ABELIAN VARIETIES AND THETA FUNCTIONS ASSOCIATED TO COMPACT RIEMANNIAN MANIFOLDS; CONSTRUCTIONS INSPIRED BY SUPERSTRING THEORY.

S. MÜLLER-STACH, C. PETERS AND V. SRINIVAS
MATH. INST., JOHANNES GUTENBERG UNIVERSITÄT MAINZ,
INSTITUT FOURIER, UNIVERSITÉ GRENOBLE I
ST.-MARTIN D'HÈRES, FRANCE AND
TIFR, MUMBAI, INDIA

RÉSUMÉ. On détaille une construction dûe Witten et Moore-Witten (qui date d'environ 2000) d'une variété abélienne principalement polarisée associée à une variété de spin. Le théorème d'indice pour l'opérateur de Dirac (associé à la structure de spin) implique qu'un accouplement naturel sur le K-groupe topologique prend des valeurs entières. Cet accouplement sert commme polarization principale sur le tôre associé.

On place la construction dans un câdre général ce qui la relie à la jacobienne de Weil mais qui suggère aussi la construction d'une jacobienne associée à n'importe quelle structure de Hodge polarisée et de poids pair. Cette dernière construction est ensuite expliquée en termes de groupes algébriques, utile pour le point de vue des catégories Tannakiennes.

Notre construction depend de paramètres, beaucoup comme dans la théorie de Teichmüller, mais en général l'application de périodes n'est que de nature analytique réelle.

ABSTRACT. We first investigate a construction of principally polarized abelian varieties attached to certain spin manifolds, due to Witten and Moore-Witten around 2000. The index theorem for the Dirac operator associated to the spin structure implies integrality of a natural skew pairing on the topological K-group. The latter serves as a principal polarization.

We place the construction in a broader framework, relate it to Weil's intermediate jacobian and point out how it leads to abelian varieties canonically associated to polarized even weight Hodge structures. The latter construction can also be explained in terms of algebraic groups which might be useful from the point of view of Tannakian categories.

The constructions depend on moduli much as in Teichmüller theory although the period maps in general are only real analytic.

Ramond-Ramond fields, spin manifold, Dirac operator, polarized Hodge structure, generalized Weil jacobian.

1. Introduction

1.1. **Motivation from Physics.** In some forms of superstring theory particular theta functions come up as partition functions. The associated abelian varieties come either from certain cohomology groups of the underlying "universe" or, in more recent theories (e.g. [Witten] [Mo-Wi]), are linked to their K-groups.

Expressed in mathematical terms, one canonically associates to the cohomology or the K-theory of an even dimensional compact spin manifold a principally polarized abelian variety. Moreover, if the dimension is 2 mod 8 a particular line bundle is singled out whose first Chern class is the principal polarization. This bundle thus has a non-zero section represented by a theta function which, after suitable normalization, is indeed the partition function of the underlying theory.

There are several types of superstring theories, e.g. type I which is self-dual and types IIA and IIB which are related via T-duality. The theories start from a space-time Y which in a first approximation can be taken to be $Y = X \times T$ where T is the time-"axis"[1] and X is some compact Riemannian manifold. In Type IIA theory the *Ramond-Ramond field* is a closed differential form $G = G_0 + G_2 + \cdots$ on X with components of all even degrees while in type IIB G is an odd degree closed differential form on X. Moreover, these forms are *integral* (that is they have integral periods over integral homology cycles). The reason is that they are Poincaré dual to certain submanifolds of X which are the "world"-part of a brane in Y. Such a field should be thought of as some configuration in the theory. The partition function assembles all possible configurations in some generating function which can in turn be used to derive further physical properties of the model. In type IIA theory this partition function is of the form $\Theta(0)/\Delta$ where Θ is some normalized theta function. While Δ is canonically associated to the Riemannian manifold X, this is no longer the case for Θ. Instead, as suggested by Witten in [Witten] and later by Moore and Witten in [Mo-Wi] one should lift the discussion up to K-theory using the Chern character. But then, in order to make a canonical choice for Θ one has to assume that the manifold has a spin structure and has dimension 2 mod 8. The background from physics is collected in § 5. It is not necessary for an understanding of the rest of the paper, but it purports to explain how physicist came to the particular jacobians and the normalized theta functions.

For algebraic geometers these constructions may look a bit esoteric at first sight, the more since they are phrased in terms foreign to most of them. For instance, one might ask: *is the construction related to the Weil jacobian?* This question was one of the motivations for the present note. Clearly, an answer entails a careful analysis of the construction proposed in [Mo-Wi].

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[1] T could be a circle in physical theories but it could even be a point ("absence of branes").

1.2. **Mathematical Contents.** The constructions from [Mo-Wi] use in a critical way the index theorems of Atiyah and Singer, a subject not too well known among algebraic geometers. On the other hand, people well versed in this topic might not have heard about Griffiths' period domains. So, to make this paper profitable for readers with widely different backgrounds, chunks of theory from several branches of mathematics have been summarized in a way adapted to the needs of this paper.

In § 2 we review some basic Riemannian geometry, in § 3 K-theory and related index theorems are summarized while in § 4 basic constructions concerning Hodge structures are reviewed. The introductory part finishes with a short motivating section (§ 5) on quantum field theory.

The basic construction implicitly used in [Mo-Wi] is really simple and given in §6.1. It is apparently well known among physicists but we could not find a reference for it in this precise form, although a variant is well known in symplectic geometry, cf. [McD-S, Prop. 2.48 (ii)]. In § 6.2 this construction has been phrased in terms of homogeneous spaces. Noteworthy is a diagram given in Th. 6.13 which summarizes this. Curiously, this fits very well with Teichmüller theory which describes moduli of compact Riemann surfaces of a given genus in terms of conformal equivalence classes of metrics.

Next, the basic construction is applied in several different situations. First of all, in § 7.1 a polarized torus is associated to even or odd cohomology of a given compact Riemannian manifold whose dimension is 2 mod 4. These constructions can be twisted by certain automorphisms of the cohomology. These can be exploited to see that the construction generalizes the construction of the Weil intermediate jacobian. So, for polarized complex algebraic manifolds this twisted version is a canonical choice.

The question arises if some other extra structure on a Riemannian manifold in a similar manner leads to a canonical choice of abelian variety. The idea is that the extra structure should be such that it comes with a natural differential operator whose index can be calculated from a unit in the cohomology ring. The action of this unit then defines the canonical twist. The main example is a spin-structure where we have the associated Dirac operator. By the Atiyah-Singer index theorem its index is calculated using the $\hat{A}$-genus which provides the unit in the rational cohomology ring. In § 7.2 the reader finds the details. In this case the abelian variety is principally polarized. This abelian variety is exactly the one from [Mo-Wi]

As a parenthesis, it should be noted that in loc. cit. no proof is offered that the polarization on this abelian variety is indeed principal. This is true and we show this *crucial fact* by reducing it to an old result [AH3] on normalized multipliers.

As suggested by the Teichmüller approach we propose as a moduli space for a given compact smooth manifold the space of conformal metrics on it. The construction then gives various period maps associated to the manifold. See Theorem 7.3 and Theorem 7.9.

Next, in § 7.3 we show how in the context of abstract Hodge structures the construction of the Weil jacobian fits in this framework. This applies to *odd* weight. However, there is an apparently new construction related to even weight which also leads to a polarized abelian variety. By means of an example with holomorphically varying Hodge structures, we show that the new abelian variety varies in general non-trivially with parameters. However, as the example shows, as in the case of Weil jacobians, the dependence is in general neither holomorphic nor anti-holomorphic. In § 7.4 this abstract construction is applied to the cohomology of Kähler manifolds (Theorem 7.13). The reader should contrast this with the tori obtained using the cohomology of general Riemannian manifolds. See Theorem 7.9.

The note ends with § 8 where, after a short digression on normalized theta functions a mathematical formulation is offered of the pertinent results of [Mo-Wi, §3]. Isolating the line bundle from the numerical equivalence class of the principal polarization uses in a crucial way some constructions from real K-theory. These are quite subtle and have been placed in appendix B. Noteworthy in this appendix is a version of the Thom-isomorphism in real K-theory (=Theorem B.1) which extends the one found in the literature for spin-manifolds whose dimension is divisible by 8. This generalization can be extracted from [At2] but is not explicitly stated there. We state and prove it in the appendix since this form of the Thom-isomorphism theorem is used in a crucial way in [Witten] to find the "right" θ-function.

Notation. For any topological space X we let $H^k(X)$ be the $\mathbb{Q}$-vector space of the k-th singular cohomology of X with *rational* coefficients. The sublattice $H^k(X)_{\mathbb{Z}}$ of images of integral cohomology classes is canonically isomorphic to integral cohomology modulo torsion. Furthermore we put

$$H^k(X)_{\mathbb{R}} := H^k(X) \otimes_{\mathbb{Q}} \mathbb{R}, \qquad H^k(X)_{\mathbb{C}} := H^k(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

If X is a manifold, we let $T(X)$ be its tangent bundle and $T(X)^{\vee}$ its dual, the co-tangent bundle of X. If X is a smooth (i.e., C^{∞}-)manifold we denote the vector space of smooth differential p-forms on X by $A^p(X)$.
We put

$$H^+ = \bigoplus_{k \in \mathbb{Z}} H^{2k}(X), \quad H^- = \bigoplus_{k \in \mathbb{Z}} H^{2k+1}(X)$$

so that we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}$-algebra

$$H^*(X) = H^+(X) \oplus H^-(X).$$

For an oriented compact connected manifold X the intersection form

$$H^*(X) \times H^*(X) \to \mathbb{Q}, \quad \langle x, y \rangle = \int_X x \wedge y$$

is a perfect pairing (Poincaré duality) and if moreover $\dim X$ is even, it induces perfect pairings on $H^{\pm}(X)$. On $H^{-}(X)$ it induces a skew form

$$\omega^{-} : H^{-}(X) \times H^{-}(X) \to \mathbb{Q}, \quad \omega^{-}(x, y) = \int_X x \wedge y,$$

but on $H^{+}(X)$ a symmetric form. We replace it by a skew form as follows. The splitting $H^{+}(X) = H^{4*}(X) \oplus H^{4*+2}(X)$ defines the involution ι with the first subspace as $+1$-eigenspace and the second as (-1)-eigenspace. Suppose next that $\dim X \equiv 2 \bmod 4$. Then the form

$$\omega^{+} : H^{+}(X) \times H^{+}(X) \to \mathbb{Q}, \quad \omega^{+}(x, y) = \int_X x \wedge \iota(y)$$

is a perfect skew pairing. Both pairings $\omega^{\pm}$ restrict to unimodular integral skew pairings on $H^{\pm}(X)_{\mathbb{Z}}$. We set

$$\omega := \omega^{+} + \omega^{-}.$$

2. Background from Riemannian Geometry

2.1. The Hodge Metric.
Let (X, g) be a compact d-dimensional Riemannian manifold. The metric g induces a metric on the co-tangent bundle and its k-th exterior powers, the bundles of k-forms. This is a fibre-wise metric.

