Two facets of Van de Ven's work: vector bundles and surfaces – a lifelong fascination.

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Theorem (Barth-van de Ven, Inventiones 1974)

Let $X \subset \mathbf{P}^n_{\mathbf{C}}$ be a degree d smooth projective variety of dimension $n-2$(=codimension 2).
If $d < \frac{1}{4}n + \frac{3}{2}$ then X is a complete intersection.

Explanation of the terms in the theorem:

Definition

A *complex projective algebraic variety* is the solution set of a collection of *homogeneous* polynomials $f_1, \ldots, f_r$. The solution set $V(f_1, \ldots, f_r)$ is to be taken in *projective n–space* $\mathbf{P}^n_{\mathbf{C}}$:

$$\begin{aligned} \mathbf{P}^n_{\mathbf{C}} &= \{\text{lines in } \mathbf{C}^{n+1} \text{ through } \mathbf{0}\} \\ &= \{\text{ratios } (x_0 : x_1 : \cdots : x_n),\ x_j \in \mathbf{C}\} \\ &= \{\text{equiv. classes } [\vec{x}], \quad \vec{x} \sim \lambda\vec{x} \quad \forall \lambda \in \mathbf{C}^*\} \\ V(f_1, \ldots, f_r) &= \{[\vec{x}] = (x_0 : \cdots : x_n) \in \mathbf{P}^n_{\mathbf{C}} \mid f_j(\vec{x}) = 0,\ j = 1, \ldots, r\}. \end{aligned}$$

A projective variety V (irreducible) need not be a *manifold*, i.e., locally "like" some open set in $\mathbf{C}^m$, but this is so around "most" points. Then $m =$*dimension* of V. If V is a manifold, we call it a *smooth variety*. The next invariant depends on the embedding:

$$\text{the degree } (V) = \text{number of pnts. in } V \cap L, \quad L \text{ a general linear space of } codim.\ m.$$

Examples:
1. A *hypersurface* $V(f) \subset \mathbf{P}^n_{\mathbf{C}}$ with f of degree d has dimension $n-1$ and degree d.
2. A variety $V(f_1, \dots, f_r)$ has dimension $\geq n-r$ (each f_j is a condition or constraint and each condition lowers the dimension at most by one). If equality holds it is by definition a *complete intersection*. In that case:

$$\deg V(f_1, \dots, f_r) = \deg f_1 \cdot \deg f_2 \cdots \deg f_r \quad \text{(theorem of Bezout)}.$$

3. The rational curve, image of $t \mapsto (1 : t : t^2 : t^3)$ is *not* a complete intersection (by Bezout).

Relation with Hartshorne's conjecture

Conjecture (Hartshorne (1974))

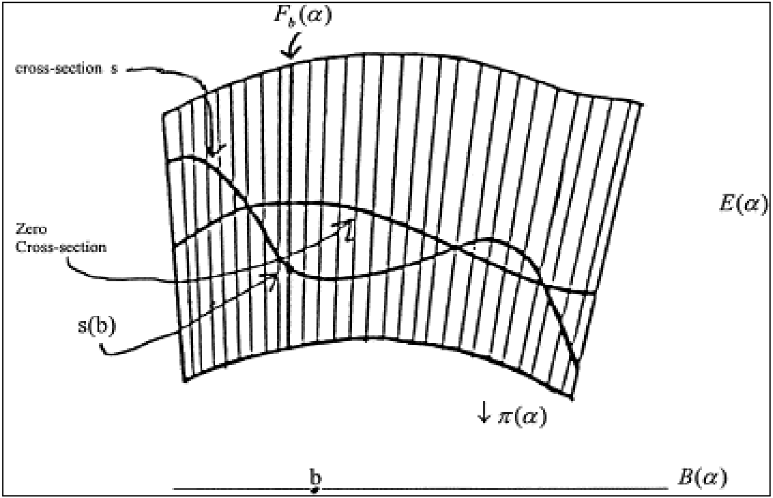
Suppose that $X \subset \mathbf{P}^n_{\mathbf{C}}$ is a smooth projective variety of dimension $> \frac{2}{3}n$. Then X is a complete intersection.

