii} \in \mathrm{End}_A(m_i\, M_i) \simeq \mathrm{Mat}_{m_i \times m_i}\big(\mathrm{End}_A(M_i)\big).$$

By Lemma 1.3.3, we have a decomposition $\mathrm{End}_A(M_i) = I_i \oplus k\, \mathrm{id}_{M_i}$, where I_i is a nilpotent ideal. This induces a homomorphism $\mathrm{End}_A(M_i) \to k$ and, in turn, a homomorphism

$$u_i : \mathrm{Mat}_{m_i \times m_i}\big(\mathrm{End}_A(M_i)\big) \longrightarrow \mathrm{Mat}_{m_i \times m_i}(k).$$

Consider the linear map

$$u : \mathrm{End}_A(M) \longrightarrow \prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k), \quad f = \sum_{i,j} f_{i,j} \longmapsto \big(u_1(f_{11}), \ldots, u_r(f_{rr})\big).$$

Clearly, u is split surjective via the natural inclusions

$$\mathrm{Mat}_{m_i \times m_i}(k) \hookrightarrow \mathrm{Mat}_{m_i \times m_i}\big(\mathrm{End}_A(M_i)\big) = \mathrm{End}_A(m_i\, M_i) \hookrightarrow \mathrm{End}_A(M).$$

We claim that u is an algebra homomorphism. Since $(gf)_{ii} = \sum_j g_{ij}\, f_{ji}$ for all $f, g \in \mathrm{End}_A(M)$, it suffices to check that $u_i(g_{ij}\, f_{ji}) = 0$ whenever $i \neq j$. For this, we may assume that $m_i = m_j = 1$; we then have to show that $gf \in I_j$ for any morphisms $f : M_j \to M_i$ and $g : M_i \to M_j$. But otherwise, gf is an automorphism of M_j, and hence f yields an isomorphism of M_j with a summand of M_i, a contradiction.

To complete the proof, it remains to show that the two-sided ideal $\mathrm{Ker}(u)$ is nilpotent. By arguing as in the proof of Lemma 1.3.3, it suffices to show that $\mathrm{Ker}(u)$ consists of nilpotent elements. Let $f = \sum_{i,j} f_{i,j} \in \mathrm{Ker}(u)$, so that no $f_{i,j}$ is an isomorphism. Let n be a positive integer and write $f^n = \sum_{i,j} (f^n)_{i,j}$, where

$$(f^n)_{i,j} = \sum_{i_1,\ldots,i_{n-1}} f_{i,i_1} f_{i_1,i_2} \cdots f_{i_{n-1},j}.$$

Each product $f_{i,i_1} f_{i_1,i_2} \cdots f_{i_{n-1},j}$ is a sum of compositions of morphisms

$$g_{i,i_1,\ldots,i_{n-1},j} : M_j \longrightarrow M_{i_{n-1}} \longrightarrow \cdots \longrightarrow M_{i_1} \longrightarrow M_i.$$

Choose $n = Nr$, where N is a positive integer. Then there exists an index ℓ that appears N times in the sequence $(i, i_1, \dots, i_{n-1}, j)$. Thus, $g_{i,i_1,\dots,i_{n-1},j}$ factors through the composition $h_1 \cdots h_N$, where $h_1, \dots, h_N$ are endomorphisms of M_ℓ. Moreover, any h_p factors in turn through $M_\ell \to M_{\ell'} \to M_\ell$ for some $\ell' = \ell'(p) \neq \ell$. Thus, h_p is not an isomorphism; it follows that $h_1 \cdots h_N = 0$ for $N \gg 0$, by Lemma 1.3.3 again. $\square$

We say that the algebra A is *of finite orbit type*, if there are only finitely many isomorphism classes of finite-dimensional modules of any prescribed dimension. By Theorem 1.3.4, this is equivalent to the finiteness of the isomorphism classes of *indecomposable* modules of any prescribed (finite) dimension.

If there are only finitely many isomorphism classes of finite-dimensional indecomposable modules (of arbitary dimensions), then we say that A is *of finite representation type.* Clearly, this implies that A is of finite orbit type; we shall see in Subsection 2.4 that the converse holds for quiver algebras.

Next, we apply Theorem 1.3.4 to the structure of a finite-dimensional algebra A, by viewing A as a module over itself via left multiplication, and using the isomorphism (1.3.3). The summands of A are easily described (see [3, Lem. 1.3.3]):

Lemma 1.3.5. *Let A be any algebra, viewed as an A-module via left multiplication.*
(i) *Every decomposition $1 = e_1 + \cdots + e_r$, where $e_1, \dots, e_r$ are orthogonal idempotents of A, yields a decomposition of (left) A-modules $A = P(e_1) \oplus \cdots \oplus P(e_r)$, where $P(e_i) := Ae_i$. This sets up a bijection between decompositions of* 1 *as a sum of orthogonal idempotents, and direct sum decompositions of the A-module A. In particular, the non-zero summands of A are exactly the (left) ideals $P(e) := Ae$, where e is an idempotent.*
(ii) *For any A-module M, we have an isomorphism of vector spaces*

$$\mathrm{Hom}_A\big(P(e), M\big) \xrightarrow{\sim} eM, \quad f \longmapsto f(e). \tag{1.3.6}$$

(iii) *There is an isomorphism of algebras*

$$\mathrm{End}_A\big(P(e)\big) \simeq (eAe)^{\mathrm{op}}, \tag{1.3.7}$$

where eAe is viewed as an algebra with unit e.
(iv) *The A-module $P(e)$ is indecomposable if and only if e is not the sum of two orthogonal idempotents; equivalently, e is the unique idempotent of eAe.*

An idempotent satisfying the assertion (iv) is called *primitive.* Also, recall that an A-module P is *projective*, if P is a direct factor of a free A-module (see [3, Lem. 1.5.2] for further characterizations of projective modules). Thus, the $P(e)$ are projective ideals of A. If A is finite-dimensional, this yields a complete description of all projective modules:

Proposition 1.3.6. *Let A be a finite-dimensional algebra, and choose a decomposition of A-modules*

$$A \simeq m_1 P_1 \oplus \cdots \oplus m_r P_r \,,$$

where $P_1, \ldots, P_r$ are indecomposable and pairwise non-isomorphic.

(i) *There is a decomposition of vector spaces $A \simeq I \oplus B$, where I is a nilpotent ideal and B is a semi-simple subalgebra, isomorphic to $\prod_{i=1}^r \mathrm{Mat}_{m_i \times m_i}(k)$.*

(ii) *Every A-module $S_i := P_i/IP_i$ is simple. Conversely, every simple A-module is isomorphic to a unique S_i.*

(iii) *Every projective indecomposable A-module is isomorphic to a unique P_i. In particular, every such module is finite-dimensional.*

(iv) *Every finite-dimensional projective A-module admits a decomposition*

$$M \simeq n_1\, P_1 \oplus \cdots \oplus n_r P_r,$$

where $n_1, \ldots, n_r$ are uniquely determined non-negative integers.

Proof. (i) follows from Theorem 1.3.4.(ii) applied to the A-module A, taking into account the isomorphism (1.3.3).

(ii) Note that $S_i \neq 0$ since the ideal I is nilpotent. Also, we may identify A/I with B, and each S_i with a B-module. By Theorem 1.3.4 and its proof, B acts on S_i via its ith factor $\mathrm{Mat}_{m_i \times m_i}(k)$. As a consequence, $S_i \simeq n_i\, k^{m_i}$ for some integer $n_i \geq 1$. Then $A/I \simeq m_1\, S_1 \oplus \cdots \oplus m_r\, S_r$ has dimension $n_1\, m_1^2 + \cdots + n_r\, m_r^2$. But $\dim(A/I) = \dim(B) = m_1^2 + \cdots + m_r^2$, and hence $n_1 = \ldots = n_r = 1$.

(iii) Let P be a projective indecomposable A-module. Then, as above, $P \neq IP$. The quotient P/IP is a semi-simple module (since so is $A/I \simeq B$) and non-trivial; thus, there exists a surjective morphism of A-modules $p : P \to S_i$ for some i. Let $p_i : P_i \to S_i$ denote the natural map. Since P is projective, there exists a morphism $f : P \to P_i$ such that $p_i f = p$. Likewise, there exists a morphism $g : P_i \to P$ such that $pg = p_i$. Then $p_i f g = p_i$, so that $fg \in \mathrm{End}_A(P_i)$ is not nilpotent: fg is invertible, i.e., P_i is isomorphic to a summand of P. Thus, $P_i \simeq P$.

(iv) Since M is finite-dimensional, there exists a surjective morphism $f : F \to M$, where the A-module F is a direct sum of finitely many copies of A. By the projectivity of M, this yields an isomorphism $F \simeq M \oplus N$ for some A-module N. Now the desired statement follows from the Krull-Schmidt theorem. $\square$

Returning to the case of the algebra of a quiver Q (possibly with oriented cycles, so that kQ may be infinite-dimensional), recall the decomposition $1 = \sum_{i \in Q_0} e_i$ into orthogonal idempotents, and consider the corresponding decomposition

$$kQ \simeq \bigoplus_{i \in Q_0} P(i), \tag{1.3.8}$$

where $P(i) := P(e_i) = kQe_i$ $(i \in Q_0)$.

Proposition 1.3.7. *Let Q be any quiver, and i a vertex.*

(i) *The vector space $P(i)$ is the linear span of all paths with source i. Moreover, the algebra $\mathrm{End}_Q\big(P(i)\big)$ is isomorphic to the linear span of all oriented cycles at i.*

(ii) *We have an isomorphism of kQ-modules*

$$P(i)/kQ_{\geq 1}P(i) \simeq S(i).$$

In particular, $P(i)$ is not isomorphic to $P(j)$, when $i \neq j$.

(iii) *The representation $P(i)$ is indecomposable; equivalently, e_i is primitive.*

(iv) *If Q has no oriented cycle, then $\mathrm{End}_Q\big(P(i)\big) \simeq k$. Moreover, every indecomposable projective kQ-module is isomorphic to a unique $P(i)$.*

Proof. (i) The first assertion is clear, and the second one is a consequence of (1.3.7).

(ii) By (i), the space $P(i)/kQ_{\geq 1}P(i)$ has basis the image of e_i.

(iii) It suffices to show that e_i is the unique idempotent of $\mathrm{End}_Q\big(P(i)\big)$. Let $a \in e_i kQ e_i$, $a \neq e_i$, and consider a path π of maximal length occuring in a. Then π is an oriented cycle at i. Thus, π^2 occurs in a^2, and hence $a^2 \neq a$.

(iv) follows from (iii) combined with Proposition 1.3.6. □

Examples 1.3.8. 1) The indecomposable finite-dimensional modules over the loop algebra $kL = k[X]$ are the quotients

$$M(\lambda, n) := k[X]/(X-\lambda)^n\, k[X],$$

where $\lambda \in k$, and n is a positive integer. For $0 \leq i \leq n-1$, denote by v_i the image of $(X-\lambda)^i$ in $M(\lambda, n)$. Then $v_0, \ldots, v_{n-1}$ form a basis of $M(\lambda, n)$ such that $\alpha v_i = \lambda v_i + v_{i+1}$ for all i, where we set $v_n = 0$. Thus, α acts on $M(\lambda, n)$ via a Jordan block of size n and eigenvalue λ. Note that $M(\lambda, 1)$ is just the simple representation $S(\lambda)$.

Clearly, $\mathrm{Hom}_L\big(M(\lambda, m), M(\mu, n)\big) = \{0\}$ unless $\lambda = \mu$. Moreover,

$$\mathrm{Hom}_L\big(M(\lambda, m), M(\lambda, n)\big) \simeq M\big(\lambda, \min(m, n)\big).$$

In particular,

$$\mathrm{End}_L\big(M(\lambda, n)\big) \simeq M(\lambda, n). \tag{1.3.9}$$

Also, $k[X]$ has a unique finitely generated, indecomposable module of infinite dimension, namely, $k[X]$ itself. It also has many indecomposable modules which are not finitely generated, e.g., all non-trivial localizations of $k[X]$.
2) The indecomposable representations of the quiver

$$K_1 : \qquad i \xrightarrow{\alpha} j$$

fall into 3 isomorphism classes:

$$S(i) : \qquad k \longrightarrow 0,$$

$$S(j) : \qquad 0 \longrightarrow k,$$

$$P(i) : \qquad k \xrightarrow{1} k.$$

of respective dimension vectors $(1, 0)$, $(0, 1)$, $(1, 1)$. Note that $P(j) = S(j)$.