Next, assume that X has an orientation. Then the Hodge $*$ operator can be defined as follows. Let $\{e_1, \ldots, e_d\}$ be an oriented orthonormal basis for $T(X)^{\vee}$. Let $I = \{i_1, \ldots, i_k\}$ be an ordered subset of the ordered set $[n] := \{1, \ldots, n\}$. Then the $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda^k T(X)^{\vee}$ give an orthormal basis for $\Lambda^k T(X)^{\vee}$ and one defines

$$* : \Lambda^k T(X)^{\vee} \quad \to \quad \Lambda^{d-k} T(X)^{\vee}$$

$$e_I \quad \mapsto \quad e_{[n]-I}.$$

This defines a linear operator $* : A^k(X) \mapsto A^{d-k}(X)$. We need the following property (see [Warn, 4.10(6)]):

Lemma 2.1. *Let* $\dim X = d$ *be even. Then* $*^2 = (-1)^k$ *on* $H^k(X)_{\mathbb{R}}$. *In particular, on odd cohomology it defines a complex structure.*

Note that for all I the element $e_I \wedge *e_I = *1$ is the volume form on X and the fibrewise metric on $\Lambda^k T(X)^{\vee}$ defines a metric on $A^k(X)$ given by

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta.$$

By definition, the operator $d^* : A^k(X) \to A^{k-1}(X)$ is the adjoint of d with respect to these metrics, i.e.,

$$\langle \alpha, d\beta \rangle = \langle d^*\alpha, \beta \rangle \text{ for all forms } \alpha, \beta.$$

A form G is *co-closed*, respectively *co-exact* if $d^*G = 0$, respectively $G = d^*H$ for some $(d+1)$-form H. A form is *harmonic* if it is closed and co-closed.

The *Hodge decomposition theorem* [Warn, Chapter 6] states that there is an orthogonal decomposition

$$A^p(X) \quad = \quad \mathsf{Har}^p(X) \quad \oplus \quad d\mathsf{A}^{p-1}(X) \quad \oplus \quad d^*\mathsf{A}^{p+1}(X)$$

$$[p\text{-forms}] \qquad [\text{harmonic } p\text{-forms}] \qquad [\text{exact } p\text{-forms}] \qquad [\text{co-exact } p\text{-forms}].$$

Moreover, the space of harmonic forms is *finite dimensional* and every De Rham cohomology class has a unique representing harmonic form. From harmonic theory it also follows (see loc. cit.) that $d \oplus d^*$ induces a self-adjoint operator on $\mathsf{Har}(X)^\perp \subset \mathsf{A}(X)$.

Finally we remark that the metric on $A^k(X)$ when restricted to the subspace of harmonic forms defines a metric on $H^k(X)_\mathbb{R}$, the *Hodge metric*.

Definition 2.2. Denote the cohomology class of a closed form α by $[\alpha]$. The *Hodge metric* on $H^*(X)_\mathbb{R}$ is the metric associated to g defined by

$$b^{(g)}([a],[b]) := \int_X \alpha \wedge *\beta, \qquad a = [\alpha],\ b = [\beta]$$

and α, β are the unique harmonic forms in the classes a, b respectively.

2.2. Kähler Manifolds. Assume X is a compact complex manifold of dimension d. Then X is a real $(2d)$-dimensional manifold with an almost complex structure J. Decompose the hermitian metric h on $T(X)$ into real and imaginary parts

$$h(x,y) = g(x,y) + \mathrm{i}\omega(x,y).$$

Then h is called *Kähler*, if the real non-degenerate skew-symmetric form ω is closed. Since $g(x,y) = \omega(x,Jy)$ and hence is determined by ω we sometimes call the pair (X,ω) a Kähler manifold. The Riemannian metric is then written g_ω.

A manifold admitting a Kähler metric is called a *Kähler manifold*. Examples include projective space with the Fubini-Study metric and projective manifolds with metric induced from the Fubini-Study metric.

2.3. Spin Manifolds. For this section the reader may consult [B-G-V, Chapters 3,4]. For spin-groups consult Appendix A. Recall that for any vector bundle E on a compact topological space X we have functorially behaving Stiefel-Whitney classes $w_k(E) \in H^k(X;\mathbb{Z}/2\mathbb{Z})$ in cohomology with $\mathbb{Z}/2\mathbb{Z}$-coeffcients. If E happens to be a complex bundle we also have the Chern classes $c_k \in H^{2k}(X;\mathbb{Z})$ in integral cohomology. They are related: $w_{2k}(E)$ is the reduction modulo two of $c_k(E)$, i.e., the image of $c_k(E)$ under the coefficient morphism $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/2\mathbb{Z})$.

Definition 2.3. (1) Let X be an oriented Riemannian manifold. An even rank vector bundle E has a *spin^c-structure* if for some *integral class* $w \in H^2(X;\mathbb{Z})$ one has $w \equiv w_2(E) \bmod 2$
(2) If $T(X)$ has a spin^c-structure, we say that *X has spin^c-structure*. Note

that in particular this implies that $\dim X$ should be even.

(3) A *spin structure* on a d-dimensional manifold X is a $\mathrm{Spin}(d)$-principal bundle $\mathrm{Spin}(X)$ on X such that

$$T(X)^\vee \simeq \mathrm{Spin}(X) \times_{\mathrm{Spin}(d)} \mathbb{R}^d.$$

Any manifold having a spin structure is said to be a *spin manifold*.

Remarks 2.4. (1) In particular, E has a spin^c-structure if $w_2(E) = 0$. It is well known that this latter condition for $E = T(X)$, i.e., $w_2(X) = 0$ is equivalent to X being spin. In particular, any spin manifold of even dimension has a spin^c-structure. It is well known that the number of inequivalent spin-structures equals the rank of $H^1(X, \mathbb{F}_2) \simeq H_1(X, \mathbb{F}_2)$ (by the universal coefficient theorem). In particular, there are at most finitely many such structures and if X is simply connected there exists at most one spin structure.

(2) The preceding remark makes it easy to find examples of spin-manifolds: compact Riemann surfaces, (real or complex) tori, complex K3-surfaces, and, more generally any complex manifold whose canonical bundle is a square.

In Appendix A we recall the notion of a Clifford algebra. This notion can be globalized to the framework of vector bundles on a compact Riemannian manifold (X, g). The cotangent bundle $T(X)^\vee$ is a metric bundle. The Clifford algebras $\mathsf{C}(T_x^\vee)$, $x \in X$ glue together to give the *Clifford algebra* $\mathsf{C}(X)$. The Riemannian structure defines a unique metric connection [2] on $T(X)$ (and on $T(X)^\vee$) without torsion, the *Levi-Civita connection*. Both g and this connection extend to the entire exterior algebra $\Lambda^* T(X)^\vee$ and, using the isomorphism of Lemma A.2 produces a g-connection and an induced Levi-Civita connection ∇^{LC} on the Clifford-algebra. The Clifford algebra has a self-adjoint Clifford action. More generally one defines:

Definition 2.5. A *Clifford bundle* is a triple (W, h, ∇) consisting of a $\mathbb{Z}_2$-graded complex $\mathsf{C}(X)$-module $W = W^+ \oplus W^-$ equipped with a hermitian metric h and an h-connection ∇ such that

i) The Clifford action on the module W, denoted $c : \mathsf{C}(X) \to \mathrm{Aut}(W)$, is graded, W^+ and W^- are mutually h-orthogonal and the action is self-adjoint with respect to h, i.e.,

$$h(c(\alpha)\sigma, \tau) + h(\sigma, c(\alpha)\tau) = 0,$$

for all differentiable sections σ, τ of W and differential 1-forms α [3];

ii) The connection is compatible with the Levi-Civita connection in the sense that for any local vector field ξ one has

$$\nabla_\xi(c(\alpha)\sigma) = c(\nabla_\xi^{\mathrm{LC}}\alpha)\sigma + c(\alpha)(\nabla_\xi\sigma),$$

for all differentiable sections σ of W, and differentiable 1-forms α.

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[2] with respect to the metric g; it is a g-connection

[3] Recall that $\alpha(x) \in T_x^\vee \subset \mathsf{C}(T_x^\vee)$.

The *Dirac operator* associated to a Clifford bundle W is a first order differential operator on the space of sections of W which is defined as follows:

$$\Gamma(W) \xrightarrow{\nabla} \Gamma(W \otimes_{\mathbb{C}} (T(X)_{\mathbb{C}}^{\vee})) \xrightarrow{c} \Gamma(W).$$

It sends sections in $W^{\pm}$ to sections in $W^{\mp}$.

Examples 2.6. 1. The original Dirac operator is Diracs answer as how to find a square root of the positive Laplacian $\nabla := -\sum_{j=1}^{4} \partial^2/\partial^2 x_j$ on classical space-time $\mathbb{R}^4$ equipped with the Lorentz metric. The Clifford algebra is known to be the algebra of (2×2)-matrices with coefficients in the quaternions $\mathbb{H} = \mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$ with real basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. A complex basis is $\{1, \mathbf{j}\}$. The Clifford bundle V in question is $\mathbb{H}^2$. The Dirac operator then is defined to be

$$\not{D} := \sum_{k=1}^{4} \gamma_k \frac{\partial}{\partial x_k},$$

where the γ_k are certain (4×4)-matrices with complex coefficients involving the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, \sigma_2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Precisely, one has

$$\gamma_1 = \mathbf{i} \cdot \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \, \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \, k = 2, 3, 4$$

for the Euclidean case. In the Lorentz case one removes the factor $\mathbf{i}$ in γ_1.
2. The bundle $\mathsf{C}(X)_{\mathbb{C}} = \mathsf{C}(X) \otimes \mathbb{C}$ with its hermitian extension of g and Levi-Civita connection ∇^{LC} is a Clifford bundle with Dirac operator $d + d^*$.
3. Assume X is spin manifold of even dimension d and let S be the irreducible complex $\mathrm{Spin}(d)$-spinor space (see (31) in Appendix A) and form the *spinor bundle*

$$\mathcal{S} = \mathrm{Spin}(X) \times_{\mathrm{Spin}(d)} \mathsf{S}.$$

It is a Clifford bundle when equipped with the Levi-Civita connection ∇^{LC} coming from restricting the usual Levi-Civita connection to the subbundle $\mathcal{S}$ of the complexified Clifford algebra $\mathsf{C}(X)_{\mathbb{C}}$. The associated Dirac operator is called *the Dirac operator* of the spin-manifold X.
Let (E, h) be any hermitian vector bundle on X with an h-metric connection ∇. The twisted bundle $W = E \otimes \mathcal{S}$ has a product hermitian structure and a natural product connection which is compatible with this metric. All Clifford bundles are of this form. The associated Dirac-operator $\not{D}_E$ is called the *Dirac operator with coefficients in E.*

3. Summary of K-theory and Index Theory

The reader may consult the excellent introduction [At1]. For a solid introduction to index theorems consult the appendices [Hir, §24-26] to Hirzebruch's classic.

