Applications of Mixed Hodge Modules to Representation Theory After Burgos-Wildeshaus

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Chapter 0

Introduction

The goal of this note is to discuss the article [Bu-Wi]. It uses rather delicate constructions from the theory of mixed Hodge modules which are applied to Shimura varieties. The article very well illustrates how one could apply these tools to obtain deep results in other fields.

The main result from loc. cit.[1] concerns representation theory of certain reductive algebraic groups, namely the ones that come up as homogeneous groups for bounded symmetric domains. This class of groups has been characterized group-theoretically a long time ago by E. Cartan for which the reader may consult [He].

To any representation V of such a group G one can associate a homogeneous polarized variation of Hodge structure over the corresponding domain. One can further divide out by the action of an arithmetic subgroup which yields a locally symmetric quotient X. The latter turns out to be a quasi-projective variety to which the variation descends. The Baily-Borel compactification X^* is a minimal compactification of X and its boundary consists of a union of similar locally symmetric quotients, say Y associated to a reductive groups G' (depending on Y) canonically associated to a certain parabolic subgroup $P \subset G$. The group G' is a quotient of P and hence the G'-module associated to this boundary component corresponds to the U-invariant subspace $V^U \subset V$. The same construction as for X now gives a polarized variation on Y. Theorem 1.7.1 tells you how this variation can be obtained using the standard operations from mixed Hodge modules defined by the inclusions $Y \hookrightarrow X^*$ and $X \hookrightarrow X^*$.

These results are indeed non-trivial: they generalize for instance earlier results of Looijenga-Rapoport [L-R] that are directly related to Looijenga's solution [L] of the Zucker conjecture for L^2-cohomology. It should be said however although the articles [L-R] as well as [Bu-Wi] simplify and clarify the proof in [L], some unavoidable hard analysis is still needed to obtain Looijenga's result and cannot be substituted by the purely algebraic treatment of [Bu-Wi].

The proof of the main result, Theorem 1.7.1 is complicated and uses a lot of background material:

First of all group theoretical background on bounded symmetric domains, the corresponding locally symmetric varieties and their compactifications, both the Baily-Borel compactifications and the various toroidal compactifications.

[1] Corresponding to Theorem 1.7.1 in these notes.

The latter are needed if one wants to work with degenerations of mixed Hodge modules on a smooth background variety along a normal crossing variety. [2]

Secondly, material on mixed Hodge modules, especially Verdier specialization plays an important role in the proof as well as mixed Hodge modules with group actions.

Thirdly there are abstract simplicial constructions and implications thereof on the level of mixed Hodge modules. This comes from the combinatorics of the toroidal strata in the toroidal compactification.

In the final step of the proof some very abstract categorical constructions from group theory are needed.

The reader can see from the list of contents which of these topics I discuss and where.

[2]This is probably not really necessary but the technical details of the actual situation needed (degeneration of mixed Hodge modules on the Baily-Borel compactification along its boundary) might very well be equivalent to what has been done in [Bu-Wi] and would therefore not simply the proof.

Notation.

- Let k be a field, $L \supset k$ a field extension and G an algebraic group defined over k. Then $G_L = G \times_k \operatorname{Spec}(L)$ denotes the extension of G to L while $G(L)$ stands for the group of points of G in L. If V is a k-vector space we accordingly write $V_L := V \otimes_k L$;
- For an algebraic $\mathbf{Q}$-group G the notation $G^0(\mathbf{R})$ means the connected component in the classical topology and G^0 is the corresponding irreducible component;
- $\mathbf{G}_m$: the $\mathbf{Q}$-group such that for $k \supset \mathbf{Q}$ one has $\mathbf{G}_m(k) = k^\times$, the multiplicative group of the field k; an algebraic k-torus is an algebraic group T such that some field extension $L \supset k$ and some integer r one has an L-isomorphism $T(L) \simeq (L^\times)^r$. If this already holds for $L = k$ we say that T is *a k-split torus*;
- $\mathbf{S}$ the $\mathbf{Q}$-group called *Deligne torus*:
$$\mathbf{S}(k) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in k; x^2 + y^2 \neq 0; \right\}$$
- the *weight cocharacter* $w : \mathbf{G}_m \to \mathbf{S}$ defined by $w(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$;
- U: the *algebraic circle group*: the $\mathbf{Q}$-subgroup of the Deligne torus given by the equation $x^2 + y^2 = 1$; its real points $\mathbf{U}(\mathbf{R})$ can be identified with the circle $\{z \in \mathbf{C} \mid z\bar{z} = 1\}$;
- For a group G the center is denoted $Z(G)$.

Chapter 1

Shimura Varieties, their Compactifications and the Main Result

1.1 Bounded Symmetric Domains

Definition 1.1.1.

1. $D \subset \mathbf{C}^n$ is called *bounded symmetric domain* if it is bounded and if, moreover, for every $x \in D$ there exists a holomorphic involution $s_x : D \to D$ which has x as isolated fixed point.

2. A *symmetric hermitian space* (X, g) is a complex manifold X equipped with a hermitian metric g such that for each $x \in X$ there are holomorphic involutions s_x as in 1.

Before discussing their structure let me recall a few notions from the theory of algebraic groups. The notions of (semi)simple groups and reductive groups are assumed to be known. See for example [Bo91]. In this note only algebraic matrix groups of this sort will be considered. Semisimple groups (over a fixed field) are isogenous to products of simple groups (over the same field).

Definition 1.1.2. Let G be an algebraic group.

- G is of *adjoint type* if the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$ (given by $g \mapsto \{X \mapsto gXg^{-1}\}$ is injective – equivalently — if $Z(G) = 1$. The adjoint group G^{ad} is the image under the adjoint representation and a group is of adjoint type if and only if $G^{\mathrm{ad}} = G$;

- G is (algebraically) *simply connected* if every isogeny $G' \to G$ with G' a connected algebraic group is an isomorphism. Its group of real points, $G(\mathbf{R})$, is simply connected if and only if this is so in the classical topology;

- An involution $\sigma : G \to G$ of a real reductive algebraic group is called a *Cartan involution* if on the Lie algebra we have $(\mathrm{Lie}(G))^{d\sigma=1} = \mathfrak{k} = \mathrm{Lie}(K)$ with K a maximal compact subgroup. Note that with $(\mathrm{Lie}(G))^{d\sigma=-1} = \mathfrak{p}$ one has

$$\mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

An important property of Cartan involutions is:

Lemma 1.1.3 ([De72, 2.8]). *An involution σ of a real reductive group G is Cartan if and only if $\sigma = \operatorname{Ad} C$, for some $C \in G(\mathbf{R})$ for which C^2 is in the center and for which there exists a faithful embedding*

$$G \hookrightarrow \operatorname{Aut}(V, b), \quad V \text{ an } \mathbf{R}\text{-space}, b \text{ a symmetric bilinear form with } b(-, C-) \text{ an inner product.}$$

For the structure of symmetric hermitian spaces, I can now quote [He, Ch. VIII, Prop. 4.4.]:

Proposition 1.1.4.

1. *A symmetric hermitian space is a product of irreducible ones; an irreducible hermitian space is either of compact, non-compact or of euclidean type.*
2. *An irreducible factor is compact, non-compact, respectively euclidean if the curvature of its metric is positive, negative, respectively zero. A non-compact factor is biholomorphic to a bounded symmetric domain.*
3. *The connected component of the group of biholomorphic automorphisms of a compact, respectively non-compact factor is a compact, respectively non-compact simple Lie group of adjoint type. It has the structure of an algebraic simple group defined over $\mathbf{R}$.*
4. *Let D be an irreducible bounded symmetric domain and G the connected component of its group of holomorphic automorphisms. The group G acts transitively on D; the isotropy group K_x of x is a maximal compact subgroup of G with connected 1-dimensional center: $Z(K_x) \simeq S^1$ and there is an algebraic homomorphism*

$$\nu_x : \mathrm{U}(\mathbf{R}) \to G, \quad \operatorname{Im}(\nu_x) = Z(K_x) \tag{1.1}$$

   with the following properties:

   (a) *$d\nu_x(\mathrm{i}) = J$, the complex structure on $(T_x D)_{\mathbf{C}}$ (coming from the embedding $D \subset \mathbf{C}^n$);*

   (b) *the adjoint operation of $\nu_x(-1)$ on $\operatorname{Lie}(G)$ is a Cartan involution of the pair (G, K_x) and induces[1] $ds_x = -\operatorname{id}$ on $T_x D = \operatorname{Lie}(G)/\operatorname{Lie}(K_x)$.*

5. *Conversely, any simple algebraic group of adjoint type with one-dimensional connected center is the connected component of the group of automorphisms of a hermitian symmetric domain.*

Let me look at the complex structure on $D = G/K$. For this, first note that the Cartan decomposition

$$\mathfrak{g} := \operatorname{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$$

identifies $T_x D = \mathfrak{p}$ and in fact, using the adjoint representation of K on $\mathfrak{p}$ the tangent bundle $T(D)$ can be written

$$T(D) = G \times_{\operatorname{Ad} K} \mathfrak{p}. \tag{1.2}$$

[1] Recall that s_x is the involution at x defining a symmetric domain.

The complex structure on $T_xD = \mathfrak{p}$ induces a splitting of the complexification $(T_xD)_{\mathbf{C}} = T_x^{1,0} \oplus T_x^{0,1}$ in the $\pm$i-eigenspaces of the complex structure $J = \nu_x(\mathrm{i})$. In fact, we get a splitting

$$\mathfrak{g}_{\mathbf{C}} \quad = \quad \mathfrak{k}_{\mathbf{C}} \oplus \underbrace{\overset{\overset{T^{1,0}}{\|}}{\mathfrak{p}^+} \oplus \overset{\overset{T^{0,1}}{\|}}{\mathfrak{p}^-}}_{\mathfrak{p}_{\mathbf{C}}}$$

which implies that $d\nu_x(z) = \mathrm{Ad}(\nu_x(z))$ acts as follows:

$$\begin{array}{ccccccc} \mathfrak{g}_{\mathbf{C}} & = & \mathfrak{k}_{\mathbf{C}} & \oplus & \mathfrak{p}^+ & \oplus & \mathfrak{p}^- \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ & & \mathrm{id} & & z^{-1} & & z. \end{array} \tag{1.3}$$

This ties in with the *compact dual* $\check{D}$ of D, a projective manifold homogeneous under $G(\mathbf{C})$ with stabilizer P at x the connected Lie group with Lie algebra

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+.$$

To see that this is a Lie algebra, note that $[\mathfrak{p}_{\mathbf{C}}, \mathfrak{p}_{\mathbf{C}}] \subset \mathfrak{k}_{\mathbf{C}}$ and $[k_{\mathbf{C}}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm}$. The tangent space $T_x\check{D}$ gets identified with $\mathfrak{p}^-$ which as a real space is isomorphic to $\mathfrak{p}$ and so the embedding $G \hookrightarrow G(\mathbf{C})$ induces an embedding $D = G/K \to G_{\mathbf{C}}/P$. The group P is indeed a parabolic subgroup: it is the stabilizer of the flag $\{\mathrm{Lie}(G_{\mathbf{C}}) \supset \mathfrak{p}^+\}$.

Such domains admit a canonical polarizable variation of Hodge structure. This can be most easily seen if you use the group theoretic definition of a pure Hodge structure which I now recall:

Definition 1.1.5. A pure Hodge structure of weight k on a rational vector space V is an algebraic morphism

$$h : \mathbf{S} \to \mathrm{GL}(V)$$

defined of $\mathbf{R}$ and such that the weight co-character

$$w_h = h \circ w : \mathbf{G}_m \xrightarrow{w} \mathbf{S} \xrightarrow{h} \mathrm{GL}(V)$$

is given by $t \mapsto t^k\,\mathrm{id}$.

