Monodromy of a Variation of Hodge Structure and Algebraic Groups

Chris PETERS

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1 Motivation

Consider the following situation. Let $\pi : X \to S$ be a smooth projective family over a quasi-projective base S which is effectively parametrized; for instance, if there is a good moduli theory for the fibers, there is an injective map $\mu(\pi) : S \hookrightarrow M$, with M a coarse moduli space for the fibres. A *deformation of π with fixed base S* and parametrized by a germ of an algebraic variety (T, o) is a smooth projective family $\pi_{T,o} : X_{T,o} \to S \times (T, o)$ which restricts to π over $S \times \{o\} = S$. Let $\mu(\pi_{\{T,o\}}) : S \times T \to M$ be the corresponding moduli map. If $\mu(\pi_{\{T,o\}})|\{s\} \times (T, o)$ is immersive for generic $s \in S$ one speaks of an effective deformation. If no effective positive dimensional deformation exists, one says that the deformation is *rigid*.

One may "linearize" the situation by passing to the corresponding cohomology groups. As is well known (see e.g.[CSP]), the primitive cohomology group $H^k_{\rm prim}(X_s)$, $X_s = \pi^{-1}(s)$ form a polarized variation of Hodge structure:

Definition 1.1. A *$\mathbb{Q}$-variation of Hodge structure* on S of weight n is a local system $\underline{W}_S$ of finite dimensional $\mathbb{Q}$-vector spaces such that each fiber over $t \in S$ admits a Hodge structure of weight n, say $\underline{W}_t \otimes \mathbb{C} = \oplus_{p+q=n} W_t^{p,q}$ and such that
— the associated Hodge flag $F_t^\bullet$, where $F_t^p = \oplus_{r \geq p} W_t^{r,s}$, depends holomorphically on t,
— the flat connection ∇ satisfies *Griffiths' transversality condition*:

$$\nabla_\xi F_t^q \subset F_t^{q-1}, \quad \xi \text{ a germ of a holomorphic tangent field at } t\ .$$

The Hodge structure is *polarized* by a flat bilinear integral form q if q induces a polarization on the Hodge structure on each fibre of $\underline{W}_S$.

These conditions can also be expressed using the period domain D which classifies the Hodge structures on W polarized by q and having the same Hodge numbers as the Hodge structure F_o on $W = \underline{W}_o$. Let me briefly recall the construction of D. On the real vector space $W_{\mathbb{R}} := W \otimes_{\mathbb{Q}} \mathbb{R}$ the form induced by q will be denoted by $q_{\mathbb{R}}$. The group

$$\mathsf{G} := \mathrm{GL}(W_{\mathbb{R}}, q_{\mathbb{R}})$$

acts transitively on D with isotropy group at $o \in D$ a compact group V corresponding to the isomorphisms fixing the Hodge structure $F_o^\bullet$ on W. So $D = \mathsf{G}/\mathsf{V}$.

The map assigning to $t \in S$ the Hodge structure $F_t^\bullet \in D$ is multivalued because of the action of the monodromy group $\Gamma \subset \mathrm{GL}(W_{\mathbb{R}}, q_{\mathbb{R}})$. Incorporating this as a left action, one gets the *period map* $p : S \to \Gamma/D$, a well-defined holomorphic map. It replaces the moduli map. As a substitute for effective families one should consider immersive period maps.

The spaces of fibre-wise endomorphisms

$$E_t := \mathrm{End}(\underline{W}_t, q_t), \quad t \in S \tag{1}$$

receive a Hodge structure of weight zero: put

$$E_t^{-p,p} = \{\text{the } q\text{-skew endomorphisms sending } E_t^{r,s} \text{ to } E_t^{r-p,s+p}\}.$$

Observe that $E := E_0 = \mathsf{G}$. It turns out that the Hodge structure induces a Cartan splitting

$$\begin{aligned} E &= \mathfrak{k} \oplus \mathfrak{p}, \\ \mathfrak{k} &:= \mathsf{G} \cap \bigoplus_{p \text{ even}} E^{-p,p} \qquad (2) \\ \mathfrak{p} &:= \mathsf{G} \cap \bigoplus_{p \text{ odd}} E^{-p,p}. \qquad (3) \end{aligned}$$

Indeed, $\mathfrak{k}$ is the Lie algebra of a maximal compact Lie subgroup K of G and the Cartan-involution is the Weil-operator of the weight 0 Hodge structure on E. Since the above splitting is invariant under the adjoint action of K one gets a real vector bundle $\mathsf{G} \times_{\mathsf{V}} \mathfrak{p}$, in fact a sub bundle of the real tangent bundle $T(D)$ mapping isomorphically to $T(\mathsf{G}/\mathsf{K})$ under the canonical projection $D = \mathsf{G}/\mathsf{V} \to \mathsf{G}/\mathsf{K}$.

To find the holomorphic tangent bundle, observe that there is a V-equivariant splitting

$$\mathsf{GV} \oplus \mathfrak{m}, \quad \mathfrak{m}_{\mathbb{C}} = \bigoplus_{p \neq 0} E^{-p,p}$$

so that

$$T(D) = \mathsf{G} \times_{\mathsf{V}} \mathfrak{m}.$$

The *holomorphic* subbundle $T^{\mathrm{hol}}D$ as a sub bundle of the complexification $T(D)_{\mathbb{C}}$ is just

$$T^{\mathrm{hol}}(D) = \mathsf{G} \times_{\mathsf{V}} \mathfrak{m}^-, \quad \mathfrak{m}^- = \bigoplus_{p>0} E^{-p,p}.$$

The *horizontal tangent bundle* $T^{\mathrm{hor}}D$ is the sub bundle

$$T^{\mathrm{hor}}(D) = \mathsf{G} \times_{\mathsf{V}} \mathfrak{p}^-, \quad \mathfrak{p}^- := \bigoplus_{p>0,\ \mathrm{odd}} E^{-p,p}.$$

Inside this bundle there is the bundle of directions which come up in period maps, defining the *strictly horizontal tangent bundle*

$$T^{\mathrm{shor}}(D) = \mathsf{G} \times_{\mathsf{V}} E^{-1,1}.$$

A holomorphic map $p : U \to D$ is called *(strictly) horizontal* if for all $u \in U$ the image $p_*[T_u U]$ belongs to the horizontal tangent bundle, respectively to the strictly horizontal tangent bundle.

Griffith's transversality is equivalent to the period map $S \to \Gamma/D$ being strictly horizontal: the latter notion is defined purely locally on the base so that the period map can be replaced by any local lifting $p_U : U \to D$ and Griffiths' transversality just states that p is strictly horizontal. Deformations and effective deformations of a period map can now be defined as for moduli maps, except that not all deformations will be relevant: we want the end-result to be a period map as well. For deformations coming from geometry one even want the total deformation to be a period map. Those will be called *strictly horizontal deformations.* One may slacken the condition slightly by looking at *horizontal deformations* only.

