Monodromy and Rigidity

Chris PETERS

Aug, 2006

1 Some group theoretic considerations

The references here are [Borel] and [Sat].

Let k be a field of characteristic zero, not necessarily closed, and let K be an algebraic closure. An algebraic k-group T is an *m-torus* if $T(K)$ is isomorphic to a direct product of m copies of the multiplicative group K^*. There is a finite Galois extension L/k so that $T(L)$ is already isomorphic to m copies of L^*. One says that T is *split over* L. A torus is *anisotropic* if it has no non-trivial characters defined over k, i.e. no homomorphisms $T(k) \to k^*$ besides $t \mapsto 1$. Any torus splits as a semi-direct product $T(k) = T_a T_s$ of a maximal split sub-torus T_s and an anisotropic torus T_a.

Let G be a connected linear algebraic k-group, i.e. it has a faithfull representation as a matrix group such that the group is defined by polynomial equations with coefficients in k. By [Borel, §11.3] all maximal tori in G are conjugate. The centralizer of a maximal torus T is called a Cartan subgroup $C(T)$. By [Borel, §12.1] a Cartan subgroup is a maximal connected nilpotent subgroup of G. One can also speak of split Cartan subgroups: by definition these have a decomposition series with successive quotients k^* or k. Equivalently, it is trigonalizable over k.

All Cartan subgroups are conjugate [Borel, §12.1]. The common dimension is called the *rank* of G. There are always maximal tori $T(k)$ and Cartan subgroups $C(k)$ defined over k, and these are conjugate [Borel, Theorem 12.1]. Hence there is a maximal split torus $T_s(k)$ over k and a maximal split connected nilpotent subgroup $C_s(k)$ of k, the k-Cartan group. By [Borel, Theorem 15.9] these are all k-conjugate. The dimension of $C_s(k)$ is called the *k-rank* of G. If it is zero G not only has no split k-tori, it does not contain any non-trivial connected nilpotent k-group. The following result is needed later on:

Lemma 1.1. *Let G be a connected linear algebraic k-group of k-rank* 0. *Then G has no non-trivial unipotent elements.*

Proof: The Zariski-closure of a non-trivial unipotent element in $G(k)$ is isomorphic to the additive group k (see [Borel, remark in § 7.3]). This is a connected nilpotent split subgroup of G and so the k-rank of G is positive.

Examples 1.2. 1. Let $k = \mathbb{R}$. Then a 1-torus is either $\mathbb{R}^*$ with trivial Galois action, or S^1, with Galois action $\theta \mapsto -\theta$. In the first case the torus is split and the rank is 1; in the second case the torus is anisotropic and the $\mathbb{R}$-Cartan group is 1 so that $\mathbb{R}$-rank is 0.

2. Let k be a finite extension of $\mathbb{Q}$ and G a k-group such that for some embedding $k \hookrightarrow \mathbb{C}$ the resulting group $G(\mathbb{C})$ is compact. Then the k-rank is 0. Indeed, if C is a k-Cartan subgroup of G, the successive quotients from a decomposition series being k^* or k imply that either G contains k^* or k and hence $G(\mathbb{C})$ contains $\mathbb{C}^*$ or $\mathbb{C}$ which is impossible for a compact group.

Next, let me recall the construction of the Weil restriction. Let K/k be a finite Galois extension of a field k of degree d and with Galois group

$$\mathrm{Gal}(K/k) = \{\sigma_i,\, i = 1, \dots, d\}.$$

Viewing K as a k-algebra, we get the regular representation $\rho : K \to M_d(k)$. Then for all positive integers m from the representation ρ one gets a new one, $\rho(m) : M_K(m) \to M_k(md)$, defined by $\rho(m)(A_{ij}) = (\rho(A_{ij})$. Suppose now that G is a K-matrix group $G \subset \mathrm{GL}_K(N)$, then the *Weil restriction* $R_{K/k}G$ is the k-group $\rho(N)(G)$. If $\dim_K G = n$, then $\dim_k R_{K/k}G = nd$. By construction, its group of K-points is a product

$$R_{K/k}G(K) = \prod_1^d G^{(i)}, \quad G^{(i)} := \{g^{\sigma_i} \mid g \in G\}.$$