3.1. **K-theory for Complex Bundles.** We let X be a topological space and we let $K(X)$ be the Grothendieck group of *complex* vector bundles on X. This is by definition the free $\mathbb{Z}$-module generated by the isomorphism classes of complex vector bundles modulo the relations $E \oplus F - E - F$. It can be seen to be generated by *virtual bundles*, i.e., differences of the form $E - F$, where E and F are any two vector bundles. The tensor product on vector bundles is compatible with these relations and so $K(X)$ becomes a ring. If $f : X \to Y$ is continuous, pull back of bundles induces a ring homomorphism $f^* : K(Y) \to K(X)$.

The *suspension SX* is obtained from the product $S^1 \times X$ by identifying all points in the subspace $\{\mathbf{1}\} \times X$ where $\mathbf{1} = (1, 0) \in S^1 \subset \mathbb{R}^2$; n-fold iterated suspension is denoted $S^n X$. One defines

$$K^{-n}X := K(S^n X).$$

Bott's periodicity theorem [Bott] can be stated as $K^{-2}(X) \simeq K(X)$ which makes it possible to define $K^n X$ for all integers n. There are natural pairings $K^n(X) \times K^m(X) \to K^{n+m}(X)$ compatible with the Bott periodicity making $\oplus_{n \in \mathbb{Z}} K^n(X)$ into a graded ring. In view of Bott's theorem the essential part of this ring is

$$K^*(X) = K(X) \oplus K^1(X)$$

with $\mathbb{Z}_2$-grading. The cohomology-ring can also be given a $\mathbb{Z}_2$-grading as $H^*(X) = H^+ \oplus H^-(X)$. The Chern character (see e.g. formula (4) below) gives $\mathbb{Z}_2$-graded isomorphisms [AH2]

$$(1) \qquad\qquad \mathrm{ch} : K^*(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X).$$

One can also define relative K groups $K^n(X, Y)$ where Y is a subset of X and these fit in exact sequences as for ordinary cohomology. One important fact is the K-theoretic version of the *Thom isomorphism theorem*: Let $B(E)$, respectively $S(E)$ be the unit disk-bundle, unit sphere bundle associated to a hermitian vector bundle (E, g) of rank r. Then

$$K^*(B(E), S(E)) \simeq K^*(X).$$

3.2. **The Index Theorem.** Let X be a differentiable manifold, E, F two hermitian vector bundles, $\mathsf{D} : \Gamma(E) \to \Gamma(F)$ a differential operator with adjoint D^*. Recall that the *index* is given by

$$(2) \qquad\qquad \mathrm{ind}(\mathsf{D}) := \dim \ker \mathsf{D} - \dim \ker \mathsf{D}^*.$$

Let us write for brevity

$$BX := B(T(X)^\vee), \qquad SX := S(TX)^\vee.$$

The symbol of D defines an element $\sigma(\mathsf{D}) \in K(BX, SX)$ whose Chern character lands in $H^*(BX, SX) \simeq H^*_c(T(X)^\vee)$. Let $\pi : T(X)^\vee \to X$ be the natural projection. Since $T(X)^\vee$ is a symplectic manifold and g a Riemannian metric on X, the bundle $T(T(X)^\vee)$ has a natural complex structure (Prop. 6.2) such that $\pi^* TX \otimes \mathbb{C} \simeq T(T(X)^\vee)$ and there is a Todd class

$$\mathrm{td}(T(X)^\vee | BX) \in H^*(BX).$$

Since $H^*(BX, SX)$ is a $H^*(BX)$-module the following formula makes sense; it defines the topological index

$$(3) \qquad \mathrm{ind}_\tau(\mathsf{D}) := \int_{T(X)^\vee} \mathrm{ch}(\sigma(\mathsf{D})) \cdot \mathrm{td}(T(X)^\vee | BX).$$

One has:

Theorem 3.1 ([AS1]). *Let X be a compact differentiable manifold and D an elliptic differential operator between complex vector bundles on X. Then the topological index (3) equals the (analytical) index (2). In particular, it is an integer.*

The K-theoretic extension comes from the remark that the right hand side of (3) makes sense if we replace $\sigma(\mathsf{D})$ by any element $d \in K(BX, SX)$. The topological index for such an element is then defined by

$$\mathrm{ind}_\tau(d) = \int_{T(X)^\vee} \mathrm{ch}(d) \cdot \mathrm{td}(T(X)^\vee | BX).$$

It can be shown that it also makes sense to speak of an *analytic index* $\mathrm{ind}(d)$ for such elements and that it is an integer:

Theorem 3.2 ([AS2]). *For a compact differentiable manifold X, the two homomorphisms*

$$\mathrm{ind}, \mathrm{ind}_\tau : K(BX, SX) \to \mathbb{Q}$$

coincide and hence take values in $\mathbb{Z}$.

This version has a relative form for differentiable locally trivial fibrations $f : X \to T$. The starting observation is the fact that $K(\mathrm{point}) = \mathbb{Z}$ so that the integer $\mathrm{ind}(\mathsf{D})$ is just the K-theoretic difference of the vector spaces $\ker \mathsf{D} - \ker \mathsf{D}^*$. For a family over T this pointwise construction gives a difference of complex bundles on T and hence an element of $K(T)$. For the topological index one has to replace $T(X)^\vee$ by the relative cotangent bundle $T(X/T)^\vee$ and one gets:

Theorem 3.3 ([AS3]). *For a differentiable family $f : X \to T$ of compact differentiable manifolds, the two homomorphisms*

$$\mathrm{ind}, \mathrm{ind}_\tau : K(B(X/T), S(X/T)) \to K(T) \otimes \mathbb{Q}$$

coincide and hence take values in $K(T)$.

3.3. **The Index Theorem for the Dirac Operator.** We start with a few preliminaries on genera. See [Hir, Ch. 1]. Start with any formal power series $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in 1 + \mathbb{Z}[z]$ whose m-th order truncation has a formal factorization

$$1 + p_1 z + \cdots + p_m z^m = (1 + \beta_1 z) \cdots (1 + \beta_m z).$$

Next we explain a certain formal procedure which uses the β_j and a second formal powerseries of the form

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots \in \mathbb{Q}[[z]]$$

as an input and whose output is the so called *q-series for p*. For some fixed m write down the m-fold product series

$$
\begin{aligned}
q(\beta_1 z) \cdot q(\beta_2 z) \cdot \ldots \cdot q(\beta_m z) \quad &= \quad 1 + Q_1(p_1) z + Q_2(p_1, p_2) z^2 + \cdots \\
&= \quad 1 + q_1 p_1 z + \left[(-2q_2 + q_1^2) p_2 - q_1 p_1^2 \right] z^2 + \cdots,
\end{aligned}
$$

where by definition the Q_j are the coefficients of z^j. These turn out to be universal polynomials of total degree j in the first j "variables" $p_1, \ldots, p_j$ with coefficients expressible in the coefficients of the formal series $q(z)$. To find these, one calculates successively, setting $m = 1$, $m = 2$ etc. The *q-series for p* is the resulting formal series

$$1 + Q_1(p_1) z + Q_2(p_1, p_2) z^2 + \cdots \in \mathbb{Q}[p_1, p_2, \ldots][[z]].$$

The corresponding q-genus is obtained by setting $z = 1$. The particular choice

$$q(z) = \frac{\frac{1}{2} z}{\sinh \frac{1}{2} z} = 1 - \frac{1}{2^2} \cdot \frac{1}{6} z + \frac{1}{2^4} \cdot \frac{7}{360} z^2 + \cdots$$

defines the $\hat{A}$-series

$$\hat{A}(z, p_1, p_2, \ldots)) = 1 - \frac{1}{2^2} \cdot \frac{1}{6} p_1 z + \frac{1}{2^4} \left(-\frac{1}{90} p_2 + \frac{7}{360} p_1^2 \right) z^2 + \cdots.$$

Next we recall that for any vector bundle F the *Pontryagin classes* are obtained from the Chern classes as follows:

$$p_i(F) := (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(X), i = 1, \ldots, m = \operatorname{rank}(F).$$

One associates to these the *Pontryagin polynomial* $p(F) := 1 + p_1(F) z + \cdots + p_m(F) z^m$. The Pontryagin classes of X are those of $T(X)$. If we now substitute $p_i(F)$ for p_i in the $\hat{A}$-series, truncate the series at order $m = \operatorname{rank}(F)$ and set $z = 1$ we obtain the *$\hat{A}$-genus* for F:

$$\hat{A}(F) := \hat{A}(1, p_1(F), \ldots, p_m(F)) \in H^{4*}(X; \mathbb{Q}).$$

In particular

$$\hat{A}(X) := \hat{A}(T(X)).$$

For any complex vector bundle F one also has the Chern character $\mathrm{ch}(F)$, defined as follows. Write formally $1 + c_1(F)x + \cdots c_m(F)x^m = (1 + \gamma_1 x) \cdots (1 + \gamma_m x)$ and evaluate

$$(4) \quad \mathrm{ch}(F) = \sum e^{\gamma_i} = m + c_1(F) + \frac{1}{2}(c_1^2(F) - c_2(F)) + \cdots \in H^{2*}(X; \mathbb{Q}).$$

Now we have

Theorem 3.4 (Atiyah-Singer index theorem, [AS1])**.** *Let X be a manifold with a spin structure, E a complex bundle on X. Let $\not{D}_E$ be the Dirac operator with coefficients in E.*[4] *We have*

$$\mathrm{ind}(\not{D}_E) = \int_X \hat{A}(X)\mathrm{ch}(E).$$

This is a special case of the general index theorem, Theorem 3.2.