1. This conjecture is still wide open. Hartshorne made some contributions but the above theorem by Barth and van de Ven is the most substantial one. It proves the conjecture in the first interesting case, that of codimension 2 *provided the degree is not too big*.
2. Barth and van de Ven and, independently, Hartshorne a bit later proved similar results for any codimension. The sharpest version is:

$$n \geq d(d-1) \implies X \text{ is a complete intersection.}$$

Ingredients of proof

Here *vector bundles* play a central role. I don't give a formal definition, just an idea of what a (complex) vector bundle α looks like – van de Ven drew this picture many times on the blackboard:



Ingredients of proof

More formally,:

- A *total space* $E = E(\alpha)$ (in this context: a complex manifold);
- a *base space* $B = B(\alpha)$ (in this context: a complex manifold);
- a *bundle map* $\pi : E \to B$ (here: a holomorphic map) such that the *fibers* $F_b = \pi^{-1}b$ are complex vector spaces of fixed dimension r, the *rank* of α;
- the fiber structure is "locally constant".

Examples

1. The *tangent bundle* T_X of a manifold X.
2. The *normal bundle* N_X of an embedded submanifold $X \subset Y$.

Terminology

A line bundle is a vector bundle of rank 1. *The normal bundle to a hypersurface* $V(f)$ *is a line bundle.*
A vector bundle of rank $r \geq 2$ *may or may not be a direct sum of line bundles. If it is, the vector bundle splits.*

Ingredients of proof

The link with vector bundles comes via the *normal bundle*:

Outline of proof

- *Codimension* 2 *submanifolds* $V \subset \mathbf{P}^n_{\mathbf{C}}$ *have rank* 2 *normal bundles* N_V *and the so called Serre construction shows that for* $n \geq 3$*:*

$$N_V \text{ extends to a rank } 2 \text{ bundle } \alpha \text{ on } \mathbf{P}^n_{\mathbf{C}}.$$

- *For extendable normal bundles* $N_V = \alpha|_V$ *one has:*

$$V \text{ complete intersection } \iff \alpha \text{ splits}.$$

- *Determine when* 2*-bundles on* $\mathbf{P}^n_{\mathbf{C}}$ *split and translate this in a condition on the degree of* V*.*

Serre's Construction

Suppose X is a complex algebraic manifold, $Y \subset X$ is locally a complete intersection (e.g. Y smooth) with the properties

- $\text{codim}_Y X = 2$;
- Let $\mathcal{I}_Y$ be the ideal sheaf of Y and set $L^{-1} = \det N_{Y/X}$. The locally free rank one sheaf $\mathcal{E}xt_{\mathcal{O}_X}(\mathcal{I}_Y, L)$ is trivial as a line bundle.

Then the sheaf-extension E

$$0 \to L \to E \to \mathcal{I}_Y \to 0$$

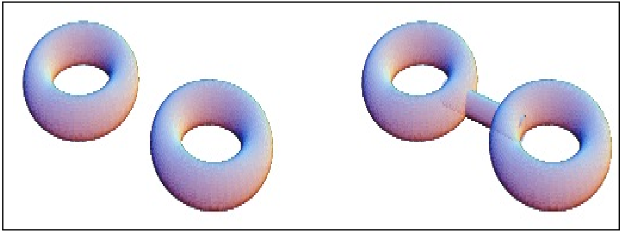
defined by the canonical generator $1 \in H^0(X, \mathcal{E}xt_{\mathcal{O}_X}(\mathcal{I}_Y, L))$ is locally free, and so defines a *vectorbundle* E on X. Moreover, the dual bundle E^* has a section vanishing exactly along Y and $E^*|_Y = N_{Y/X}$, i.e. E^* is an extension of the normal bundle of Y to all of X.

Theorem (Van de Ven 1966)

1. The smooth manifold $\underbrace{\mathbf{P}^2_{\mathbf{C}} \# \cdots \# \mathbf{P}^2_{\mathbf{C}}}_{n}$ *admits an almost complex structure if and only if n is odd.*

2. It admits a complex structure only for $n = 1$.

Explanation of the notation: If X and Y are two smooth manifolds the *connected sum* $X \# Y$ is a smooth manifold obtained by cutting out discs from X and Y and glueing in a tube:



Terminology

Definition

X has an *almost complex structure* $\iff$ there is a smoothly varying "complex multiplication" J_x on the tangent space $T_{X,x}$:

$$J_x : T_{X,x} \to T_{X,x}, \quad J_x^2 = -\text{identity}.$$

Obviously: X complex manifold $\implies$ X has an almost complex structure, but not conversely: The above theorem claims that a connected sum of **three** copies of $\mathbf{P}^2_{\mathbf{C}}$ has an almost complex structure but not a complex structure.