Example 1.3. Let $k = \mathbb{R}$, $K = \mathbb{C}$, $G = \mathbb{C}^*$. The Galois group of $\mathbb{C}/\mathbb{R}$ consists of the identity and the complex conjugation σ. The map $\rho : \mathbb{C} \to M_2(\mathbb{R})$ is just the map sending $z = x + iy$ to the matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and $\rho(\mathbb{C}^*)$ is just the product of the unit circle S^1 and the half line $\mathbb{R}^*$. The conjugation preserves both factors. It acts as $-\mathrm{id}$ on the first factor and as id on the second. Hence $S := R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ is just $\mathbb{C}^*$ with the standard Galois-action and standard real structure. With the standard embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ one can identify $S(\mathbb{C})$ with the pairs (u, v) with $u^2 \neq -v^2$, i.e. with the pairs $(u + iv, u - iv) \in \mathbb{C}^* \times \mathbb{C}^*$. So $S(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ with Galois action interchanging the two factors.

From example 1.2 ii) one gets the following obvious but useful result.

Lemma 1.4. *Let H be an algebraic group defined over a number field K and let $G = R_{K/\mathbb{Q}}H$ be its Weil-restriction. Suppose that $G(\mathbb{R})$ decomposes as a direct product of real groups $G = \prod_i G_i$. Then the $\mathbb{Q}$-rank of G is at most* $\max_i(\mathbb{R}\text{-rank } G_i)$. *In particular, if some G_i is compact, the $\mathbb{Q}$-rank of G is zero.*

Let me now discuss some classical examples. Fix a field k and a division algebra D over k with center F. The opposite division algebra is denoted D^0. Let V be a k-vector space with right D-action, or, equivalenty, a left D^0-action. The algebra of D-transformations of V is isomorphic to the matrix algebra $M_n(D)$ where n is the rank of V over D. The invertible matrices form the group $\mathrm{GL}_D(V)$. The determinant of an invertible matrix belongs to the units $D^\times$ of D and using the norm map $N : D^\times \to F$ the special linear group $\mathrm{SL}_D(V)$ consists of invertible elements $A \in M_n(D)$ with $N(\det(A)) = 1$.

Suppose that D admits a (generalized) *involution*, i.e. an anti-automorphism $a \mapsto a^\sigma$ of order 1 or two. Denote the resulting involution on D^0 also by σ. Let $\epsilon = \pm 1$. A D-valued k-bilinear form on V is called *ϵ-hermitian with respect to D*, if

$$\left.\begin{array}{rcl} h(v, v'a) & = & h(v, v')a, \\ h(v', v) & = & \epsilon h(v, v')^\sigma \end{array}\right\} \qquad v, v' \in V,\ a \in D. \tag{1}$$

One says that h is *non-degenerate*, if in some D-basis for V the corresponding matrix for h is invertible. By definition, for such h the associated *unitary group* and *special unitary group* are

$$\begin{array}{rcl} \mathrm{U}(V, h) & = & \{g \in \mathrm{GL}_D(V) \mid h(gv, gv') = h(v, v'), \quad \forall v, v' \in V\} \\ \mathrm{SU}(V, h) & = & \mathrm{U}(V, h) \cap \mathrm{SL}_D(V). \end{array}$$

Example 1.5. Let $k = \mathbb{R}$. Then $\mathbb{R}$, $\mathbb{C}$ and the quaternions $\mathbb{H}$ are the only non-trivial division algebras over $\mathbb{R}$ with $F = \mathbb{R}, \mathbb{C}, \mathbb{R}$ respectively. Take for σ the identity, the complex conjugation, the standard involution $a + bi + cj + dk \mapsto a - bi - cj - dk$. For $\epsilon = 1$ one can assume that h is of the form $(\vec{x}, \vec{y}) \mapsto \mathrm{Tr}\, \vec{x}^\sigma \vec{x}$ and one gets the real compact groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, respectively $\mathrm{SU}(n, \mathbb{H})$. These have $\mathbb{R}$-rank 0.