Remark 3.5. There is another important index theorem for X a compact complex manifold carrying a complex vector bundle E. To explain it, let $\Lambda^{p,q}(E)$ be the *bundle* of complex E-valued forms of type (p, q) and denote its sections (the corresponding *forms*) by $A^{p,q}(X)$. Decompose the d-operator in the usual way as

$$d = \partial + \bar{\partial} : A^{p,q}(E) \to A^{p+1,q}(E) \oplus A^{p,q+1}(E).$$

Choose a hermitian metric h on X and h' on E and let ∂^*, $\bar{\partial}^*$ be the (h, h')-adjoints of ∂ and $\bar{\partial}$. The bundle

$$\bigoplus_q \Lambda^{0,q}(E) = \bigoplus_k \Lambda^{0,2k+1}(E) \oplus \bigoplus_k \Lambda^{0,2k}(E)$$

is a Clifford bundle with $\bar{\partial} + \bar{\partial}^*$ acting on its global sections as Dirac operator. Applying the index theorem yields the Hirzebruch-Riemann-Roch theorem

$$\int_X \mathrm{td}(X)\mathrm{ch}(E) = \mathrm{ind}(\bar{\partial} + \bar{\partial}^*)$$

and so the left hand side is an integer as well.

Let us see what the above index theorems gives if E is real. Since $c_i(\bar{E}) = (-1)^i c_i(E)$ we see from (4) that the complex conjugation fixes the terms in $\mathrm{ch}(E)$ of degree divisible by 4 while it acts as minus the identity on the other terms. So the Atiyah-Singer index theorem implies

Corollary 3.6. [5] *If* $\dim X \equiv 2 \mod 4$ *one has* $\mathrm{ind}(\not{D}_{\bar{E}}) = -\mathrm{ind}(\not{D}_E)$. *In particular, if E is real, the index of the Dirac operator with values in E. vanishes.*

---

[4]It also makes sense to speak of the Dirac operator with values in a virtual bundle $[E] \in K(X)$ and the same formula for its index holds. It is this version which we need below.

[5]It is exactly at this point that passing to real operators and real K-theory yields finer invariants. See Theorem B.2.

4. Background From Hodge Theory

We recall a number of general facts about polarized Hodge structures [PS, Chapter 2.1].

4.1. **Hodge Structures.** Recall that a *rational Hodge structure of weight* k consists of a rational vector space W together with a decomposition

$$W_{\mathbb{C}} := W \otimes \mathbb{C} = \oplus_{p+q=k} W^{p,q}$$

such that $W^{q,p} = \overline{W^{p,q}}$. Its *Hodge numbers* are

$$h^{p,q} = \dim_{\mathbb{C}} W^{p,q}$$

Sometimes we write just W for the structure. It can alternatively be described as an algebraic representation of the (real) matrix group

$$\mathbf{G}(\mathbb{R}) = \{s(a,b) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a,b \in \mathbb{R},\ a^2 + b^2 \neq 0\},$$

say $h : \mathbf{G}(\mathbb{R}) \to \operatorname{Aut} W_{\mathbb{R}}$, $W_{\mathbb{R}} = W \otimes \mathbb{R}$ with the extra two properties of being defined over the rationals and for which $h(t) = t^k$, encoding the weight. The Hodge decomposition can then be found back as follows. First identify $s(a,b)$ with the complex number $z = a + \mathrm{i}b$ establishing an isomorphism $\mathbf{G}(\mathbb{R}) \simeq \mathbb{C}^*$ of real algebraic groups. Then $W^{p,q}$ is the subspace of $W_{\mathbb{C}} = W \otimes \mathbb{C}$ on which $z \in \mathbb{C}^*$ acts as $z^p \bar{z}^q$.

The simplest example making use of this description is the one-dimensional *Tate Hodge structure* $\mathbb{Q}(-k)$, the one-dimensional representation of $\mathbf{G}(\mathbb{R})$ given by the character $h(z) = |z|^k$. It is pure of type (k,k).

The Weil operator $C_W = h(\mathrm{i})$ is the real operator acting as i^{p-q} on $W^{p,q}$. Note that $C_W^2 = (-1)^k$ and so defines a complex structure if the weight is odd. In general the Weil operator alone does not suffice to determine the Hodge structure since it does not determine the $W^{p,q}$. But we have

Lemma 4.1. *Giving a weight* 1 *Hodge structure on* W *is equivalent to giving a complex structure on* $W_{\mathbb{R}} := W \otimes \mathbb{R}$.

Proof: The space $W^{1,0}$, respectively $W^{0,1}$ is the eigenspace of C_W with eigenvalue i, respectively $(-\mathrm{i})$. This shows that one must have $h(a + b\mathrm{i}) = a\mathbf{1} + bC_W$. Conversely, given a complex structure J on a rational vector space W defines a weight 1 Hodge structure by defining $h(a + b\mathrm{i}) = a\mathbf{1} + bJ$. $\square$

4.2. **Polarizations.** A *polarization* for a weight k Hodge structure W is a non-degenerate $\mathbb{Q}$-valued bilinear form Q on W which is symmetric if k is even and skew-symmetric otherwise and whose complex-linear extension $Q_{\mathbb{C}}$ satisfies the two Riemann conditions [6]

(1) $Q_{\mathbb{C}}(x,y) = 0$ if $x \in W^{p,q}$, $y \in W^{r,s}$ and $(p,q) \neq (s,r)$;
(2) $Q_{\mathbb{C}}(C_W x, \bar{x}) > 0$ if $x \neq 0$.

---

[6]Weil demands instead that $Q(x, C_W x) > 0$ for $x \neq 0$. The difference changes the sign for Q by $(-1)^k$.

Note that the first condition implies that the Weil operator preserves the polarization: $Q(C_W x, C_W y) = Q(x, y)$. Conversely, in weight 1 we have:

Lemma 4.2. *For a weight 1 Hodge structure W, the first Riemann bilinear relation relative to a skew form Q on W is equivalent to C_W being Q-symplectic.*

Proof: If $x, y \in W^{1,0}$ one has $Q_{\mathbb{C}}(x, y) = Q_{\mathbb{C}}(C_W x, C_W y) = -Q_{\mathbb{C}}(x, y)$. $\quad\square$

One can easily see that the datum of a polarization is equivalent to giving a morphism

$$S : W \otimes W \to \mathbb{Q}(-k), \qquad S = (2\pi i)^{-k} Q$$

of weight $(2k)$-Hodge structures for which the bilinear form defined by $(x, y) \mapsto (2\pi i)^k S(C_W x, y)$ is symmetric and positive definite on W. It follows that a polarization induces an isomorphism

$$\hat{Q} : W \xrightarrow{\sim} W^\vee(-k)$$

of weight k Hodge structures.

If $W = W_{\mathbb{Z}} \otimes \mathbb{Q}$, i.e., W is an *integral Hodge structure*, we speak of an *integral* polarization Q if Q takes integral values on $W_{\mathbb{Z}}$. This is inspired by the fact that integral weight one polarized Hodge structures (W, Q) are the same as polarized abelian varieties: to the Hodge structure associate the real torus $W/W_{\mathbb{Z}}$ equipped with complex structure induced by the Weil operator C_W and polarization induced by Q. The associated polarized abelian variety is then denoted $J(W, Q)$. The polarization is a principal polarization precisely when Q is a unimodular.

The standard example of a polarized Hodge structure comes from the cohomology of a Kähler manifold as we now recall briefly. See e.g. [Weil2] for details.

Example 4.3. Let (X, ω) be a compact Kähler manifold of dimension d.

Let L be the real operator on cohomology which is cup product with the class defined by ω. The weak Lefschetz theorem states that $L^k : H^{d-k}(X)_{\mathbb{R}} \xrightarrow{\sim} H^{d+k}(X)_{\mathbb{R}}$. The kernel of L^{k+1} by definition is the *primitive cohomology*

$$H^{d-k}_{\text{prim}}(X)_{\mathbb{R}} := \ker\{L^{k+1} : H^{d-k}(X)_{\mathbb{R}} \to H^{d+k+2}(X)_{\mathbb{R}}\}.$$

By definition $H^k_{\text{prim}}(X)_{\mathbb{R}} = 0$ when $k > d$. The Lefschetz decomposition theorem tells us how to build up cohomology from the primitive parts:

$$H^k(X)_{\mathbb{R}} = \bigoplus_{k \geq r} L^m H^{k-2r}_{\text{prim}}(X)_{\mathbb{R}}.$$

Write the primitive decomposition for $x, y \in H^k(X)_{\mathbb{R}}$ as $x = \sum L^r x_r$, $y = \sum L^s y_s$. Then the *Riemann-form* [Weil2, p. 77] can be written as:

$$
\begin{aligned}
(5) \qquad Q_\omega(\textstyle\sum_r L^r x_r, \sum_s L^s y_s) &:= \epsilon_k \sum_r (-1)^r \mu_r \int_X L^{d-k+2r}(x_r \wedge y_r), \\
\epsilon_k &:= (-1)^{\frac{1}{2}k(k+1)}, \quad \mu_r := \frac{r!}{(d-k+r)!}.
\end{aligned}
$$

Riemann's bilinear relations tell us that Q_ω is symmetric and non-degenerate if k is even and symplectic if k is odd.

From Weil's formula [Weil2, Chap 1, Th. 2] for the $*$ operator in terms of the Weil operator C for the Hodge structure on cohomology:

$$(6) \qquad *(L^r x_r) = \epsilon_k (-1)^r \mu_r L^{d-k+r} C x_r, \qquad x_r \in H^{k-2r}(X)_{\mathrm{prim}}$$

we see that the Hodge metric is given by

$$(7) \qquad\qquad b^{(g_\omega)}(x, y) = Q_\omega(x, Cy), \quad x, y \in H^k(X).$$

If X is a projective manifold, the class ω can be taken to be rational so that the Lefschetz decomposition as well as the form Q_ω is rationally defined. Remark however that even if we choose for ω an integral class, formula (5) shows that the polarization becomes a priori only rational on $H^k(X)_{\mathbb{Z}}$. Not only a denominator is introduced but also, even if x is integral, the primitive constituents are in general only rational. To remedy this, using a basis, one can always find an integer N such that $N Q_\omega$ becomes integral on $H^k(X)_{\mathbb{Z}}$. Taking N minimal when varying over all possible bases gives an intrinsically defined integer, say N_ω so that $N_\omega Q_\omega$ becomes integral.

4.3. Griffiths Domains.

We also make use of the Griffiths period domains [Gr] which parametrize polarized real Hodge structures of given weight and Hodge numbers.