Let me now pass to a bounded symmetric domains $D = G(\mathbf{R})/K_x$ with G an algebraic group *defined over* $\mathbf{Q}$. Note that if G is connected and of adjoint type this can always be assumed: see [Bo91, 7.9]. We choose some faithful representation as a matrix group, say $G \subset \mathrm{GL}(W)$. Using Proposition 1.1.4, one gets an algebraic morphism

$$u : \mathbf{U}(\mathbf{R}) \xrightarrow{\nu_x} G(\mathbf{R}) \subset \mathrm{GL}(W_{\mathbf{R}})$$

and hence a weight zero Hodge structure on W. Similarly, one obtains a weight zero Hodge structure on *any* representation $\rho : G \to \mathrm{GL}(V)$

$$h = h_x : \mathbf{S}(\mathbf{R}) \xrightarrow{z \mapsto z/\bar{z}} \mathbf{U}(\mathbf{R}) \xrightarrow{u} G(\mathbf{R}) \xrightarrow{\rho} \mathrm{GL}(V_{\mathbf{R}}). \tag{1.4}$$

This structure evidently depends on $x \in D$.

As an example, one has $\mathfrak{g} = \mathrm{Lie}(G)$. We have a decomposition according to the characters $1, \bar{z}/z, z/\bar{z}$ of $\mathbf{S}(\mathbf{C})$ respectively:

$$\begin{array}{ccccccc} & & \mathfrak{k}_{\mathbf{C}} & & \mathfrak{p}^- & & \mathfrak{p}^+ \\ & & \| & & \| & & \| \\ \mathfrak{g}_{\mathbf{C}} & = & \mathfrak{g}^{0,0} & \oplus & \mathfrak{g}^{-1,1} & \oplus & \mathfrak{g}^{1,-1}. \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ & & \mathrm{id} & & \bar{z}/z & & z/\bar{z} \end{array}$$

with corresponding Hodge flag

$$\begin{array}{ccc} F^\bullet_x & = & \{F^{-1} \supset F^0 \supset F^1\} \\ & & \quad\| \qquad\quad \| \qquad\quad \| \\ & & \mathfrak{g}_{\mathbf{C}} \supset \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+ \supset \mathfrak{p}^+. \end{array}$$

Letting x vary we get a G-homogeneous variation of Hodge structure on D. Indeed, this is a holomorphically varying flag and Griffiths transversality is a direct consequence of (1.2) and(1.3). Moreover, by Lemma 1.1.3 it is a polarized variation of Hodge structure. Similarly. any G-representation gives a weight zero G– polarized variation of Hodge structure over D, but now the Hodge numbers may be different.

Lemma–Definition 1.1.6. *Let G be of adjoint type and let $\rho : G \to \mathrm{GL}(W)$ be an algebraic representation defined over a subfield $F \subset \mathbf{R}$. Then $\rho \circ \nu_x$ defines a weight 0 F-Hodge structure. The resulting local system on D defines a polarizable F-variation of Hodge structures, $\tilde{\mu}(\rho)$ of pure weight 0; it is called the standard construction. It yields a functor*

$$\tilde{\mu} : G\text{-Reps}_F \to \{\text{polarizable } F\text{-VHS on } D = G/K.\}$$

1.2 Shimura Data

The passage to Shimura varieties starts with the observation that a symmetric space can be written in many ways as $D = G/K$ with G a semisimple or even a reductive algebraic group. For the purpose of representations this is an important remark, since representations of G^{ad} do not always lift to representations of G.

To elaborate on this let me describe the points of an irreducible $D = G/K$ with G connected and of adjoint type differently. By Prop. 1.1.4, the algebraic homomorphisms $\nu_x : \mathbf{U}(\mathbf{R}) \to G$ satisfy

U1 Only the characters $\{1, z, z^{-1}\}$ of $\mathbf{U}(\mathbf{R})$ occur in the representation $\mathrm{Ad} \circ \nu_x : \mathbf{U}(\mathbf{R}) \to \mathrm{GL}(\mathfrak{g}_{\mathbf{C}})$

U2 $\mathrm{Ad} \circ \nu_x(-1)$ is a Cartan involution.

Instead, one can also use the associated homomorphisms $h = h_x : \mathbf{S}(\mathbf{R}) \to G(\mathbf{R})$ from (1.4). These satisfy

S1 Only $\{1, z/\bar{z}, \bar{z}/z\}$ occur in the representation $\mathrm{Ad} \circ h : \mathbf{S}(\mathbf{R}) \to \mathrm{GL}(\mathfrak{g}_C)$

S2 $\mathrm{Ad} \circ h(\mathrm{i})$ is a Cartan involution.

Then one can see that for a connected group of adjoint type, one has:

$$\begin{aligned} D &= \{G(\mathbf{R})\text{-conjugates of non-const. } \nu : \mathbf{U}(\mathbf{R}) \to G(\mathbf{R}) \mid \text{ U1 and U2 hold.}\} \\ &= \{G(\mathbf{R})\text{-conjugates of non-const. } h : \mathbf{S}(\mathbf{R}) \to G(\mathbf{R}) \mid \text{ S1 and S2 hold.}\} \end{aligned}$$

As a hint as to why, note that ν being non-constant is equivalent to G and hence D being not compact and for ν_x it implies that $\mathrm{Im}(\nu_x) = Z(K_x)$ and hence

$$g(x) = x \text{ for some } g \in G \iff g \in K_x \iff g\nu_x(u)g^{-1} = \nu_x(u) \text{ for all } u \in \mathbf{U}.$$

To extend this to reductive G defined over $\mathbf{Q}$, one considers likewise $h : \mathbf{S} \to G$ obeying S1 and S2, but this does not suffice. The problem is that the maximal semi-simple quotient G^{ss} might have simple factors which do not correspond to irreducible bounded Hermitian domains. To correct this, one demands an extra axiom

S3 G^{ss} has no simple $\mathbf{Q}$-factor on which h is constant.

This leads to the introduction of the homogeneous domain

$$X = \left\{G^{\mathrm{ad}}(\mathbf{R})\text{-conjugates } h : \mathbf{U}(\mathbf{R}) \to G(\mathbf{R}) \mid \text{S1, S2 and S3 hold.}\right\}$$

See [Mi04, Chapter 5] for details. The resulting *pair* (G, X) or (G, K) is then called a *Shimura datum*.

Remark. Note that X may or may not be connected. See Example 1.2.2 below. If instead, one only considers morphisms $h : \mathbf{S} \to G^{\mathrm{ad}}(\mathbf{R})^0$, the resulting X is necessarily connected and one calls $(G, X)^0$ the corresponding *connected Shimura datum*. Note that if G is connected, simple and of adjoint type, $G(\mathbf{R}) = G^{\mathrm{ad}}(\mathbf{R})^0$ and the corresponding Shimura datum is automatically connected.

Clearly, for a given Shimura datum the weight

$$w(G, X) := w_h : \mathbf{G}_m(\mathbf{R}) \to G(\mathbf{R}) \tag{1.5}$$

is independent of $[h]$, the class of h and is called the *weight of the Shimura datum*. It implies that for a given irreducible representation ρ of G the composition $\rho\circ h$ defines a Hodge structure which is not necessarily of weight 0. It turns out [De77, Part1], [Mi04, Chapter5] to be still polarizable and when x varies we still get a polarizable variation of Hodge structure on X.

Lemma–Definition 1.2.1. *Given a Shimura datum (G, X) and a representation $\rho : G \to \mathrm{GL}(V)$, the polarizable variation on X which at $[h]$ is given by*

$$\rho\circ h : \mathbf{S} \to \mathrm{GL}(V)$$

is called the standard construction for the representation $\rho : G \to \mathrm{GL}(V)$.

Remark. Since ρ is no longer assumed to be irreducible, the variation splits as a direct sum of variations of possibly different weights. Let me still call such a variation a variation of Hodge structures.

In this broader set-up indeed non-connected domains may arise since the group G is not assumed connected.

Example 1.2.2. Consider the Siegel upper half space $\mathbf{H}_g = \mathrm{Sp}(g;\mathbf{R})/\mathrm{U}(g)$. The group $Sp(g;\mathbf{R})$ is simple but not of adjoint type: its center is $\pm I_{2g}$. In the theory of Shimura varieties also the union of upper and lower half planes play a role. This is the non-connected domain $C\,\mathrm{Sp}(g;\mathbf{R})/C\mathrm{U}(g)$ where the C stands for "similitudes"; the group $C\,\mathrm{Sp}(g;\mathbf{R})$ is reductive with one-dimensional center. Here homogeneous weight one Hodge structures come up. For instance, the tautological variation on the upper half plane (and the same on the lower half plane) is such a variation. These already appear for the symplectic group itself (due to the non-trivial center).

1.3 Quotients: Locally Symmetric Varieties

Let (G, X) be a Shimura datum. We choose a fixed representation of G as a matrix subgroup of $\mathrm{GL}(n)$.

Definition 1.3.1. 1. A subgroup Γ of $G(\mathbf{Q})$ is called *arithmetic* if it is commensurable with $G_{\mathbf{Z}} := G(\mathbf{Q}) \cap \mathrm{GL}(n;\mathbf{Z})$ (i.e. $\Gamma \cap G_{\mathbf{Z}}$ has finite index in both Γ and G_Z);
2. A *congruence subgroup* of $G(\mathbf{Q})$ is a subgroup of $G(\mathbf{Q})$ containing

$$G(\mathbf{Q}) \cap \{g \in \mathrm{GL}(n;\mathbf{Z}) \mid g \equiv \mathrm{id} \bmod N\}$$

as a subgroup of finite index. A congruence subgroup is arithmetic;
3. A subgroup of $G(\mathbf{Q}) \subset \mathrm{GL}(n;\mathbf{Q})$ is *neat* if for any given element its eigenvalues generate a torsion free subgroup of $\mathbf{C}^\times$ (in particular it cannot have finite order).

It is a fact [Bo69, 17.4] that arithmetic subgroups contain neat congruence subgroups of finite index.

Let Γ be any neat congruence subgroup of $G(\mathbf{Q})$. Then

$$\boxed{X(\Gamma) := \Gamma\backslash X}$$

is a smooth quasi-projective variety. Such varieties are also called *locally symmetric varieties*.

Lemma–Definition 1.3.2. *Let there be given a Shimura datum (G, X), a torsion free congruence subgroup Γ of $G(\mathbf{Q})$ and a representation $\rho : G \to \mathrm{GL}(V)$ defined over a subfield F of $\mathbf{R}$. Then the polarizable variation from Lemma-Definition 1.3.2 descends to $X(\Gamma)$ which defines the* standard construction for the representation $\rho : G \to \mathrm{GL}(V)$ *on $X(\Gamma)$ and is denoted $\mu(\rho)$.*

It yields a functor

$$\boxed{\mu : G\text{-}\mathrm{Reps}_F \to \{\text{polarizable } F\text{-VHS on } X(\Gamma).\}}$$

Remark 1.3.3 (The adelic description). Let $\mathbf{A}_f$ be the ring of the finite adèles. For simplicity I shall only consider the case of a simply connected and simple group G. Then congruence subgroups of $G(\mathbf{Q})$ are given by compact and open subgroups of $G(\mathbf{A}_f)$: if K is such a subgroup

$$\Gamma = K \cap G(\mathbf{Q})$$

is congruence and conversely, any congruence subgroup can be written in this way. One can see [Mi04, Prop. 4.18] that one gets a bijection

$$\begin{array}{rcl} X(\Gamma) & \longrightarrow & (G(\mathbf{Q})\backslash X) \times (G(\mathbf{A}_f)/K) \\ x & \longmapsto & [x,1]. \end{array}$$

The resulting variety is a connected Shimura variety. A similar assertion holds for general Shimura data, leading to (in general non-connected) Shimura varieties. The reason for this is number theoretical: if one wants to find models defined over a *fixed* number field which work for all arithmetic subgroups Γ at the same time, one uses the Galois action to permutes components which makes the union of these components defined over a smaller field. To descend all the way to such a canonically defined number field (the reflex field) one also needs to consider *all* congruence subgroups at the same time. That this indeed would work was completely unexpected at the time and it was shown by Shimura, justifying the terminology! See [Mi04] for pertinent references; it also serves as a good introduction.