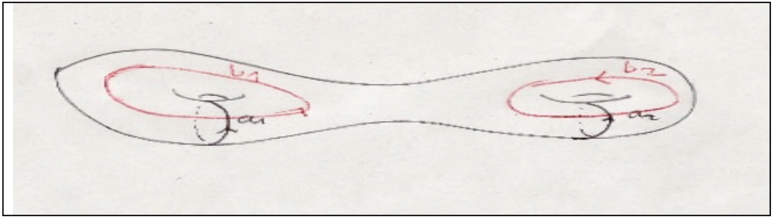
Ingredients for the proof:

One needs **topological invariants**:

1. The Euler number $e(X)$; for instance $e(\mathbf{P}^2_{\mathbf{C}}) = 3$ and one knows

$$e(X \# Y) = e(X) + e(Y) - 2 \implies e(\underbrace{\mathbf{P}^2_{\mathbf{C}} \# \cdots \# \mathbf{P}^2_{\mathbf{C}}}_{n}) = n + 2.$$

2. The intersection form $I(X)$ in middle cohomology. Example: for a genus 2 surface C_2:



the form $I(C_2)$ is a *skew symmetric form*.

Ingredients for the proof

By contrast, if $\dim_{\mathbf{R}} X \equiv 0 \bmod 4$, the form $I(X)$ is a *symmetric form* and can be diagonalized over $\mathbf{Q}$, say $I(X)(x_1, \ldots, x_n) = \sum a_j x_j^2$ and one defines

$$\sigma(X) = (\text{number of } a_j \text{ with } a_j > 0) - (\text{number of } a_j \text{ with } a_j < 0).$$

For example

$$I(\underbrace{\mathbf{P}^2_{\mathbf{C}} \# \cdots \# \mathbf{P}^2_{\mathbf{C}}}_{n}) = \text{the diagonal form}$$

$$\vec{x} = (x_1, \ldots, x_n) \longmapsto \sum_j x_j^2 \text{ of signature } n.$$

Ingredients for the proof

For the proof of 1. one uses:

Lemma (Wu and G. Reeb (1952))

Let X be a compact connected oriented smooth 4-manifold and let w_2 be the second Stiefel-Withney class of X. Then:

X has an almost complex structure	$\iff$	$(*)$	$\exists k \in H^2(X)$, $k \equiv w_2 \bmod 2$ with $I(X)(k) = 3\sigma(X) + 2e(X)$.

In case $X = \mathbf{P}^2_{\mathbf{C}} \# \cdots \# \mathbf{P}^2_{\mathbf{C}}$ (n copies) this condition $(*)$ boils down to

$$I(X)(\vec{x}) = \sum_{j=1}^{n} x_j^2 = 4 + 5n \text{ has a}$$

solution $\vec{x} = (x_1, \ldots, x_n)$ in *odd* integers x_j .

Such a solution exists if and only if $n = 2m + 1$, e.g. $(3, 3, 1, 3, 1, \ldots, 3, 1)$ (m ones).

For the proof of 2. one uses much deeper tools:

Theorem (Bogomolov/Miyaoka/Yau, BMY–inequality)

Let X be a 4-manifold with a complex structure (=complex surface). Let $\sigma(X)$ = signature of $I(X)$. Assume X is not a ruled surface. Then one has the inequality

$$\sigma(X) \leq \frac{1}{3} e(X).$$

Application: for $\underbrace{\mathbf{P}^2_{\mathbf{C}} \# \cdots \# \mathbf{P}^2_{\mathbf{C}}}_{n}$ one has $e = 2 + n$, $\sigma = n$ and so

$$3\sigma = 3n \leq e = n + 2 \implies n = 1,$$

which proves 2.

Historical Remarks about this section

Van de Ven's 1966 article is a very special case of a more general study of almost complex four-manifolds. His construction of a four-manifold with an almost complex structure not coming from a complex structure is different. In fact the above example could not be treated with the methods of the 1966 article. In it one finds a weaker version of the BMY-inequality. To explain this, recall the usual formulation of the BMY–inequality:

$$K^2 := 3\sigma + 2e \implies K^2 \leq 3e.$$

Historical remarks (cont,)

In the 1966 article one finds the inequality

$$K^2 \leq 8e,$$

good enough for the alternative example. Only much later, in 1974 the details of the proof appeared, shortly before Bogomolov (in 1975) ameliorated it to $K^2 \leq 4e$. Van de Ven suggested that the "true" bound should be $K^2 \leq 3e$. This is proven a little later by Miyaoka (1977) and, independently, by Yau (1977).