By the rigidity theorem [Sch, Theorem 7.24] the sub algebra

$$\mathsf{E} = \mathrm{End}_{\Gamma}(W, q) \subset E = \mathsf{G} \tag{4}$$

of endomorphisms skew with respect to q which commute with the monodromy action inherits a Hodge structure of weight 0 which is constant in the following sense: a section $s \in \mathsf{E}$ is a global flat section of the bundle $\mathcal{E}$ of fibre wise endomorphisms whose fibre is E_t (see (1)); the space of these flat sections form a constant sub variation of Hodge structure of weight 0 inside the variation of Hodge structures of weight zero on $\mathcal{E}$. Note that E is canonically associated to the variation, or, what is the same, its period map p. The relevance of this space to the problem of finding the deformations to a period map is the following slight generalization of [Pe90, Theorem 3.2 and Proposition 3.6]:

Theorem 1.2. *The tangent space to the horizontal deformations of p is $\mathsf{E}\cap\mathfrak{p}^- = \mathsf{E}\cap\mathfrak{p}$ (since E is real) and the tangent space to the strictly horizontal deformations of p is $\mathsf{E}\cap E^{-1,1}$. So if $\mathsf{E}\cap\mathfrak{p} = 0$ the period map is horizontally rigid and if $\mathsf{E} \cap E^{-1,1} = 0$ the period map is strictly horizontally rigid.*

It is therefore clear that a rigidity study should include a systematic study of the above endomorphism algebra. As a first step, note that for any variation of Hodge structures the monodromy representation is complete reducible:

Theorem 1.3 ([Del71, 4.2.6])**.** *A polarized variation of Hodge structures over a quasi-projective manifold is a direct sum of irreducible ones.*

Here one has to pay attention: an irreducible variation of Hodge structure is *not* in general irreducible as a local system, but the direct sum of all irreducible systems isomorphic to it. In other words, the irreducible variations correspond to the isotypical local systems.

2 Isotypic Representations

Motivated by theorem 1.3 I now study isotypic representations in general. More precisely, fix a field k of characteristic zero, a group G and a k-vector space which is a completely reducible right G-module. Let V be an irreducible submodule and let W be the sum of all the submodules isomorphic to V. We set

$$\begin{aligned} W &:= \underbrace{V \oplus \cdots \oplus V}_{m} \\ D &:= \operatorname{End}_G V \\ U &:= \operatorname{Hom}_G(V, W) \\ Z &:= \operatorname{Center}(D). \end{aligned}$$

By Schur's lemma D is division algebra over its center. Hence Z is a finite extension field of k. We put

$$\begin{aligned} \dim_Z D &:= r^2 \\ d &:= [Z : k]. \end{aligned}$$

The algebra D acts naturally by composition on U from the right and from the left on V. Hence there is an action on $U \otimes_k V$ and if we divide out by the subspace generated by the elements $u \circ \gamma \otimes v - u \otimes \gamma(v)$ we get a k-vector space denoted by convention $U \otimes_D V$. Note that V and U both being D-modules are also Z-vector spaces. Hence W is.

The composition makes the module D into a left-right D-module. The opposite algebra D^o is the same as D but the product is reversed. For instance $M_d(D)$, the full matrix algebra of matrices of size m with coefficients in D is isomorphic as a D-module to $\operatorname{End}_{D^o}(D)$. Usually one thinks of D^o as the collection γ^o, $\gamma \in D$ where $\gamma \mapsto \gamma^o$ is some *fixed* anti-isomorphism. For example, if D is the full matrix algebra $M_d(k)$, the transpose map is such an anti-automorphism. This can be used to switch between left and right D-modules.

The following result describes the position of the G-endomorphisms with inside the full algebra of Z-endomorphisms:

Lemma 2.1 ([Sat, Ch. IV, § 1]). *1. U is a D-module of rank m and $W = U \otimes_D V$. If rank$_D V = n$, we have $\dim_Z V = nr^2$, $\dim_Z W = mnr^2$.*

2. There is a natural identification

$$\operatorname{End}_Z W = \operatorname{End}_{D^o} U \otimes_Z \operatorname{End}_D V,$$

the action of $\gamma \in G$ on W can be written $\mathrm{id} \otimes \gamma | V$ and

$$\operatorname{End}_G W = \operatorname{End}_{D^o} U \otimes \mathrm{id}_V .$$

From now on we assume that G is an algebraic group defined over k. Since the G-action is entirely on the V-factor, the G-action restricts Z-linearly on V :

$$G \hookrightarrow \mathrm{GL}_Z(V).$$

It follows that G "comes from" an algebraic group $\tilde{G}$ defined over Z. More precisely:

Lemma 2.2. *There is an algebraic group $\tilde{G} \subset \mathrm{GL}(V)$ defined over Z such that*

$$G = R_{Z/k}\tilde{G}.$$

The centralizer $C(G)$ is the Weil-restrictriction of the Z-group $C(\tilde{G})$.

Remark 2.3. Note that $\operatorname{End}_G W$ is the commutant of the Lie-algebra of G inside the Lie-algebra $\operatorname{End}_k W$. In other words, the Lie group of $C(G)$ is $\operatorname{End}_G(W)$.

I refer to appendix A for the notion of Weil-restriction

3 Invariant bilinear forms

Start out with a division algebra D over a field k with centre Z equipped with an involution $a \mapsto a^*$, $a \in D$. This means that $(ab)^* = b^* a^*$ and that $[a^*]^* = a$. The involution is not necessarily Z-linear: it induces an involution on Z which may or may not be the identity. If it is, we say that the **involution is of the first kind** and otherwise that it is of the **second kind**.

Any non-degenerate Z-bilinear form $q : D \times D \to D$ defines an involution of the first kind: just take the transpose with respect to q. The theorem of Skolem-Noether implies that any involution of the first kind on D is of this sort. Since D is a simple algebra over Z the form q is either symmetric or anti-symmetric:

$$q(x,y) = \pm q(x,y), \quad \forall x, y \in D.$$

and we denote this sign by ϵ_*. It is called the **sign of the involution** $*$.

Let T be a k-vector space which is a *right* D-module of rank say n. One has $\operatorname{End}_D T \simeq M_n(D)$ and one puts

$$\begin{aligned} \mathrm{GL}_D(T) &= \text{the group of units in } \operatorname{End}_D T\,(\simeq \mathrm{GL}(n,D)) && (5)\\ \mathrm{SL}_D(T) &= \text{the elements of norm 1 in } \mathrm{GL}_D(T)\,(\simeq \mathrm{SL}(n,D)). && (6) \end{aligned}$$

One says that a k-bilinear map

$$h : T \times T \to D$$

is (D, ϵ_T)-**hermitian with respect to** $*$ if

$$\left.\begin{array}{rcl} h(x,y)\delta &=& h(x,y)\delta \\ h(y,x) &=& \epsilon_T\,[h(x,y)]^* \end{array}\right\}, \quad \forall x, y \in T, \quad \delta \in D. \tag{7}$$

Such a form defines an involution $*_h$ on $\operatorname{End}_D T$.
The **unitary group** $\mathrm{U}_D(T,h)$ and the **special unitary group** $\mathrm{SU}_D(T,h)$ are defined in the usual way.

Examples 3.1. Over $\mathbb{R}$ we only have 2 central division algebras $\mathbb{R}$ (Case (R1)) and $\mathbb{H}$ (Case (R2)).
(R1). $D = M_r(\mathbb{R})$, and $Z = \mathbb{R}$. Any involution, automatically of the first kind, is of the form $A \mapsto A^* := Q^{-1}\,{}^{\mathsf{T}}\!AQ$ for some quadratic form Q on $\mathbb{R}^r$.
Let $T = M_{r,rn}(\mathbb{R}) = D^n \simeq \mathbb{R}^{r^2 n}$ viewed as column vectors $\underline{X} = \begin{pmatrix} X_1 \\ \dots \\ X_n \end{pmatrix}$,
$X_j \in M_r(\mathbb{R})$ on which D acts from the right.
Write the form $h(\underline{X},\underline{Y}) = X_1^* Y_1 \cdots X_p^* Y_p - X_{p+1}^* Y_{p+1} - \cdots - X_{p+q}^* Y_{p+q}$ $(p+q=n)$ on D^n as $\mathbf{1}_p \oplus -\mathbf{1}_q$. It is an example of a $(D,+)$-hermitian form on $T = D^n$. The group $U(T,h)$ is denoted $\mathrm{OG}_D(p,q)$; for $r=1$ this gives the standard quadratic form $\mathbf{1}_p \oplus -\mathbf{1}_q$ with corresponding group $\mathrm{OG}_{\mathbb{R}}(p,q)$. If Q is symmetric, say $Q = \mathbf{1}_a \oplus \mathbf{1}_b$, then the induced trace form ${}^{\mathsf{T}}{}_{D/\mathbb{R}}h$ is equal to $p[Q \otimes Q] - q[Q \otimes Q]$, a form of signature $[(a^2+b^2)p - 2abq, (a^2+b^2)q - 2abq]$. If Q is symplectic, then r is even and the form ${}^{\mathsf{T}}{}_{D/\mathbb{R}}h$ is symplectic as well.
For n is even and h the symplectic form