1.4 Baily-Borel Compactification

From now on I assume that G is a connected non-compact simple group defined over $\mathbf{Q}$ (not necessarily of adjoint type), D the associated irreducible Hermitian domain and (G,D) the corresponding Shimura datum.

Let $\Gamma \subset G(\mathbf{Q})$ a neat arithmetic subgroup and $D(\Gamma)$ the associated quasi-projective Shimura variety. It admits a canonical compactification $D(\Gamma)^*$, the Baily-Borel compactification of $D = G/K$. Let me describe how this works. To start, $D \subset \check{D}$, the compact dual and we let $\bar{D}$ be the closure of D in $\check{D}$. Points in the boundary that can be joined by the holomorphic image of a unit disk in $\bar{D}$ generate an equivalence relation on $\bar{D}$ and the equivalence classes are called *boundary components*. Note that D itself is also a boundary component. It is called an improper boundary component; the others are the proper ones. For any proper $\mathbf{Q}$-rational parabolic subgroup $P \subset G$ there is a unique proper boundary component D_P normalized by P, called the *rational boundary component* associated to P. The group P turns out to stabilize a flag $W_\bullet$ in V where $G \subset \mathrm{GL}(V)$). To describe it, recall that giving a grading $V_\bullet$ on V is the same as giving a homomorphism $\mu : \mathbf{G}_m \to V$: the character space for the character $\lambda \to \lambda^k$ is V_k. The associated grading is defined by $W_\ell = \bigoplus_{k \le \ell} V_k$. We have:

Proposition 1.4.1 ([Mi90, V.2])**.** *Let $D = G/K$ be hermitian symmetric with G a rational reductive group. Given a $\mathbf{Q}$-parabolic subgroup P of G there is a unique homomorphism*

$$\lambda_P : \mathbf{G}_m \to P$$

such that for all *representations $\rho : P \to \mathrm{GL}(V)$ the 2 data*

- *the filtration W on V obtained from the grading defined by $\rho\circ\lambda_P$;*
- *the Hodge filtration defined by $\rho\circ\mu_x$*

define a mixed Hodge structure on V.

I shall now deduce from this that there exists a nice filtration on P. First introduce

$$W_{-k}P := \{g \in P \mid \rho_g = \text{id on } W_\bullet(\rho)/W_{\bullet-k}(\rho) \text{ for all } \rho\}.$$

It turns out [Mi04, p. 82] that this yields indeed a 2-step filtration in which figures

$$U = \text{ the unipotent radical of } P:$$

$$\boxed{\begin{array}{ccccc} Z(U) & \subset & U & \subset & P \\ \| & & \| & & \| \\ W_{-2}P & \subset & W_{-1}P & \subset & W_0P = P \end{array}}.$$

It follows that P/U is semisimple; it is the centralizer of λ_P. Moreover, $Z(U)$ as well as $U/Z(U)$ are abelian and will be identified with their Lie algebras. These define two real vector spaces that play a role later on:

$$E := U(\mathbf{R})/Z(U)(\mathbf{R}). \tag{1.6}$$
$$F := Z(U)(\mathbf{R})(-1) \subset Z(U)(\mathbf{C}). \tag{1.7}$$

Let me isolate the subgroup of automorphisms of D_P that act as the identity

$$\boxed{G'_P = \{\text{Centralizer in } P/U \text{ of } D_P\}.}$$

Then, since P/U is semi-simple, there is a factorization $P/U = G_P \cdot G'_P$ (this is an almost direct product) with G_P semi-simple. Note that P/U has a non-trivial center A_P which is also the center of G'_P; in fact one has (as algebraic groups over $\mathbf{Q}$)

$$A_P := Z(G/U) \simeq \mathbf{G}_m. \tag{1.8}$$

The inverse image P_1 of G_P in P plays a special role. Summarizing, one has

$$\boxed{P/U = G_P \cdot G'_P,\ P_1 := G_P \cdot U \lhd P,} \tag{1.9}$$

The connected reductive group P_1 is such that $D_P = P_1/K_1$, where K_1 is a connected maximally compact subgroup. In other words, (D_P, P_1) is a (connected) Shimura datum. Note that one could also has an *equivalent* Shimura datum (D_P, G_P) with G_P simple (but not in general of adjoint type).

Let me now introduce the following open subset in the compact dual:

$$D(P) = Z(U)(\mathbf{C}) \cdot D \subset \check{D},$$

i.e. the set of translates of D by elements of $Z(U)(\mathbf{C})$. This set contains D_P and all rational boundary components having D_P in its closure such as D. Since $Z(U)$ is normal in U, and U normal in P, this set is acted on by $Z(U)(\mathbf{C}) \times P(\mathbf{R})$. It turns out to be fibered over D_P in the obvious way: first divide out by the (left) action under the abelian group $Z(U)(\mathbf{C})$ and next under the remaining action of $U(\mathbf{R})/Z(U)(\mathbf{R})$. It turns out that both fibrations are trivial and so one may identify equivariantly

$$D(P) = Z(U)(\mathbf{C}) \times E \times D_P. \tag{1.10}$$

Since $Z(U) = W_{-2}P$, the vector space underlying $\mathrm{Lie}(Z(U))$ has a polarized weight -2 Hodge structure and hence the real vector space $F := \mathrm{Lie}(Z(U))(-1) \subset Z(U_{\mathbf{C}})$ admits a polarized weight 0 Hodge structure. The adjoint action of $G'_P(\mathbf{R})$ on $\mathrm{Lie}(Z(U))$ induces an action on F and the cone

$$\boxed{C(P) := \{G'_P(\mathbf{R})\text{-orbit of } 1 \in Z(U)\} \subset F.} \tag{1.11}$$

This cone is clearly homogeneous under the adjoint action of $G'_P(\mathbf{R})$ and it turns out to be self-dual with respect to the polarization on F.

There is a family of $\mathbf{R}$-bilinear forms

$$h_t : E \times E \to F, \quad t \in D_P$$

depending in a real-analytic fashion on t such that, setting

$$\begin{array}{rcl} \Phi_P : D(P) & \to & F \subset F_{\mathbf{C}} \\ (z, v, t) & \mapsto & 2\pi \mathrm{i}[\mathrm{Im}(z) - h_t(v, v)], \end{array} \tag{1.12}$$

one has the *realization of D as a Siegel domain of the third kind*

$$D = \Phi_P^{-1} C(P).$$

For a proof see e.g. [Mi90, V.3].

1.5 Two Weight Filtrations

On the P-module V there is a canonical weight filtration W, as explained in Prop. 1.4.1. There is a second weight filtration directly related *any* operator

$$T \in Z(U)(\mathbf{Q}) \text{ such that } \pm \frac{1}{2\pi \mathrm{i}} T \in C(P).$$

Note that $N := \log T \in W_{-2}(\mathrm{Lie}(P))$ is nilpotent and via $d\rho$ acts nilpotently on the representation space (V, ρ) of P. Moreover, it acts on the W-filtration since N has weight (-2). For every integer k the operator N defines a unique filtration $W_\bullet(N)[k]$ the *monodromy weight filtration centered at k* characterized by

1. $NW_p(N) \subset W_{p-2}(N)$ for all p;
2. there are isomorphisms $N^\ell : \mathrm{Gr}^{W(N)}_{k+\ell} \xrightarrow{\simeq} \mathrm{Gr}^{W(N)}_{k-\ell}$ for all $\ell \geq 0$.

A priori this has nothing to do with W. Suppose now that the two are linked as follows:

1. $NW_p \subset W_{p-2}$ for all p;
2. The filtration induced by W on $\mathrm{Gr}^{W(N)}_k$ is the monodromy weight filtration centered at k.

Then this is the only such filtration and is called the *weight filtration of N relative to W*. In the situation at hand this is indeed the case, basically because the mixed Hodge structure defined by W and $\rho \circ \mu_x$ define a split mixed Hodge structure (see Prop. 1.4.1):

Proposition 1.5.1 ([Wi2, Prop. 1.3]). *The weight filtration of N on V relative to W exists and is the same as $W(N)$. In particular, the latter* does not depend on the choice of T *provided* $\pm\frac{1}{2\pi\mathrm{i}}T \in C(P)$.

1.6 Interlude: Homological Algebra and Groups

In this section I review two concepts: group cohomology and twisted representations. I shall only look at the classical situation. In [Bu-Wi, Section 3.] this is extended to group actions on Abelian categories (such as those of pure Hodge structures). This requires some rather abstract considerations on derived functors.

Group Cohomology

If G is an abstract group and V a finite dimensional G-representation (over a field F), one defines group cohomology usually using right resolutions of $\mathbf{Z}$ by free $\mathbf{Z}G$-modules as follows. Fix such a resolution $F_\bullet$. Then

$$H^p(G,V) := H^p(\mathrm{Hom}(F_\bullet, V)).$$

In particular, $H^0(G,V) = V^G$, the G-invariant subspace of V. Suppose now that U is a subgroup of G and consider the functor

$$G\text{-Reps}_F \to (G/U)\text{-Reps}_F, \quad V \mapsto V^U.$$

Its derived version inv^U produces out of a G-module V an object in the derived category $D^b((G/U)\text{-Reps}_F)$ and hence is representable by a *complex* of G/U-modules and this holds similarly, for complexes of G-modules as representing an object in the derived category $D^b(G\text{-Reps}_F)$. Concretely:

$$(\mathrm{inv}^U V^\bullet)^n = \bigoplus_{-p+q=n} [\mathrm{Hom}(F_p, V^q)]^U .$$

Twisted representations of abstract groups

First note that if $K \lhd G$ is normal in G and V is any K-representation, conjugation by $\gamma \in G$ on K defines new representation:

$$(V^\gamma, \rho^\gamma);\ \rho^\gamma_g(v) = \rho_{\gamma\cdot g\cdot\gamma^{-1}}(v),\ \forall g \in K\,, v \in V.$$

Definition 1.6.1. Let G be an abstract group, $H \subset P$ a subgroup, $K \lhd G$ a normal subgroup and V an F-representation for K. An *H-twisting* for V consists of a collection of isomorphisms in K-Reps$_F$

$$f_\gamma : V^\gamma \xrightarrow{\sim} V, \quad \gamma \in H$$

such hat

1. $f_\gamma(v) = \rho_{\gamma^{-1}} v$ whenever $\gamma \in K$;
2. the co-cycle condition holds: $f_{\gamma\gamma'} = f_\gamma \circ f^{(\gamma)}_{\gamma'} : V^{\gamma\gamma'} \xrightarrow{\sim} V$.

Morphisms between H-twistings are defined in the obvious way; the resulting category is denoted

$(K\text{-Reps}_F, H)$: *the category of F-representations for K with an H-twisting.*

Remark 1.6.2. 1. Whenever one has a G-representation V, by restriction to K one obtains a K-representation with an obvious H-twisting (for *any* subgroup $H \subset G$: just take $f_\gamma := \rho_{\gamma^{-1}}$.
2. Next, consider the following situation: Suppose $G = P/U$ for some abstract group P and $U \lhd P$ and one starts out with a P-module W. Then the invariant submodule $V := W^U$ is a G-module . Suppose now moreover that all G-representations are fully reducible (i.e. G is reductive). Then the spectral sequence for the functor "taking U-invariants" degenerates at E_2 and reads:

$$\mathrm{inv}^U(W^\bullet) \xrightarrow{\sim} \bigoplus_q H^q(U, W^\bullet)[-q], \quad W^\bullet \in D^b(P\text{-Reps}_F). \tag{1.13}$$

Here one should see $H^q(U, W^\bullet)$ as a complex which at place n has $H^q(U, W^n)$ and so $H^q(U, W^\bullet)[-q]$ has at place n the module $H^q(U, W^{n-q})$.
3. Suppose that moreover $G = K \cdot L$, an almost direct product with K, L both normal in G such that L acts trivially on V so that V is an fact a K-representation. Then for any subgroup H of G the preceding lines apply to (G, K, H): V is an H-twisted K-representation in an obvious way. Then (1.13) takes place in the category $(K\text{-Reps}_F, H)$.