To explain this roughly, let (W, Q) be a given real Hodge structure of weight k. It is characterized by a Hodge flag $F = \{F^k \subset F^{k-1} \subset \cdots \subset F^\ell\}$ where $F^k = H^{k,0}$, $F^{k-1} = H^{k,0} + H^{k-1,1}$, etc. If $k = 2\ell - 1$ (odd case) or $k = 2\ell$ (even case) the flag stops at stage ℓ. The first Riemann condition is an algebraic condition while the second is open in the complex topology. The group $G = \mathrm{SO}(W, Q)$ acts transitively on flags satisfying these conditions and the stabilizer of the given flag F is a compact subgroup $H \subset G$ so that the period domain can be written as $D = G/H$. Below, in § 7.3 we recall this in some more detail.

The standard example of a polarized Hodge structure of weight k is the primitive k-th cohomology of any Kähler variety X as recalled in Example 4.3. If moreover X varies in an algebraic family $\{X_t\}_{t \in T}$, the assigment $t \mapsto k$-th primitive cohomology of X_t induces a holomorphic *period map* $p : T \to D$. This map is in general multi-valued because of monodromy. For details see e.g. [CSP, Chap. IV.3].

5. Physics background

The constructions are in fact inspired by quantum field theory. Start with a Riemannian manifold Y. Fields on Y are sections in certain vector bundles on Y. These come in three types: scalar fields, fermionic fields and gauge fields. The first type is just a function on Y, the second one is a smooth section in a spinor-bundle on Y, and the last are Lie-algebra valued forms on Y. Experimentally observed are gauge fields and fermionic

fields. A scalar field has not been observed yet, but physicist are desperately trying to observe it (the search for the Higgs boson, for which the new LHC accelarator has been built).

Given the fields, the physics is deduced from a Lagrangian density. For a free scalar field ϕ the Lagrangian reads (using Einstein summation)

$$L = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\phi(x)\phi(x).$$

Its integral over Y is the action:

$$S = \int_Y \mathrm{d}^n x\, L,$$

where $\mathrm{d}^n x$ is some suitable n-dimensional measure on Y, $n = \dim Y$. The action obviously depends on the field.

The manifolds Y occurring in physics are called space-time worlds and they are to have a *time component*; moreover, one assumes that there is a Lorentzian metric on Y which is a positive definite metric on world-sheets. To change it to a positive definite metric on *all* of Y, there is a trick, called *Wick rotation*: one formally replaces the time parameter t by it. Note that this replaces S by iS as well. Below we explain why this is a useful trick.

Once this has been done, by definition, the partition function is the integral over all fields, where each field is weighted by $\exp(\mathrm{i}S)$. For a free scalar field one gets $Z = \int \mathcal{D}\phi \cdot \exp(\mathrm{i}S)$. Such an integral, a path integral, is mathematically ill-defined. In physics literature it has been interpreted in analogy with similar integrals in probability theory and gets replaced by an ordinary, but complicated integral over the so-called configuration space of Y. It turns out that this integral can be approximated in a very particular way up to any given order in a way described by the associated Feynman graphs and Feynman rules for the fields. To actually calculate the integrals in the resulting expansion, one has to rewrite them so as to involve certain contour integrals over paths which are in the complex plane (thanks to the Wick rotation explained before). Residue theory then makes it possible to calculate these. For several quantum field theories the thus calculated path integral yields results which are surprisingly close to the experimentally observed values.

The theory that is important here is inspired by gauge fields on 4-manifolds Y. These have a nice geometric reformulation for the action. A gauge field can be viewed as a connection one-form on a given principal fibre bundle on Y. This is a Lie algebra valued one-form on Y. Its covariant derivative $F = DA$, i.e., the curvature of the fibre bundle is an ordinary 2-form on Y. The action for the gauge fields can be written in this geometric language as $S = \int F \wedge *F$, where $*F$ is the Hodge-star of F, another two-form on Y.

In string theory one replaces the ordinary 4-dimensional time-world Y by some other variety of dimension 10. There are several types of superstring theories: type I which is self-dual, and types IIA and IIB which are each others dual. In this note we mainly consider type IIA theories. For such a

theory a gauge field F, an integral closed form, is replaced by an arbitrary even degree closed form $G = G_0 + G_2 + ...$ with integral periods, a *Ramond-Ramond field* and, in analogy to ordinary gauge theory, the action is

$$S = \int_Y G \wedge *G.$$

Next, one wants to define a partition function of the form

$$(8) \qquad\qquad Z = \int \mathcal{D}C \cdot \exp(\mathrm{i}S),$$

where $\int \mathcal{D}C$ is the path integral over all C with $dC = G$ and where G runs over the even degree closed forms with integral periods. In order to make this more precise, one assumes that $Y = X \times T$ where T is the "time-axis"-which may or may not be compact and represents time (usually $T = \mathbb{R}$, but $T =$ a point is also a possibility) and $X = (X, g)$ is a Riemannian manifold of dimension d which is assumed to be compact or at least one on which the Hodge decomposition theorem holds (§ 2). Recall that $d \oplus d^*$ induces a self-adjoint operator on $\mathsf{Har}(X)^\perp \subset \mathsf{A}(X)$. In the theory of Ramond-Ramond fields one is only interested in even degree forms that are already closed. So one writes down the decomposition $\mathsf{A}(X) = \mathsf{A}^+(X) \oplus \mathsf{A}^-(X)$ into even and odd degree forms and to understand the even exact forms, one looks at the associated Dirac operator:

$$(9) \qquad\qquad \mathsf{D} := \begin{pmatrix} 0 & d\big|_{\mathsf{Har}^-(X)^\perp} \\ d^*\big|_{\mathsf{Har}^+(X)^\perp} & 0 \end{pmatrix},$$

where $\mathsf{Har}^+(X)$, $(\mathsf{Har}^-(X))$ are the even (odd) degree harmonic forms on X. By [B-G-V, §9.6] using ζ-functions there is an exact way to define its regularized determinant $\det \mathsf{D}$ and its square root

$$(10) \qquad\qquad \Delta := \sqrt{\det(\mathsf{D})}$$

is called the *determinant of the non-zero modes.* In view of the form (9) for the operator D, the determinant of the non-zero modes can be viewed as the determinant of the operator d on odd degree forms.

Return now to the integral (8). Fix $\alpha \in H^{2*}(X)_{\mathbb{Z}}$ and consider all possible closed forms G_α representing α. If G_α^0 is the unique harmonic representative in the class α we then can write

$$G_\alpha = G_\alpha^0 + dC, \quad \langle G_\alpha^0, dC \rangle = 0,$$

where the inner product has been introduced in §2.1. It follows that the action for the field G_α is a sum

$$S(G_\alpha) = \underbrace{\langle G_\alpha^0, G_\alpha^0 \rangle}_{S_\alpha^{\mathrm{cl}}} + \underbrace{\langle dC, dC \rangle}_{S^{\mathrm{q}}},$$

where the first term depends on α, the "classical" contribution and the second term, the "quantum contribution" does not. All possible C and α

together describe all the field-configurations; the partition function becomes

$$\int \mathcal{D}C \exp{(\mathrm{i}S)} = \underbrace{\left(\int \mathcal{D}C \exp(\mathrm{i}S^{\mathrm{q}})\right)}_{Z_{\mathrm{quantum}}} \cdot \underbrace{\sum_\alpha \exp(\mathrm{i}S_\alpha^{\mathrm{cl}})}_{Z_{\mathrm{classical}}}$$

The relevant calculations have been carried out in detail in [H-N-S] [7] and they can than be summarized as follows:

- the classical contribution is of the form

$$Z_{\mathrm{classical}} = \text{Anomalous pre-factor} \cdot \Theta(0)$$

  where $\Theta(z)$ is some classical theta function.
- in the total partition function the anomalous pre-factor together with Z_{quantum} gives the factor Δ^{-1} where Δ is the determinant of non-zero modes (10):

$$Z = \frac{1}{\Delta}\Theta(0).$$

Remark. In loc. cit. there is given no clue as to which Θ-function should be used. This is one of the issues which the two articles [Witten, Mo-Wi] address and which we want to discuss below in § 8.

6. The basic construction

6.1. A Linear Algebra Construction.
To motivate the construction, re-call how one defines a hermitian metric on a complex vector space in terms of real geometry.

Definition 6.1. Let V be a real vector space V equipped with a complex structure J. A *hermitian metric* on (V, J) is given by a real bilinear form $h : V \times V \to \mathbb{C}$ such that, writing h in real and imaginary parts as $h = b + \mathrm{i}\omega$ one has

(1) h satisfies $h(Jx, Jy) = h(x, y)$ for all $x, y \in V$;
(2) b is a metric and ω is a symplectic form, i.e., a non-degenerate skew-symmetric real form;
(3) $\omega(x, y) = b(Jx, y)$ for all $x, y \in V$ – or, equivalently $b(x, y) = \omega(x, Jy)$;

The form ω is called the *metric form* and b the *underlying real metric*. Condition 3) states that the metric form is uniquely determined by the underlying real metric and the complex structure.

A symplectic form ω on V which satisfies the following two weaker conditions

(1) $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$;
(2) $\omega(x, Jx) > 0$ if $x \neq 0$

is said to be *tamed by J*.

---

[7]Actually, in [H-N-S] the odd middle degree differential forms on a compact manifold of even dimension $d \equiv 2 \bmod 4$ are studied. However, exactly the same calculations apply in the setting of the article [Mo-Wi].

One can ask whether given a (real) metric b and a non-degenerate skew-symmetric real form ω determine a complex structure so that the two come from a hermitian metric. Note that one should then have that (3) holds. This means that in any case the form $b_{\omega,J}(-,-) = \omega(-,J-)$ should define a metric. Below we show that, replacing (3) by this weaker condition, there does exists a unique complex structure such that $b_{\omega,J} + i\omega$ is hermitian with respect to J. However $b_{\omega,J}$ rarely coincides with the original metric b. If this happens, we speak of a coherent pair (b, ω) (see Definition 6.6). We have the following result.