In contrast, K_2 admits infinitely many indecomposable representations; for example,

$$k \underset{\lambda}{\overset{1}{\rightrightarrows}} k \qquad (\lambda \in k)$$

of dimension vector $(1, 1)$.
3) One may show that the indecomposable representations of the quiver

$$S_2 : \qquad i_1 \xrightarrow{\alpha_1} j \xleftarrow{\alpha_2} i_2$$

fall into 6 isomorphism classes:

$$S(i_1) : \qquad k \longrightarrow 0 \longleftarrow 0,$$

$$S(j) : \qquad 0 \longrightarrow k \longleftarrow 0,$$

$$S(i_2) : \qquad 0 \longrightarrow 0 \longleftarrow k,$$

$$P(i_1) : \qquad k \xrightarrow{1} k \longleftarrow 0,$$

$$P(i_2) : \qquad 0 \longrightarrow k \xleftarrow{1} k,$$

and finally

$$M(j) : \qquad k \xrightarrow{1} k \xleftarrow{1} k,$$

of respective dimension vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 1, 1)$. Also, $P(j) = S(j)$.

1.4 The standard resolution

Throughout this subsection, we fix a quiver $Q = (Q_0, Q_1, s, t)$.

Proposition 1.4.1. *For any left kQ-module M, we have an exact sequence of kQ-modules*

$$(1.4.1) \qquad 0 \longrightarrow \bigoplus_{\alpha \in Q_1} P\big(t(\alpha)\big) \otimes_k e_{s(\alpha)} M \xrightarrow{u} \bigoplus_{i \in Q_0} P(i) \otimes_k e_i M \xrightarrow{v} M \longrightarrow 0,$$

where the maps u and v are defined by

$$u(a \otimes m) := a\alpha \otimes m - a \otimes \alpha m \quad \big(a \in P\big(t(\alpha)\big),\ m \in e_{s(\alpha)} M\big)$$

and

$$v(a \otimes m) := am \quad \big(a \in P(i),\ m \in e_i M\big).$$

Here each $P(j) \otimes_k e_i M$ is a kQ-module via $a(b \otimes m) = ab \otimes m$, where $a \in kQ$, $b \in P(j)$, and $m \in e_i M$.

Proof. Note that the map

$$kQ \otimes_{kQ} M \longrightarrow M, \quad a \otimes m \longmapsto am$$

is an isomorphism of (left) kQ-modules, where the tensor product is taken for kQ viewed as a right kQ-module. Since the algebra kQ is generated by the e_i and the α, the vector space $kQ \otimes_{kQ} M$ is the quotient of $kQ \otimes_k M$ by the linear span of the elements $ae_i \otimes m - a \otimes e_i m$ and $a\alpha \otimes m - a \otimes \alpha m$, where $a \in kQ$, $m \in M$, $i \in Q_0$, and $\alpha \in Q_1$. Moreover,

$$kQ \otimes_k M = \bigoplus_{i,j \in Q_0} P(i) \otimes_k e_j M$$

and the linear span of the $ae_i \otimes m - a \otimes e_i m$ ($a \in kQ$, $m \in M$, $i \in Q_0$) is the partial sum $\bigoplus_{i \neq j} P(i) \otimes_k e_j M$. It follows that v is surjective, and its kernel is generated by the image of u.

It remains to show that u is injective. Recall that each space $P\big(t(\alpha)\big)$ has a basis consisting of all paths π such that $s(\pi) = t(\alpha)$. So u is given by

$$\sum_{\alpha \in Q_1} \sum_{\pi,\, s(\pi) = t(\alpha)} \pi \otimes m_{\alpha,\pi} \longmapsto \sum_{\alpha,\, \pi} (\pi\alpha \otimes m_{\alpha,\pi} - \pi \otimes \alpha m_{\alpha,\pi}).$$

If the left-hand side is non-zero, then we may choose a path π of maximal length such that $m_{\alpha,\pi} \neq 0$. Then the right-hand side contains $\pi\alpha \otimes m_{\alpha,\pi}$ but no other component on $\pi\alpha \otimes e_{s(\alpha)} M$. This proves the desired injectivity. □

The exact sequence (1.4.1) is called the *standard resolution* of the kQ-module M; it is a projective resolution of length at most 1.

As a consequence, *each left ideal I of kQ is projective*, as follows by applying Schanuel's lemma [3, Lem. 1.5.3] to the standard resolution of the quotient kQ/I and to the exact sequence

$$0 \longrightarrow I \longrightarrow kQ \longrightarrow kQ/I \longrightarrow 0.$$

This property defines the class of *(left) hereditary* algebras; we refer to [3, Sec. 4.2] for more on these algebras and their relations to quivers.

Next, recall the definition of the groups $\mathrm{Ext}^i_Q(M,N)$, where M and N are arbitrary kQ-modules. Choose a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0. \tag{1.4.2}$$

Taking morphisms to N yields a complex

$$\mathrm{Hom}_Q(P_0,N) \longrightarrow \mathrm{Hom}_Q(P_1,N) \longrightarrow \mathrm{Hom}_Q(P_2,N) \longrightarrow \cdots$$

The homology groups of this complex turn out to be independent of the choice of the resolution (1.4.2); the ith homology group is denoted by $\mathrm{Ext}^i_Q(M,N)$ (see e.g. [3, Sec. 2.4]).

Clearly, $\mathrm{Ext}^0_Q(M,N) = \mathrm{Hom}_Q(M,N)$. Also, recall that $\mathrm{Ext}^1_Q(M,N)$ is the set of equivalence classes of *extensions of M by N*, i.e., of exact sequences of kQ-modules

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

up to isomorphisms that induce the identity maps on N and M (see [3, Sec. 2.6]).

Using the standard resolution and the isomorphism (1.3.6), we readily obtain the following:

Corollary 1.4.2. *For any representations $M = (V_i, f_\alpha)$ and $N = (W_i, g_\alpha)$ of a quiver Q, the map*

$$\begin{array}{rcl} c_{M,N} : \prod_{i\in Q_0} \mathrm{Hom}(V_i, W_i) & \longrightarrow & \prod_{\alpha\in Q_1} \mathrm{Hom}(V_{s(\alpha)}, W_{t(\alpha)}), \\ (u_i)_{i\in Q_0} & \longmapsto & \big(u_{t(\alpha)} f_\alpha - g_\alpha u_{s(\alpha)}\big)_{\alpha\in Q_1} \end{array}$$

has kernel $\mathrm{Hom}_Q(M,N)$ *and cokernel* $\mathrm{Ext}^1_Q(M,N)$.

Moreover, $\mathrm{Ext}^j_Q(M,N) = 0$ *for all* $j \geq 2$.

In particular, there is a four-term exact sequence

$$0 \to \mathrm{End}_Q(M) \to \prod_{i\in Q_0} \mathrm{End}(V_i) \to \prod_{\alpha\in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)}) \to \mathrm{Ext}^1_Q(M,M) \to 0 \tag{1.4.3}$$

which will acquire a geometric interpretation in Subsection 2.2. The space $\mathrm{Ext}^1_Q(M,M)$ is called the *space of self-extensions of* M.

Taking dimensions in Corollary 1.4.2 yields:

COROLLARY 1.4.3. *For any finite-dimensional representations* M*,* N *of* Q *with dimension vectors* $(m_i)_{i\in Q_0}$*,* $(n_i)_{i\in Q_0}$*, we have*

$$\dim \mathrm{Hom}_Q(M,N) - \dim \mathrm{Ext}^1_Q(M,N) = \sum_{i\in Q_0} m_i\, n_i - \sum_{\alpha\in Q_1} m_{s(\alpha)}\, n_{t(\alpha)}. \tag{1.4.4}$$

In particular, $\dim \mathrm{Ext}^1_Q\big(S(i),S(j)\big)$ is the number of arrows with source i and target j, for all vertices i and j. For example, the dimension of the space of self-extensions $\mathrm{Ext}^1_Q\big(S(i),S(i)\big)$ is the number of oriented cycles at i.

Also, note that the left-hand side of (1.4.4) only depends on the dimension vectors of M, N, and is a bi-additive function of these vectors. This motivates the following:

DEFINITION 1.4.4. The *Euler form* of the quiver Q is the bilinear form $\langle,\rangle_Q$ on $\mathbb{R}^{Q_0}$ given by

$$\langle \underline{m}, \underline{n}\rangle_Q = \sum_{i\in Q_0} m_i\, n_i - \sum_{\alpha\in Q_1} m_{s(\alpha)}\, n_{t(\alpha)} \tag{1.4.5}$$

for any $\underline{m} = (m_i)_{i\in Q_0}$ and $\underline{n} = (n_i)_{i\in Q_0}$.

Note that the assignment $(\underline{m},\underline{n}) \mapsto \langle \underline{n}, \underline{m}\rangle_Q$ is the Euler form of the *opposite quiver*, obtained from Q by reverting all the arrows. Thus, the Euler form is generally non-symmetric (e.g., for the quiver K_r).

DEFINITION 1.4.5. The quadratic form associated to the Euler form is the *Tits form* q_Q. In other words,

$$q_Q(\underline{n}) := \langle \underline{n}, \underline{n}\rangle_Q = \sum_{i\in Q_0} n_i^2 - \sum_{\alpha\in Q_1} n_{s(\alpha)}\, n_{t(\alpha)} \tag{1.4.6}$$

for any $\underline{n} = (n_i)_{i\in Q_0}$.

By (1.4.4), we have

$$q_Q(\underline{\dim}\, M) = \dim \mathrm{End}_Q(M) - \dim \mathrm{Ext}^1_Q(M,M) \tag{1.4.7}$$

for any finite-dimensional representation M.

Also, note that the Tits form depends only on the underlying undirected graph of Q, and determines that graph uniquely. For example, if Q has type A_r, then

$$q_Q(x_1,\ldots,x_r) = \sum_{i=1}^{r} x_i^2 - \sum_{i=1}^{r-1} x_i\, x_{i+1}.$$

The positivity properties of the Tits form are closely related with the shape of Q. For instance, if Q contains a (possibly non-oriented) cycle with vertices $i_1, \ldots, i_r$, then $q_Q(\varepsilon_{i_1} + \cdots + \varepsilon_{i_r}) \le 0$. Together with [3, Prop. 4.6.3], this implies:

Proposition 1.4.6. *For a quiver Q, the following conditions are equivalent:*

(i) *The Tits form q_Q is positive definite.*

(ii) $q_Q(\underline{n}) \ge 1$ *for any non-zero* $\underline{n} \in \mathbb{N}^{Q_0}$.

(iii) *The underlying undirected graph of each connected component of Q is a simply-laced Dynkin diagram.*

As a consequence, Theorem 1.1.7 may be rephrased as follows: *the quivers of finite orbit type are exactly those having a positive definite Tits form.* This version of Gabriel's theorem will be proved in the next section.

2 Quiver representations: the geometric approach

In this section, we study the representations of a prescribed quiver having a prescribed dimension vector from a geometric viewpoint: the isomorphism classes of these representations are in bijection with the orbits of an algebraic group (a product of general linear groups) acting in a representation space (a product of matrix spaces).

Subsection 2.1 presents general results on representation spaces of quivers, and orbits of algebraic groups in algebraic varieties. As an application, we obtain a proof of the "only if" part of Gabriel's theorem (Thm. 1.1.7).

In Subsection 2.2, we describe the isotropy groups of representation spaces and we study the differentials of the corresponding orbit maps. In particular, the normal space to an orbit is identified with the space of self-extensions of the corresponding representation.

The main result of Subsection 2.3 asserts that the orbit closure of every point in a representation space contains the associated graded to any filtration of the corresponding representation. Further, the filtrations for which the associated graded is semi-simple yield the unique closed orbit.

In Subsection 2.4, we prove the "if" part of Gabriel's theorem by combining the results of the previous subsections with a key technical ingredient. An alternative proof via purely representation-theoretic methods is exposed in [3, Sec. 4.7].

The prerequisites of this section are basic notions of affine algebraic geometry (Zariski topology on affine spaces, dimension, morphisms, Zariski tangent spaces, differentials); they may be found e.g. in the book [8].