1.7 Main Result and Implications

Suppose now that Γ is an arithmetic subgroup of $G(\mathbf{Q})$. It acts not only on D but also on the rational boundary components. I shall use the convention

$$\Gamma_{H/I} := (\Gamma \cap H)/(\Gamma \cap I) \subset H/I,\ H \text{ a subgroup of } G,\ I \lhd H.$$

With this convention, the group

$$\Gamma_P := \Gamma_{P/U}/\Gamma_{G'_P}$$

is an arithmetic subgroup of G_P which acts freely and discontinuously on D_P and there is a closed embedding

$$i_P : D_P(\Gamma_P) \subset D(\Gamma)^*, \tag{1.14}$$

where $D(\Gamma)^*$ is the Baily-Borel compactification [B-B] of $D(\Gamma)$. The latter is a projective variety, in general highly singular, which contains $D(\Gamma)$ as a Zariski-dense open set:

$$j : D(\Gamma) \hookrightarrow D(\Gamma)^*. \tag{1.15}$$

The morphisms i_P and j play the crucial role in the statement of the main result from [Bu-Wi]. These induce exact functors i_P^* and j_* on the level of polarizable mixed Hodge modules. Following Saito's convention, I'll denote their derived functors by the same symbols.

As a first step, one has to see the functor μ defined in Lemma-Definition 1.2.1 as a functor with target the polarizable mixed Hodge modules. Next, one observes that this functor is exact and hence defines a functor in the derived category:

$$\mu_G : D^b(G\text{-Reps}_F) \to D^b(\mathsf{MHM}_F D(\Gamma).$$

Any given G-representation V gives by restriction a P_1-representation , denoted

$$\mathrm{Res}^G_{P_1} V,$$

respectively, and the subspace of U-invariants $[\mathrm{Res}^G_{P_1} V]^U$ is a P_1/U-representation, or, what is the same, a G_P-representation. By what has been recalled in § 1.6 this construction works in the derived categories as well (see also [Wi3, Theorem 2.2, 2.3]) and hence gives a functor

$$\begin{array}{rcl} \mathrm{inv}^U : D^b(G\text{-Reps}_F) & \longrightarrow & D^b(G_{P_1}\text{-Reps}_F) \\ V^\bullet & \longmapsto & ([\mathrm{Res}^G_{P_1} V^\bullet]^U. \end{array}$$

The main result states that the only obvious natural relation one could guess indeed holds:

Theorem 1.7.1. *Recall that*

$$c = \mathrm{codim}_{D(\Gamma)^*} D_P(\Gamma_P).$$

One has an equality of functors

$$\mu_{P_1} \circ \mathrm{inv}^U = (i_P^* \circ j_*)[-c] \circ \mu_G : D^b(G\text{-Reps}_F) \to D^b(\mathsf{MHM}_F D_P(\Gamma_P),$$

i.e. the diagram

$$\boxed{\begin{array}{ccc} D^b(G\text{-Reps}_F) & \xrightarrow[\mu_G]{} & D^b(\mathsf{MHM}_F D(\Gamma)) \\ {\scriptstyle \mathrm{inv}^U}\big\downarrow & & \big\downarrow{\scriptstyle i_P^* j_*[-c]} \\ D^b(P_1\text{-Reps}_F) & \xrightarrow[\mu_{P_1}]{} & D^b(\mathsf{MHM}_F D_P(\Gamma_P)) \end{array}}$$

is commutative.

Despite the apparent simplicity of this statement, it has quite deep consequences which I now briefly discuss. Firstly, since by (1.5) the weight of a connected Shimura datum is fixed, the above isomorphism sends an irreducible G-representation to a direct sum of pure VHS on the boundary components:

Corollary 1.7.2. *For any $V \in G\text{-Reps}_F)$ the MHVS $\mu_{P_1} \circ \mathrm{inv}^U(Res^G_{P_1} V)$ is a direct sum of pure VHS (in general of different weights).*

Use now Remark 1.6.2, 3 with K replaced by G_P and H the group [2]

$$\boxed{\Gamma_{G'_P} := \Gamma_{G'_P U}/\Gamma_U \subset G'_P(\mathbf{Q}).} \tag{1.16}$$

It is not the spectral sequence (1.13) that is of interest, but rather the one which is implied by a Theorem 1.7.1:

$$E_2^{p,q} = \mu_{P_1} \circ H^p(\Gamma_{G'_P}, H^q(U, (V^\bullet)^U) \implies H^{p+q-c} i^* j_* \circ \mu_G(V^\bullet).$$

It follows that this spectral sequence also degenerates and one deduces

[2]Of course, one may, if one wishes, take instead twisting by subgroup of $G/U(\mathbf{Q})$ generated by $\Gamma_{G'_P}$ and the group $G_P(\mathbf{Q})$; this is done in [Bu-Wi]; it is there called $\tilde{H}_Q$

Corollary 1.7.3. *There is a canonical and functorial isomorphism of MVHS on $D_P(\Gamma_P)$*

$$H^n i_P^* j_* \mu_G(V) \xrightarrow{\sim} \bigoplus_{p+q=n+c} \mu_{G_P} H^p(\Gamma_{G'_P}, H^q(U, \mathrm{Res}^G_{P_1} V)),\ V \in G\text{-Reps}_F.$$

Recall that $A_P = Z(P/U)$ is one-dimensional (1.8). Choose a lift $A_P \hookrightarrow P$ of the inclusion $A_P \hookrightarrow G_P$. Its action on any P_1-representation space W gives rise to a splitting of W into character spaces W_{χ^m} on which $t \in A_P$ acts through multiplication with t^m. Using abstract representation theory, one deduces from Cor 1.7.2 that this is compatible with weights (see [Bu-Wi, Cor. 2.10]:

Corollary 1.7.4. *There is a canonical and functorial isomorphism*

$$\mathrm{Gr}^W_m\, H^n i_P^* j_* \mu_G(V) \xrightarrow{\sim} \bigoplus_{p+q=n+c} \mu_{G_P} H^p(\Gamma_{G'_P}, (H^q(U, \mathrm{Res}^G_{P_1} V))_{\chi^m}).$$

Remark 1.7.5 (The adelic description–continued)**.** I continue to use the notation of Remark 1.3.3. Let me point out the following dictionary:

Notation in [Bu-Wi]	My notation
G, Q, P_1, W_1, G_1	G, P, P_1, U, G_P
$\mathfrak{H}, \mathfrak{H}_1$	D, D_P
$M^K(G, \mathfrak{H}),\ M^{K_1}(G_1, \mathfrak{H}_1)$	$D(\Gamma),\ D_P(\Gamma_P)$
$H_Q, H_C, \Delta_1, \Delta$	$\Gamma_{G'_P} \cdot P_1(\mathbf{Q}),\ \Gamma_{G'_P} \cdot U(\mathbf{Q}),\ \Gamma_{G'_P},\ \{1\}$
$\bar{H}_Q, \bar{H}_C$	$\Gamma_{G'_P} \cdot G_P(\mathbf{Q}),\ \Gamma_{G'_P}$

Note that in [Bu-Wi] due to the appearance of non-connected Shimura varieties some complications occur so that the group Δ in general is no longer trivial.

Chapter 2

Tools From Mixed Hodge Modules

2.1 Nearby and Vanishing Cycle Functors

Let X be a smooth manifold, $f : X \to \mathbf{C}$ a holomorphic function with image the unit disk Δ and 0 as the only possible critical value. We let $X_0 = f^{-1}(0)$, $i : X_0 \hookrightarrow X$ the natural embedding, and assume that it is normal crossing divisor. Complexes of sheaves of $\mathbf{Q}$-vector spaces on X on which the monodromy operator T around 0 acts are called *monodromical complexes*. Recall that with the universal cover $\tilde{\Delta}$ = upper half plane $\to \Delta^*$,

$$X_\infty \quad \stackrel{\text{def}}{=} \quad X \times_{\Delta^*} \tilde{\Delta}$$

and $k : X_\infty \to X$ the natural projection (which factors as $X_\infty \xrightarrow{k'} X^* \xrightarrow{j} X$), one has *the nearby cycle map*

$$\psi_f(K) \stackrel{\text{def}}{=} i^*(k_*((k')^*K))), \quad K \in D^b(\mathbf{Q}_{X^*})$$

So, by the very definition of the nearby cycle functor

$$\psi_f : D^b(\mathbf{Q}_{X^*}) \to D^b(\mathbf{Q}_{X_0})$$

sends any bounded complex of sheaves of $\mathbf{Q}$-vector spaces on X to a monodromical sheaf. From adjunction $K \to k_*(k^*K)$, one obtains the *specialization morphism*

$$\mathrm{sp}_{X_0|X}(K) : i^*K \to i^*(k_*(k^*K))) = \psi_f(j^*K)$$

which is used to define the *vanishing cycle functor*:

$$\phi_f(K) \stackrel{\text{def}}{=} \mathrm{Cone}(\mathrm{sp}_{X_0|X}(K)),$$

and T operates on such cones as $T(x,y) = (x, Ty)$, $x \in K[1], y \in \psi_f K$. Set $\mathrm{var}(x,y) = Ty - y$ and $\mathrm{can}(y) = (0,y)$. It follows that we have homomorphisms of complexes

$$\begin{tikzcd} \phi_f K \arrow[r, bend left, "\mathrm{var}"] & \psi_f K. \arrow[l, bend left, "\mathrm{can}"] \end{tikzcd} \tag{2.1}$$

To do this for perverse complexes we need shifts:

$$\psi_f[-1] : \mathrm{Perv}_{\mathbf{Q}} X^* \to \mathrm{Perv}_{\mathbf{Q}} X_0$$

and

$$\phi_f[-1] : \mathrm{Perv}_{\mathbf{Q}} X^* \to \mathrm{Perv}_{\mathbf{Q}} X_0.$$

2.2 Quivers

By definition, (2.1) gives an example of a *quiver* on the abelian category $\mathfrak{A} = \mathrm{Perv}_{\mathbf{Q}} X_0$: a directed graph with for each vertex v an object K_v in $\mathfrak{A}$ and for each arrow a a morphism f_a in the category.