$$h(\underline{X},\underline{Y}) = \begin{pmatrix} X_1^* & \dots & X_n^* \end{pmatrix} J_n \begin{pmatrix} Y_1 \\ \dots, \\ Y_n \end{pmatrix} \quad J = \begin{pmatrix} 0 & -\mathbf{1}_{\frac{1}{2}n} \\ \mathbf{1}_{\frac{1}{2}n} & 0 \end{pmatrix}.$$

we call the group $\mathrm{U}_D(h)$ the *symplectic group* which will be denoted *operatornameSp*$_D(\frac{1}{2}n)$. For a symmetric form Q we get a symplectic trace

form while for Q symplectic, the trace form becomes a symmetric form of signature $(\frac{1}{2}r^2n, \frac{1}{2}r^2n)$.
All groups considered so far are *real* algebraic groups.
(R2). Then r is even and $D = M_{\frac{1}{2}r}(\mathbb{H})$. Any involution is of the first kind. Let $a \mapsto a^*$ be the standard involution on $\mathbb{H}$ (the one that sends $a + b\mathsf{i} + b\mathsf{j} + c\mathsf{k}$ to $a - b\mathsf{i} - b\mathsf{j} - c\mathsf{k}$). Then an arbitrary involution of the first kind on D is of the form $A \mapsto Q^{-1}\,{}^{\mathsf{T}}A^*Q$, where Q is a bilinear form. Let us consider $Q = \mathbf{1}_r$ and the form $\mathbf{1}_p \oplus -\mathbf{1}_q$ on $\mathbb{H}^{\frac{1}{2}rn}$, $\frac{1}{2}rn = p + q$. The corresponding group is $U_{\mathbb{H}}(p, q)$ and the corresponding trace form leads to the *real* unitary groups $U_{\mathbb{C}}(4p, 4q)$. A $(D, -)$-form is the *real* algebraic group $\mathrm{U}_n(\mathbb{H})^- = \{g \in GL_n(\mathbb{H}) \mid g^*(\mathsf{j}\mathbf{1}_n)g = \mathsf{j}\mathbf{1}_n\}$.
(C). Over $\mathbb{C}$ the only division algebra is $\mathbb{C}$. This is called "Case (C)". There are two involutions, the identity (first kind) and the complex conjugation (second kind).
$(\mathrm{C})_1$. This case is largely similar to (R1) except that the trace form now gives the *complex* algebraic groups $\mathrm{OG}_{\mathbb{C}}(p, q) \simeq \mathrm{OG}_{\mathbb{C}}(n)$ and the complex symplectic groups $\mathrm{Sp}_{\mathbb{C}}(\frac{1}{2}n)$.
$(\mathrm{C})_2$. This case leads to unitary groups $\mathrm{U}_D(p, q)$ whose trace forms give *real* algebraic groups $\mathrm{U}_{\mathbb{C}}(p, q)$ both for a $(D, +)$-hermitian) or a $(D, -)$-hermitian form. This last is seen using the skew-hermitian form form $-\mathsf{i}\mathbf{1}_p \oplus \mathsf{i}\mathbf{1}_q$.

Return now to the situation of § 2 where $W = U \otimes_D V$ is supposed to have a non-degerate k-bilinear form q which is $\pm$-symmetric and G-invariant. Denote this sign by ϵ_q. The form q defines an involution on $\mathrm{End}_k W$ (just take the transpose of endomorphisms with respect to q) and hence there is an involution on $\mathrm{End}_G W$ as well and its center D. The latter is denoted $*$. We have

Lemma 3.2 ([Sat, Ch IV. §2])**.** *There exists a (D, ϵ_U)-hermitian form $h_U : U \times U \to D$ (with respect to $*$) and a $(D^o, \epsilon_U\epsilon_q)$-hermitian form $h_V : V \times V \to D^o$ (with respect to $*^o$) such that*

$$q(u \otimes_D v, u' \otimes_D v') = {}^{\mathsf{T}}{}_{D/k}\left(h_U(u, u')\left[h_V(v, v')\right]^o\right).$$

This is abbreviated by writting

$$q = {}^{\mathsf{T}}{}_{D/k}(h_U \cdot h_V^o).$$

If the involution $*$ on D is of the first kind one has $\epsilon_U = \epsilon_*$ and if $*$ is of the second kind ϵ_U can be chosen freely.

Observe that the group G acting only on the V factor in the decomposition $W = U \otimes_D V$, we have an embedding $\tilde{G} \hookrightarrow U(h_V)$ which is defined over Z (use the trace map ${}^{\mathsf{T}}{}_{D/Z}$), i.e. there is an embedding

$$G = R_{Z/k}\tilde{G} \hookrightarrow R_{Z/k}U(h_V). \tag{8}$$

Likewise there is a natural *isomorphism*

$$G' := C_{\mathrm{GL}(W,q)}(G) = R_{Z/k} C_{\mathrm{GL}(W,q)}(\tilde{G}) \simeq R_{Z/k} U(h_U). \tag{9}$$

For later use we observe:

Lemma 3.3. *The Lie-algebra of G' consists of the k-linear endomorphisms α of W which preserve q in the sense that $q(\alpha(v), w) + q(v, \alpha(w) = 0$ for all $v, w \in W$. In other words, using the notation* (4) *we have Lie*$(G') = \mathsf{E}$.

4 Field Extensions

Suppose that D is a simple algebra over k with center Z and suppose

$$\dim_Z D = r^2.$$

Fix any extension field K of k and an algebraic closure $\bar{K}$ of K. Then

$$Z \otimes_k K = \bigoplus_{\sigma \in \Sigma} Z^\sigma, \quad d' = [Z^\sigma : K] < \infty,$$

where Σ is the set of $\mathrm{Gal}(\bar{K}/K)$-orbits in the finite set T of k-embeddings $i_\tau : Z \hookrightarrow \bar{K}$ and the image of Z^σ in $\bar{K}$ equals $i_\tau(Z)K$, where τ represents $\sigma \in \Sigma$. We have

$$\begin{array}{rcl} d & = & \#\Sigma \cdot d' \\ \dim_K D \otimes_Z Z^\sigma & = & d' \dim_Z D = d' r^2. \end{array} \tag{10}$$

In the situation of § 2 we have

$$\begin{array}{rcl} V_K := V \otimes_k K & = & \bigoplus_{\sigma \in \Sigma} Z^\sigma V, \quad Z^\sigma V \simeq V \otimes_Z Z^\sigma, \\ W_K := W \otimes_k K & = & \bigoplus_{\sigma \in \Sigma} Z^\sigma W, \quad Z^\sigma W \simeq W \otimes_Z Z^\sigma, \\ \mathrm{End}_G(Z^\sigma V) & \simeq & D \otimes_Z Z^\sigma. \end{array}$$