As in the previous section, we only make the first steps in the geometry of quiver representations. For further results, including Kac's broad generalization of Gabriel's theorem, a very good source is [11]. The invariant theory of quiver representations is studied in [12] over a field of characteristic zero, and [7] in arbitrary characteristics. Moduli spaces of representations of finite-dimensional algebras are constructed in [10]; the survey [17] reviews this construction in the setting of quivers, and presents many developments and applications. In another direction, degenerations of representations (equivalently, orbit closures in representation spaces) are intensively studied, see e.g. [20].

2.1 Representation spaces

Throughout this section, we fix a quiver $Q = (Q_0, Q_1, s, t)$ and a dimension vector $\underline{n} = (n_i)_{i \in Q_0}$.

Recall that a representation of Q with dimension vector $\underline{n}$ assigns a vector space V_i of dimension n_i to every vertex i, and a linear map $f_\alpha : V_i \to V_j$ to every arrow $\alpha : i \to j$. Choosing bases, we may identify each V_i to k^{n_i}; then each f_α is just a matrix of size $n_j \times n_i$. This motivates the following:

Definition 2.1.1. The *representation space* of the quiver Q for the dimension vector $\underline{n}$ is

$$\operatorname{Rep}(Q, \underline{n}) := \bigoplus_{\alpha: i \to j} \operatorname{Hom}(k^{n_i}, k^{n_j}) = \bigoplus_{\alpha: i \to j} \operatorname{Mat}_{n_j \times n_i}(k). \tag{2.1.1}$$

This is a vector space of dimension $\sum_{\alpha: i \to j} n_i n_j$.

Here $\sum_{\alpha: i \to j}$ denotes (abusively) the summation over all $\alpha \in Q_1$, to simplify the notation. Likewise, a point of $\operatorname{Rep}(Q, \underline{n})$ will be denoted by $x = (x_\alpha)_{\alpha: i \to j}$.

For any positive integer n, we denote by $\operatorname{GL}(n)$ the group of invertible $n \times n$ matrices with coefficients in k, and by id_n the identity matrix. The group

$$\operatorname{GL}(\underline{n}) := \prod_{i \in Q_0} \operatorname{GL}(n_i)$$

acts linearly on each space $\operatorname{Mat}_{n_j \times n_i}(k)$ by

$$(g_i)_{i \in Q_0} \cdot x_\alpha := g_j x_\alpha g_i^{-1}. \tag{2.1.2}$$

Hence $\operatorname{GL}(\underline{n})$ acts on $\operatorname{Rep}(Q, \underline{n})$ by preserving the decomposition (2.1.1). The subgroup

$$k^* \operatorname{id}_{\underline{n}} := \{(\lambda \operatorname{id}_{n_i})_{i \in Q_0} \mid \lambda \in k^*\}$$

is contained in the center of $\operatorname{GL}(\underline{n})$, and acts trivially on $\operatorname{Rep}(Q, \underline{n})$. Thus, the action of $\operatorname{GL}(\underline{n})$ factors through an action of the quotient group

$$\operatorname{PGL}(\underline{n}) := \operatorname{GL}(\underline{n}) / k^* \operatorname{id}_{\underline{n}}.$$

Clearly, any point $x \in \mathrm{Rep}(Q, \underline{n})$ defines a representation M_x of Q. Moreover, any two such representations M_x, M_y are isomorphic if and only if x, y are in the same orbit of $\mathrm{GL}(\underline{n})$ or, equivalently, of $\mathrm{PGL}(\underline{n})$. This yields the following fundamental observation.

LEMMA 2.1.2. *The assignment $x \mapsto M_x$ sets up a bijective correspondence from the set of orbits of $\mathrm{GL}(\underline{n})$ in $\mathrm{Rep}(Q, \underline{n})$ to the set of isomorphism classes of representations of Q with dimension vector $\underline{n}$. The isotropy group*

$$\mathrm{GL}(\underline{n})_x := \{g \in \mathrm{GL}(\underline{n}) \mid g \cdot x = x\}$$

is isomorphic to the automorphism group $\mathrm{Aut}_Q(M_x)$.

EXAMPLE 2.1.3. Consider the quiver H_r of Example 1.2.3.4, and choose the dimension vector $\underline{n} := (1, n)$. Then $\mathrm{Rep}(H_r, \underline{n})$ consists of all tuples $(v, x_1, \ldots, x_r)$, where $v \in k^n$ and $x_1, \ldots, x_r \in \mathrm{Mat}_{n \times n}(k)$. Further, $\mathrm{GL}(\underline{n}) = k^* \times \mathrm{GL}(n)$ acts on $\mathrm{Rep}(H_r, \underline{n})$ via

$$(t, g) \cdot (v, x_1, \ldots, x_r) := (tgv, g x_1 g^{-1}, \ldots, g x_r g^{-1}).$$

So the orbits are those of $\mathrm{PGL}(\underline{n}) \simeq \mathrm{GL}(n)$, acting by simultaneous multiplication on v and conjugation on the x_i's.

Each point $(v, x_1, \ldots, x_r) \in \mathrm{Rep}(Q, \underline{n})$ defines a representation

$$\varphi : k\langle X_1, \ldots, X_r \rangle \longrightarrow \mathrm{Mat}_{n \times n}(k), \quad X_i \longmapsto x_i,$$

together with a point $v \in k^n$. Moreover, the orbits of $\mathrm{GL}(n)$ parametrize the isomorphism classes of such pairs (φ, v).

We say that a tuple $(v, x_1, \ldots, x_r)$ is *cyclic*, if v generates k^n as a module over $k\langle X_1, \ldots, X_r \rangle$; we denote by $\mathrm{Rep}(H_r, \underline{n})^{\mathrm{cyc}}$ the subset of $\mathrm{Rep}(H_r, \underline{n})$ consisting of cyclic tuples. Clearly, $\mathrm{Rep}(H_r, \underline{n})^{\mathrm{cyc}}$ is stable under the action of $\mathrm{GL}(n)$, and the isotropy group of each cyclic tuple is trivial. Moreover, *the orbit space* $\mathrm{Rep}(H_r, \underline{n})^{\mathrm{cyc}} / \mathrm{GL}(n)$ *may be identified with the set of all left ideals of codimension n in $k\langle X_1, \ldots, X_r \rangle$*. Indeed, to any tuple $(v, x_1, \ldots, x_r)$, we assign the ideal

$$I(v, x_1, \ldots, x_r) := \{P \in k\langle X_1, \ldots, X_r \rangle \mid P(x_1, \ldots, x_r) v = 0\}.$$

which depends only on the orbit of $(v, x_1, \ldots, x_r)$. Conversely, to any left ideal $I \subset k\langle X_1, \ldots, X_r \rangle$ of codimension n, we assign the isomorphism class of the pair (φ, v), where φ is the representation of $k\langle X_1, \ldots, X_r \rangle$ in the quotient $k\langle X_1, \ldots, X_r \rangle / I \simeq k^n$, and v is the image of the unit 1 in this quotient. One readily checks that these assignments are mutually inverse bijections.

Returning to the case of an arbitrary quiver Q, we denote by $\mathcal{O}_M$ the orbit in $\mathrm{Rep}(Q, \underline{n})$ associated with a representation M of Q. Thus, $\mathcal{O}_M = \mathrm{GL}(\underline{n}) \cdot x$, where $x \in \mathrm{Rep}(Q, \underline{n})$ is any point such that $M \simeq M_x$.

In particular, Q is of finite orbit type if and only if $\mathrm{Rep}(Q, \underline{n})$ contains only finitely many orbits of $\mathrm{GL}(\underline{n})$ for any dimension vector $\underline{n}$. This observation will yield a quick proof of the "only if" part of Gabriel's theorem. To present that proof, we need some general notions and results on algebraic group actions.

DEFINITION 2.1.4. An *(affine) algebraic group* is an (affine) algebraic variety G, equipped with a group structure such that the multiplication map

$$\mu : G \times G \longrightarrow G, \quad (g, h) \longmapsto gh$$

and the inverse map

$$\iota : G \longrightarrow G, \quad g \longmapsto g^{-1}$$

are morphisms of varieties.

EXAMPLES 2.1.5. 1) The *general linear group*, denoted by $\mathrm{GL}(n)$, is the open affine subset of $\mathrm{Mat}_{n \times n}(k)$ (an affine space of dimension n^2) where the determinant is non-zero. The corresponding algebra of regular functions is generated by the matrix coefficients and the inverse of their determinant. Since the coefficients of the product of matrices (resp. of the inverse of a matrix) are polynomials in the matrix coefficients (and in the inverse of the determinant), $\mathrm{GL}(n)$ is an affine algebraic group; it is an irreducible variety of dimension n^2.

As a consequence, any closed subgroup of $\mathrm{GL}(n)$ is an affine algebraic group; such a subgroup is called a *linear algebraic group*. In fact, all affine algebraic groups are linear (see [19, 2.3.7]).

For example, the group $k^* \,\mathrm{id}_n$ of scalar invertible matrices is a closed central subgroup of $\mathrm{GL}(n)$, isomorphic to the multiplicative group

$$\mathbb{G}_m := \mathrm{GL}(1).$$

The quotient group

$$\mathrm{PGL}(n) := \mathrm{GL}(n)/k^* \,\mathrm{id}_n$$

is isomorphic to the image of $\mathrm{GL}(n)$ in the automorphism group of the vector space $\mathrm{Mat}_{n \times n}(k)$, where $\mathrm{GL}(n)$ acts by conjugation. This image is a closed subgroup of $\mathrm{GL}(n^2)$ of dimension $n^2 - 1$, as follows from Corollary 2.1.8 below. (Alternatively, $\mathrm{PGL}(n)$ is the automorphism group of the algebra $\mathrm{Mat}_{n \times n}(k)$ by the Skolem-Noether therem; see [3, Prop. 1.3.6]. This realizes $\mathrm{PGL}(n)$ as a subgroup of $\mathrm{GL}(n^2)$ defined by quadratic equations.) Thus, $\mathrm{PGL}(n)$ is an irreducible linear algebraic group of dimension n^2-1, the *projective linear group*.

2) More generally, $\mathrm{GL}(\underline{n})$ is the subset of

$$\mathrm{End}(\underline{n}) := \prod_{i\in Q_0} \mathrm{End}(k^{n_i}) = \prod_{i\in Q_0} \mathrm{Mat}_{n_i\times n_i}(k)$$

consisting of those families $(g_i)_{i\in Q_0}$ such that $\prod_{i\in Q_0} \det(g_i)$ is non-zero. This realizes $\mathrm{GL}(\underline{n})$ as a principal open subset of the affine space $\mathrm{End}(\underline{n})$ or, alternatively, as a closed subgroup of $\mathrm{GL}(\sum_{i\in Q_0} n_i)$. Thus, $\mathrm{GL}(\underline{n})$ is an irreducible linear algebraic group of dimension $\sum_{i\in Q_0} n_i^2$, and the subgroup $k^*\,\mathrm{id}_{\underline{n}}$ is closed. As in the preceding example, one shows that the quotient

$$\mathrm{PGL}(\underline{n}) := \mathrm{GL}(\underline{n})/k^*\,\mathrm{id}_{\underline{n}}$$

is an irreducible linear algebraic group of dimension $\left(\sum_{i\in Q_0} n_i^2\right) - 1$.

Definition 2.1.6. An *algebraic action* of an algebraic group G on a variety X is a morphism

$$\varphi : G\times X \longrightarrow X, \quad (g,x)\longmapsto g\cdot x$$

of varieties, which is also an action of the group G on X.

For instance, the action of $\mathrm{GL}(\underline{n})$ on $\mathrm{Rep}(Q,\underline{n})$ is algebraic by (2.1.2). Since the subgroup $k^*\,\mathrm{id}_{\underline{n}}$ acts trivially, this action factors through a linear action of $\mathrm{PGL}(\underline{n})$, which is also algebraic by Corollary 2.1.8 below.

Proposition 2.1.7. *Let X be a variety equipped with an algebraic action of an algebraic group G and let $x\in X$.*

(i) *The isotropy group*

$$G_x := \{g\in G \mid g\cdot x = x\}$$

is closed in G.

(ii) *The orbit*

$$G\cdot x := \{g\cdot x,\ g\in G\}$$

is a locally closed, non-singular subvariety of X. All connected components of $G\cdot x$ have dimension $\dim G - \dim G_x$.