Example 2.2.1. (1) Any n-truncated cubical $\mathbf{Q}$-vector space yields a quiver and in particular those that are at the same time n-truncated cubical and co-cubical vector spaces. Let me call these *n bi-cubical quivers.* An explicit description is as follows: start with the ordered set $[n] = \{1, 2, \ldots, n\}$ and for each subset $J \subset [n]$ one has a vector space V_J (*also* for $J = \varnothing$) and for each pair $I, J \subset [n]$ such that $I \subset J$ there are two morphisms $c_{IJ} : V_I \to V_J$, $v_{JI} : V_J \to V_I$ with the obvious compatibilities coming from the inclusions. For $n = 3$ this gives:

$$\begin{tikzcd}
 & V_{123} \arrow[rr, shift left, "v_{123,23}"] \arrow[dl, shift left] \arrow[dd, shift left] & & V_{23} \arrow[ll, shift left, "c_{23,123}"] \arrow[dl, shift left] \arrow[dd, shift left] \\
V_{13} \arrow[rr, shift left] \arrow[ur, shift left] \arrow[dd, shift left] & & V_3 \arrow[ll, shift left] \arrow[ur, shift left] \arrow[dd, shift left] & \\
 & V_{12} \arrow[rr, shift left] \arrow[uu, shift left] \arrow[dl, shift left] & & V_2 \arrow[ll, shift left] \arrow[uu, shift left] \arrow[dl, shift left] \\
V_1 \arrow[rr, shift left] \arrow[uu, shift left] \arrow[ur, shift left] & & V_\varnothing \arrow[ll, shift left] \arrow[uu, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

The compatibilities imply that the quiver is commutative in the obvious sense. This implies that it is completely determined by specifying the neighboring subquivers

$$V_J \underset{v_{J-j,J}}{\overset{c_{J,J-j}}{\rightleftarrows}} V_{J-j}. \tag{2.2}$$

Note that the compositions

$$N_j = c_{\varnothing j} \circ v_{j \varnothing} : V_j \to V_j$$

define endomorphism on V_j and similarly, using the above subquivers (2.2) we get endomorphisms $N_j = c_{J-j,J} \circ v_{J,J-j}$.
(2) For $n = 1$ there is a standard such example: Take $X = \Delta$, $f = \mathrm{id}$, $j : \Delta^* \hookrightarrow \Delta$ the embedding, and take for the perverse complex $Rj_*\mathbf{V}[1]$, $\mathbf{V}$ a local system on Δ^*. Then $\mathbf{V}$ can be given as a pair (V, T) with V the stalk of $\mathbf{V}$ at some point of Δ^*. Then $\psi_{\mathrm{id}}(j_*\mathbf{V}) = \phi_{\mathrm{id}}(Rj_*\mathbf{V}) = V$, $\mathrm{var} = T - \mathrm{id}$ and $\mathrm{can} = \mathrm{id}$.

Recall at this point that T is quasi-unipotent. Assume for simplicity that it is unipotent and put

$$\boxed{N := \frac{\log T}{2\pi \mathrm{i}}}.$$

In the above let me replace $T - \mathrm{id}$ by N. This gives

$$\boxed{\begin{tikzcd} V = V_1 \arrow[r, bend left, "N"] & V = V_\varnothing. \arrow[l, bend left, "\mathrm{id}"] \end{tikzcd}}$$

(3) Let me now pass to higher dimension: Replace Δ^* by $U = (\Delta^*)^n$, $X = \Delta^n$ and let $(z_1, \ldots, z_n)$ the coordinates. Start with a perverse complex K on X. The local monodromy operators $T_j, j = 1, \ldots, n$ around the coordinate hyperplanes $Z_j = \{z_j = 0\}$ act on K as automorphisms. We assume that these are all unipotent. We let $N_j = \log(T_j) \in \mathrm{End}(\phi_{z_j} K)$. For every $j \in [n]$ consider the quiver on $\mathrm{Perv}_{\mathbf{Q}} Z_j$ given by

$$\begin{tikzcd} \psi_{z_j} K \arrow[rr, bend left, "v_j = \mathrm{var}"] & & \phi_{z_j} K, \arrow[ll, bend left, "c_j = \mathrm{can}"] \end{tikzcd} \quad c_j \circ v_j = N_j. \tag{2.3}$$

These are the building blocks for n bi-cubical quivers where for $I = \{i_1, \ldots, i_k\}$ and $[n] - I = \{j_1, \ldots, j_{n-k}\}$ we put

$$V_I = \psi_{z_{i_1}} \cdot \cdots \cdot \psi_{z_{i_k}} \cdot \phi_{z_{j_1}} \cdot \cdots \cdot \phi_{z_{j_{n-k}}} K,$$

which is a vector space since it is supported on the origin in X. Since T_i and T_j commute, also N_i and N_j commute so that the order in which one takes the vanishing and nearby functors in defining V_I does not matter. Pick $j \in J$ and write $V_I = \phi_{z_j} W$; then the quiver (2.3) induces the neighboring quivers

$$\begin{tikzcd} V_I = \phi_{z_j} W \arrow[rr, bend left, "v_j"] & & V_{I \cup j} = \psi_{z_j} W, \arrow[ll, bend left, "c_j"] \end{tikzcd} \quad c_j \circ v_j = N_j.$$

Such n bi-cubical quivers are called *monodromical bi-cubical quivers.*

For instance, if $\mathbf{V}$ is a local system on U, take $K = Rj_*\mathbf{V}$ where $j : U \hookrightarrow X$ is the inclusion. Then $V_I = V$, the stalk at any point of U, $v_j = N_j$ and $c_j = \mathrm{id}$, which generalizes what has been done in dimension 1.

2.3 Comparison of Weight Filtrations

Let me continue with the situation of monodromical bi-cubical quivers,coming from local systems $\mathbf{V}$ on $U = (\Delta^*)^n$. For each subset $J \subset \{1, \ldots, n\}$ set

$$Z_J = \bigcap_{j \in J} Z_j, \quad N_J = \sum_{j \in J} N_j.$$

Then N_J is a nilpotent endomorphism of the restriction of $\mathbf{V}$ to a small neighborhood of $z \in Z_J$ in U. A priori the weight filtration $W(N_J)$ on V depends on the choice of z. If it does not and the filtrations glue to a global filtration (over U) of $\mathbf{V}$ by local subsystems, one says that the resulting pair $(\mathbf{V}, W_\bullet)$ is a *uniform weight filtration*. The same terminology can be used with respect to the shifted weight filtrations $W(N_J)[k]$ and then $(\mathbf{V}, W[k])$ is called a *uniform monodromy weight filtration*.

With the obvious modifications this local notion can be made global, i.e. where U is a complex manifold embedded in a complex manifold X such that $Z = X - U$ is a normal crossing divisor whose components are smooth.

The main auxiliary result is:

Theorem 2.3.1. *Let X be a complex manifold and $Z \subset X$ a normal crossing divisor whose components are smooth. Suppose $g : X \to \mathbf{C}$ is a holomorphic function such that*

$$Z = \{x \in X \mid g(x) = 0\}.$$

Let $j : U := X - Z \hookrightarrow X$ be the inclusion. Assume that $\mathbf{V}$ is a local system on U underlying a polarized weight k variation of Hodge structure.

Suppose that there exists a uniform monodromy weight filtration $(\mathbf{V}, W[k])$. Then the following two filtrations on the perverse complex $\psi_g Rj_* \mathbf{V}[-1]$ coincide:

1. *the filtration[1] $W(N)[k]$ where N is the monodromy operator associated to the monodromy around $g^{-1}(0)$;*
2. *the filtration induced by the given uniform monodromy weight filtration.*

The proof uses quivers as described in § 2.2 as well as the Cattani-Kaplan-Schmid results [C-K, Sch]. The theorem then is used in the following model example to which the later weight comparison is applied (see § 4.2).

Example 2.3.2. Start with a G-representation V. Suppose that there is a compactification $D(\Gamma)^\sharp$ by normal crossing divisors Z, say $Z \twoheadrightarrow D_P(\Gamma_P)$. Let X be an open neighborhood of Z in $D(\Gamma)^\sharp$ such that $D := Z \cap U$ is the fiber at 0 of a holomorphic function $g : X \to \mathbf{C}$ Assume for simplicity that the variation μ_G has pure weight k. Consider the local system $\mathbf{V}$ underlying the restriction to U of this variation. Then by Proposition 1.5.1 the canonical weight filtration on $\mathrm{Res}^G_P V$ defines a uniform weight filtration on this system and hence by Theorem 2.3.1 coincides with the weight filtration defined by the monodromy along the divisor D shifted by k.

[1]See § 1.5.

2.4 Verdier Specialization

Let X be an algebraic variety and $i : Z \hookrightarrow X$ a closed sub variety. Let

$$N_{Z|X} : \text{ \textit{the normal cone to} } Z.$$

If Z is smooth then this nothing but the total space of the normal bundle of Z in X. In general it is an affine bundle over Z with a natural inclusion $i_Z : Z \hookrightarrow N_{Z|X}$. Introduce

$$E_{Z|X} := \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{C}) - \mathrm{Bl}_Z X$$

This variety is fibered over $\mathbf{C}$ and the fiber over 0 is precisely $N_{X|Z}$. It turns out that the part of the fibration over $\mathbf{C}^*$ is naturally isomorphic to $X \times \mathbf{C}^*$ and there is a commutative diagram

$$\begin{tikzcd}
Z \arrow[d, hook, "i_Z"'] \arrow[rrr, hook, bend left, "i"] & & & \\
N_{Z|X} \arrow[r, hook] \arrow[d] & E_{Z|X} \arrow[d, "t"'] & X \times \mathbf{C}^* \arrow[l, hook'] \arrow[r, "\pi_X"'] \arrow[d] & X \\
\{0\} \arrow[r, hook] & \mathbf{C} & \mathbf{C}^* \arrow[l, hook'] &
\end{tikzcd}$$

which shows that t defines a deformation of $N_{Z|X}$ to X. Let F be a constructible sheaf on X. Pull it to $X \times \mathbf{C}^*$ via the projection π_X. Recall that the nearby cycle functor produces from a constructible sheaf on the complement of the special fiber a constructible sheaf on the special fiber itself. So, ψ_t yields a constructible sheaf on the special fiber of t which is the normal cone. This succession of operations defines the *Verdier specialization*

$$\mathrm{sp}_{Z|X}(F) := \psi_t(\pi_X^* F).$$

Its derived functor preserves perversity:

$$\mathrm{sp}_{Z|X} : \mathrm{Perv}_{\,\mathbf{Q}} X \to \mathrm{Perv}_{\,\mathbf{Q}} N_{Z|X}$$

The same formula defines Verdier specialization mixed Hodge modules and the two are compatible:

$$\begin{tikzcd}
\mathsf{MHM}_Q X \arrow[r, "\mathrm{sp}_{Z|X}"'] \arrow[d, "\mathrm{rat}"'] & \mathsf{MHM}_Q N_{Z|X} \arrow[d, "\mathrm{rat}"'] \\
\mathrm{Perv}_{\,\mathbf{Q}} X \arrow[r, "\mathrm{sp}_{Z|X}"'] & \mathrm{Perv}_{\,\mathbf{Q}} N_{Z|X}
\end{tikzcd}$$

Moreover, one has:

Proposition 2.4.1. *1. Let M be a polarized mixed Hodge module on X. Then $\mathrm{sp}_{Z|X} M$ is naturally polarized.*
2. One has

$$i^* = i_Z^* \circ \mathrm{sp}_{Z|X} : \mathsf{MHM}_{\mathbf{Q}} X \to \mathsf{MHM}_{\mathbf{Q}} Z.$$

3. If Z is a divisor given by a single equation $g = 0$, then $N_{Z|X} = Z \times \mathbf{C}$ and using the maps $s : z \mapsto (z, 1)$, $j : X^* = Z \times \mathbf{C}^* \hookrightarrow X$, one has

$$\psi_g = s^* \circ \mathrm{sp}_{Z|X} \circ j^*.$$

In particular, the notion specialization of § 2.1) corresponds to the notion of Verdier specialization.

2.5 Hodge Modules with Group Actions

Let X be a reduced $\mathbf{C}$-scheme which is *locally* of finite type and H be an abstract group acting on X through algebraic morphisms. Fix a subfield F of $\mathbf{R}$.

Definition 2.5.1. Let M be a an F-mixed Hodge module on X. An *H-twisting* for M consists of a collection of isomorphisms in $\mathsf{MHM}_F X$

$$f_\gamma : \gamma^* M \xrightarrow{\sim} M, \quad \gamma \in H$$

such that the co-cycle condition holds: $f_{\gamma\gamma'} = f_\gamma \circ f_{\gamma'} : \gamma\gamma'^* M \xrightarrow{\sim} M$. Morphisms between H-twistings are defined in the obvious way; the resulting category is denoted

$(\mathsf{MHM}_F X, H)$: *the category of F-mixed Hodge modules on X with an H-twisting*.