So the $Z^\sigma V$ are the isotypic components of V_K, say of irreducible type V^σ and the $Z^\sigma W$ are the isotypic components of W_K. The algebra $\mathrm{End}_G V^\sigma$ is a K-division algebra with center Z^σ. Put

$$D^\sigma = \mathrm{End}_G V^\sigma, \quad Z^\sigma = \mathrm{Center}(D^\sigma)$$

$$\dim_{Z^\sigma} D^\sigma = r_\sigma^2 \tag{11}$$

$$\mathrm{rank}_{D^\sigma} V^\sigma = n_\sigma, \tag{12}$$

$$\mathrm{End}_G(Z^\sigma V) \simeq M_{s_\sigma}(D^\sigma). \tag{13}$$

One can identify $Z^\sigma V$ with the matrices of size $s_\sigma \times n_\sigma$ on which $M_{s_\sigma}(D^\sigma)$ acts from the left. The s_σ rows correspond to the s_σ isotypic components. If we identify the D^σ-module of the first row vectors with V^σ we can write

$$Z^\sigma V = \epsilon_{11} V^\sigma \oplus \cdots \oplus \epsilon_{s_\sigma 1} V^\sigma,$$

where $e_{ij} \in M_{s_\sigma}(D^\sigma)$ denotes the matrix with 1 on the (i,j)-th entry and zero elsewhere. Now one has $\mathrm{Hom}_G(V^\sigma, V_K) = Z^\sigma \epsilon_{11} U$ and

$$\begin{aligned} U^\sigma &:= \mathrm{Hom}_G(V^\sigma, V_K) \simeq Z^\sigma U, \\ Z^\sigma W &= U^\sigma \otimes_{D^\sigma} V^\sigma. \end{aligned}$$

Recalling (10), (11), (12) and (13), a short calculation gives:

Lemma 4.1. *We have* $rank_{D^\sigma} U^\sigma = m s_\sigma$, $rank_{D^\sigma} V^\sigma = n s_\sigma$ *and* $r = r_\sigma s_\sigma$.

Example 4.2. In the case $K = \mathbb{R}$ the division algebra D^σ equals $\mathbb{R}$ (Case R1), $\mathbb{H}$ (case R2) or $\mathbb{C}$ (Case C).

Suppose for example that $k = \mathbb{Q}$ and that Z is a totally real algebraic extension (of degree d). Then $d' = 1$, $d = t$. Here only cases (R1) and (R2) are possible. In case (R1), one has $r_\sigma = 1$ and $s = r = s_\sigma$, $U^\sigma = \mathbb{R}^{mr}$, $V^\sigma = \mathbb{R}^{nr}$. In case (R2), one has $r_\sigma = 2$, $Z_\sigma = \mathbb{R} = K$ and $s = \frac{1}{2} r = s_\sigma$. In particular r must be even and $U = \mathbb{H}^{\frac{1}{2}mr}$, $V = \mathbb{H}^{\frac{1}{2}nr}$.

If Z is an imaginary extension of a totally real number field, we have case (C) with $Z_\sigma = \mathbb{C}$ and hence $d' = 2$. In this case $r_\sigma = 1$, so that $U^\sigma = \mathbb{C}^{rm}$, $V^\sigma = \mathbb{C}^{rn}$.

In the situation of § 3 one has [Sat, Ch. IV §3]:

Lemma 4.3. *Let there be given a non-degenerate k-bilinear form q on $W = U \otimes_D V$ inducing the involution $*$ on D, with product decomposition $q = \mathsf{T}_{D/k}(h_U \otimes h_V^o)$. Let q^σ the induced bilinear form on W^σ and let $*_\sigma$ the induced involution on D^σ. Then there are signs ϵ_σ such that h_U induces a $(D^\sigma, \epsilon_\sigma \epsilon_U)$-hermitian form h_{U^σ} on U^σ (with respect to $*_\sigma$) and a $([D^\sigma]^o, \epsilon_\sigma \epsilon_V \epsilon_q)$-hermitian form h_{V^σ} on V^σ (with respect to $*_\sigma$) such that*

$$q^\sigma = \mathsf{T}_{D^\sigma/K}[h_{U^\sigma} h_{V^\sigma}^o].$$

Fix some embedding $\sigma : Z \hookrightarrow \bar{K}$, a non-degenate (D^σ, η)-hermitian form h_U^σ on U^σ. From the decomposition (17) for the group $\tilde{G}$ given by (8) one gets :

$$G(K) = R_{Z/k} \tilde{G}(K) = \prod_\sigma \tilde{G}^\sigma(\sigma Z), \tag{14}$$

$$G'(K) = R_{Z/k} \tilde{G}'(K) = \prod_\sigma \tilde{G'}^\sigma(\sigma Z). \tag{15}$$

Example 4.4. Continuing with the situation of example 4.2, examples 3.1 we take $k = \mathbb{Q}$ and suppose that Z is either a totally real extension of $\mathbb{Q}$ of finite degree (Case (R)), or an imaginary extension thereof (Case (C)). In the cases (R) we suppose that the involution is the "standard involution" with fixed field $\mathbb{R}$ while for the case (C) we suppose it is the complex conjugation. Again we take $K = \mathbb{R}$ and we have the following possibilities for the component G^σ in (14). A similar analysis holds for G'^σ.

R1 $D^\sigma = \mathbb{R}$, $D \otimes_Z Z^\sigma = M_{s_\sigma}(\mathbb{R})$, $U = \mathbb{R}^{mr}$, $V = \mathbb{R}^{nr}$ and according to the sign η of h^σ_U we have

R1$_+$ $\tilde{G}^\sigma(\mathbb{R}) = \mathrm{O}(p,q)$ with $p + q = mr$ if $\eta = 1$

R1$_-$ $\tilde{G}^\sigma = \mathrm{Sp}(\frac{1}{2}mr)$ if $\eta = -1$. In this case mr is even.

or

R2 $D^\sigma = \mathbb{H}$, r is even, $D \otimes_Z Z^\sigma = M_{s_\sigma}(\mathbb{H})$, $U = \mathbb{H}^{\frac{1}{2}mr}$, $V = \mathbb{H}^{\frac{1}{2}nr}$ and according to the sign η of h^σ_U we have

R2$_+$ $\tilde{G}^\sigma(\mathbb{R}) = \mathrm{U}_\mathbb{H}(p,q)$ with $p + q = mr$ if $\eta = 1$.

R2$_-$ $\tilde{G}^\sigma(\mathbb{R}) = \mathrm{U}_\mathbb{H}(n)^-$ if $\eta = -1$.

C $D^\sigma = \mathbb{C}$, $D \otimes_Z Z^\sigma = M_{s_\sigma}(\mathbb{C})$, $U = \mathbb{C}^{mr}$, $V = \mathbb{C}^{nr}$ and $\tilde{G}^\sigma(\mathbb{R}) = \mathrm{U}_\mathbb{C}(p,q)$, $p + q = mr$, irrespective of the sign.

5 Monodromy Representations of a Variation of Hodge structure

Let (S,o) be a arc-wise connected pointed topological space and $\underline{W}_S$ a local system of $\mathbb{Q}$-vector spaces on S. The stalk $W = \underline{W}_o$ at $o \in S$ is a left $\pi_1(S,o)$-module under the monodromy action. Suppose that W comes equipped with a non-degenerate $\mathbb{Q}$-bilinear form q, preserved by the monodromy action and let $G \subset \mathrm{GL}(W,q)$ the connected component of the Zariski-closure of the monodromy group. It is called the *(connected) algebraic monodromy group.*

In studying rigidity questions, one may replace $\underline{W}_S$ by its pull back under a finite unramified cover of S and hence one may assume that the Zariski-closure of the monodromy group itself is connected so that the considerations of the previous sections can be applied to this situation. For instance, $\mathrm{End}_G(W)$ is the algebra of global endomorphisms of the local system $\underline{W}_S$. Suppose next that $\underline{W}_S$ admits a q-polarized $\mathbb{Q}$-variation of Hodge Recall that W is the fiber of $\underline{W}_S$ considered as G-module where G is the algebraic monodromy group. One has [Del71, Cor. 4.2.9]:

Lemma 5.1. *G is semi-simple; in particular, its center is finite.*

The fiber $V = \underline{V}_o$ is an isotypical component and $D = \mathrm{End}_G(V)$ so that we can apply the considerations of § 2–5.