(iii) *The orbit closure $\overline{G\cdot x}$ is the union of $G\cdot x$ and of orbits of smaller dimension ; it contains at least one closed orbit.*

(iv) *The variety G is connected if and only if it is irreducible; then the orbit $G\cdot x$ and its closure are irreducible as well.*

PROOF. (i) Consider the orbit map

$$\varphi_x : G \longrightarrow X, \quad g \longmapsto g \cdot x.$$

This is a morphism of varieties with fibers being the left cosets gG_x, $g \in G$; hence these cosets are closed.

(ii) The orbit $G \cdot x$ is the image of the morphism φ_x, and hence is a constructible subset of X; thus, $G \cdot x$ contains a dense open subset of its closure (see [8, p. 311]). Since any two points of $G \cdot x$ are conjugate by an automorphism of X, it follows that $G \cdot x$ is open in its closure. Likewise, $G \cdot x$ is non-singular, and its connected components have all the same dimension. The formula for this dimension follows from a general result on the dimension of fibers of morphisms (see again [8, p. 311]).

(iii) By the results of (ii), the complement $\overline{G \cdot x} \setminus G \cdot x$ has smaller dimension; being stable under G, it is the union of orbits of smaller dimension. Let $\mathcal{O}$ be an orbit of minimal dimension in $\overline{G \cdot x}$, then $\overline{\mathcal{O}} \setminus \mathcal{O}$ must be empty, so that $\mathcal{O}$ is closed.

(iv) Note that G is an orbit for its action on itself by left multiplication. Thus, it is non-singular by (ii), so that connectedness and irreducibility are equivalent. Finally, if G is irreducible, then so is its image $G \cdot x$ under the orbit map. This implies the irreducibility of $\overline{G \cdot x}$. $\square$

COROLLARY 2.1.8. *Let $\varphi : G \to H$ be a homomorphism of algebraic groups. Then the kernel* $\operatorname{Ker}\varphi$ *and image* $\operatorname{Im}\varphi$ *are closed in G resp. H, and we have* $\dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi = \dim G$.

PROOF. Consider the action of G on H by $g \cdot h := \varphi(g)h$. This action is algebraic and its orbits are the right cosets $(\operatorname{Im}\varphi)h$, where $h \in H$; these orbits are permuted transitively by the action of H on itself via right multiplication. By Proposition 2.1.7, there exists a closed coset. Thus, all cosets are closed; in particular, $\operatorname{Im}\varphi$ is closed. On the other hand, the isotropy group of any point of H is $\operatorname{Ker}\varphi$. So this subgroup is closed by Proposition 2.1.7, which also yields the equality $\dim \operatorname{Im}\varphi = \dim G - \dim \operatorname{Ker}\varphi$. $\square$

We may now prove the "only if" part of Gabriel's theorem. For any quiver Q, note the equality

$$\dim \mathrm{GL}(\underline{n}) - \dim \operatorname{Rep}(Q, \underline{n}) = \sum_{i \in Q_0} n_i^2 - \sum_{\alpha : i \to j} n_i n_j. \tag{2.1.3}$$

Equivalently,

$$\dim \mathrm{PGL}(\underline{n}) - \dim \operatorname{Rep}(Q, \underline{n}) = q_Q(\underline{n}) - 1, \tag{2.1.4}$$

where q_Q denotes the Tits form defined in (1.4.6). Together with Proposition 2.1.7, it follows that $q_Q(\underline{n}) \geq 1$ whenever $\operatorname{Rep}(Q, \underline{n})$ contains an open orbit of the group $\mathrm{PGL}(\underline{n})$.

By Proposition 2.1.7 again, this assumption holds if $\mathrm{Rep}(Q, \underline{n})$ contains only finitely many orbits of that group. Thus, if Q is of finite orbit type, then $q_Q(\underline{n}) \geq 1$ for all non-zero $\underline{n} \in \mathbb{N}^{Q_0}$. By Proposition 1.4.6, it follows that q_Q is positive definite.

2.2 Isotropy groups

We begin with a structure result for automorphism groups of representations. To formulate it, we say that an algebraic group is *unipotent* if it is isomorphic to a closed subgroup of the group of upper triangular $n \times n$ matrices with diagonal coefficients 1.

PROPOSITION 2.2.1. *Let M be a finite-dimensional representation of Q.*
(i) *The automorphism group $\mathrm{Aut}_Q(M)$ is an open affine subset of $\mathrm{End}_Q(M)$. As a consequence, $\mathrm{Aut}_Q(M)$ is a connected linear algebraic group.*
(ii) *There exists a decomposition*

$$\mathrm{Aut}_Q(M) = U \rtimes \prod_{i=1}^{r} \mathrm{GL}(m_i), \tag{2.2.1}$$

where U is a closed normal unipotent subgroup and $m_1, \ldots, m_r$ denote the multiplicities of the indecomposable summands of M.

PROOF. (i) Just note that $\mathrm{Aut}_Q(M)$ is the subset of $\mathrm{End}_Q(M)$ where the determinant is non-zero.

(ii) The decomposition (1.3.5) yields a split surjective homomorphism of algebras

$$\mathrm{End}_Q(M) \longrightarrow \mathrm{End}_Q(M)/I \simeq \prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$$

and, in turn, a split surjective homomorphism of algebraic groups

$$\mathrm{Aut}_Q(M) \longrightarrow \prod_{i=1}^{r} \mathrm{GL}(m_i)$$

with kernel

$$\mathrm{id}_M + I := \{\mathrm{id}_M + f, \quad f \in I\}$$

(indeed, $\mathrm{id}_M + f$ is invertible for any $f \in I$, since f is nilpotent). Thus, $\mathrm{id}_M + I$ is a closed connected normal subgroup of $\mathrm{Aut}_Q(M)$.

It remains to show that the group $\mathrm{id}_M + I$ is unipotent. For this, we consider the linear action of that group on the subspace $k \,\mathrm{id}_M \oplus I$ by left multiplication. Since the orbit of id_M is isomorphic to the affine space $\mathrm{id}_M + I$, this action yields a closed embedding

$$\mathrm{id}_M + I \hookrightarrow \mathrm{GL}(k \,\mathrm{id}_M \oplus I).$$

Moreover, the powers I^n form a decreasing filtration of the vector space $k\,\mathrm{id}_M \oplus I$, and $I^n = 0$ for $n \gg 0$; any I^n is stable under the group $\mathrm{id}_M + I$, and the latter group fixes pointwise the quotients I^n/I^{n+1} and $(k\,\mathrm{id}_M \oplus I)/I$. This realizes $\mathrm{id}_M + I$ as a unipotent subgroup of $\mathrm{GL}(k\,\mathrm{id}_M \oplus I)$, by choosing a basis of $k\,\mathrm{id}_M \oplus I$ compatible with the filtration $(I^n)_{n\geq 1}$. □

The decomposition (2.2.1) yields a criterion for the indecomposability of a representation, in terms of its automorphism group:

COROLLARY 2.2.2. *Let $x \in \mathrm{Rep}(Q, \underline{n})$, then the representation M_x is indecomposable if and only if the isotropy group $\mathrm{GL}(\underline{n})_x$ is the semi-direct product of a unipotent subgroup with the group $k^*\,\mathrm{id}_{\underline{n}}$; equivalently, $\mathrm{PGL}(\underline{n})_x$ is unipotent.*

Next, we obtain the promised geometric interpretation of the four-term exact sequence (1.4.3):

THEOREM 2.2.3. *Let $x = (x_\alpha)_{\alpha: i\to j} \in \mathrm{Rep}(Q, \underline{n})$ and denote by M the corresponding representation of Q.*

(i) *We have an exact sequence*

$$0 \longrightarrow \mathrm{End}_Q(M) \longrightarrow \mathrm{End}(\underline{n}) \xrightarrow{c_x} \mathrm{Rep}(Q, \underline{n}) \longrightarrow \mathrm{Ext}^1_Q(M, M) \longrightarrow 0, \tag{2.2.2}$$

where $c_x\big((f_i)_{i\in Q_0}\big) = \big(f_j x_\alpha - x_\alpha f_i\big)_{\alpha: i\to j}$.

(ii) *c_x may be identified with the differential at the identity of the orbit map*

$$\varphi_x : \mathrm{GL}(\underline{n}) \longrightarrow \mathrm{Rep}(Q, \underline{n}), \quad g \longmapsto g\cdot x.$$

(iii) *The image of c_x is the Zariski tangent space $T_x\big(\mathrm{GL}(\underline{n})\cdot x\big)$ viewed as a subspace of $T_x\big(\mathrm{Rep}(Q, \underline{n})\big) \simeq \mathrm{Rep}(Q, \underline{n})$.*

PROOF. (i) is a reformulation of (1.4.3).

(ii) Since the algebraic group $\mathrm{GL}(\underline{n})$ is open in the affine space $\mathrm{End}(\underline{n})$, the Zariski tangent space to this group at $\mathrm{id}_{\underline{n}}$ may be identified with the vector space $\mathrm{End}(\underline{n})$. Likewise, by Proposition 2.2.1, the tangent space to $\mathrm{Aut}_Q(M)$ at $\mathrm{id}_{\underline{n}}$ may be identified with $\mathrm{End}_Q(M)$. Now the assertion follows from the definition (2.1.2) of the action of $\mathrm{GL}(\underline{n})$. Consider indeed an arrow $\alpha : i \to j$ and a matrix $x_\alpha \in \mathrm{Mat}_{n_j\times n_i}(k)$. Then the differential at $(\mathrm{id}_{n_i}, \mathrm{id}_{n_j})$ of the morphism

$$\mathrm{GL}(n_i) \times \mathrm{GL}(n_j) \longrightarrow \mathrm{Mat}_{n_j\times n_i}(k), \quad (g_i, g_j) \longmapsto g_j x_\alpha g_i^{-1}$$

is easily seen to be the map

$$\mathrm{Mat}_{n_i\times n_i}(k) \times \mathrm{Mat}_{n_j\times n_j}(k) \longrightarrow \mathrm{Mat}_{n_j\times n_i}(k), \quad (f_i, f_j) \longmapsto f_j x_\alpha - x_\alpha f_i.$$

(iii) By Proposition 2.1.7, we have

$$\dim T_x\big(\,\mathrm{GL}(\underline{n})\cdot x\big) = \dim \mathrm{GL}(\underline{n})\cdot x = \dim \mathrm{GL}(\underline{n}) - \dim \mathrm{GL}(\underline{n})_x.$$

Together with Proposition 2.2.1, it follows that

$$\dim T_x\big(\,\mathrm{GL}(\underline{n})\cdot x\big) = \dim \mathrm{End}(\underline{n}) - \dim \mathrm{End}_Q(M).$$

On the other hand, the differential of the orbit map has kernel $\mathrm{End}_Q(M)$ by (2.2.2). Thus, its image is the whole space $T_x\big(\,\mathrm{GL}(\underline{n})\cdot x\big)$. □

Remarks 2.2.4. 1) By Theorem 2.2.3, the differential at the identity of the orbit map $\mathrm{GL}(\underline{n}) \to \mathrm{GL}(\underline{n})\cdot x$ is surjective for any $x \in \mathrm{Rep}(Q,\underline{n})$. In other words, orbit maps for quiver representations are *separable* (for this notion, see e.g. [19, 3.2]).

This holds in fact for any algebraic group action in characteristic zero by [loc. cit.], but generally fails in characteristic $p > 0$. For example, the additive group of k acts algebraically on the affine line via $t \cdot x = t^p + x$, and the differential of each orbit map is 0.

2) The exact sequence (2.2.2) may also be interpreted in terms of Lie algebras. We briefly review the relevant definitions and results from algebraic groups, refering to [19, 3.3] for details.

Let G be an algebraic group with identity element e. Consider the commutator map

$$G \times G \longrightarrow G, \quad (x,y) \longmapsto xyx^{-1}y^{-1}.$$

Its differential at (e,e) yields the Lie bracket

$$T_e(G) \times T_e(G) \longrightarrow T_e(G), \quad (x,y) \longmapsto [x,y]$$

which endows $T_e(G)$ with the structure of a Lie algebra, denoted by $\mathrm{Lie}\, G$. The assignment $G \mapsto \mathrm{Lie}(G)$ is clearly functorial. For example, if H is a closed subgroup of G, then $\mathrm{Lie}(H)$ is a Lie subalgebra of $\mathrm{Lie}(G)$ via the identification of $T_e(H)$ to a subspace of $T_e(G)$.