Assume next that the action of H on X is free and proper so that $H\backslash X$ is a reduced $\mathbf{C}$-scheme and let $\pi : X \to H\backslash X$ be the quotient map. One shows [Bu-Wi, Section 4]:

Proposition 2.5.2. *The induced morphism*

$$\pi^* \mathsf{MHM}_F H\backslash X \to (\mathsf{MHM}_F X, H)$$

is an equivalence of categories with a canonical pseudo-inverse. The same is true for the bounded derived categories.

Chapter 3

Other Tools

3.1 Toroidal Compactifications

Let me recall briefly the ingredients of a toroidal compactification of $D(\Gamma)$.

The Relevant Torus Fibration

The starting point is the fibration

$$D(P) = Z(U)(\mathbf{C}) \cdot D \to Z(U)(\mathbf{C})\backslash D(P)$$

Now divide out by the action of the discrete group

$$\Gamma'_P := Z(U)(\mathbf{C}) \cap \Gamma$$

which acts on this fibration fibre-wise. Note that

$$T(P) := \Gamma'_P\backslash Z(U)(\mathbf{C}) \qquad (3.1)$$

is an algebraic torus. So one gets a $T(P)$-torsor $\Gamma'_P\backslash D(P) \longrightarrow Z(U)(\mathbf{C})\backslash D(P)$.

For later use, recall that on $Z(U)(\mathbf{C})\backslash D(P)$ there is a further action of $U(\mathbf{R})/Z(U)(\mathbf{R})$ which makes the projection $Z(U)(\mathbf{C})\backslash D(P) \to D_P(\Gamma_P)$ a relative Abelian scheme. Summarizing

$$\pi_P : \Gamma'_P\backslash D(P) \xrightarrow{\pi_{\text{torus}}} Z(U)(\mathbf{C})\backslash D(P) \xrightarrow{\pi_{\text{Ab}}} D_P(\Gamma_P), \qquad (3.2)$$

where π_{torus} is a T_P-torsor and π_{Ab} is a relative Abelian scheme.

Associated Torus Embeddings

Recall now the self dual cone $C(P) \subset F$ (see (1.11)). The group $Z(U)(\mathbf{R}) \cap \Gamma$ gives a lattice $F_{\mathbf{Z}}$ in F and hence one gets a rational structure on F. The group G'_P is the automorphism group of $(F, C(P))$ and hence the trace of Γ in it which precisely the group $\Gamma_{G'_P}$ – defined by (1.16) – acts properly discontinuously on the pair $(F, C(P))$.

Definition 3.1.1. Set $\boxed{\bar{\Gamma} = \Gamma_{G'_P}}$. A *$\bar{\Gamma}$-admissible fan* Σ on F is a collection of rational cones together with all of its faces such that 2 cones overlap only in common faces and such that

1. $\bar{\Gamma}$ preserves Σ;
2. modulo the action of $\bar{\Gamma}$ there are only finite many cones in Σ;
3. the support $|\Sigma| := \{\bigcup \sigma \mid \sigma \in \Sigma\}$ is a *contractible set*[1] with the property

$$C(P) \subset |\Sigma| \subset \overline{C(P)}$$

The fan Σ on F defines a torus embedding $T(P) \subset T(P)_\Sigma$ with $T(P)$ the algebraic torus with character space $Z(U)(\mathbf{C}) \cap \Gamma = \Gamma'_P$. This torus is indeed the one from (3.1).

Let me briefly pause at this point to explain how the combinatorics of the strata relate to the group $\bar{\Gamma}$. Consider Σ as an index set and for each $\sigma \in \Sigma$ put

$$\mathrm{St}(\sigma) = \bigcup_{\tau < \sigma} \text{open cone } |\tau|^0 \subset |\Sigma|.$$

These given an open covering of $|\Sigma|$ indexed by Σ. By assumption the group $\bar{\Gamma}$ acts on this covering. Since $|\Sigma|$ is contractible, one has:

Lemma 3.1.2. *The natural augmentation*

$$C_\bullet(\{\mathrm{St}(\sigma)\}_{\sigma\in\Sigma}\,; \mathbf{Z}) \to \mathbf{Z}$$

of the Čech chain complex for the covering is a free $\mathbf{Z}[\bar{\Gamma}]$*-resolution.*

Partial Compactification

Let me from now on write Σ_P instead of Σ to stress the dependence on P.

The $T(P)$-torsor $D(\Gamma'_P)(P) \to Z(U)(\mathbf{C})\backslash D(P)$ can be fibre wise compactified by taking the corresponding $T(P)_{\Sigma_P}$-torsor. This gives a fibration

$$(\Gamma'_P\backslash D(P))_{\Sigma_P} \to Z(U)(\mathbf{C})\backslash D(P)$$

of toroidal varieties. There remains the second projection from (3.2). In total

$$\pi_P : (\Gamma'_P\backslash D(P))_\Sigma \xrightarrow{\text{toroidal scheme}} Z(U)(\mathbf{C})\backslash D(P) \xrightarrow{\text{Abelian scheme}} D_P(\Gamma_P) \tag{3.3}$$

Note that $\Gamma'_P\backslash D \subset \Gamma'_P\backslash D(P)$ and set

$$D(\Gamma'_P)_{\Sigma_P} := \mathrm{Int}\left[\overline{\Gamma'_P\backslash D} \subset (\Gamma'_P\backslash D(P))_{\Sigma_P}\right]. \tag{3.4}$$

One has a natural extension of the canonical map $D(\Gamma'_P) \to D(\Gamma)$ fitting in the commutative diagram:

$$\begin{tikzcd}
Z_P := p_P^{-1}[D_P(\Gamma_P)] \arrow[r, two heads] \arrow[d, hook, "i_P"'] & D_P(\Gamma_P) \arrow[d, hook] \\
D(\Gamma'_P)_{\Sigma_P} \arrow[r, "p_P"'] \arrow[d, hook, "j_{\Sigma_P}"'] & D(\Gamma)^* \arrow[d, hook] \\
D(\Gamma'_P) \arrow[r, two heads] & D(\Gamma)
\end{tikzcd} \tag{3.5}$$

[1] This demand is not standard; it is put here for technical reasons; obviously, by adding some orbits this can be assumed.

The variety Z_P is a locally finite union of algebraic varieties on which $\bar{\Gamma}$ acts freely and discretely. In particular, it is not an algebraic variety (but the quotient is). However, by (3.3) it is a smooth relative torus-embedding and hence affine.

The next step is to divide out by the action of Γ_P. One shows that the action on $D(\Gamma'_P)_{\Sigma_P}$ is by a proper discontinuous and (under the assumption of Γ being neat) free action of the quotient Γ_P/Γ'_P so that

$$D(\Gamma)_{\Sigma_P} := \Gamma_P \backslash D(\Gamma'_P)_{\Sigma_P} \tag{3.6}$$

is a complex manifold. By construction, the top half of diagram (3.5) can be completed by inserting the intermediate quotient varieties:

$$\begin{tikzcd}
Z_P \arrow[r, two heads, "\tilde{q}_P"'] \arrow[d, hook, "i_P"'] \arrow[rr, bend left, "\tilde{p}_P"] & \bar{Z}_P \arrow[r, two heads, "p_P"'] \arrow[d, hook, "i_P"'] & D_P(\Gamma_P) \arrow[d, hook] \\
D(\Gamma'_P)_{\Sigma_P} \arrow[r, "q_P"'] & D(\Gamma)_{\Sigma_P} \arrow[r, "p_P"'] & D(\Gamma)^*
\end{tikzcd} \tag{3.7}$$

Glueing Partial Compactifications

Definition 3.1.3. A family $\mathfrak{S} = \{\Sigma_P\}$ of fans Σ_P, one for each rational boundary component D_P is called *Γ-admissible* if it has the property that

1. Σ_P is $\bar{\Gamma}$-admissible (recall that $\bar{\Gamma} = \Gamma_{G'_P}$ and so depends on P);

2. for all $\gamma \in \Gamma$ one has $\gamma\Sigma_P = \Sigma(\gamma P \gamma^{-1})$;

3. one has compatibility with inclusions: if $D_{P'} \subset D_P$, one has $\Sigma(P') = \{\sigma \cap C(P') \mid \sigma \in \Sigma_P\}$.

Next, one may divide out by the action of Γ and obtain the toroidal compactification $D(\Gamma)_{\mathfrak{S}}$. More precisely, one has [AMRT, p. 253–310]

Theorem 3.1.4. *1. For every Γ-admissible family of fans $D(\Gamma)_{\mathfrak{S}}$ is a compact analytic space. It is glued from the $D(\Gamma)_{\Sigma_P}$ (3.6) which are open in $D(\Gamma)_{\mathfrak{S}}$. This procedure induces an analytic inclusion of $D(\Gamma) \hookrightarrow D(\Gamma)_{\mathfrak{S}}$.*

2. There exists Γ-admissible families of fans such that $(D(\Gamma))_{\mathfrak{S}}$ is a smooth projective variety. The above identifications are then algebraic and $(D(\Gamma))_{\mathfrak{S}}$ contains $D(\Gamma)$ as a Zariski-dense open set.

3. There is a natural proper morphism $p : D(\Gamma)_{\mathfrak{S}} \to D(\Gamma)^*$ restricting to the identity on the Zariski dense open subsets $D(\Gamma)$.

Let me summarize some of these constructions in the following diagram

$$
\begin{tikzcd}
D(\Gamma) \arrow[r, hook, "j_{\mathfrak{S}}"] \arrow[ddd, dash, bend right, shift right, "="'] & D(\Gamma)_{\mathfrak{S}} & D_P(\Gamma_P)_{\Sigma_P} \arrow[l, hook', "i_{\mathfrak{S}}"'] \arrow[d, equal] \arrow[ddd, bend left, "p_P"] \\
 & D(\Gamma)_{\Sigma_P} \arrow[u, hook, "\text{open}"] & D_P(\Gamma_P)_{\Sigma_P} \arrow[l, hook'] \\
D(\Gamma'_P) \arrow[r, hook, "j_{\Sigma_P}"] \arrow[uu, two heads] \arrow[d, two heads] & D(\Gamma'_P)_{\Sigma_P} \arrow[u, two heads, "q_P"] \arrow[d] & Z_P \arrow[l, hook', "i_{Z_P}"'] \arrow[u, two heads, "\tilde{q}_P"] \arrow[d, "\tilde{p}_P"'] \\
D(\Gamma) \arrow[r, hook, "j"] & D(\Gamma)^* & D_P(\Gamma_P). \arrow[l, hook', "i_P"']
\end{tikzcd}
\tag{3.8}
$$

One can further show:

Lemma 3.1.5 ([Bu-Wi, §8]). *Recall* (3.2)*: one has a commutative diagram*

$$
\begin{tikzcd}
Z_P \arrow[d, hook] \arrow[dr, "\tilde{p}"] & \\
Z(U)(\mathbf{C})\backslash D(P) \arrow[r, "\pi_{\mathrm{Ab}}"'] & D_P(\Gamma_P)
\end{tikzcd}
$$

The image of Z_P in $Z(U)(\mathbf{C})\backslash D(P)$ is a closed union of strata of the torus embedding π_{torus}.