Since the Lie-algebra of the real points of the group G' defined by (9) is precisely E Theorem 1.2 implies that the variation is horizontally rigid, i.e $\mathfrak{p}_{\mathbb{C}} \cap \mathsf{E} = 0$ precisely if the group G' happens to be compact:

Corollary 5.2. *The real group $G'(\mathbb{R})$ is compact precisely if the variation is horizontally rigid.*

Remark 5.3. 1. In particular, the $\mathbb{Q}$-rank of W is divisible by $r^2[Z : \mathbb{Q}]$, $r^2 = \dim_Z D$. So if this rank is a square-free number, $D = Z$ is a field and $D = D^{0,0}$. In particular, if in this case the local system is irreducible, the system is (strongly) rigid. This is automatically the case if the rank of the system (which equals $mnr^2[Z : \mathbb{Q}]$) is a prime number.
2. A local system which remains irreducible after extending to $\mathbb{C}$ is said to be **absolutely irreducible**. If such a system carries a polarizable variation of Hodge structure it is strongly rigid, since then $\mathrm{End}(W) \otimes \mathbb{C} = \mathbb{C}$ must be pure of type $(0, 0)$. Examples include the variation given by the primitive cohomology of Lefschetz pencils of complete intersections (except for a small number of obvious exceptions).

The division algebra D is of a special kind: the Weil operator on the center Z is the identity and hence the fact that q polarizes the Hodge structure implies the involution on Z is positive. Then as in [S-Zu, Thm. 2.4.1] Albert's classification implies:

Theorem 5.4. *Suppose that $\underline{W}_S$ underlies an isotypical q-polarized variation of Hodge structures with irreducible component $\underline{V}_S$ and let D be the algebra of flat endomorphisms of $\underline{V}_S$. Then for the center Z of the divsion algebra D one has two possibilities*

(R) *$Z = Z_0$ is either a totally real number field and $*$ is of the first kind,*

(C) *Z is a quadratic extension of a totally real number field Z_0 and $*$ is the complex conjugation on Z.*

Let me use this to study what happens when I extend scalars to $K = \mathbb{R}$. Our variation of Hodge structure on $\underline{W}_S$ splits over the reals as

$$(W, q) \otimes \mathbb{R} \simeq \bigoplus_{\sigma} (W, q)^{\sigma}. \tag{16}$$

and the classification of Example 4.4 applies. In particular, $W^{\sigma} = U^{\sigma} \otimes_{D^{\sigma}} V^{\sigma}$ is isotypical of type V^{σ}. Recall also (Prop. 3.2) that if the induced hermitian form on U^{σ} has sign ϵ_{σ} then the induced form on V^{σ} has sign $-\epsilon_q \epsilon_{\sigma}$. In this way, one thus obtains:

Proposition 5.5. *Let $\sigma : Z_0 \hookrightarrow \mathbb{R}$ be any of the real embeddings of Z_0. For each of the types (R1)$_+$, (R2)$_+$ respectively (C), let (p_σ, q_σ), (p'_σ, q'_σ) be the signatures of the corresponding hermitian forms on U^σ, respectively V^σ. Then with the notation* (14), (15) *and the conventions of Appendix B, one has*

$$\begin{aligned} G(\mathbb{R}) \quad = \quad & \prod_{\substack{\sigma \text{ of type } R1_+ \\ p_\sigma + q_\sigma = mr}} \mathrm{O}(p_\sigma, q_\sigma) \times \prod_{\sigma \text{ of type } R1_-} \mathrm{Sp}_{\mathbb{R}}(\frac{1}{2}mr) \times \\ & \prod_{\substack{\sigma \text{ of type } R2_+ \\ p_\sigma + q_\sigma = \frac{1}{2}mr}} \mathrm{U}_{\mathbb{H}}(p_\sigma, q_\sigma) \times \prod_{\sigma \text{ of type } R2_-} \mathrm{U}_{\mathbb{H}}(\frac{1}{2}mr)^- \times \prod_{\substack{\sigma \text{ of type } C \\ p_\sigma + q_\sigma = mr}} \mathrm{U}_{\mathbb{C}}(p_\sigma, q_\sigma), \end{aligned}$$

and if $\epsilon_q = 1$, respectively $\epsilon_q = -1$, one then has correspondingly

$$\begin{aligned} G'(\mathbb{R}) \quad = \quad & \prod_{\substack{\sigma \text{ of type } R1_- \\ p'_\sigma + q'_\sigma = nr}} \mathrm{O}(p_\sigma, q_\sigma)) \times \prod_{\sigma \text{ of type } R1_+} \mathrm{Sp}_{\mathbb{R}}(\frac{1}{2}nr) \times \\ & \prod_{\substack{\sigma \text{ of type } R2_- \\ p'_\sigma + q'_\sigma = \frac{1}{2}nr}} \mathrm{U}_{\mathbb{H}}(p'_\sigma, q'_\sigma) \times \prod_{\sigma \text{ of type } R2_+} \mathrm{U}_{\mathbb{H}}(\frac{1}{2}nr)^- \times \prod_{\substack{\sigma \text{ of type } C \\ p'_\sigma + q'_\sigma = nr}} \mathrm{U}_{\mathbb{C}}(p'_\sigma, q'_\sigma), \end{aligned}$$

respectively,

$$\begin{aligned} G'(\mathbb{R}) \quad = \quad & \prod_{\substack{\sigma \text{ of type } R1_+ \\ p'_\sigma + q'_\sigma = nr}} \mathrm{O}(p_\sigma, q_\sigma) \times \prod_{\sigma \text{ of type } R1_-} \mathrm{Sp}_{\mathbb{R}}(\frac{1}{2}nr) \times \\ & \prod_{\substack{\sigma \text{ of type } R2_+ \\ p'_\sigma + q'_\sigma = \frac{1}{2}nr}} \mathrm{U}_{\mathbb{H}}(p'_\sigma, q'_\sigma) \times \prod_{\sigma \text{ of type } R2_-} \mathrm{U}_{\mathbb{H}}(\frac{1}{2}nr)^- \times \prod_{\substack{\sigma \text{ of type } C \\ p'_\sigma + q'_\sigma = nr}} \mathrm{U}_{\mathbb{C}}(p'_\sigma, q'_\sigma), \end{aligned}$$

From the above decomposition and Cor. 5.2 we deduce:

Corollary 5.6. *A variation of Hodge structure is (horizontally) rigid if and only if the following conditions hold simultatiously:*

- $p'_\sigma q'_\sigma = 0$ *for type (R1)$_-$, (R2)$_-$ and (C) for even weight, and* $p'_\sigma q'_\sigma = 0$ *for type (R1)$_+$, (R2)$_+$ and (C) for the case of odd weight,*
- *no factor of type (R1)$_+$, (R2)$_+$ for even weight, and no factor of type (R1)$_-$, (R2)$_-$ for odd weight.*

It turns out that the indices of the various hermitian forms are strongly related to the type of Hodge structure on the various irreducible constituents V^σ of the isotypical parts W^σ from (16). As an illustration we mention 2 examples.