For $\epsilon = -1$ the situation is more complicated. For $B = \mathbb{R}$ one can take for h the standard symplectic group, and so $n = 2m$ and one gets ^{m}. Its $\mathbb{R}$-rank is m. For $B = \mathbb{C}$ one can take for h the diagonal matrix $(-i\mathbf{1}_p, i\mathbf{1}_q)$, $p + q = n$. The resulting group is $\mathrm{SU}(p, q)$ whose $\mathbb{R}$-rank is $r = \min(p, q)$. It is only compact for $r = 0$ since isomorphic to $\mathrm{SU}(n)$. Finally, if $B = \mathbb{H}$, one can take $j\mathbf{1}_n$ and the resulting group is usually denoted $\mathrm{SU}(n, \mathbb{H})^-$. It has $\mathbb{R}$-rank $[n/2]$. For $n = 1$ it is the group S^1 which is compact. The $\mathbb{R}$-rank follows since it is equal to the dimension of a maximal isotropic subspace.

Lemma 1.6. *Let G be any of the classical groups ^{m}, $\mathrm{SU}(p, q, \mathbb{C})$ or $\mathrm{SU}(n, \mathbb{H})^-$ and let $g \in G$ be unipotent. Then $(g - \mathbf{1})^\ell = 0$, where ℓ is the $\mathbb{R}$-rank of G.*

Proof:

2 Two Groups Associated to Monodromy Representations

A local system $\underline{H}_S$ of k-vector spaces on any topological space S can be considered as a left Γ-module H, where Γ is the image of $\pi_1(S, s_0)$ in the group $\mathrm{Aut}(H)$, where H is the fiber of $\underline{H}_S$ at s_0. If H comes equipped with a non-degenerate k-bilinear form Q, preserved by the monodromy action, one has $\Gamma \subset \mathrm{Aut}(H, Q)$. The form Q is supposed to be ϵ-hermitian with respect to the trivial involution on k. This just means that for $\epsilon = 1$, respectively -1 the form Q is symmetric, respectively skew-symmetric.

We assume now that the *representation is completely reducible* so that one can group together all the irreducible constituents which are isomorphic. More generally, let Γ be any group with a finite-dimensional representation in $\mathrm{Aut}(H, Q)$, and assume that the representation is *isotypical*, i.e. there is an *irreducible* Γ-module V such that H is a direct sum of copies of V. Put

$$\begin{array}{rr} D := \mathrm{End}_\Gamma(V) & E := \mathrm{End}_\Gamma(H), \\ F := \mathrm{Center}(D) & U := \mathrm{Hom}_\Gamma(V, H). \end{array} \tag{2}$$

Then D is a k-division algebra (this is just Schur's Lemma), and so F is a finite extension field over k. By construction we have :

Lemma 2.1. *The algebra D is central and simple over F and hence for some integer r we have $\dim_F D = r^2$.*

Now D acts on the left on V and on the right on U (by composition) and

$$H = U \otimes_D V, \tag{3}$$

where the tensor product is $U \otimes_F V$ modulo the subspace generated by $(u \circ \alpha \otimes v) - (u \otimes \alpha \circ v)$, $\alpha \in D$, $u \in U, v \in V$.

Lemma 2.2 ([Sat, Ch. IV, Lemma 1.1])**.** *Suppose that as a D-module U has rank a and V has rank b. Then*

$$\dim_k H = abr^2[F : k] \tag{4}$$

and the number of irreducible constituents isomorphic to V in H is equal to a.

The form Q induces an involution $a \mapsto a^*$ on E by setting

$$Q(x, ay) = Q(a^* x, y), \quad \forall x, y \in H.$$

On the other hand, Q makes H and hence V self-dual and so, by [Sat, IV, Lemma 2.2] there is an involution $b \mapsto b^\sigma$ on D preserving the center F such that σ coincides with $*$ on F. Moreover, by [Sat, IV, Theorem 2.3]:

Proposition 2.3. *There is a non-degenerate $\tilde{\epsilon}$-hermitian form h_V on V with respect to the* opposite *involution σ on D^0 (remember that V has a left D-action, hence a right D^0-action), and a $(-\epsilon\tilde{\epsilon})$-hermitian form h_U on U such that*

$$Q(u \otimes_D v, u' \otimes_D v') = {}^{\mathsf{T}}{}_{D/F}(h_U(u,u')[h_V(v,v')]^0).$$

As to signs, one needs to distinguish how $*$ acts on F. If $* = \mathrm{id}$ one calls $*$ (and also σ) of the *first kind*, and of the *second kind* otherwise. Introduce:

$$\begin{aligned} G_U &:= R_{F/\mathbb{Q}}\,\mathrm{SU}(U,h_U) \quad (\textit{monodromy deformation group}) && (5)\\ G_V &:= R_{F/\mathbb{Q}}\,\mathrm{SU}(V,h_V) \quad (\textit{algebraic monodromy group}). && (6)\end{aligned}$$

Then we have:

Lemma 2.4. 1. *The Lie algebra of the group $G_U(\mathbb{R})$ is equal to* $\mathrm{End}(H,Q)\otimes\mathbb{R}$.