3.2 Simplicial Constructions

Let Σ be an index set, and let $S(\Sigma)$ be the associated simplicial set: $S(\Sigma)_p = \Sigma^{p+1}$ the collections of $(p+1)$-tuples $(\sigma_0, \ldots, \sigma_p)$ in Σ and for each increasing map $f : [q] \to [p]$ a map $S(\Sigma)_f : \Sigma^p \to \Sigma^q$ satisfying the obvious compatibilities. Let Z be a complex algebraic variety. Then one may form $Z \times S(\Sigma)$ and one may consider mixed Hodge modules over this simplicial variety. such a simplicial mixed Hodge module $H_\bullet$ consist of mixed Hodge modules $H_{\sigma_0,\ldots,\sigma_p}$ over Z indexed by $S(\Sigma)$ together with morphisms $H(f) : H_{f(\sigma_0),\ldots,f(\sigma_q)} \to H_{\sigma_0,\ldots,\sigma_p}$ for increasing maps $f : [q] \to [p]$. It may happen that for all $i \in [q]$ one has $f(\sigma_i) \in [p]$ and vice versa. One says that the simplicial Hodge module $H_\bullet$ is *reduced*, if for such $f, [p]$ and $q]$ the morphism $H(f)$ is an isomorphism. Set

$$
\begin{aligned}
(\mathsf{MHM}_F Z)^{S(\Sigma)} \quad := \quad & \big\{ \text{full abelian subcategory of } \mathsf{MHM}_F Z \times S(\Sigma) \\
& \text{of reduced simplicial Hodge modules.} \big\}.
\end{aligned}
$$

I need a standard construction that links the two categories $\mathsf{MHM}_F Z$ and $(\mathsf{MHM}_F Z)^{S(\Sigma)}$, the one associated to forming the normalized chain complex: Let $H_\bullet$ be a simplicial mixed Hodge module over Z. Then one has morphisms $d_j : H_{\sigma_0,\ldots\sigma_p} \to H_{\sigma_0,\ldots\sigma_{p-1}}$, one for each $j \in [0, \ldots, p]$ which can be combined to

$$
d = \sum (-1)^j d_j : H_{\sigma_0,\ldots\sigma_p} \to H_{\sigma_0,\ldots\sigma_{p-1}}
$$

These turn out to give a complex of mixed Hodge modules on Z. From this complex one extracts a new complex with the same cohomology, called *normalized sub complex.* See [Sel, 8.6]. This construction is functorial and extends to the derived categories upon taking total complexes first and then pass to the normalized sub complex. This yields:

$$\mathrm{Tot} : D^b((\mathsf{MHM}_F Z)^{S(\Sigma)}) \to D^b(\mathsf{MHM}_F Z).$$

Example 3.2.1. Suppose that for all σ one has $Z_\sigma = Z$. In that case one has an isomorphism

$$\mathrm{Tot} : D^b((\mathsf{MHM}_F Z)^{S(\Sigma)}) \xrightarrow{\sim} D^b(\mathsf{MHM}_F Z).$$

Next, suppose that one has a covering $S = \bigcup_{\sigma\in\Sigma} S_\sigma$ by closed subvarieties. Then one also has a functor in the other direction:

$$\begin{array}{rcl} S_\bullet : \mathsf{MHM}_F Z & \to & (\mathsf{MHM}_F Z)^{S(\Sigma)} \\ H & \mapsto & (i_{\sigma_0,\dots,\sigma_p})_* \circ (i_{\sigma_0,\dots,\sigma_p})^* H. \end{array}$$

Actually, this functor is not defined by the above formulas; some adjustments have to be made. See [Bu-Wi, Section 5]. One has:

Proposition 3.2.2. *[Bu-Wi, Prop. 5.6] There is a canonical isomorphism of functors*

$$\mathrm{Tot} \circ S_\bullet \simeq \mathrm{Id} : D^b(\mathsf{MHM}_F Z) \longrightarrow D^b(\mathsf{MHM}_F Z)$$

Chapter 4

Sketch of the Proof

4.1 Transport to a Toroidal Compactification

To show the equality of functors from Theorem 1.7.1, one needs to reinterpret $i_P^* j_* \circ \mu_G$. The first step is to lift this composition of functors to a suitable toroidal compactification of $D(\Gamma)$. Here I use the notation and results from § 3.1.

Recall the diagram (3.8). The closed subvariety $Z_P \subset D(\Gamma'_P)_{\Sigma_P}$ lies over the boundary component $D_P(\Gamma_P)$. It is only locally an algebraic variety; it has infinitely many components. On it the group $\Gamma_{G'_P}$ acts freely and properly and the quotient yields one of the divisors in the boundary of $D(\Gamma)_{\mathfrak{S}}$. It is contained as a closed sub variety in the open subset $D(\Gamma)_{\Sigma_P} \subset D(\Gamma)_{\mathfrak{S}}$ which is the quotient of $D(\Gamma'_P)_{\Sigma_P}$ under the free action of $\Gamma_{G'_P}$.

Remark. 1. Since Z_P is only locally an algebraic variety one has a serious problem: the yoga of the Grothendieck functors cannot be directly applied to mixed Hodge modules.
2. Another technical problem is that Z_P is not in general a normal crossing variety. However, after a suitable barycentric subdivision of $\mathfrak{S}$ this can always be achieved [Pink90, Proof of Prop. 9.20]. So in what follows this will always be assumed.

One needs to study the normal cones to various strata of $D(\Gamma)_{\mathfrak{S}}$. Especially those in $\Gamma_{G'_P} \backslash Z_P$ which come from those of Z_P. One can show that.

$$\Gamma_{G'_P} \backslash N_{P|D(\Gamma'_P)_{\Sigma_P}} \simeq N_{D_P(\Gamma_P)_{\mathfrak{S}}|D(\Gamma)_{\mathfrak{S}}}.$$

From Prop. 2.5.2 one has:

$$D^b(\mathsf{MHM}_F D(\Gamma'_P)_{\Sigma_P}) = D^b(\mathsf{MHM}_F Z_P, \Gamma_{G'_P})$$

and

$$D^b(\mathsf{MHM}_F N_{D_P(\Gamma_P)_{\mathfrak{S}}|D(\Gamma)_{\mathfrak{S}}}) = D^b(\mathsf{MHM}_F N_{Z_P|D(\Gamma'_P)_{\Sigma_P}}, \Gamma_{G'_P})$$

Set

$$i_0 : D_P(\Gamma_P)_{\mathfrak{S}} \hookrightarrow N_{D_P(\Gamma_P)_{\mathfrak{S}}|D(\Gamma)_{\mathfrak{S}}}$$

and recall (§ 2.4) the Verdier specialization functor:

$$\mathrm{sp}_1 := \mathrm{sp}_{D(\Gamma)|D_P(\Gamma_P)_{\mathfrak{S}}} : D^b(\mathsf{MHM}_F D_P(\Gamma_P)_{\mathfrak{S}}) \to D^b(\mathsf{MHM}_F N_{D_P(\Gamma_P)_{\mathfrak{S}}|D(\Gamma)_{\mathfrak{S}}})$$

and property 2 from Theorem 2.4.1. Combining the two preceding remarks, there is a commutative diagram

$$
\begin{tikzcd}
 & D^b(\mathsf{MHM}_F N_{Z_P|D(\Gamma'_P)_{\Sigma_P}}, \Gamma_{G'_P}) \arrow[d, equal] \arrow[ddd, bend left=60, "i_0^*"] \\
D^b(\mathsf{MHM}_F D(\Gamma)_{\mathfrak{S}}) \arrow[r, "\mathrm{sp}_1"] \arrow[dr, "i_{\mathfrak{S}}^*"'] & D^b(\mathsf{MHM}_F N_{D_P(\Gamma_P)_{\mathfrak{S}}/D(\Gamma)_{\mathfrak{S}}}) \arrow[d, "i_0^*"] \\
 & D^b(\mathsf{MHM}_F D_P(\Gamma_P)_{\mathfrak{S}}) \arrow[d, equal] \\
 & D^b(\mathsf{MHM}_F Z_P, \Gamma_{G'_P}).
\end{tikzcd}
\tag{4.1}
$$

Combine diagrams (3.8) and (4.1):

$$
i_P^* j_* = p_{P*} i_{\mathfrak{S}}^* (j_{\mathfrak{S}})_* = p_* i_0^* \, \mathrm{sp}_1 (j_{\mathfrak{S}})_* \tag{4.2}
$$

4.2 Local Comparison in the Toroidal Compactification

Recall from diagram (3.8) the affine morphism

$$
j_{\Sigma_P} : D(\Gamma) \hookrightarrow D(\Gamma'_P)_{\Sigma_P}.
$$

Now, even if $D(\Gamma'_P)_{\Sigma_P}$ is only locally of finite type, the fact that j_{Σ_P} is affine allows to define an exact functor

$$
(j_{\Sigma_P})_* : \mathsf{MHM}_F D_P(\Gamma_P) \to \mathsf{MHM}_F D(\Gamma'_P)_{\Sigma_P}.
$$

In the same way there is a specialization functor

$$
\mathrm{sp}_2 := \mathrm{sp}_{Z_P} : \mathsf{MHM}_F D(\Gamma'_P)_{\Sigma_P} \to \mathsf{MHM}_F N_{Z|D(\Gamma'_P)_{\Sigma_P}}.
$$

Both functors admit $\Gamma_{G'_P}$-equivariant versions. Recall (3.6) that there is a holomorphic map

$$
\tilde{q} : D(\Gamma'_P)_{\Sigma_P} \twoheadrightarrow D(\Gamma)_{\Sigma_P} = \Gamma_{G'_P} \backslash D(\Gamma'_P)_{\Sigma_P} \subset D(\Gamma)_{\mathfrak{S}}
$$

which is local biholomorphism near Z_P.

The following result is one of the *crucial results of the paper*. It allows to replace $\mathrm{sp}_1 \circ (j_{\mathfrak{S}})_*$ in (4.2) by $\mathrm{sp}_2 \circ (j_{\Sigma_P})_*$.

Proposition 4.2.1. *There is a natural commutative diagram*

$$
\begin{tikzcd}
D^b(\mathrm{Reps}_F G) \arrow[r, "\mu_G"] \arrow[d, "\mathrm{Res}^G_P"'] & D^b(\mathsf{MHM}_F D(\Gamma)) \arrow[dddd, "j_{\mathfrak{S}\,*}"] \\
D^b(\mathrm{Reps}_F P) \arrow[d] & \\
D^b(\mathrm{Reps}_F P_1, \Gamma_{G'_P}\cdot P_1(\mathbf{Q})) \arrow[d, "\mu_P"'] & \\
D^b(\mathsf{MHM}_F D(\Gamma'_P), \Gamma_{G'_P}) \arrow[d, "j_{\Sigma_P\,*}"'] & \\
D^b(\mathsf{MHM}_F D(\Gamma'_P)_{\Sigma_P}, \Gamma_{G'_P}) \arrow[d, "\mathrm{sp}_2"'] & D^b(\mathsf{MHM}_F D(\Gamma)_{\mathfrak{S}}) \arrow[d, "\mathrm{sp}_1"] \\
D^b(\mathsf{MHM}_F N_{Z_P|D(\Gamma'_P)_{\Sigma_P}}, \Gamma_{G'_P}) \arrow[r, equal] & D^b(\mathsf{MHM}_F N_{D_P(\Gamma_D)|D(\Gamma)_{\Sigma_P}})
\end{tikzcd}
$$

Now combine this diagram with (4.2):

Corollary 4.2.2. *One has*

$$i_P^* j_* \mu_G = p_{P*} \circ i_0^* \circ \mathrm{sp}_2 \circ (j_{\Sigma_P})_* \circ \mu_{P_1} \circ \mathrm{Res}^G_P . \tag{4.3}$$

Sketch of the proof of Prop. 4.2.1. Let V be a G-representation. The operation μ_{P_1} in the diagram requires a bit of explanation. The group $P_1 = PU \subset G$ acts on D and since $\mathrm{Res}^G_P V$ defines a twisted P_1-representation under $G'_P U$, one gets a variation of Hodge structure on D with a $G'_P U$-twisting on D and since $\Gamma'_P \subset Z(U)(\mathbf{C}) \subset P_1('C)$ this further descends to $D(\Gamma'_P)$.