Examples 5.7. 1. K3-surfaces. In [S-Zu] one finds that the transcendental variation $\underline{W}$ associated to a family of K3's which is non-isotrivial and non-rigid (i.e. there exist no non-trivial *strictly* horizontal deformations) has a very particular structure: D itself is a quaternion-algebra with center a totally real field and $W = V$ is of rank one over D. In particular, $\underline{W}$ must have rank $4d$, i.e. divisible by 4. Moreover, D splits over exactly one place so that

$$\begin{aligned} D \otimes_Z \mathbb{R} &= M_2(\mathbb{R})) \times \underbrace{\mathbb{H} \times \cdots \times \mathbb{H}}_{d-1} \\ G(\mathbb{R}) = G'(\mathbb{R}) &= \mathrm{SL}_{\mathbb{R}}(2) \times \underbrace{\mathrm{U}_{\mathbb{C}}(2) \times \cdots \times \mathrm{U}_{\mathbb{C}}(2)}_{d-1}. \end{aligned}$$

This has remarkable consequences for the monodromy group: if the monodromy group has a non-trivial unipotent element T such that $(T - \mathrm{id})^N \neq 0$ for $N = n_0$ while higher powers are zero, one must have $n_0 = 2$ and then the whole monodromy group embeds in $2, \mathbb{Q}$. This follows since as soon as $G(\mathbb{R})$ contains a compact factor, the $\mathbb{Q}$-rank of G is zero and G does not contain a non-trivial unipotent element defined over $\mathbb{Q}$ (Appendix A). In particular, the assumption then implies that in the above decomposition of $G(\mathbb{R})$ one has $d = 1$.
2. Weight one. In [Sa] one finds that in this case, the above orthogonal groups and the quaternionic unitary groups all must be definite. This implies $G(\mathbb{R})$ has t compact factors and $d/d' - t$ non-compact factors, the opposite is true for $G'(\mathbb{R})$. If if the variation is non-isotrivial and non-rigid one can show that $t > 0$ and $d/d' - t > 0$. In particular, since $G(\mathbb{R})$ then has compact factors, the $\mathbb{Q}$-rank must be zero. By Appendix A this implies that one cannot have a non-trivial $T \in G(\mathbb{Q})$ which is unipotent.

6 The role of the Mumford-Tate group

The Hodge group of the fibres of a variation of Hodge structure vary, but one can easily show that there is a dense open subset over which it stays constant. The resulting group is called the **generic Hodge group**, or the Hodge group of the variation and denoted $\mathrm{Hg}_{\mathrm{gen}}$. It is known that G, the algebraic monodromy group, is a normal subgroup of the generic Hodge group. In fact, one has [André, Theorem 1]:

Lemma 6.1. $G \lhd \mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}$*, the "semi-simple part" of the generic Hodge group i.e its commutator subgroup* $[\mathrm{Hg}_{\mathrm{gen}}, \mathrm{Hg}_{\mathrm{gen}}]$.

I say that *the generic Hodge group is as small as possible* equality holds in lemma 6.1, i.e. $G = \mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}$. The motivation behind this terminogy is as follows. One *fixes* a local system over S which carries at least one polarizable variation of Hodge structure and hence the group G is considered to be fixed while the generic Hodge group depends on which varation of Hodge structure one puts on the local system. Minimality thus refers to a theoretical minimal possible generic Hodge group among all possible variations on the given local system.

Rigidity forces contraints on the generic Hodge group:

Proposition 6.2. *If the generic Hodge group is as small as possible, then the period map p is horizontally rigid. Conversely, if p is horizontally rigid and moreover $\mathfrak{k} \cap \mathfrak{m}$ is abelian, then the generic Hodge group is as small as possible.*

Proof: The action of the Hodge group on $\mathrm{End}(W) = W \otimes W^\vee$ factors its semi-simple quotient since the central torus acts trivially. This semi-simple quotient contains the Weil-operator which acts as $-\mathrm{id}$ on vectors in $\mathrm{End}(W)$ of pure type $(-p, p)$ for p odd. So these cannot be invariant under $\mathrm{Hg}^{\mathrm{ss}}(V)$. Suppose that o is very general so that $\mathrm{Hg}^{\mathrm{ss}}(V) = \mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}$ and suppose that moreover $\mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}} = G$. So then $\mathrm{End}(W)$ has no vectors of this type invariant under the monodromy group, i.e. $\mathsf{E} \otimes \mathbb{C} = \bigoplus_q \mathsf{E}^{-2q,2q}$ and in particular the period map is horizontally rigid.

For the converse, let me first analyze the position of $G(\mathbb{R})$ and $\mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}(\mathbb{R})$ inside the group $\mathsf{G} = \mathrm{GL}(W_\mathbb{R}, q_\mathbb{R})$. These groups are all semi-simple and so they can be written as the semi-direct product of their simple factors. In particular $\mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}(\mathbb{R})$ can be written as a semi-direct product $\mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}(\mathbb{R}) = G(\mathbb{R}) \cdot H$ where H is a normal subgroup of the centralizer $G'(\mathbb{R})$ of G in G.

The Lie-algebra of the centralizer of $\mathrm{Hg}^{\mathrm{ss}}_{\mathrm{gen}}(\mathbb{R})$ inside G is precisely $\mathsf{E}^{0,0}_\mathbb{R}$. It follows that the Lie-algebra of H is entirely contained in $\mathfrak{m}$. Write $H = H_\mathrm{c} \cdot H_\mathrm{nc}$, where H_c, H_nc are the products of the compact, respectively the non-compact factors of H. The Lie-algebra of H_c is contained in $\mathsf{E} \cap \mathfrak{k} \cap \mathfrak{m}$ while the Lie-algebra of H_nc is contained in $\mathfrak{p} \cap \mathsf{E}$. Although in general $H_\mathrm{c} \neq 1$, if $\mathfrak{k} \cap \mathfrak{m}$ is abelian this group must be trivial since then the semi-simple group H_c^+ is abelian. Since $\mathfrak{p} \cap \mathsf{E}$ are the tangents to horizontal deformations, the result follows.

Example 6.3. The condition that $\mathfrak{k} \cap \mathfrak{m}$ is abelian is trivially fulfilled for weight 0-variations and for weight 2-variations with $h^{2,0} = 1$, i.e. for K3-variations, since for those $\mathfrak{k} = \mathsf{E}^{0,0}_\mathbb{R}$ and hence $\mathfrak{k} \cap \mathfrak{m} = \{0\}$. For these variation horizontal rigidity and strict horizontal rigidity also coincide.

A Useful Facts on Algebraic Groups

See [Sat, Bor66, Bor69, Hum] for further details.

Let k be a field of characteristic zero, $\bar{k}$ an algebraic closure of k. Algebraic group are supposed to be defined over k. Moreover they are to be *connected*, and *linear* in the sense that it has a faithful representation as an algebraic matrix group. An algebraic k-group T is an *m-torus* if $T(\bar{k})$ is isomorphic to a direct product of m copies of the multiplicative group $\bar{k}^*$. There is a finite Galois extension L/k so that $T(L)$ is already isomorphic to m copies of L^*. One says that T is *split over* L. A torus is *anisotropic* if it has no non-trivial characters defined over k, i.e. no homomorphisms $T(k) \to k^*$ besides $t \mapsto 1$. Any k-torus T can be written uniquely as $T = T_{\mathrm{spl}} \cdot T_{\mathrm{an}}$ where T_{spl} is largest k-split subtorus of T and T_{an} is the largest k-anisotropic subtorus. The k-**rank** of T is the dimension of the maximal k-split subtorus of T. This rank is 0 if and only if T is anisotropic.