The Lie algebra of the general linear group $\mathrm{GL}(n)$ is the space $\mathrm{Mat}_{n\times n}(k)$ endowed with the standard Lie bracket $(x,y) \mapsto xy - yx$. It follows that $\mathrm{Lie}\,\mathrm{GL}(\underline{n}) = \mathrm{End}(\underline{n})$ endowed with the same Lie bracket, where $\mathrm{End}(\underline{n})$ is viewed as a subalgebra of $\mathrm{End}(\sum_i n_i)$. Likewise, the Lie algebra of $\mathrm{Aut}_Q(M)$ is $\mathrm{End}_Q(M)$ viewed as a Lie subalgebra of $\mathrm{End}(M)$. Moreover, the representation of $\mathrm{GL}(\underline{n})$ in $\mathrm{Rep}(Q,\underline{n})$ differentiates to a representation of the Lie algebra $\mathrm{End}(\underline{n})$ given by $f \cdot x := c_x(f)$.

We now obtain a representation-theoretic interpretation of the Zariski tangent spaces to orbits, and also of their normal spaces; these are defined as follows. Let X be a variety,

Y a locally closed subvariety, and x a point of Y. Then the Zariski tangent space $T_x(Y)$ is identified to a subspace of $T_x(X)$; the quotient

$$N_x(Y/X) := T_x(X)/T_x(Y)$$

is the *normal space at x to Y in X.*

COROLLARY 2.2.5. *With the notation of Theorem 2.2.3, we have isomorphisms*

$$T_x(\mathcal{O}_M) \simeq \operatorname{End}(\underline{n})/\operatorname{End}_Q(M), \quad N_x\big(\mathcal{O}_M/\operatorname{Rep}(Q,\underline{n})\big) \simeq \operatorname{Ext}^1_Q(M,M). \tag{2.2.3}$$

Moreover, $\mathcal{O}_M$ is open in $\operatorname{Rep}(Q,\underline{n})$ *if and only if* $\operatorname{Ext}^1_Q(M,M) = 0$*; then the orbit $\mathcal{O}_M$ is uniquely determined by the dimension vector $\underline{n}$.*

PROOF. The isomorphisms (2.2.3) follow readily from Theorem 2.2.3, and the second assertion is a consequence of the lemma below. For the uniqueness assertion, just recall that any two nonempty open subsets of $\operatorname{Rep}(Q,\underline{n})$ meet, whereas any two distinct orbits are disjoint. □

LEMMA 2.2.6. *Let X be a variety, and Y a non-singular locally closed subvariety. Then Y is open in X if and only if $N_x(Y/X) = 0$ for some $x \in Y$.*

PROOF. Note that $\dim T_x(Y) = \dim(Y)$ for all $x \in Y$, whereas $\dim T_x(X) \geq \dim(X)$. Thus, if $N_x(Y/X) = 0$, then $\dim(X) = \dim(Y)$ and hence Y is open in X. The converse is obvious. □

2.3 Orbit closures

The points in the closure of an orbit $\mathcal{O}_M$ may be viewed as geometric degenerations of the representation M. The following fundamental result constructs some of these degenerations in algebraic terms.

THEOREM 2.3.1. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{2.3.1}$$

be an exact sequence of finite-dimensional representations of Q. Then the closure of $\mathcal{O}_M$ contains $\mathcal{O}_{M'\oplus M''}$. Moreover, the exact sequence (2.3.1) splits if and only if $\mathcal{O}_M = \mathcal{O}_{M'\oplus M''}$.

PROOF. Let the representation M be given by the spaces V_i and the maps f_α; then the subrepresentation M' yields subspaces $V_i' \subset V_i$, $i \in Q_0$, such that $f_\alpha(V_i') \subset V_j'$ for all $\alpha : i \to j$. Choosing bases for the V_i' and completing them to bases of the V_i, we obtain a point $x = (x_\alpha) \in \mathrm{Rep}(Q, \underline{n})$ such that $M_x \simeq M$ and $x_\alpha(k^{n'_i}) \subset k^{n'_j}$ for all $\alpha : i \to j$. Here $\underline{n}' = (n'_i)_{i \in Q_0}$ denotes the dimension vector of M', and

$$k^{n_i} = k^{n'_i} \oplus k^{n''_i} \tag{2.3.2}$$

is the obvious decomposition (of vector spaces), so that $\underline{n}'' = (n''_i)_{i \in Q_0}$ is the dimension vector of M''. Then the family of restrictions $x' := (x'_\alpha : k^{n'_i} \to k^{n'_j})$ satisfies $M_{x'} \simeq M'$. Moreover, the family of quotient maps $x'' := (x''_\alpha : k^{n'_i} \to k^{n'_j})$ satisfies $M_{x''} \simeq M''$.

Define a homomorphism of algebraic groups

$$\lambda : \mathbb{G}_m \longrightarrow \mathrm{GL}(\underline{n}), \quad t \longmapsto \big(\lambda_i(t)\big)_{i \in Q_0},$$

where

$$\lambda_i(t) := \begin{pmatrix} t\,\mathrm{id}_{n'_i} & 0 \\ 0 & \mathrm{id}_{n''_i} \end{pmatrix}$$

in the decomposition (2.3.2). We have

$$x_\alpha = \begin{pmatrix} x'_\alpha & y_\alpha \\ 0 & x''_\alpha \end{pmatrix}$$

for some y_α, so that

$$\lambda_j(t)\, x_\alpha\, \lambda_i(t)^{-1} = \begin{pmatrix} x'_\alpha & t y_\alpha \\ 0 & x''_\alpha \end{pmatrix}.$$

As a consequence, the morphism

$$\lambda_x : \mathbb{G}_m \longrightarrow \mathrm{GL}(\underline{n}) \cdot x, \quad t \longmapsto \lambda(t) \cdot x$$

extends to a morphism

$$\overline{\lambda}_x : \mathbb{A}^1 \longrightarrow \overline{\mathrm{GL}(\underline{n}) \cdot x}, \quad 0 \longmapsto \begin{pmatrix} x'_\alpha & 0 \\ 0 & x''_\alpha \end{pmatrix}.$$

It follows that $\overline{\mathcal{O}_M}$ contains $M' \oplus M''$.

To complete the proof, it suffices to show that the exact sequence (2.3.1) splits if $M \simeq M' \oplus M''$ as representations of Q. For this, consider the *left* exact sequence

$$0 \longrightarrow \mathrm{Hom}_Q(M'', M') \longrightarrow \mathrm{Hom}_Q(M'', M) \longrightarrow \mathrm{Hom}_Q(M'', M'') \longrightarrow 0 \tag{2.3.3}$$

induced by (2.3.1). Since $M \simeq M' \oplus M''$, we have

$$\dim \mathrm{Hom}_Q(M'', M) = \dim \mathrm{Hom}_Q(M'', M') + \dim \mathrm{Hom}_Q(M'', M'').$$

It follows that the sequence (2.3.3) is also right exact. Thus, the identity of M'' extends to a morphism of representations $M'' \to M$, which provides a splitting of (2.3.1). □

An iterated application of Theorem 2.3.1 yields:

Corollary 2.3.2. *Let M be a finite-dimensional representation of Q equipped with a filtration*

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_r M = M$$

by subrepresentations. Then the closure of $\mathcal{O}_M$ contains $\mathcal{O}_{\operatorname{gr} M}$, where

$$\operatorname{gr} M := \bigoplus_{i=1}^{r} F_i M / F_{i-1} M$$

denotes the associated graded representation.

Next, consider a *composition series*

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_r M = M,$$

that is, a filtration such that all subquotients $F_i M / F_{i-1} M$ are simple representations. Then, by the Jordan-Hölder theorem (see [3, Thm. 1.1.4]), these subquotients are independent of the series, up to reordering; in other words, the semi-simple representation $\operatorname{gr} M$ depends only on M. We put

$$M^{\mathrm{ss}} := \operatorname{gr} M.$$

Then $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{M^{\mathrm{ss}}}$ by Corollary 2.3.2.

We shall show that $\mathcal{O}_{M^{\mathrm{ss}}}$ is the unique closed orbit in $\overline{\mathcal{O}_M}$; this will yield another proof of the uniqueness of M^{ss}. For this, we need the following auxiliary result.

Lemma 2.3.3. *Let N, N' be finite-dimensional semi-simple representations of Q. Then $N \simeq N'$ (as representations) if and only if $\det(a_N) = \det(a_{N'})$ for all $a \in kQ$, where a_N denotes the map $N \to N$, $x \mapsto ax$.*

Proof. The implication ($\Rightarrow$) is clear. For the converse, denote by A the image of kQ in $\operatorname{End}(N \oplus N')$. Then A is a finite-dimensional semi-simple algebra, since $N \oplus N'$ is a finite-dimensional semi-simple A-module. Hence there exist positive integers $m_1, \ldots, m_r$ such that

$$A \simeq \prod_{i=1}^{r} \operatorname{Mat}_{m_i \times m_i}(k).$$

Moreover, $N \simeq \bigoplus_{i=1}^{r} n_i\, k^{m_i}$ and $N' \simeq \bigoplus_{i=1}^{r} n'_i\, k^{m_i}$ where the multiplicities $(n_1, \ldots, n_r)$ and $(n'_1, \ldots, n'_r)$ are uniquely determined. By our assumption,

$$\prod_{i=1}^{r} \det(x_i)^{n_i} = \prod_{i=1}^{r} \det(x_i)^{n'_i}$$

for all $x_i \in \operatorname{Mat}_{m_i \times m_i}(k)$. Taking $x_i = \lambda_i \operatorname{id}_{m_i}$ where $\lambda_1, \ldots, \lambda_r$ are arbitrary scalars, it follows that $n_i = n'_i$ for all i. Equivalently, $N \simeq N'$. $\square$

THEOREM 2.3.4. *Let M be a finite-dimensional representation of Q. Then $\mathcal{O}_{M^{\mathrm{ss}}}$ is the unique closed orbit in the closure of $\mathcal{O}_M$.*

PROOF. For any representation N such that $\mathcal{O}_N \subset \overline{\mathcal{O}_M}$, we have

$$\mathcal{O}_{N^{\mathrm{ss}}} \subset \overline{\mathcal{O}_N} \subset \overline{\mathcal{O}_M}.$$

Thus, it suffices to show that $N^{\mathrm{ss}} = M^{\mathrm{ss}}$. By Lemma 2.3.3, this is equivalent to checking the equalities

$$\det(a_{N^{\mathrm{ss}}}) = \det(a_{M^{\mathrm{ss}}}) \text{ for all } a \in kQ. \tag{2.3.4}$$

Let $x, y \in \mathrm{Rep}(Q, \underline{n})$ such that $M \simeq M_x$ and $N \simeq M_y$. Then $y \in \overline{\mathrm{GL}(\underline{n}) \cdot x}$. On the other hand, any $a \in kQ$ defines a map

$$a_z := a_{M_z} \in \mathrm{End}(M_z)$$

for any $z \in \mathrm{Rep}(Q, \underline{n})$; the matrix coefficients of a_z depend polynomially on z. Moreover, we have $a_{g\cdot z} = g a_z g^{-1}$ for all $g \in \mathrm{GL}(\underline{n})$ and $z \in \mathrm{Rep}(Q, \underline{n})$. Thus, the map

$$f_a : \mathrm{Rep}(Q, \underline{n}) \longrightarrow k, \quad z \longmapsto \det(a_z)$$

is polynomial and invariant under $\mathrm{GL}(\underline{n})$; hence f_a is constant on orbit closures. It follows that $f_a(x) = f_a(y)$, that is, (2.3.4) holds with M^{ss}, N^{ss} being replaced with M, N. Applying this to $\mathcal{O}_{M^{\mathrm{ss}}} \subset \overline{\mathcal{O}_M}$ and, likewise, for N completes the proof of (2.3.4). □

COROLLARY 2.3.5. *The orbit $\mathcal{O}_M$ is closed if and only if the representation M is semi-simple.*

REMARK 2.3.6. If Q has no oriented cycle, then any semi-simple representation with dimension vector $\underline{n}$ is isomorphic to $\bigoplus_{i\in Q_0} n_i\, S(i)$ by Proposition 1.3.1. Therefore, 0 is the unique point $x \in \mathrm{Rep}(Q, \underline{n})$ such that M_x is semi-simple, and hence *every orbit closure contains the origin.* This can be seen directly, as follows. We may choose a function $h : Q_0 \to \mathbb{N}$ such that $h(i) < h(j)$ whenever there exists an arrow $\alpha : i \to j$. Now consider the homomorphism

$$\lambda : \mathbb{G}_m \longrightarrow \mathrm{GL}(\underline{n}), \quad t \longmapsto (t^{h(i)}\, \mathrm{id}_{n_i})_{i\in Q_0}.$$

Then λ acts on $\mathrm{Rep}(Q, \underline{n})$ via

$$\lambda(t) \cdot x_\alpha := t^{h(j)-h(i)} x_\alpha$$

for any $\alpha : i \to j$. Thus, for any $x = (x_\alpha)_{\alpha: i\to j} \in \mathrm{Rep}(Q, \underline{n})$, the morphism

$$\lambda_x : \mathbb{G}_m \longrightarrow \mathrm{GL}(\underline{n}) \cdot x, \quad t \longmapsto \lambda(t) \cdot x$$

extends to a morphism

$$\overline{\lambda}_x : \mathbb{A}^1 \longrightarrow \operatorname{Rep}(Q, \underline{n}), \quad 0 \longmapsto 0.$$

So the closure of $\operatorname{GL}(\underline{n}) \cdot x$ contains 0.