Proposition 6.2. *Let (V, b, ω) be a finite dimensional $\mathbb{R}$-vector space equipped with a (positive definite) metric b and a non-degenerate $\mathbb{R}$-valued skew- symmetric form ω. There exists a unique complex structure J on V such that*

    (1) *$b(Jx, Jy) = b(x, y)$ for all $x, y \in V$;*
    (2) *$\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$;*
    (3) *the form $b_{\omega,J}$ defined by $b_{\omega,J}(x, y) := \omega(x, Jy)$ is a (positive definite) metric.*

Proof: Define $A \in \mathrm{Gl}(V)$ by

$$(11) \qquad\qquad \omega(x, y) = b(Ax, y).$$

Then A is b-skew adjoint: $A^* = -A$, where $*$ means the b-adjoint. Hence $P = A^*A = AA^* = -A^2$ is self-adjoint and positive definite with respect to b. In particular, V has a b-orthonormal basis of P-eigenvectors so that the matrix of P becomes diagonal with positive entries, say $\lambda_i > 0$, on the diagonal. Replacing these by the positive root $\sqrt{\lambda_i}$ defines the root $Q = P^{\frac{1}{2}}$ of P. Now write

$$(12) \qquad\qquad A = QJ, \quad J := Q^{-1}A.$$

Since $Q^2 = P = -A^2$ and $Q^* = Q$ we have

$$b(Jx, Jy) = b(Q^{-1}Ax, Q^{-1}Ay) = -b(Ax, A^{-1}y) = b(x, y)$$

so that J is b-orthogonal. Since Q is self-adjoint and positive definite, with respect to b this implies that (12) is the unique b-*polar decomposition* of A. Moreover, since $A^*A = AA^*$, one easily deduces that J and Q (and also A and Q) commute. It follows that

$$J^2 = (AQ^{-1})^2 = A^2Q^{-2} = -\,\mathrm{id}_V$$

and hence J is a complex structure. Next, $\omega(Jx, Jy) = b(AJx, Jy) = b(JAx, Jy) = b(Ax, y) = \omega(x, y)$ since J and A also commute.
Finally, $\omega(x, Jy) = \omega(Jx, J^2y) = -\omega(Jx, y) = \omega(y, Jx)$ and hence $b_{\omega,J}$ is symmetric. To show that it is positive definite write $\omega(x, Jy) = b(Ax, Jy) = b(QJx, Jy) = b(JQx, Jy = b(Qx, y) = b(Q^{\frac{1}{2}}x, Q^{\frac{1}{2}}y)$. Then, using a b-orthogonal basis of Q-eigenvectors one sees that $b(Qx, x) > 0$ if $x \neq 0$.
To show uniqueness, note that A is uniquely defined by b and ω and hence so is its polar decomposition. We only have to see that J as characterized

by 1)–3) gives a polar decomposition $A = RJ$. First of all $AJ = JA$ as can easily be seen from 1) and 2). But then $R := AJ^{-1} = J^{-1}A$ is seen to be self-adjoint. From this and 3) it follows that R is positive definite which finishes the proof of uniqueness. $\square$

Corollary 6.3. *Under the hypotheses of 6.2 the form ω is a Riemann-form for (V, J):*

    (1) *$\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$;*
    (2) *the form $\omega(x, Jy) = b_{\omega, J}(x, y)$ is a symmetric $\mathbb{R}$-valued positive definite bilinear form on V for which J is orthogonal.*

It follows that the $\mathbb{C}$-valued form

$$(13) \qquad\qquad h(x, y) := \omega(x, Jy) + \mathrm{i}\omega(x, y)$$

is hermitian (with respect to J) and positive definite. Hence:

Corollary 6.4. *Suppose that $V = \Lambda \otimes \mathbb{R}$ for some lattice $\Lambda \subset V$ of maximal rank and that ω is integer valued. Then the torus V/Λ is an abelian variety with polarization h given by equation (13). This is a principal polarization if ω is unimodular.*

Conversely, if (Λ, ω) is a lattice equipped with a non-degenerate integral skew form such that $V = \Lambda \otimes \mathbb{R}$ admits a complex structure J compatible with ω and such that $\omega(x, Jx) > 0$ for $x \neq 0$, the form ω is a Riemann form for the complex torus V/Λ. Its complex structure comes from the unique complex structure of the lemma with respect to ω and b, where $b(x, y) = b_{\omega, J}(x, y) = \omega(x, Jy)$.

Definition 6.5. The abelian variety just constructed is denoted $J(\Lambda, b, \omega)$. This can equivalently be phrased in terms of Hodge theory: the triple (Λ, b, ω) defines a unique polarized weight one Hodge structure whose jacobian is $J(\Lambda, b, \omega)$.

To stress that $b \neq b_{\omega, J}$ in general, we recall the notion of conformal equivalence and introduce a new notion.

Definition 6.6. 1) Two (indefinite) metrics b, b' on a vector space are *conformally equivalent*, if for some positive constant λ one has $b(x, y) = \lambda b'(x, y)$ for all $x, y \in V$. [8]We say that b is conformal to b', or b' is conformal to b;
2) The pair (b, ω) is called a *coherent* pair if the metric $b_{\omega, J}$ is conformal to b. If b' is conformal to b and (b, ω) is a coherent pair, then also (b', ω) is a coherent pair.
3) More generally, if we say that (b, ω) and (b', ω') are conformally equivalent if for some positive constants λ, μ we have $b' = \lambda b$ and $\omega' = \mu\omega$.

Remarks 6.7. 1. Observe that while conformally equivalent (b, ω) and (b', ω') give the same complex structure, the pair $(b, -\omega)$ gives $-J$.

---

[8]For the usual notion of conformal equivalence of metrics on a manifold in this definition the constant has to be replaced by a function.

2. One can rephrase Proposition 6.2 in terms of symplectic geometry as follows: *A symplectic structure on a finite dimensional euclidean vector space is tamed by a unique complex structure compatible with the metric.*

We also want to record how coherent pairs behave under the obvious group actions:

Lemma 6.8. *The group* $\mathrm{Gl}^+(V)$ *operates on metrics and symplectic forms:*

$$b_\gamma(x,y) := b(\gamma x, \gamma y), \quad \omega_\gamma(x,y) := \omega(\gamma x, \gamma y), \quad \gamma \in \mathrm{Gl}^+(V).$$

If (b,ω) *is coherent, then so is* $(b_\gamma, \omega_\gamma)$. *The pair* (b_γ, ω) *is coherent, precisely if* $\gamma \in \mathrm{Aut}(V, \omega)$.

6.2. **Interpretation in Terms of Homogeneous Spaces.** In what follows one chooses a basis for V to identify V with $\mathbb{R}^{2n}$ and we assume that the symplectic form in this basis is the standard symplectic form $\omega_o(x,y) = {}^{\mathsf{T}}y\mathbf{J}x$ where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}.$$

With b the standard metric on V, the pair (b, ω_o) is a coherent couple with complex structure on $\mathbb{R}^n$ given by the matrix $\mathbf{J}$. The symplectic group $\mathrm{Sp}(n)$ is the group of $2n \times 2n$-matrices T with ${}^{\mathsf{T}}T\mathbf{J}T = \mathbf{J}$.

The set $\mathcal{J}_n^+$ of complex structures J on $\mathbb{R}^{2n}$ which preserve a given orientation forms a homogeneous space under conjugation by elements of $\mathrm{Gl}^+(2n, \mathbb{R})$. This gives an effective group action since every complex structure on $\mathbb{R}^{2n}$ can be written as a conjugate of the standard complex structure structure $\mathbf{J}$. Furthermore, the isotropy group of the latter is just $\mathrm{Gl}(n, \mathbb{C})$ under the identification

(14) $$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \mathrm{i}B \in \mathrm{Gl}(n, \mathbb{C}).$$

Hence

$$\mathrm{Gl}^+(2n, \mathbb{R})/\mathrm{Gl}(n, \mathbb{C}) \xrightarrow{\simeq} \mathcal{J}_n^+, \quad [T] \mapsto T^{-1}\mathbf{J}T.$$

The following is a reformulation of the definition:

Lemma 6.9. *A complex structure* $J \in \mathcal{J}_n^+$ *is symplectic if and only if* $\omega_o(Jx, Jy) = \omega_o(x, y)$ *for all* $x, y \in \mathbb{R}^{2n}$.

While all complex structures are conjugate to the standard one under the action of $\mathrm{Gl}^+(2n, \mathbb{R})$, this is no longer the case under the action of $\mathrm{Sp}(n, \mathbb{R})$. Indeed, with $\mathbf{D}_k = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{-1, \ldots, -1}_{n-k})$, the matrices $\begin{pmatrix} \mathbf{0}_n & -\mathbf{D}_k \\ \mathbf{D}_k & \mathbf{0}_n \end{pmatrix} \in$ $\mathrm{Sp}(n, \mathbb{R})$ are in different orbits under the symplectic group. However, *tamed* almost complex structures are conjugate to the standard one by symplectic matrices. Indeed, pick any basis which at the same time is symplectic and g_J-orthonormal. The last condition precisely means that in this basis J is given by the matrix $\mathbf{J}$. We in fact have:

Lemma 6.10. *The subset $\mathcal{J}_n^{\omega_o} \subset \mathcal{J}_n$ of complex structures tamed by[9] ω_o form a homogeneous space*

$$\mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n) \xrightarrow{\simeq} \mathcal{J}_n^{\omega_o}, \quad [T] \mapsto T^{-1}\mathbf{J}T.$$

Moreover, the matrix

$$^{\mathsf{T}}TT = {}^{\mathsf{T}}J{\circ}\mathbf{J} = b_{\omega,J}$$

is positive definite, i.e., defines a metric. In fact $b_{\omega_o,J}$ is the metric associated to the symplectic form ω_o and complex structure J in accordance with the definition from Proposition 6.2.

Proof: One has

(15) $^{\mathsf{T}}J\mathbf{J}J \;\; = \;\; \mathbf{J}$

(16) $^{\mathsf{T}}J\mathbf{J} \qquad$ is symmetric and positive definite.