2. *The monodromy representation factors over the natural representation $G_V \to \mathrm{Aut}(H,Q)$.*

3 Variations of Hodge Structure

Suppose next that $k = \mathbb{Q}$ and that $\underline{H}_S$ admits a $\mathbb{Z}$-structure which underlies a Q-polarized variation of Hodge structures. Let me briefly recall the definition.

Definition 3.1. A *variation of Hodge structure* on S of weight n is a local system $\underline{H}_S$ of free $\mathbb{Z}$-modules of finite rank on S such that each fiber over $t \in S$ of the complexification admits a Hodge structure of weight n and such that
— the associated Hodge flag $F_t^\bullet$ depends holomorphically on t (this is the holomorphicity of the period map)
— the flat connection ∇ satisfies *Griffiths' horizontality condition*:

$$\nabla_\xi F_t^q \subset F_t^{q-1}, \quad \xi \text{ a germ of a holomorphic tangent field at } t\ .$$

(this last condition is the horizontality of the period map).

The Hodge structure is *polarized* by a flat bilinear integral form Q if Q induces a polarization on the Hodge structures on each fibres of $\underline{H}_S$.

For any variation of Hodge structures the monodromy representation is complete reducible:

Theorem 3.2 ([Del71, 4.2.6])**.** *A polarized variation of Hodge structures over a quasi-projective manifold is direct sum of irreducible ones.*

This implies that we can apply the considerations of 2. As in [S-Zu, Thm. 2.4.1] one shows:

Lemma 3.3. *Suppose that $k = \mathbb{Q}$ and that $\underline{H}_S$ underlies a Q-polarized variation of Hodge structures. Assume that the local system is isotypical and let D be the algebra of Q-endomorphisms of $\underline{H}_S$. One has two possibilities for the center F of D:*

(R) *F is either a totally real number field and $*$ is of the first kind,*

(C) *F is a quadratic extension of a totally real number field F_0 and $*$ is the complex conjugation on F.*

Proposition 3.4 ([,]). *Suppose that (H, Q) underlies a variation of Hodge structure. Then* $\mathrm{End}(H, Q$ *inherits a weight* 0 *Hodge structure and (H, Q) is rigid as a variation of Hodge structure if* $\mathrm{End}^{-1,1}(H, Q) = 0$*. This is in particular the case if G_U is* 0*-dimensional.*

We next consider what happens when we extend to $\mathbb{R}$. From Lemma 2.1 we know that D is a central simple algebra over F and that $\dim_F D = r^2$. Let $[F_0 : \mathbb{Q}] = t$, let $\sigma_i : F_0 \to \mathbb{R}$, $i = 1, \dots, t$ be the distinct real embeddings. Our variation of Hodge structure on $\underline{H}_S$ splits over the reals as

$$(H, Q) \otimes \mathbb{R} \simeq \bigoplus_{i}^{t} (H, Q)^{(i)},$$

and this gives the following restriction:

Lemma 3.5. *For an absolutely irreducible representation we must have $F = F_0 = \mathbb{Q}$.*

In any case, we have:

Lemma 3.6. *$D^{(i)} := D \otimes_{\sigma_i} \mathbb{R}$ is a matrix algebra over one of the three simple real division algebras $\mathbb{R}, \mathbb{H}$ or $\mathbb{C}$:*

Case R1 *$F \otimes_{\sigma_i} \mathbb{R} = \mathbb{R}$, $D^{(i)} = M_r(\mathbb{R})$*

Case R2 *$F \otimes_{\sigma_i} \mathbb{R} = \mathbb{R}$, r is even and $D^{(i)} = M_{r/2}(\mathbb{H})$,*

Case C *$F \otimes_{\sigma_i} \mathbb{R} = \mathbb{C}$ and $D^{(i)} = M_r(\mathbb{C})$.*