Examples 2.3.7. 1) Applying Theorem 2.3.4 to the loop, we see that the closure of the conjugacy class of an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ contains a unique closed conjugacy class, that of $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. In particular, the closed conjugacy classes are exactly those of diagonalizable matrices.

More generally, the simultaneous conjugacy class of an r-tuple of $n \times n$ matrices $(x_1, \ldots, x_r)$ is closed if and only if the $k\langle X_1, \ldots, X_r\rangle$-module k^n is semi-simple, where each X_i acts on k^n via x_i.

2) Consider the quiver H_r and the dimension vector $\underline{n} = (1, n)$ as in Example 2.1.3. Then the closure of the orbit of a tuple $(v, x_1, \ldots, x_r)$ contains $(0, x_1, \ldots, x_r)$. Thus, the closed orbits are exactly those of the tuples $(0, x_1, \ldots, x_r)$ such that the $k\langle X_1, \ldots, X_r\rangle$-module k^n is semi-simple.

On the other hand, one may show that the subset $\operatorname{Rep}^{\mathrm{cyc}}(H_r, \underline{n})$ of cyclic tuples is open in $\operatorname{Rep}(H_r, \underline{n})$; clearly, each $\operatorname{GL}(n)$-orbit in that subset is closed there. Moreover, the quotient

$$\operatorname{Rep}^{\mathrm{cyc}}(H_r, \underline{n}) \longrightarrow \operatorname{Rep}^{\mathrm{cyc}}(H_r, \underline{n}) / \operatorname{GL}(n)$$

is a principal $\operatorname{GL}(n)$-bundle, and $\operatorname{Rep}^{\mathrm{cyc}}(H_r, \underline{n}) / \operatorname{GL}(n)$ is a non-singular quasi-projective variety of dimension $(r-1)n^2 + n$. By Example 2.1.3, this variety parametrizes the (left) ideals of codimension n in the free algebra $k\langle X_1, \ldots, X_r\rangle$; it is the *non-commutative Hilbert scheme* introduced in [15] and studied further in [16].

2.4 Schur representations and Gabriel's theorem

Definition 2.4.1. A representation M of the quiver Q is called a *Schur representation* (also known as a *brick*), if $\operatorname{End}_Q(M) = k \operatorname{id}_M$.

Clearly, any Schur representation is indecomposable. The converse does not hold for an arbitrary quiver Q, e.g., for the loop, in view of (1.3.9). However, the converse does hold if the Tits form of Q is positive definite. This is in fact the main step for proving the "if" part of Gabriel's theorem, and is a direct consequence of the following result of Ringel (see [18, pp. 148–149]):

Lemma 2.4.2. *Let M be an indecomposable representation of Q. If M is not Schur, then it has a Schur subrepresentation N such that $\operatorname{Ext}^1_Q(N, N) \neq 0$.*

Proof. We begin with a construction of non-trivial extensions between quotients and submodules of M. Consider an exact sequence of representations

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and write $M' = \bigoplus_{i=1}^r M'_i$, where $M'_1, \ldots, M'_r$ are indecomposable. Then we claim that

$$\mathrm{Ext}^1_Q(M'', M'_i) \neq 0 \quad (i = 1, \ldots, r).$$

Indeed, we have an exact sequence

$$0 \longrightarrow M'_i \longrightarrow M/\bigoplus_{j \neq i} M'_j \longrightarrow M'' \longrightarrow 0$$

which splits if $\mathrm{Ext}^1_Q(M'', M'_i) = 0$. This yields a complement to M'_i in $M/\bigoplus_{j\neq i} M'_j$, and hence a complement to M'_i in M, contradicting the indecomposability of M.

Next, we construct an indecomposable submodule N of M having non-zero self-extensions. Let $f \in \mathrm{End}_Q(M)$ and consider the associated exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow M \xrightarrow{f} \mathrm{Im}(f) \longrightarrow 0.$$

By our assumption and Lemma 1.3.3, we may choose f to be nilpotent and non-zero; we may further assume that $\mathrm{Im}(f)$ has minimal dimension among all such f. Then $f^2 = 0$, i.e., $\mathrm{Im}(f) \subset \mathrm{Ker}(f)$. Thus, we may choose an indecomposable summand N of $\mathrm{Ker}(f)$ such that the projection

$$p : \mathrm{Im}(f) \longrightarrow N$$

is non-zero. Then p is in fact injective, since the composition

$$M \xrightarrow{f} \mathrm{Im}(f) \xrightarrow{p} N \hookrightarrow M$$

is a non-zero endomorphism with image $\mathrm{Im}(p)$ of dimension $\leq \dim \mathrm{Im}(f)$. Moreover,

$$\mathrm{Ext}^1_Q\big(\mathrm{Im}(f), N\big) \neq 0$$

by the first step of the proof. Next, consider the exact sequence

$$0 \longrightarrow \mathrm{Im}(f) \xrightarrow{p} N \longrightarrow C := \mathrm{Coker}(p) \longrightarrow 0.$$

By [3, Prop. 2.5.2], it yields a long exact sequence of Ext groups

$$\cdots \longrightarrow \mathrm{Ext}^1_Q(N, N) \longrightarrow \mathrm{Ext}^1_Q\big(\mathrm{Im}(f), N\big) \longrightarrow \mathrm{Ext}^2_Q(C, N) \longrightarrow \cdots$$

Since $\mathrm{Ext}^2_Q(C, N) = 0$ by Corollary 1.4.2, we see that $\mathrm{Ext}^1_Q(N, N) \neq 0$.

If N is Schur, then the proof is complete; otherwise, we replace M with N and conclude by induction. $\square$

We may now complete the proof of Gabriel's theorem, in a more precise form:

THEOREM 2.4.3. *Assume that the Tits form q_Q is positive definite. Then:*
(i) *Every indecomposable representation is Schur and has no non-zero self-extensions.*
(ii) *The dimension vectors of the indecomposable representations are exactly those $\underline{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\underline{n}) = 1$.*
(iii) *Every indecomposable representation is uniquely determined by its dimension vector, up to isomorphism.*
(iv) *There are only finitely many isomorphism classes of indecomposable representations of Q. In other words, Q is of finite representation type.*

PROOF. Consider an indecomposable representation M. If M is not Schur, let N be a subrepresentation satisfying the assertions of Lemma 2.4.2. Then we have by (1.4.7):

$$q_Q(\underline{\dim}\, N) = 1 - \dim \operatorname{Ext}^1_Q(N, N) \leq 0$$

which contradicts our assumption on q_Q. Thus, M is a Schur representation; moreover, $\operatorname{Ext}^1_Q(M, M) = 0$, so that $q_Q(\underline{\dim}\, M) = 1$. This proves (i) and one half of (ii).

For the other half, consider $\underline{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\underline{n}) = 1$, and a representation M such that $\underline{\dim}\,(M) = \underline{n}$ and the orbit $\mathcal{O}_M$ has maximal dimension; in particular, $\mathcal{O}_M$ is not contained in the closure of another orbit. If $M \simeq M' \oplus M''$ is a non-trivial decomposition of representations, then every extension of M'' by M' splits by Proposition 2.1.7 and Theorem 2.3.1. Thus, $\operatorname{Ext}^1_Q(M'', M') = 0$ and likewise, $\operatorname{Ext}^1_Q(M', M'') = 0$. Hence, setting $\underline{n}' := \dim M'$ and $\underline{n}'' := \dim M''$, we obtain

$$q_Q(\underline{n}) = q_Q(\underline{n}' + \underline{n}'') = q_Q(\underline{n}') + \langle \underline{n}', \underline{n}'' \rangle_Q + \langle \underline{n}'', \underline{n}' \rangle_Q + q_Q(\underline{n}'')$$

where $q_Q(\underline{n}') \geq 1$, $q_Q(\underline{n}'') \geq 1$, $\langle \underline{n}', \underline{n}'' \rangle_Q = \dim \operatorname{Hom}_Q(M', M'') \geq 0$ and likewise, $\langle \underline{n}'', \underline{n}' \rangle_Q \geq 0$. So $q_Q(\underline{n}) \geq 2$, a contradiction.

(iii) follows from (i) combined with Corollary 2.2.5.

Finally, (iv) is a consequence of (iii) together with the assumption that q_Q is positive definite, which implies that there are only finitely many $\underline{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\underline{n}) = 1$. $\square$

The tuples $\underline{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\underline{n}) = 1$ are called the *positive roots* of Q. Thus, Theorem 2.4.3 sets up a bijection between the isomorphism classes of indecomposable representations and the positive roots.

Note that the set of positive roots only depends on the underlying undirected graph of Q. For example, if Q is of type A_r, then the positive roots are exactly the partial sums $\sum_{\ell=i}^{j} \varepsilon_\ell$, where $1 \leq i \leq j \leq r$. When $r = 2$, this gives back the classification of indecomposable representations of S_2 presented in Example 1.3.8.3.

3 Representations of finitely generated algebras

In this section, we generalize some of the results of Section 2 to the setting of representations of finitely generated algebras. The latter include of course quiver algebras and finite-dimensional algebras, but also group algebras $k[G]$, where G is any finitely generated group.

In this setting, it is natural to consider a more advanced object than the representation space of quiver theory; namely, the representation scheme $\mathcal{R}ep(A,n)$ parametrizing the homomorphisms from a finitely generated algebra A to the algebra of $n \times n$ matrices. This is an affine scheme of finite type, that we discuss in Subsection 3.1.

The natural action of the general linear group on the representation scheme is considered in Subsection 3.2. In particular, the normal space to the orbit of a representation is again identified with the space of self-extensions.

Subsection 3.3 is devoted to a variant of the representation scheme, in the setting of algebras equipped with orthogonal idempotents. As an application, we describe the representation scheme of any quiver algebra, in terms of the representation spaces of the quiver for fixed dimension vectors. Additional details, and further results concerning finite-dimensional algebras, may be found in Bongartz's article [4] and its references.

As prerequisites, we shall assume some familiarity with affine schemes, refering to the book [9] for an introduction to that topic.

3.1 Representation schemes

Let A be a finitely generated algebra. Choose a presentation

(3.1.1)
$$A = k\langle X_1, \ldots, X_r \rangle / I$$

where $k\langle X_1, \ldots, X_r \rangle$ denotes the free algebra on $X_1, \ldots, X_r$, and I is a two-sided ideal. We denote by $a_1, \ldots, a_r$ the images of $X_1, \ldots, X_r$ in A.

Definition 3.1.1. For any positive integer n, let $\mathrm{Rep}(A,n)$ be the set of all representations of A on the vector space k^n.