An algebraic group is **semi-simple** if it has no nontrivial closed normal commutative subgroups. The corresponding Lie-algebra is semi-simple and is the direct sum of its simple ideals. This translates as follows: a semi-simple group is the almost direct product of its simple components, by definition the closed normal non-abelian subgroups which have no non-trivial closed normal subgroups themselves. Any closed normal subgroup of a semi-simple group is a product of some of its simple components.

An algebraic group is **reductive** if it contains no normal subgroups that are solvable. It contains maximal tori defined over k and they are all conjugate by [Bor69, §11.3]. The k-**rank of** G is the k-rank of any of its maximal tori. If this rank is zero G is called **anisotropic**.

A reductive group G can be written as an almost direct product $G = Z(G) \cdot G_{\mathrm{ss}}$ of its center and its commutator subgroup $G_{\mathrm{ss}} = [G, G]$ which is semi-simple. The center $Z(G)$ is an algebraic torus.

Example A.1. Let $k = \mathbb{R}$. Then a 1-torus is either $\mathbb{R}^*$ with trivial Galois action, or S^1, with Galois action $\theta \mapsto -\theta$. In the first case the torus is split and the rank is 1; in the second case the torus is anisotropic and the $\mathbb{R}$-Cartan group is 1 so that $\mathbb{R}$-rank is 0.

Remark A.2. Let G be a reductive algebraic group in characteristic zero is anisotropic if and only if G has no non-trivial characters over k and no unipotent elements $g \in G(k)$, $g \neq 1$. See [Bor66, § 6.4] for this assertion.

Here is a sketch of the proof. The kernel N of a non-trivial character $\chi : G \to k^*$ is a normal subgroup and clearly $[N, N] = N$. Hence N is semi-simple and $G = Z(G) \cdot N$ with $Z(G)$ a central torus to which χ restricts non-trivially contrary to the assumption that there are no k-split tori. For the second assertion, any non-trivial unipotent element corresponds to a nilpotent element X in the Lie-algebra of the semi-simple part. It can be completed to a *standard triple* $\{H, X, Y\}$, i.e. a triple k-isomorphic to $\mathfrak{sl}(2; k)$ and contained in G_{ss} (essentially by using the Jordan-Chevalley theorem and the non-degeneracy of the Killing form on the semi-simple part). Hence the Lie group must contain a non-trivial split k-subtorus contrary to the assumption that G is anisotropic.

For the converse, first observe that the first assumption (no non-trivial k-characters exist for G) implies that $Z(G)$ can not have split k-tori. As before, using standard triples, one can see that any split k-torus in the semi-simple part of G gives rise to an embedded copy of $2, k$ and hence a non-trivial unipotent element in $G(k)$ contrary to the second assumption.

Example A.3. Let k be a finite extension of $\mathbb{Q}$ and G a k-group such that for some embedding $\sigma : k \hookrightarrow \mathbb{R}$ the resulting group G^σ is compact. Then the k-rank is 0. Indeed, any character χ for G induces a character χ_σ for G^σ and any unipotent $g \in G$ gives a unipotent element g_σ in G^σ. Hence $\chi_\sigma = 1$, $g_\sigma = 1$ and also $\chi = 1$, $g = 1$.

Next, let me recall the construction of the Weil restriction [Weil, 1.3]. Let Z/k be a *finite* Galois extension of a field k of degree d. Suppose that X_Z is a variety defined over Z and let X_k be the same set viewed as a k-variety, i.e. the Z-points of X_Z form the k-points of X_k. However, one would like to find a variety Y_Z defined over Z such that its k-points $Y_Z(k)$ are in one-to one correspondence to the k-points $X_k(k)$ of X_k. The correspondence should be "natural" in that any algebraic structure on Y should be inherited by the one on X. If such Y exists we say that X is the Weil restriction of Y, denoted $X = R_{Z/k}Y$.

For algebraic groups there is a direct construction as follows. Viewing Z as a k-algebra, we get the regular representation $\rho : Z \to M_d(k)$. Then for all positive integers m from the representation ρ one gets a new one, $\rho(m) : M_Z(m) \to M_k(md)$, defined by $\rho(m)(A_{ij}) = (\rho(A_{ij}))$. Suppose now that G is a Z-matrix group $G \subset \mathrm{GL}_Z(N)$, then the *Weil restriction* $R_{Z/k}G$ is the k-group $\rho(N)(G)$. If $\dim_Z G = n$, then $\dim_k R_{Z/k}G = nd$. By construction, its group of Z-points is a product

$$R_{Z/k}G(Z) = \prod_\sigma G^\sigma(Z), \quad G^\sigma := G \otimes_{\sigma(Z)} \bar{k}, \tag{17}$$

where σ runs over the set of k-linear *embeddings* $\sigma : Z \hookrightarrow \bar{k}$ of Z in some algebraic closure of k. There are d distinct such embeddings.

From example A.3 one gets the following obvious but useful result.

Lemma A.4. *Let H be an algebraic group defined over a number field Z and let $G = R_{Z/\mathbb{Q}}H$ be its Weil-restriction. Suppose that $G(\mathbb{R})$ decomposes as a direct product of real groups $G = \prod_i G_i$. Then the $\mathbb{Q}$-rank of G is at most* $\max_i(\mathbb{R}\text{-rank } G_i)$*. In particular, if some G_i is compact, the $\mathbb{Q}$-rank of G is zero.*

Example A.5. The Weil restriction $\mathsf{S} = R_{\mathbb{C}/R}\mathbb{C}^*$ is the algebraic group $\mathbb{C}^*$ viewed as a real group. Its complex points consists of $\mathbb{C}^* \times \mathbb{C}^*$ with complex conjugation acting as $(z, w) \mapsto (\bar{w}, \bar{z})$. The embedding $\mathbb{R}^* \hookrightarrow \mathbb{C}^*$ induces a homomorphism of algebraic groups $w : \mathbf{G}_m \to \mathsf{S}$.

A representation of S on a real vector space V such that $w(t)$ acts as multiplication by t^k is a real Hodge structure where $V^{p,q}$ is the subspace of $V \otimes \mathbb{C}$ on which $(z,w) \in \mathsf{S}(\mathbb{C})$ acts as $z^p w^q$. If $w(\mathbb{C}^*)$ is defined over $\mathbb{Q}$ we have a rational Hodge structure. The **Mumford-Tate group** is the algebraic closure MT of the image of S inside $\mathrm{GL}(V)$. The unitary subgroup of $\mathbb{C}^*$ gives rise to a morphism $\mathrm{U}(1) \to \mathsf{S}$. The complex points of its image is the subgroup of $\mathsf{S}(\mathbb{C})$ of points of the form $(z, \bar{z}^{-1})$. Its Zariski-closure in $\mathrm{GL}(V)$ is called the **Hodge group** Hg of the Hodge structure. These groups by construction are *connected* and $\mathrm{MT} = \mathbf{G}_m \cdot \mathrm{Hg}$. If the Hodge structure on V is polarized, the definitions imply that the Hodge group preserves the polarization. In fact, if q is a polarization, one has $\mathrm{Hg} = \mathrm{MT} \cap V, q$. In that situation it is known that the Mumford-Tate group is reductive [DMOS] It follows that is the product of its center and a semi-simple group $\mathrm{Hg}^{\mathrm{ss}} = [\mathrm{Hg}, \mathrm{Hg}]$.

B Classical groups

Over $\mathbb{C}$ one has the following classical groups:

1. $n; \mathbb{C}$ the group of complex $n \times n$ matrices of determinant 1 and the corresponding Lie algebra $\mathfrak{sl}(n;\mathbb{C})$.