Lemma 3.7. *Put*

$$(V, Q)^{(i)} := (V, Q) \otimes_{\sigma_i} \mathbb{R}, \quad U^{(i)} := U \otimes_{\sigma_i} \mathbb{R}, \quad H^{(i)} := U^{(i)} \otimes_{D^{(i)}} V^{(i)}.$$

Recall that $\dim_D U = a$ *and that* $\dim_D V = b$ *we have*

Case R1 *$U^{(i)} = \mathbb{R}^{ar}$, $V^{(i)} = \mathbb{R}^{br}$,*

Case R2 $U^{(i)} = \mathbb{H}^{ar/2}$, $V^{(i)} = \mathbb{H}^{br/2}$,

Case C $U^{(i)} = \mathbb{C}^{ar}$, $V^{(i)} = \mathbb{C}^{br}$.

Recall (Prop. 2.3) that if the induced hermitian form on $U^{(i)}$ has sign ϵ_i then the induced form on $V^{(i)}$ has sign $-\epsilon\epsilon_i$. In this way, one thus obtains:

Lemma 3.8. *Let $\sigma_i : F_0 \hookrightarrow \mathbb{R}$, $i = 1, \ldots, t$ be the real embeddings of F_0, as before and let $R1^{\pm}$, $R2^{\pm}$, respectively $C^{\pm}$ be the numbers of embeddings σ_i of type $(R1$, sign $\epsilon_i = \pm 1)$, $(R2$, sign $\epsilon_i = \pm 1)$, $(C$, sign $\epsilon_i = \pm 1)$ respectively. For each of $(C$, sign $\epsilon_i = -1)$, let (c_i, d_i), $i = 1, \ldots, C^-$, respectively (c'_i, d'_i), $i = 1, \ldots, C^-$ be the signatures of the corresponding hermitian forms on $U^{(i)}$, respectively $V^{(i)}$. Then with the notation* (5), *and* (6) *one has*

$$\begin{aligned} G_U(\mathbb{R}) &= \prod^{R1^+} \mathrm{SO}_{ar} \times \prod^{R1^-} \mathrm{Sp}_{ar/2} \times \prod^{R2^+} \mathrm{SU}_{ar/2}(\mathbb{H}) \times \\ &\quad \prod^{R2^-} \mathrm{SU}_{ar/2}(\mathbb{H})^- \times \prod^{C^+} \mathrm{SU}_{ar} \times \prod_{i=1}^{C^-} \mathrm{SU}(b_i, c_i), \\ &\qquad (c_i + d_i = ar), \end{aligned}$$

and if $\epsilon = 1$, respectively $\epsilon = -1$, one then has correspondingly

$$\begin{aligned} G_V(\mathbb{R}) &= \prod_{i=1}^{R1^+} \mathrm{SO}_{br} \times \prod_{i=1}^{R1^-} \mathrm{Sp}_{br/2} \times \times \prod^{R2^+} \mathrm{SU}_{br/2}(\mathbb{H}) \times \\ &\quad \prod^{R2^-} \mathrm{SU}_{br/2}(\mathbb{H})^- \times \prod^{C^+} \mathrm{SU}_{br} \times \prod^{C^-} \mathrm{SU}(c'_i, d'_i), \\ G_V(\mathbb{R}) &= \prod_{i=1}^{R1^-} \mathrm{SO}_{br} \times \prod_{i=1}^{R1^+} \mathrm{Sp}_{br/2} \times \times \prod^{R2^-} \mathrm{SU}_{br/2}(\mathbb{H}) \times \\ &\quad \prod^{R2^+} \mathrm{SU}_{br/2}(\mathbb{H})^- \times \prod^{C^-} \mathrm{SU}_{br} \times \prod^{C^+} \mathrm{SU}(c'_i, d'_i), \\ &\qquad (c'_i + d'_i = br). \end{aligned}$$

respectively.

Corollary 3.9. *Suppose that $a = r = 1$. Then $G_U(\mathbb{R})$ is compact abelian, hence its Lie-algebra is of pure type $(0, 0)$. The variation is in particular rigid.*

There are two (?) important examples:

Examples 3.10.
1. The monodromy representation is irreducible (so that $a = 1$) and $\dim_{\mathbb{Q}} H$ is square free. Since by (4) we have $\dim_{\mathbb{Q}} H = abtr^2$, we then must have $r = 1$ also. Hence the variation is rigid.