In other words, $\mathrm{Rep}(A,n)$ is the set of all algebra homomorphisms

$$\varphi : A \longrightarrow \mathrm{Mat}_{n\times n}(k).$$

Such a homomorphism is given by the images

$$x_i := \varphi(a_i) \in \mathrm{Mat}_{n\times n}(k) \quad (i = 1, \ldots, r),$$

satisfying the relations

$$P(x_1, \ldots, x_r) = 0 \quad \text{for all } P \in I. \tag{3.1.2}$$

Thus, $\mathrm{Rep}(A,n)$ is the closed algebraic subset of $\mathrm{Mat}_{n\times n}(k)^r$ (the space of r-tuples of matrices of size $n \times n$, an affine space of dimension rn^2) defined by the polynomial equations (3.1.2). We denote by $R(A,n)$ the algebra of regular functions on $\mathrm{Rep}(A,n)$, that is, the quotient of the polynomial ring in the coefficients of r matrices of size $n \times n$, by the ideal of polynomials vanishing at all points of $\mathrm{Rep}(A,n)$. For any $x \in \mathrm{Rep}(A,n)$, we denote by M_x the corresponding A-module.

Next, we introduce a schematic version of $\mathrm{Rep}(A,n)$. Let R be any commutative algebra. Consider the set of representations of A on the vector space R^n which are compatible with the R-module structure, i.e., the set of algebra homomorphisms

$$\varphi : A \longrightarrow \mathrm{Mat}_{n\times n}(R).$$

Such a homomorphism is still given by the images $x_i := \varphi(a_i)$ $(1 \leq i \leq r)$ satisfying the relations (3.1.2). In other words, we have an isomorphism

$$\mathrm{Hom}\,\big(A, \mathrm{Mat}_{n\times n}(R)\big) \simeq \mathrm{Hom}\,\big(\mathcal{R}(A,n), R\big), \tag{3.1.3}$$

where $\mathcal{R}(A,n)$ denotes the quotient of the polynomial algebra in the coefficients of r matrices $x_1, \ldots, x_r$ of size $n \times n$, by the two-sided ideal generated by the coefficients of the matrices $P(x_1, \ldots, x_r)$ with $P \in I$. (The algebra $\mathcal{R}(A,n)$ may contain non-zero nilpotent elements, see Example 3.1.3.2 below.)

Definition 3.1.2. The affine scheme

$$\mathcal{R}ep(A,n) := \mathrm{Spec}\,\mathcal{R}(A,n)$$

is the *scheme of representations of A on k^n*.

The set of R-valued points of the scheme $\mathcal{R}ep(A,n)$ is $\mathrm{Hom}\,\big(\mathcal{R}(A,n), R\big)$ (see [9, Thm. I-40]). Together with (3.1.3), it follows that $\mathcal{R}ep(A,n)$ *represents the covariant functor $R \to \mathrm{Hom}\,\big(A, \mathrm{Mat}_{n\times n}(R)\big)$ from commutative algebras to sets.* Thus, $\mathcal{R}ep(A,n)$ is independent of the presentation (3.1.1) of the algebra A.

In particular, the set of k-valued points of $\mathcal{R}ep(A,n)$ is

$$\mathcal{R}ep(A,n)(k) = \mathrm{Hom}\,\big(A, \mathrm{Mat}_{n\times n}(k)\big) = \mathrm{Rep}(A,n).$$

In other words, $\mathrm{Rep}(A,n)$ is the reduced scheme $\mathcal{R}ep(A,n)_{\mathrm{red}}$ (as defined in [9, p. 25]). Equivalently, $R(A,n)$ is the quotient of $\mathcal{R}(A,n)$ by its ideal consisting of all nilpotent elements. As a consequence, $\mathrm{Rep}(A,n)$ is also independent of the presentation of A; of course, this may be seen directly.

EXAMPLES 3.1.3. 1) Let A be the free algebra on r generators $X_1, \ldots, X_r$. Then $\mathcal{R}ep(A,n)$ is the affine space $\mathrm{Mat}^r_{n\times n}$ of dimension rn^2, and hence $\mathcal{R}ep(A,n) = \mathrm{Rep}(A,n)$.
2) Let $A = k[X]/X^m\, k[X]$, where $m \geq n \geq 2$. Then $\mathrm{Rep}(A,n)$ consists of $n\times n$ matrices x such that $x^m = 0$, i.e., of all nilpotent matrices since $m \geq n$. As a consequence, the trace map $\mathrm{Mat}_{n\times n}(k) \to k$, $x \mapsto \mathrm{Tr}(x)$ vanishes identically on $\mathrm{Rep}(A,n)$. (In fact, the ideal of $\mathrm{Rep}(A,n)$ is generated by the coefficients of the characteristic polynomial of x.)

On the other hand, the algebra $\mathcal{R}(A,n)$ is the quotient of the polynomial algebra in the coefficients of the $n\times n$ matrix x, by the ideal generated by the coefficients of the matrix x^m. Since the latter coefficients are homogeneous polynomials of degree $m \geq 2$, the image of the trace map in $\mathcal{R}(A,n)$ is a non-zero element t. Moreover, since the algebra $R(A,n)$ is the quotient of $\mathcal{R}(A,n)$ by its ideal of nilpotents, we see that t is nilpotent. Thus, the scheme $\mathcal{R}ep(A,n)$ is not reduced.

3.2 The action of the general linear group

The group $\mathrm{GL}(n)$ acts on $\mathrm{Rep}(A,n)$ by conjugation:

$$(g\cdot\varphi)(a) = (g\varphi g^{-1})(a)$$

for all $g \in \mathrm{GL}(n)$, $\varphi \in \mathrm{Hom}\big(A, \mathrm{Mat}_{n\times n}(k)\big)$, and $a \in A$. Viewing $\mathrm{Rep}(A,n)$ as a closed subset of $\mathrm{Mat}_{n\times n}(k)^r$, this action is the restriction of the action by simultaneous conjugation. In particular, $\mathrm{GL}(n)$ acts algebraically via its quotient $\mathrm{PGL}(n)$. The orbits are the isomorphism classes of n-dimensional A-modules; for such a module M, we denote the corresponding orbit by $\mathcal{O}_M$. The description of the isotropy groups (Proposition 2.2.1) extends without change to this setting, as well as all the results of Subsection 2.3.

Likewise, $\mathcal{R}ep(A,n)$ is a closed subscheme of the affine space $\mathrm{Mat}^r_{n\times n}$, stable under the action of $\mathrm{GL}(n)$ by simultaneous conjugation; the induced action of $\mathrm{GL}(n)$ on $\mathcal{R}ep(A,n)$ is compatible with that on the reduced subscheme $\mathrm{Rep}(A,n)$.

We now describe the tangent spaces to $\mathcal{R}ep(A,n)$ and to its orbits in terms of derivations.

Given an A-module M, we say that a map $D : A \to \mathrm{End}(M)$ is a *k-derivation*, if D is k-linear and satisfies the Leibniz rule

$$D(ab) = D(a)b_M + a_M D(b)$$

for all $a, b \in A$, where a_M denotes the multiplication by a in M. The set of derivations is a subspace of $\mathrm{Hom}\big(A, \mathrm{End}(M)\big)$, denoted by $\mathrm{Der}\big(A, \mathrm{End}(M)\big)$.

Any $f \in \mathrm{End}(M)$ defines a derivation

$$\mathrm{ad}\, f : A \longrightarrow \mathrm{End}(M), \quad a \longmapsto f a_M - a_M f$$

called an *inner derivation*. The image of the resulting map

$$\mathrm{ad} : \mathrm{End}(M) \longrightarrow \mathrm{Der}\big(A, \mathrm{End}(M)\big)$$

will be denoted by $\mathrm{Inn}\big(A, \mathrm{End}(M)\big)$. Clearly, the kernel of ad is $\mathrm{End}_A(M)$; in other words, we have an exact sequence

$$0 \longrightarrow \mathrm{End}_A(M) \longrightarrow \mathrm{End}(M) \xrightarrow{\mathrm{ad}} \mathrm{Inn}\big(A, \mathrm{End}(M)\big) \longrightarrow 0. \tag{3.2.1}$$

THEOREM 3.2.1. *Let $x \in \mathrm{Rep}(A, n)$ with corresponding A-module $M = M_x$. Then there is a natural isomorphism*

$$T_x\big(\mathcal{R}ep(A, n)\big) \simeq \mathrm{Der}\big(A, \mathrm{End}(M)\big)$$

which restricts to an isomorphism

$$T_x\big(\,\mathrm{GL}(n) \cdot x\big) \simeq \mathrm{Inn}\big(A, \mathrm{End}\, M)\big).$$

PROOF. Denote by

$$k[\varepsilon] = k[X]/X^2\, k[X]$$

the algebra of dual numbers. Then $T_x\big(\mathcal{R}ep(A, n)\big)$ is the set of those $k[\varepsilon]$-points of the scheme $\mathcal{R}ep(A, n)$ that lift the k-point x, i.e., the set of those algebra homomorphisms

$$\varphi : \mathcal{R}(A, n) \longrightarrow k[\varepsilon]$$

that lift the homomorphism

$$x : \mathcal{R}(A, n) \longrightarrow k = k[\varepsilon]/k\varepsilon$$

(see [9, VI.1.3]). By (3.1.3), this identifies $T_x\big(\mathcal{R}(A, n)\big)$ with the set of all linear maps $D : A \to \mathrm{Mat}_{n\times n}(k)$ such that $x + \varepsilon D : A \to \mathrm{Mat}_{n\times n}(k[\varepsilon])$ is an algebra homomorphism; equivalently, D is a k-derivation. This proves the first isomorphism; the second one follows from the fact that the differential at id_n of the orbit map $g \mapsto g\varphi g^{-1}$ is the map $f \mapsto f\varphi - \varphi f$. □

Next, using these descriptions, we generalize Theorem 2.2.3 and Corollary 2.2.5 to the setting of representations of algebras.

COROLLARY 3.2.2. *For any $x \in \mathrm{Rep}(A, n)$ with corresponding A-module $M = M_x$, we have an exact sequence*

$$0 \longrightarrow \mathrm{End}_A(M) \longrightarrow \mathrm{End}(M) \longrightarrow T_x\big(\mathcal{R}ep(A, n)\big) \longrightarrow \mathrm{Ext}^1_A(M, M) \longrightarrow 0.$$

PROOF. In view of the exact sequence (3.2.1) and of Theorem 3.2.1, it suffices to show that the quotient $\mathrm{Der}\big(A, \mathrm{End}(M)\big)/\,\mathrm{Inn}\big(A, \mathrm{End}(M)\big)$ is isomorphic to $\mathrm{Ext}^1_A(M, M)$.

For this, recall that $\mathrm{Ext}^1_A(M, M)$ parameterizes the self-extensions of the A-module M. Given $D \in \mathrm{Der}\big(A, \mathrm{End}(M)\big)$, we let A act on the vector space $M \oplus M$ by setting

$$a \cdot (m_1, m_2) := (am_1 + D(a)m_2, am_2). \tag{3.2.2}$$

For that action, $M \oplus M$ is an A-module that we denote E_D. Further, $M \oplus 0$ is a submodule, isomorphic to M, and the quotient module is also isomorphic to M. In other words, we obtain a self-extension

$$0 \longrightarrow M \longrightarrow E_D \longrightarrow M \longrightarrow 0.$$

Conversely, any self-extension $0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$ is isomorphic to some E_D: to see this, choose a splitting $E \simeq M \oplus M$ as vector spaces, and check that the resulting action of A on $M \oplus M$ satisfies (3.2.2) for a unique derivation D. One also checks that the induced map $\mathrm{Der}\big(A, \mathrm{End}(M)\big) \to \mathrm{Ext}^1_A(M, M)$ is linear with kernel $\mathrm{Inn}\big(A, \mathrm{End}(M)\big)$. $\square$

COROLLARY 3.2.3. *The normal space at x to the orbit $\mathrm{GL}(n) \cdot x$ in $\mathcal{R}ep(A, n)$ is isomorphic to $\mathrm{Ext}^1_A(M, M)$.*

As a consequence, the orbit $\mathrm{GL}(n) \cdot x$ is open in $\mathcal{R}ep(A, n)$ if and only if $\mathrm{Ext}^1_A(M, M) = 0$. In that case, $\mathcal{R}ep(A, n)$ is non-singular, and hence equal to $\mathrm{Rep}(A, n)$, in a neighborhood of that orbit.