Note that the first condition states that $J \in \mathrm{Sp}(n, \mathbb{R})$ and it implies that $^{\mathsf{T}}J\mathbf{J}$ is symmetric. If $T \in \mathrm{Sp}(n)$, then $^{\mathsf{T}}T = -\mathbf{J}T^{-1}\mathbf{J}$ and hence

$$^{\mathsf{T}}TT = -\mathbf{J}T^{-1}\mathbf{J}T = -\mathbf{J}J = {}^{\mathsf{T}}J{\circ}\mathbf{J} = b_{\omega,J}$$

is positive definite, which proves the second assertion.   □

Now note that the collection Met_{2n} of all metrics on $\mathbb{R}^{2n}$ form a homogeneous space under $\mathrm{Gl}^+(2n, \mathbb{R})$ where the group action is by sending b to $^{\mathsf{T}}TbT$ and

$$\mathrm{Gl}^+(2n, \mathbb{R})/\mathrm{SO}(2n) \xrightarrow{\simeq} \mathsf{Met}_{2n}, \quad [T] \mapsto {}^{\mathsf{T}}TT.$$

Corollary 6.11. *The inclusion of the symplectic group in the general linear group induces an equivariant inclusion of homogeneous spaces:*

(17) $\mathcal{J}_n^{\omega_o} = \mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n) \;\; \hookrightarrow \;\; \mathrm{Gl}^+(2n, \mathbb{R})/\mathrm{SO}(2n) = \mathsf{Met}_{2n}$

(18) $J = T^{-1}\mathbf{J}T \;\; \mapsto \;\; {}^{\mathsf{T}}TT = {}^{\mathsf{T}}J{\circ}\mathbf{J} = b_{\omega,J}.$

Any other symplectic form ω is of the form $\omega = {}^{\mathsf{T}}\gamma\omega_o\gamma$ with $\gamma \in \mathrm{Gl}^+(2n, \mathbb{R})$. Then $\mathcal{J}_n^{\omega} = \gamma^{-1}\mathcal{J}_n^{\omega_o}\gamma$ and the latter gets embedded in the space of metrics by sending ω to the metric $b_{\omega,J}$ associated to ω and the complex structure J.

Introduce the map

$$r : \mathsf{Met}_{2n} \to \mathcal{J}_n^{\omega_o}, \quad b \mapsto J_b,$$

where J_b is the unique complex structure compatible with b and the symplectic form ω_o as given by Prop. 6.2. We then have

Lemma 6.12.   (1) *The map r is a retraction for the inclusion* (17).
     (2) *Write $J_b = T^{-1}\mathbf{J}T$ with T symplectic, the fibre $r^{-1}J_b$ consists of $^{\mathsf{T}}T\mathcal{K}_nT$ with*

(19) $\mathcal{K}_n = \{Z \in \mathbb{C}^{n \times n} \mid Z^* = Z, \mathrm{Re}(Z) > 0\}.$

---

[9]recall that this means in addition to compatibilty $\omega_o(Jx, Jy) = \omega_o(x, y)$ for all $x, y \in \mathbb{R}^{2n}$ one demands $\omega_o(x, Jx) > 0$ for $x \neq 0$.

In other words, r exhibits Met_{2n} as a fibre bundle over $\mathcal{J}_n^{\omega_o}$ with fibres translates of the n^2-dimensional real cone $\mathcal{K}_n$ under the natural action of the symplectic group on the space of metrics Met_{2n}.

Proof: (1) Uniqueness of J_b implies that if one starts from a complex structure $J = T^{-1}\mathbf{J}T$ with T symplectic, as above, applying r to $b_{\omega_o,J} = {}^{\mathsf{T}}TT$ gives back the complex structure J. Hence r is indeed a retraction.
(2) The proof of Prop. 6.2 gives an explicit expression for $r(b)$, $b \in \mathsf{Met}_{2n}$ a metric. Indeed, by (11) we have to apply the b-polar decomposition to the matrix $T := b^{-1}\mathbf{J}$ and hence

(20) $$r(b) = \left(-(b^{-1}\mathbf{J})(b^{-1}\mathbf{J})\right)^{-\frac{1}{2}}(b^{-1}\mathbf{J}).$$

The fibre $r^{-1}J_b$ consists of positive definite symmetric matrices B for which $J_B = J_b$. Formula (20) for B implies that $\mathbf{J}BJ_B = J_B\mathbf{J}B$ and so, if $J_B = J_b$ one has

$$[\mathbf{J}B]J_b = J_b[\mathbf{J}B].$$

This means that $\mathbf{J}B$ is complex linear for the complex structure J_b. Conversely, one sees that if the above equation holds, $r(b) = r(B) = J_b$.

Now perform the change of basis which transforms J_b into $\mathbf{J}$, i.e., with T the change of basis matrix, $\mathbf{J}B$ gets transformed into

$$T\mathbf{J}BT^{-1} = \mathbf{J}\tilde{B}, \quad \tilde{B} := {}^{\mathsf{T}}T^{-1}BT^{-1}.$$

The above equality follows since T is symplectic. Hence $\tilde{B}$ is also a symmetric matrix. Writing $\mathbf{J}\tilde{B}$ out in blocks according to the identification (14), one finds that $\tilde{B} \in \mathcal{K}_n$ and so $J_B = J_b$ if and only if $B \in {}^{\mathsf{T}}T\mathcal{K}_nT$. $\square$

6.3. Summary.

Suppose now that V is any real vector space of dimension $2n$ and let $\mathsf{Met}(V)$ be the space of metrics on it and let $(\Lambda^2V^\vee)^0 \subset \Lambda^2V^\vee$ denote the open subset of non-degenerate 2-forms. Recall (Def. 6.6) the notion of coherent pairs on V. Let us denote the set of coherent pairs $\mathsf{CohPair}(V)$. Recalling also Lemma 6.8, the above discussion then leads to

Theorem 6.13. *We have a commutative diagram*

$$
\begin{array}{ccccc}
\mathsf{Met}(V) \times (\Lambda^2V^\vee)^0 & \longleftarrow & \mathsf{Met}(V) \times \{\omega\} & \overset{\cong}{\longrightarrow} & \mathrm{Gl}^+(2n,\mathbb{R})/\mathrm{SO}(2n) \\
\uparrow \downarrow{\scriptstyle\rho} & & \uparrow{\scriptstyle r} & & \uparrow \\
\mathsf{CohPair}(V) & \longleftarrow & \mathrm{Im}(\beta) & \overset{\cong}{\longrightarrow} & \mathrm{Im}(j) \\
& & {\scriptstyle\beta}\uparrow{\scriptstyle\cong} & & {\scriptstyle j}\uparrow{\scriptstyle\cong} \\
& & \mathcal{J}^\omega & \overset{\cong}{\longrightarrow} & \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n).
\end{array}
$$

The map β associates to the complex structure J tamed by ω the metric $b_{\omega,J}$ from Lemma 6.10 while $r(b,\omega)$ is the complex structure of Prop. 6.2 and $\rho = b\circ r$ associates to (b,ω) the coherent pair $(b_{\omega,J},\omega)$ where $J = r(b,\omega)$.

The group $\mathrm{Gl}^+(2n,\mathbb{R})\times\mathrm{Gl}^+(2n,\mathbb{R})$ *operates naturally on* $\mathsf{Met}(V)\times(\Lambda^2 V^\vee)^0$. *The diagonal subgroup permutes middle columns for the various symplectic forms while the subgroup* $\mathrm{Sp}(n,\mathbb{R})\times\mathrm{id}$ *preserves the middle column for* $\omega=\omega_o$, *the standard symplectic form. The map* r *is a retraction for the inclusion map defined by* β. *Its fibres are translates of a real* n^2-*dimensional cone isomorphic to* (19).
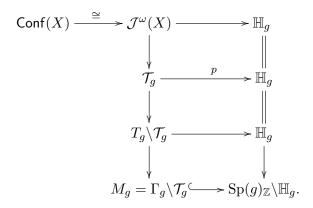
Remarks 6.14. 1. Note that for $n=1$ giving an orientation is the same as giving a positive 2-form up to a positive multiple. Such a form is a symplectic form and so any orientation preserving complex structure is automatically compatible with this symplectic form. The above then says that giving such a complex structure is equivalent to giving a metric up to a scalar, i.e., a class of a conformal metric.

This is related to Teichmüller theory as follows. Let us recall briefly the ingredients. One starts with a compact oriented (real) surface X of genus g. For simplicity, assume $g > 1$. Any almost complex structure on X is known to be integrable. One lets $\mathcal{J}^\omega(X)$ be the set of all complex structures on X compatible with the orientation. This space does not have the structure of a manifold but is some contractible subset in an infinite dimensional vector space. By the above remarks this space is the same as the space $\mathsf{Conf}(X)$ for the conformal equivalence classes of metrics on X. Points of $\mathcal{J}^\omega(X)$ can be seen as equivalence classes of pairs (C, f) consisting of a genus g curve together with a diffeomorphism[10] $f : C \to X$, where $(C, [f])$ and $(C', [f'])$ are equivalent if $f^{-1}{\circ}(f')$ is a biholomorphic map $C' \to C$. The group $\mathrm{Dif}^+(X)$ of orientation preserving diffeomorphisms of X acts on $\mathcal{J}^\omega(X)$. An orbit is an isomorphism class of genus g curves and so the quotient $\mathrm{Dif}^+(X)\backslash\mathcal{J}^\omega(X)$ is the moduli space M_g of genus g curves. The quotient of $\mathcal{J}^\omega(X)$ by the subgroup of those orientation preserving diffeomorphisms that are isotopic to the identity is the *Teichmüller space* $\mathcal{T}_g$. This turns out to be a complex manifold (of dimension $3g-3$), in fact it is biholomorphic to some ball in $\mathbb{C}^{3g-3}$. By definition, a point in $\mathcal{T}_g$ consists of an equivalence class of a pair $(C, [f])$ of a genus g curve C together with an isotopy class $[f]$ of an oriented diffeomorphism $f : C \to X$, a so-called *Teichmüller structure* on C. Equivalence for pairs is defined as above. The isotopy classes of orientation preserving diffeomorphisms $f : X \to X$ form a group, the *Teichmüller group* or *mapping class group*, denoted Γ_g. It acts on the Teichmüller structure by composition (but does not change C) and $M_g = \Gamma_g\backslash\mathcal{T}_g$. Alternatively, by the Dehn-Nielsen theorem, the group Γ_g is an index 2 subgroup inside the quotient group $\mathrm{Aut}(\pi_1(X))/\{$inner automorphisms$\}$. It follows that giving a Teichmüller structure $[f]$ on C is the same as giving the class of the induced group isomorphism $f_* : \pi_1(C) \xrightarrow{\sim} \pi_1(X)$ up to inner automorphisms of the target. If we use the standard presentation of $\pi_1(X)$ we get the standard symplectic intersection form on the resulting basis of the 1–cycles so that the natural homomorphism $\pi_1(X) \to H_1(X)$ induces a homomorphism $\Gamma_g \to$

---

[10]classically, one considers homeomorphisms, but this does not matter.

$\mathrm{Sp}(g)_{\mathbb{Z}}$ whose kernel T_g is called the *Torelli group*. A *marking* for C is an isometry $H^1(C) \xrightarrow{\sim} H^1(X)$. The symplectic group acts on markings by composition. Over $\mathcal{J}^\omega(X)$ one has a tautological family of genus g curves. This family descends to Teichmüller space and since $\mathcal{T}_g$ is contractible, it is differentiably trivial. Hence the local system of 1-cohomology groups can be trivialized on Teichmüller space and by construction, also on its quotient by the Torelli group. Markings thus globalize over these spaces. Of course this holds also over $\mathcal{J}^\omega(X)$.