Corollary 3.11. *Let $\gamma \in \Gamma$ be a non-trivial unipotent of order ℓ. Then if $\epsilon = 1$, respectively -1, one has $R1^+ = R2^+ = C^+ = 1$, $R1^- = R2^- = C^- = 1$, respectively, and $\ell \leq br/2$, $\ell \leq \min(c'_i, d'_i)$.*

Up to now, very little use has been made of the fact that Suppose now that the weight of the variation of k and that the Hodge numbers $h^{p,q}$ are zero for $p < 0$. Then all the above modules inherit Hodge structures: E, D get Hodge structures of weight 0 with Hodge numbers $h^{p,-p}$ non-zero if $|p| > k$, and V and U get weight k Hodge structures. This can be shown to have serious restriction on the possible signs $\epsilon, \epsilon_Q, \epsilon_i$ that occur.

Examples 3.12.

1. Let $k = 1$ and suppose that the variation is irreducible, i.e. $m = 1$ and that $\dim_{\mathbb{Q}}(V) = 2g$. Then, by [Sat, IV, § 6] only the following possibilities may occur

   - $G_U(\mathbb{R}) = \prod^{N_3} \mathrm{Sp}_g, \quad G_V(\mathbb{R}) = \mathrm{id}$,
   - g is even, say $g = 2h$, $G_U(\mathbb{R}) = \prod^{N_3} \mathrm{Sp}_g \times \prod^{M_1} \mathrm{SU}_h(\mathbb{H})$, $G_V(\mathbb{R}) = \prod^{N_3} \mathrm{SO}_2 \times \prod^{M_1} S^1$,
   - $g = 2h$ is even, and $G_U(\mathbb{R}) = \prod^{N_1} \mathrm{SO}_h \times \prod^{N_4} \mathrm{SU}_g(\mathbb{H})^-$, $G_V(\mathbb{R}) = \prod^{M_1} \mathrm{SO}_2 \times \prod^{N_3} S^1$,
   - For some factor r of $2g$ and some (c_i, d_i), (c'_i, d'_i) with $c_i + d_i = 2g$, $c'_i + d'_i = r$, one has $G_U(\mathbb{R}) = \prod^{M_1} \mathrm{SU}_{2g} \times \prod_{i=1}^{M_2} \mathrm{SU}(p_i, q_i)$, $G_V(\mathbb{R}) = \prod_{i=1}^{M_2} \mathrm{SU}(p_i, q_i) \times \prod^{M_1} \mathrm{SU}_r$.

   From this, one can deduce [Sa, Theorem 8.1] that the variation is non-isotrivial and non-rigid there must be at least two factors and in both groups there must be compact and non-compact factors. This is only possible in the last case and then one must also have $r \geq 2$. One deduces that the $\mathbb{Q}$-rank of G_V is zero and hence, there can't be any non-trivial unipotent element in the monodromy group.

2. $k = 2$, $h^{2,0} = 1$. In the irreducible situation, one [S-Zu, Theorem 5.2.3] deduces that $G_U(\mathbb{R}) = \mathrm{SL}_2 \times \prod_i^{M_1} \mathrm{SU}_2$, $G_V(\mathbb{R}) = \mathrm{SL}_2 \times \prod_i^{M_1} \mathrm{SU}_2$. From this it also follows that the variation is rigid if there is a unipotent element in the monodromy group of order of nilpotency ≥ 3.

References

[Borel] Borel, A.: *Linear algebraic groups*, W.A¿ Benjamin, Inc. New-York, Amsterdam (1969).

[Del71] Deligne, P.: Théorie de Hodge II, Publ. Math. I.H.E.S, **40**, 5–58 (1971)

[Sa] Saito, M.-H.: Classification of non-rigid families of abelian varieties, Tohoku Math. J. **45** (1993) 159–189.

[S-Zu] Saito, M.-H. and S. Zucker: Classification of non-rigid families of K3-surfaces and a finiteness theorem of Arakelov type, Math. Ann. **289** (1991) 1–31.

[Sat] Satake, I.:*Algebraic structure of symmetric domains*, Publ. of Math. Soc. Japan **14**, Iwanami Shoten and Princeton Univ. press (1980).