Next put

$$\mathbf{V}_G := \mu_G(V), \quad \mathbf{V}_P := \mu_{P_1}(\mathrm{Res}^G_P V)$$

The open set $D(\Gamma'_P)_{\Sigma_P}$ contains $D_P(\Gamma_P)$ as well as $D(\Gamma'_P)$ so that via $\tilde{q}_P$ one can pull both up $\mathbf{V}_G$ and $\mathbf{V}_P$ to $D(\Gamma'_P)_{\Sigma_P}$. The first is the pull up of $(j_{\mathfrak{S}})_* \mu_G(V)$; the second is the pull up of $(j_{\Sigma_P})_* \mu_{P_1}(\mathrm{res}^G_P V)$. The underlying local systems are canonically isomorphic on $D(\Gamma'_P)_{\Sigma_P}$, say

$$\alpha : \tilde{q}_P^{-1} \mathbf{V}_G \xrightarrow{\sim} \tilde{q}_P^{-1} \mathbf{V_P}.$$

It is not hard to see that the Hodge filtrations on both variations correspond under α (the representation determines the Hodge filtration from the action of the Deligne torus inside G and since it also sits in P, the restriction functor preserves Hodge filtrations). Then apply $(j_{\Sigma_P})_*$. By [Sa, Proof of Theorem 3.27] the Hodge filtrations on a mixed Hodge modules $(j_{\Sigma_P})_* M$ only depends on the Hodge filtration on M. So through α the Hodge filtrations on $(j_{\Sigma_P})_* q_P^{-1} \mathbf{V}_G$ and $(j_{\Sigma_P})_* q_P^{-1} \mathbf{V_P}$ coincide. Next, apply the two specialization functors sp_1 and sp_2. By the very definition of the mixed Hodge module structure on these (see [Sa, 2,3, 2.30]) these are completely determined by those on $(j_{\Sigma_P})_* q_P^{-1} \mathbf{V}_G$ and $(j_{\Sigma_P})_* \tilde{q}_P^{-1} \mathbf{V_P}$ respectively. So under the identification α the Hodge structures on $\mathrm{sp}_2 (j_{\Sigma_P})_* \tilde{q}_P^{-1} \mathbf{V}_G$ and $\mathrm{sp}_2 (j_{\Sigma_P})_* \tilde{q}_P^{-1} \mathbf{V_P}$ coincide.

Next, the weight filtrations. Assume for simplicity that $\mu_G V$ has pure weight k. Let me consider $\tilde{q}_P^{-1} \mathbf{V}_P$ with its canonical weight filtration. Then by Theorem 2.3.1 and Example 2.3.2 this weight filtration is the uniform weight filtration and coincides with the weight filtration of the monodromy around Z_P shifted by k.

The latter, again by [Sa, 2.3] defines the weight filtration on $\mathrm{sp}_2(j_{\Sigma_P})_* \mathbf{V}_G$. ☐

4.3 Passing to Simplicial Level

Let me summarize in a diagram the morphisms discussed so far and which are going to play a role: [1]

$$\begin{tikzcd}
 & \Gamma'_P\backslash D(P) \arrow[rr, "\text{rel. dim}=c", "\pi_P"'] \arrow[d, hook] & & D_P(\Gamma_P) \arrow[dd, equal] \\
D(\Gamma'_P) \arrow[r, hook, "j_{\Sigma_P}"] & D(\Gamma'_P)_{\Sigma_P} & & \\
 & Z_P \arrow[u, hook, "i_P"] \arrow[r, two heads, "\text{finite}", "\tilde{q}_P"'] \arrow[rr, bend left, "\tilde{p}_P"] & \bar{Z}_P \arrow[r, two heads, "p_P"'] \arrow[d, hook] & D_P(\Gamma_P) \arrow[d, hook] \\
 & Z_\sigma \arrow[u, hook, "i_\sigma"] \arrow[r, "\sim"] & Z_\sigma \arrow[r, two heads] & p_P(Z_\sigma).
\end{tikzcd} \tag{4.4}$$

To understand the bottom line, recall Lemma 3.1.5 where it is stated that Z_P is a $\Gamma_{G'_P}$-equivariant torus embedding which is covered by the closures $Z_\sigma, \sigma \in \Sigma_P$ of strata of the torus-embedding. The image $\bar{Z}_P$ is covered by closed sets $\tilde{q}_P(\bar{Z}_\sigma) \simeq Z_\sigma$. So, if you replace Σ_P by its finite image, the finite set $\bar{\Sigma}_P$ under $\tilde{q}_P$ one may write

$$Z_P = \bigcup_{\sigma\in\Sigma_P} Z_\sigma, \quad \bar{Z}_P = \bigcup_{\sigma\in\bar{\Sigma}_P} Z_\sigma.$$

Then, one uses the geometry of the situation: the strata Z_σ of Z_P go isomorphically to their images in $\bar{Z}_P$ and intersections of strata Z_σ in $\bar{Z}_P$ are either empty or some stratum Z_τ with $\tau \prec \sigma$. Use now the notation and results of §3.2. The induced simplicial set $S(\bar{\Sigma}_P)$ consists of singletons only with morphisms either the identity or the zero. The associated covering of $\bar{Z}$ is not necessarily the trivial one but the covering contains exactly one closed stratum $Z_{\sigma^{\mathrm{top}}} = \bar{Z}$ and this stratum maps onto the component $D_P(\Gamma_P)$ of the Baily-Borel compactification. For this reason one may (and does) take the trivial cover on $D_P(\Gamma_P)$ defining $(\mathsf{MHM}_F\bar{Z}_P)^{\bar{\Sigma}_P}$. Because of Example 3.2.1 one has

$$\mathrm{Tot} : (\mathsf{MHM}_F D_P(\Gamma_P))^{\bar{\Sigma}_P} \xrightarrow{\sim} \mathsf{MHM}_F D_P(\Gamma_P)$$

and there is a corresponding morphism

$$(p_{P*})^{\bar{\Sigma}_P} : (\mathsf{MHM}_F\bar{Z}_P)^{\bar{\Sigma}_P} \longrightarrow (\mathsf{MHM}_F D_P(\Gamma_P))^{\bar{\Sigma}_P}.$$

[1] Recall that $c = \mathrm{codim}_{D_P(\Gamma_P)|D(\Gamma)^*}$

Lemma 4.3.1. *The functor* $(p_{P*})^{\bar{\Sigma}_P}$ *is right exact*[2] *(and hence passes to the derived categories). It* defines *the derived* p_* *as follows [Bu-Wi, Prop. 9.7]:*

$$\begin{tikzcd} D^b(\mathsf{MHM}_F \bar{Z}_P) \arrow[d, "p_*"'] \arrow[r, bend right, "S_\bullet"'] & D^b((\mathsf{MHM}_F \bar{Z}_P)^{\bar{\Sigma}_P}) \arrow[l, "\mathrm{Tot}"'] \arrow[d, "(p_*)^{\bar{\Sigma}_P}"] \\ D^b(\mathsf{MHM}_F D_P(\Gamma_P)) \arrow[r, bend right, "S_\bullet"'] & D^b((\mathsf{MHM}_F D_P(\Gamma_P)^{\bar{\Sigma}_P}). \arrow[l, "\simeq"'] \end{tikzcd}$$

Hence, one gets

$$\begin{aligned} i_P^* j_* \mu_G &= p_* \circ i_0^* \circ \mathrm{sp}_2 \circ (j_{\Sigma_P})_* \circ \mu_{P_1} \circ \mathrm{Res}_P^G \\ &= \mathrm{Tot} \circ \underbrace{(p_*)^{\bar{\Sigma}_P} \circ S_\bullet \circ i_0^* \circ \mathrm{sp}_2 \circ (j_{\Sigma_P})_* \circ \mu_{P_1}}_{\nu_P} \circ \mathrm{Res}_P^G . \end{aligned} \tag{4.5}$$

4.4 Passing to U-invariants

The starting point is the observation that $\Gamma'_P \backslash D(P)$ is homogeneous under P_1 and so the canonical construction μ_{P_1} yields a mixed Hodge module μ_{P_1} on $\Gamma'_P \backslash D(P)$.

Secondly, without giving the details for which I refer to [Bu-Wi, §9] , there is a simplicial variant of the canonical construction:

$$\mu_{P_1}^{\Sigma_P} : (\mathrm{Reps}_F G_1, \Gamma_{G'_P}) \longrightarrow (\mathsf{MHM}_F D_P(\Gamma_P), \Gamma_{G'_P})^{\Sigma_P} = (\mathsf{MHM}_F D_P(\Gamma_P))^{\bar{\Sigma}_P}$$

The crucial assertion is:

Proposition 4.4.1 ([Bu-Wi, Prop. 9.9])**.** *One has*

$$\nu_P[-c] = \mu_{P_1}^{\Sigma_P} \circ \mathrm{inv}^U .$$

Hint of the Proof. Let V be an F-vector space which is a G-module and $V_1 = \mathrm{Res}_P^G V$. One first shows that (cf. also Prop. 2.4.1, 2)

$$(\tilde{p} \circ i_I)_* \circ \underbrace{i_{0I}^* \circ \mathrm{sp}_2}_{i_I^*} \circ (j_{\Sigma_P})_* \circ \mu_{P_1}(V_1)$$

is the component of $\nu_P V_1$ with index $I := (\sigma_0, \ldots, \sigma_p)$. Note that the geometry of the situation as summarized in diagram (4.4) tells me that whenever $Z_I \neq \varnothing$ one has $(\tilde{p} \circ i_I)_* \circ i_I^* \circ (j_{\Sigma_P})_* = \pi_*$ and hence

$$(\tilde{p} \circ i_I)_* \circ i_I^* \circ \mathrm{sp}_2 \circ (j_{\Sigma_P})_* \circ \mu_{P_1} V_1 = \pi_* \circ \mu_{P_1} V_1 .$$

Since π is a morphism with fibres of dimension $c = \mathrm{codim}_{D_P(\Gamma_P)|D(\Gamma)^*}$ one expects

$$\mu_{P_1}(V_1)^U = \pi_* \mu_{P_1}(V_1)[-c].$$

which thus would complete the proof upon going to strata. The actual proof is complicated by the fact that in first instance one knows this equality only at the level of cohomology. It requires the full abstract treatment of representation theory as given in [Bu-Wi, §4] to complete the proof. □

[2]Note that p_* is neither left nor right exact)

Combining the preceding Proposition with (4.5) one has

$$i_P^* j_* \mu_G[-c] = \mathrm{Tot} \circ \mu_{P_1}^{\Sigma_P} \circ \mathrm{inv}^U \circ \mathrm{Res}_P^G \,. \tag{4.6}$$

4.5 Replacing Simplicial Functors; Conclusion

In view of (4.6), suffices to prove:

Proposition 4.5.1. *One has* $\mathrm{Tot} \circ \mu_{P_1}^{\Sigma_P} = \mu_{P_1} \circ \mathrm{inv}^{\Gamma_{G'_P}}$.

Sketch of Proof. Recall that by Lemma 3.1.2 the combinatorics of the toroidal strata define the free $\mathbf{Z}\Gamma_{G'_P}$-resolution

$$C_\bullet(\{\mathrm{St}(\sigma)\}_{\sigma \in \Sigma}\,; \mathbf{Z}) \to \mathbf{Z}$$

and then, by abstract group theory [Bu-Wi, §4] one concludes the following equality for objects $V^\bullet$ in the derived category:

$$\mathrm{inv}^{\Gamma_{G'_P}}(V^\bullet) = \mathrm{Hom}(C_\bullet(\{\mathrm{St}(\sigma)\}_{\sigma \in \Sigma}\,; \mathbf{Z}), (V^\bullet))^{\Gamma_{G'_P}} .$$

On the other hand, the definition of the functor $\mu_{P_1}^{\Sigma_P}$ (which I did not give here) is just made up such that (after some minor adjustments)

$$\mathrm{Tot} \circ \mu_{P_1}^{\Sigma_P} \,\mathrm{Res}\, V_P^G = \mu_{P_1} \circ \mathrm{Hom}(C_\bullet(\{\mathrm{St}(\sigma)\}_{\sigma \in \Sigma}\,; \mathbf{Z}), (V^\bullet))^{\Gamma_{G'_P}} . \qquad \square$$

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