2. $\mathrm{O}(n;\mathbb{C})$ the group of complex $n \times n$ orthogonal matrices A, i.e. for which ${}^{\mathsf{T}}AA = \mathbf{1}$; the corresponding Lie algebra is $\mathfrak{o}(n;\mathbb{C})$. There is only one non-degenerate symmetric form on $\mathbb{C}^n$ represented by the matrix $\mathbf{1}_n$.

3. $\mathrm{Sp}(n;\mathbb{C})$ the group of complex $2n \times 2n$ symplectic matrices A, i.e. those for which ${}^{\mathsf{T}}AJA = J$, $J = \begin{pmatrix} 0_n & \mathbf{1}_n \\ -\mathbf{1}_n & 0_n \end{pmatrix}$. The corresponding Lie-algebra is $\mathfrak{sp}(n;\mathbb{C})$. Up to isomorphism J is the unique skew form on $\mathbb{C}^{2n}$.

The real forms of these classical groups either use real, complex or quaternion vector spaces in their description:

1. $\mathrm{SL}(n;\mathbb{R})$ with Lie algebra $\mathfrak{sl}(n;\mathbb{R})$;

2. $\mathrm{O}(p,q)$ the group of real $n \times n$ matrices A with ${}^{\mathsf{T}}A\mathbf{1}_{p,q}A = \mathbf{1}_{p,q}$ where $\mathbf{1}_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$, $p + q = n$. The corresponding Lie-algebra $\mathfrak{o}(p,q)$ consists of complex $n \times n$ matrices A with ${}^{\mathsf{T}}A\mathbf{1}_{p,q} + \mathbf{1}_{p,q}A = 0$. Up to isomorphism there is a unique non-degenerate symmetric form of signature (p,q) represented by the matrix $\mathbf{1}_{p,q}$.

3. $\mathrm{Sp}(n;\mathbb{R})$ the group of real $2n \times 2n$ symplectic matrices with Lie algebra $\mathfrak{sp}(n;\mathbb{R})$.

4. $\mathrm{U}_{\mathbb{C}}(p,q)$ the group of complex $n \times n$ matrices A with ${}^{\mathsf{T}}\bar{A}\mathbf{1}_{p,q}A = \mathbf{1}_{p,q}$ where $\mathbf{1}_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$, $p+q=n$. The corresponding Lie-algebra $\mathfrak{u}_{\mathbb{C}}(p,q)$ consists of complex $n \times n$ matrices A with ${}^{\mathsf{T}}\bar{A}\mathbf{1}_{p,q} + \mathbf{1}_{p,q}A = 0$. Up to isomorphism there is a unique hermitian form of signature (p,q) represented by the matrix $\mathbf{1}_{p,q}$. The unique skew hermitian form is represented by $\mathrm{i}\mathbf{1}_{p,q}$.

5. $\mathrm{U}_{\mathbb{H}}(p,q)$ the group of quaternionic $n \times n$ matrices A with ${}^{\mathsf{T}}A^*\mathbf{1}_{p,q}A = \mathbf{1}_{p,q}$ where $\mathbf{1}_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$, $p+q=n$. The corresponding Lie-algebra $\mathfrak{u}_{\mathbb{H}}(p,q)$ consists of quaternionic $n \times n$ matrices A with ${}^{\mathsf{T}}A^*\mathbf{1}_{p,q} + \mathbf{1}_{p,q}A = 0$. Here A^* is the matrix where each entry $a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$ is replaced by its conjugate $a - \mathrm{i}b - \mathrm{j}c - \mathrm{k}d$. Up to isomorphism there is a unique quaternionic hermitian form of signature (p,q) represented by the matrix $\mathbf{1}_{p,q}$.

6. $\mathrm{U}_n(\mathbb{H})^-$, the group of quaternionic $n \times n$ matrices A with $A^*(\mathrm{j}\mathbf{1}_n)A = \mathrm{j}\mathbf{1}_n$ and with Lie-algebra $\mathfrak{u}(n;\mathbb{H})^-$. There is a unique skew-hermitian form on $\mathbb{H}^n$ represented by $\mathrm{j}\mathbf{1}_n$.

C Nilpotent and Unipotent Elements in Classical Groups

Here the base field is $\mathbb{R}$ but many arguments also work for any field of characteristic zero. A nilpotent matrix N is said to have **nilpotency-index** k if $N^k = 0$ but $N^{k-1} \neq 0$. The nilpotency index $\nu(\mathfrak{g})$ of a matrix Lie algebra $\mathfrak{g}$ is the maximal nipotency index which occurs among the nilpotent $N \in \mathfrak{g}$. For instance $\nu(\mathfrak{sl}(n;\mathbb{R})) = n$ as exemplified by the Jordan matrix

$$N_n = \left.\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\} n$$

of size n.

If A is a unipotent matrix, its **index of unipotency** is the index of nilpotency of $A - I$ and the unipotency index $\nu(G)$ of a matrix Lie group G is the maximal index of unipotency occurring among the unipotent $A \in G$. It is clearly equal to $\nu(\mathrm{Lie}(G))$.

To calculate $\nu(\mathrm{Lie}(G))$ one proceeds as follows. A nilpotent $X \in \mathrm{Lie}(G)$ can be completed to a *standard triple* (H, X, Y) defining a Lie sub algebra isomorphic to $\mathfrak{sl}(2;\mathbb{R})$. If $\mathrm{Lie}(G) \subset \mathrm{End}(V)$, the vector space V thus is an $\mathfrak{sl}(2;\mathbb{R})$-representation and hence completely decomposes into irreducible ones, say $V=\bigoplus_{j=1}^{k} V_j$ and $X=(X_1,\ldots,X_k)$ corresponds to the partition $(r_1,\ldots,r_k)$ of $n=\dim V$ which follows the dimensions $r_j=\dim V_j$ of the irreducible constituents. Any irreducible representation is completely given by its highest weight vector. Here V_j is the irreducible representation of $\mathfrak{sl}(2;\mathbb{R})$ of highest weight (r_j-1)which means that it is isomorphic to $\mathbb{R}^{r_j}$ on which X acts as N_{r_j}, H as the diagonal matrix with entries $(r_j, r_j-2,\ldots,-r_j+2,-r_j)$ and Y as the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \mu_1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \mu_{r_j-1} & 0 \end{pmatrix}$$

where $\mu_i=i(r_j-i)$, $i=1,\ldots,r_j-1$. It follows that X has index of nilpotency $\nu(X)=\max\{r_j\}$. Hence triples containing a nilpotent X correspond to partitions of n and one can read of the index of nilpotency of X from the partition. There is a systematic way of describing which partitions occur for the classical groups. See for instance [Go-McGo, Chapter 5, Chapter 9.3]. From this classifcation one deduces easily:

Theorem C.1. *The unipotency index of the classical groups is given as follows:*

1. *For* $\mathrm{SL}(n;\mathbb{C})$ *and* $\mathrm{SL}(n;\mathbb{R})$ *it is* n*;*
2. *for* $\mathrm{Sp}(n;\mathbb{C})$ *and* $\mathrm{Sp}(n;\mathbb{R})$ *it is* $2n$*;*
3. *for* $\mathrm{O}(n;\mathbb{C})$ *it is* n *if* n *is odd, but* $(n-1)$ *if* n *is even;*
4. *for* $\mathrm{O}(p,q)$, $\mathrm{U}_{\mathbb{C}}(p,q)$ *and for* $\mathrm{U}_{\mathbb{H}}(p,q)$ *with* $p>q$ *it is* $2q+1$ *and when* $p=q$ *it equals* $n=2p$ *(in particular it equals* 1 *in the definite case);*
5. *for* $\mathrm{U}_n(\mathbb{H})^-$ *it equals* n*.*

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