As an application of these results, we describe the representation schemes of finite-dimensional semi-simple algebras:

PROPOSITION 3.2.4. *Consider the algebra $A := \prod_{i=1}^r \mathrm{Mat}_{m_i \times m_i}(k)$. Then each connected component of the scheme $\mathcal{R}ep(A, n)$ is an orbit of $\mathrm{GL}(n)$; in particular, $\mathcal{R}ep(A, n)$ is non-singular. Its components are indexed by the tuples $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that $m_1 n_1 + \cdots + m_r n_r = n$. The isotropy group of the orbit associated with $\underline{n}$ is isomorphic to $\mathrm{GL}(\underline{n}) = \prod_{i=1}^r \mathrm{GL}(n_i)$ embedded into $\mathrm{GL}(n)$ via its natural representation in*

$$(k^{m_1} \otimes k^{n_1}) \oplus \cdots \oplus (k^{m_r} \otimes k^{n_r}) = k^n.$$

PROOF. Since A is semi-simple and finite-dimensional, all A-modules are projective. Thus, $\mathrm{Ext}^1_A(M, M) = 0$ for any A-module M. By Corollary 3.2.3, each orbit of $\mathrm{GL}(n)$ in $\mathcal{R}ep(A, n)$ is open. Since the complement of an orbit is a union of orbits, it follows that each orbit is closed. And since orbits under $\mathrm{GL}(n)$ are connected (Proposition 2.1.7), they form the connected components of $\mathcal{R}ep(A, n)$. As a consequence, $\mathcal{R}ep(A, n)$ is non-singular; in particular, it coincides with $\mathrm{Rep}(A, n)$.

Next, recall that every A-module of dimension n is isomorphic to a direct sum $n_1\, k^{m_1} \oplus \cdots \oplus n_r\, k^{m_r}$, where $n_1, \ldots, n_r$ are uniquely determined and satisfy $m_1\, n_1 + \cdots + m_r\, n_r = n$. Moreover, we have an isomorphism of algebras

$$\mathrm{End}_A(n_1\, k^{m_1} \oplus \cdots \oplus n_r\, k^{m_r}) \simeq \prod_{i=1}^{r} \mathrm{Mat}_{n_i \times n_i}(k),$$

and hence an isomorphism of algebraic groups

$$\mathrm{Aut}_A(n_1\, k^{m_1} \oplus \cdots \oplus n_r\, k^{m_r}) \simeq \prod_{i=1}^{r} \mathrm{GL}(n_i)$$

embedded in $\mathrm{Mat}_{n\times n}(k)$ resp. $\mathrm{GL}(n)$ as claimed. $\square$

3.3 Representations with a prescribed dimension vector

Consider a finitely generated algebra A equipped with a sequence

$$\underline{e} := (e_1, \ldots, e_r)$$

of orthogonal idempotents (i.e., $e_i^2 = e_i \neq 0$, and $e_i e_j = 0$ whenever $i \neq j$) such that $e_1 + \cdots + e_r = 1$. To any finite-dimensional A-module M, we associate its *$\underline{e}$-dimension vector*

$$\underline{e} - \underline{\dim}\, M := (\dim e_i M)_{i=1,\ldots,r} = (n_1, \ldots, n_r) =: \underline{n}$$

and we set

$$|\underline{n}| := n_1 + \cdots + n_r.$$

Note that $M \simeq \bigoplus_{i=1}^{r} e_i M$, so that $|\underline{n}| = \dim M$.

Given $\underline{n} \in \mathbb{N}^r$, we define the set $\mathrm{Rep}(A, \underline{e}, \underline{n})$ of those algebra homomorphisms $\varphi : A \to \mathrm{Mat}_{n\times n}(k)$ such that $n = |\underline{n}|$ and each $\varphi(e_i)$ equals the projection $p_i : k^n \to k^{n_i}$ to the ith summand of the corresponding decomposition of vector spaces

$$k^n = k^{n_1} \oplus \cdots \oplus k^{n_r}.$$

Then $\mathrm{Rep}(A, \underline{e}, \underline{n})$ is the *representation space of A for the $\underline{e}$-dimension vector $\underline{n}$*; this is a closed subset of $\mathrm{Rep}(A, n)$, which depends on the choice of $\underline{e}$. (When A is finite-dimensional, an intrinsic notion of representation space is defined in [4]).

Next, we introduce the *representation scheme $\mathcal{R}ep(A, \underline{e}, \underline{n})$*. Given a presentation (3.1.1) of A, choose also representatives $P_i(X_1, \ldots, X_r)$ of the e_i's in $k\langle X_1, \ldots, X_r\rangle$. Let $\mathcal{R}(A, \underline{e}, \underline{n})$ be the quotient of the polynomial algebra in the coefficients of r matrices $x_1, \ldots, x_r$ of size $n \times n$, by the ideal generated by the coefficients of the matrices $P(x_1, \ldots, x_r)$ for all $P \in I$, and by the images of the coefficients of the matrices $P_i(x_1, \ldots, x_r) - p_i$ for $i = 1, \ldots, r$. Finally, put

$$\mathcal{R}ep(A, \underline{e}, \underline{n}) := \mathrm{Spec}\, \mathcal{R}(A, \underline{e}, \underline{n}).$$

Then, like in Subsection 3.1, the affine scheme $\mathcal{R}ep(A, \underline{e}, \underline{n})$ represents the functor assigning to each commutative algebra R the set of those algebra homomorphisms

$$\varphi : A \longrightarrow \mathrm{Mat}_{n\times n}(R)$$

such that each $\varphi(e_i)$ is the matrix of the ith projection $R^n \to R^{n_i}$. Moreover, $\mathrm{Rep}(A, \underline{e}, \underline{n})$ may be identified to the reduced scheme $\mathcal{R}ep(A, \underline{e}, \underline{n})_{\mathrm{red}}$, equivariantly for the natural action of the closed subgroup $\mathrm{GL}(\underline{n}) := \prod_{i=1}^{r} \mathrm{GL}(n_i)$ of $\mathrm{GL}(n)$. (Again, $\mathcal{R}ep(A, \underline{e}, \underline{n})$ depends on the choice of $\underline{e}$, and we refer to [4] for an intrinsic notion in the setting of finite-dimensional algebras).

We now show how to build the representation scheme $\mathcal{R}ep(A, n)$ from the schemes $\mathcal{R}ep(A, \underline{e}, \underline{n})$, where $\underline{n} \in \mathbb{N}^r$ satisfies $|\underline{n}| = n$. To formulate the result, we introduce the following:

Definition 3.3.1. The *space of decompositions* is the set $\mathrm{Dec}(n)$ of all vector space decompositions

$$k^n = E_1 \oplus \cdots \oplus E_r\,.$$

The *type* of such a decomposition is the sequence

$$\underline{n} = (\dim E_1, \ldots, \dim E_r) \in \mathbb{N}^r.$$

A decomposition $E_1 \oplus \cdots \oplus E_r = k^n$ is *standard* if each E_i is spanned by a subset of the standard basis vectors $v_1, \ldots, v_n$ of k^n.

The group $\mathrm{GL}(n)$ acts on $\mathrm{Dec}(n)$ via its linear action on k^n. The orbits are the subsets $\mathrm{Dec}(\underline{n})$ of decompositions of a fixed type, with representatives being the standard decompositions; the isotropy group of such a decomposition $k^n = k^{n_1} \oplus \cdots \oplus k^{n_r}$ is $\mathrm{GL}(\underline{n})$.

By Proposition 3.2.4 (and its proof), $\mathrm{Dec}(n)$ is the set of k-points of the representation scheme $\mathcal{R}ep(k^r, n)$, where k^r denotes the semi-simple algebra $\prod_{i=1}^{r} k$. Moreover, $\mathcal{R}ep(k^r, n)$ is non-singular, and its connected components are the orbits of $\mathrm{GL}(n)$.

The injective homomorphism of algebras

$$k^r \longrightarrow A, \quad (t_1, \ldots, t_r) \longmapsto t_1\, e_1 + \cdots + t_r\, e_r$$

yields a $\mathrm{GL}(n)$-equivariant morphism $\Phi : \mathcal{R}ep(A, n) \to \mathcal{R}ep(k^r, n)$, by composition. Denoting (abusively) $\mathcal{R}ep(k^r, n)$ by $\mathrm{Dec}(n)$, we thus obtain:

Proposition 3.3.2. *There is a natural* $\mathrm{GL}(n)$*-equivariant morphism*

$$\Phi : \mathcal{R}ep(A, n) \longrightarrow \mathrm{Dec}(n).$$

The fiber of Φ *at each standard decomposition is isomorphic to* $\mathcal{R}ep(A, \underline{e}, \underline{n})$*, equivariantly for the action of* $\mathrm{GL}(\underline{n})$*.*

Next, consider the case where $A = kQ$ for a quiver Q, and the idempotents e_i are those associated to the vertices. Then the representation scheme $\mathcal{R}ep(kQ, \underline{e}, \underline{n})$ is easily seen to be the affine space $\operatorname{Rep}(Q, \underline{n})$. Since the action of $\operatorname{GL}(\underline{n})$ on this space is linear, the preimage in $\operatorname{Rep}(A, n)$ of the orbit $\operatorname{Dec}(\underline{n}) \simeq \operatorname{GL}(n)/\operatorname{GL}(\underline{n})$ has the structure of a homogeneous vector bundle over that orbit. This shows the following:

Corollary 3.3.3. *For any quiver Q and for any positive integer n, the representation scheme $\mathcal{R}ep(kQ, n)$ is non-singular. Its components are the homogeneous vector bundles over the homogeneous spaces* $\operatorname{GL}(n)/\operatorname{GL}(\underline{n})$ *associated with the representations* $\operatorname{Rep}(Q, \underline{n})$ *of* $\operatorname{GL}(\underline{n})$*, where $\underline{n} \in \mathbb{N}^{Q_0}$ and $|\underline{n}| = n$.*

References

[1] I. Assem, A. Skowroński and D. Simson, *Elements of the Representation Theory of Associative Algebras I: Techniques of Representation Theory*, London Math. Soc. Student Texts **65**, Cambridge University Press, 2006.

[2] M. Auslander, I. Reiten and S. O. Smalo, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Math. **36**, Cambridge University Press, 1995.

[3] D. J. Benson, *Representations and cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Math. **30**, Cambridge University Press, 1991.

[4] K. Bongartz, *A geometric version of the Morita equivalence*, J. Algebra **139**, 159–171.

[5] W. Crawley-Boevey, *Lectures on Representations of Quivers*, notes available at www.maths.leeds.ac.uk/˜pmtwc/quivlecs.pdf.

[6] H. Derksen and J. Weyman, *Quiver representations*, Notices Amer. Math. Soc. **52** (2005), 200–206.

[7] S. Donkin, *Polynomial invariants of representations of quivers*, Comment. Math. Helv. **69** (1994), 137–141.

[8] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Math. **150**, Springer-Verlag, New York, 1995.

[9] D. Eisenbud, J. Harris, *The Geometry of Schemes*, Graduate Texts in Math. **197**, Springer-Verlag, New York, 2000.

[10] A. King, *Moduli of representations of finite-dimensional algebras,* Quart. J. Math. Oxford Seri. (2) **45** (1994), 515–530.

[11] H. Kraft and C. Riedtmann, *Geometry of representations of quivers*, in: Representations of algebras (Durham, 1985), 109–145, London Math. Soc. Lecture Note Ser. **116**, Cambridge Univ. Press, Cambridge, 1986.

[12] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598.

[13] S. Mukai, *An Introduction to Invariants and Moduli*, Cambridge Studies in Advanced Math. **81**, Cambridge University Press, 2003.

[14] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, TIFR Lectures on Mathematics and Physics **51**, Narosa, New Delhi, 1978.

[15] M. V. Nori, *Appendix to the paper by C. S. Seshadri: Desingularization of the moduli space of vector bundles over curves*, in: Proceedings of the International Symposium on Algebraic Geometry, 155–184, Kinokuniya, Tokyo, 1978.

[16] M. Reineke, *Cohomology of non-commutative Hilbert schemes*, Algebr. Represent. Theory **8** (2005), 541–561.

[17] M. Reineke, *Moduli of Representations of Quivers*, in : Trends in Representation Theory of Algebras and Related Topics, 589–638, EMS Series of Congress Reports, Zürich, 2008.

[18] C. Ringel, *Four papers on problems in linear algebra*, in: Representation theory, 141–156, London Math. Soc. Lecture Note Ser. **69**, Cambridge Univ. Press, Cambridge, 1982.

[19] T. A. Springer, *Linear Algebraic Groups, Second edition*, Progress in Math. **9**, Birkhäuser, Basel, 1998.

[20] G. Zwara, *Degenerations of finite-dimensional modules are given by extensions*, Compositio Math. **121** (2000), 205–218.

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