Next note that $J(C)$, the jacobian of C is just obtained via the construction of Corr. 6.4 using the metric on $H^1(X, \mathbb{R})$ induced by the conformal class of the metric on C and the cup-product pairing on $H^1(X, \mathbb{Z})$. Using the global marking, the assigment $C \mapsto J(C)$ defines the period map $\mathsf{Conf}(X) \to \mathbb{H}_g$ which descends to a holomorphic map $p : \mathcal{T}_g \to \mathbb{H}_g$ and further down to $T_g \backslash \mathcal{T}_g$. Torelli's theorem states that it descends to an injective morphism $M_g \to \mathrm{Sp}(g)_{\mathbb{Z}} \backslash \mathbb{H}_g$. Summarizing, we have

$$
\begin{array}{ccc}
\mathsf{Conf}(X) & \xrightarrow{\cong} \mathcal{J}^\omega(X) \longrightarrow & \mathbb{H}_g \\
\downarrow & & \| \\
& \mathcal{T}_g \xrightarrow{\ p\ } & \mathbb{H}_g \\
\downarrow & & \| \\
& T_g \backslash \mathcal{T}_g \longrightarrow & \mathbb{H}_g \\
\downarrow & & \downarrow \\
M_g = \Gamma_g \backslash \mathcal{T}_g & \hookrightarrow & \mathrm{Sp}(g)_{\mathbb{Z}} \backslash \mathbb{H}_g.
\end{array}
$$

2. The homogeneous space $\mathrm{Sp}(n, \mathbb{R}) / \mathrm{U}(n)$ equals the Siegel upper half space $\mathbb{H}_n$ parametrising marked[11] polarized abelian varieties of dimension n with polarization given by the symplectic form $\mathbf{J}$. The action of the symplectic group is given by

$$
Z \mapsto \langle T \rangle \cdot Z := (A + CZ)^{-1}(B + DZ), \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})
$$

Writing $Z = X + \mathrm{i}Y$, one verifies that

$$
\langle T \rangle \cdot \mathrm{i}\mathbf{1}_n = Z, \qquad T := \begin{pmatrix} Y^{-\frac{1}{2}} & Y^{-\frac{1}{2}}X \\ 0 & Y^{\frac{1}{2}} \end{pmatrix}.
$$

---

[11]See the previous remark.

Moreover, the corresponding complex structure and compatible metric are given by

$$
\begin{aligned}
J &= T^{-1}\mathbf{J}T = \begin{pmatrix} -XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & Y^{-1}X \end{pmatrix} \\
b_{\omega,J} &= {}^{\mathsf{T}}TT = \begin{pmatrix} Y & Y^{-1}X \\ XY^{-1} & XY^{-1}X + Y \end{pmatrix}.
\end{aligned}
$$

7. Examples

7.1. Cohomology of Riemannian Manifolds.

Middle Odd Cohomology. [12]
Let (X, g) be a compact oriented Riemannian manifold of dimension $4q + 2$ and consider the lattice of integral middle cohomology modulo torsion, $H^{2q+1}(X)_{\mathbb{Z}}$ equipped with the cup-product pairing ω^-. On $H^{2q+1}(X)_{\mathbb{R}}$ we put the Hodge metric (Def. 2.2). The abelian variety

$$
J^{q+1}(X, g) := J(H^{2q+1}(X)_{\mathbb{Z}}, b^{(g)}, \omega^-)
$$

is an intrinsic invariant of the pair (X, g). The polarization is a principal polarization. The complex structure is just coming from the $*$ operator on middle cohomology. Note that the pair $(b^{(g)}, \omega^-)$ is coherent in the sense of Def. 6.6.2.

When X is a compact topological surface ($q = 0$) by Remark 6.14 this construction gives back the classical jacobian of the associated Riemann surface and in fact, since the $*$ operator in cohomology is the complex structure on the tangent space of the classical jacobian and the principal polarization comes from the usual cup product, by the classical Torelli theorem, giving $J^1(X, g)$ is completely equivalent to giving the isomorphism class of the associated Riemann surface.

Odd Cohomology. Suppose that (X, g) is an even $2d$-dimensional compact connected oriented Riemannian manifold. On $H^*(X)_{\mathbb{R}}$ we put the Hodge metric $b^{(g)}$ (Def. 2.2) while on $H^-(X)$ we have the intersection pairing ω^-. It is integral and unimodular on $H^-(X)_{\mathbb{Z}}$ and thus one has unique complex structure associated to $H^-(X)_{\mathbb{R}}$. Indeed, this complex structure is given by the Hodge $*$ operator which on odd cohomology indeed is a complex structure. The pair $(b^{(g)}, \omega^-)$ is a coherent pair in the sense of Def. 6.6.2.

Note that ω^- pairs $H^{2j+1}(X)_{\mathbb{Z}}$ perfectly to $H^{2d-2j-1}(X)_{\mathbb{Z}}$. So depending on whether d is odd or even two cases arise both of which are coherent:

(1) $\dim(X) = 4q = 2d$:

$$
J^-(X, g) := J(H^-(X)_{\mathbb{Z}}, b^{(g)}, \omega^-) = J^1 \times J^2 \cdots \times J^q,
$$

where $J^k = J^k(X, g) = J(H^{2k-1}(X)_{\mathbb{Z}} \oplus H^{4q-2k+1}(X)_{\mathbb{Z}}, b^{(g)}, \omega^-)$.

---

[12]This example has been known a long time under the name of *Lazzeri's jacobian* and has been studied in detail by Elena Rubei [Ru].

(2) $\dim(X) = 4q + 2$:

$$J^-(X, g) = J(H^-(X)_{\mathbb{Z}}, b^{(g)}, \omega^-) = J^1 \times J^2 \cdots \times J^{q+1},$$

where all but the last factors are as before and the last factor is the invariant $J^{q+1}(X, g)$ we previously considered. [13]

Even Cohomology. Next, consider the even cohomology. Here we assume that $\dim X = 4q + 2$. Again we take the Hodge metric on $H^+(X)_{\mathbb{R}}$ but now we take the symplectic form ω^+.

The resulting principally polarized abelian variety is denoted $J^+(X, g)$. It is a product whose factors are principally polarized abelian varieties associated to two summands of the form $H^{2k}(X) \oplus H^{4q-2k+2}(X)$. The complex structure on such a factor comes from $\begin{pmatrix} 0 & -* \\ * & 0 \end{pmatrix}$ since $*^2 = \mathrm{id}$ on even cohomology. Again this gives a coherent example (Def. 6.6.2).

Twisted versions. We now consider a twisted version of the above. Given a vector space isomorphism γ of $H^*(X)$ preserving even and odd cohomology, put:

(21) $H^\pm(X) \times H^\pm(X) \ni (x, y) \mapsto \omega_\gamma^\pm(x, y) := \omega^\pm(\gamma(x), \gamma(y)).$

This new pairing is again symplectic, but it need no longer be integral on $H^\pm(X)_{\mathbb{Z}}$ but one may choose $N = N(\gamma)$ to be a minimal integer such that $N\omega_\gamma^\pm$ becomes integral. This gives a canonical integral twist.

We can likewise use an $\mathbb{R}$-vector space isomorphism τ of $H^*(X)_{\mathbb{R}}$ to modify the Hodge metric: we define

(22) $b_\tau^{(g)}(x, y) := \int_X \tau(x) \wedge *\tau(y)$

Of course now in general the pairs $(b_\tau^{(g)}, \omega_\gamma^\pm)$ need no longer be coherent, but Lemma 6.8 states:

Lemma 7.1. *The pair $(b_\tau^{(g)}, \omega_\gamma^\pm)$ is coherent if and only if $\gamma^{-1}\tau$ is a positive multiple of an $\omega^\pm$-symplectic map.*

So we have:

Proposition 7.2. *Let (X, g) be a compact oriented Riemannian manifold of dimension $2 \bmod 4$ and let γ be a $\mathbb{Q}$-vector space isomorphism of $H^*(X)$ preserving even and odd degree, and τ any $\mathbb{R}$-isomorphism of $H^*(X)_{\mathbb{R}}$. Using the notation of Definition 6.5, the above construction yields an abelian variety*

$$J(H^\pm(X)_{\mathbb{Z}}, b_\tau^{(g)}, N(\gamma)\omega_\gamma^\pm),$$

---

[13]Note that except for the middle dimension, the odd cohomology groups are not in general even dimensional, hence the need to combine H^{2k-1} with its "dual" $H^{4q-2k+3}$. This is in contrast with odd cohomology for Kähler manifolds. See Remarks 7.14 (2).

canonically associated to (X, g, γ, τ).

The pair $(b_\tau^{(g)}, \omega_\gamma^\pm)$ *is coherent if and only if* $\tau\gamma^{-1}$ *is a positive multiple of an* $\omega^\pm$-*symplectic map.*

Moduli Interpretation. For simplicity we only consider manifolds whose dimension is 2 mod 4. One can give a similar statement for any even-dimensional manifold when one restricts only to odd cohomology.

Theorem 7.3. *Let* X *be a fixed smooth compact oriented manifold of dimension* $2d = 4q+2$. *Let* $\mathsf{Conf}(X)$ *be the space of classes of conformal metrics on* X *and* $\mathsf{Conf}(H^*(X)_\mathbb{R} \times \Lambda^2[H(X)^\vee]^0)$ *the set of conformal equivalence classes of pairs (metrics, symplectic forms) on* $H^*(X)_\mathbb{R} \times H^*(X)$. *The space of symplectic forms on* $H^*(X)$ *is denoted* $\Lambda^2[H(X)^\vee]^0$. *Let* G *be the group of* $\mathbb{R}$-*vector space isomorphisms of* $H^*(X)_\mathbb{R}$ *which preserve even and odd degree classes,* $G_\mathbb{Q}$ *the rationally defined isomorphisms and let* G_ω *be the subgroup of isometries preserving* $\omega = \omega^+ + \omega^-$. *Finally, let* $g_1 := \sum_{j=1}^q b_{2j-1} + \frac{1}{2}b_{2q+1}$, $g_2 := \frac{1}{2} \dim H^+(X)$.

   The construction of Prop. 6.2 applied to $(b^{(g)}, \omega)$ *defines* r *in the following commutative diagram*

$$
\begin{array}{ccc}
\mathsf{Conf}(X) \times \{\omega\} & \xrightarrow{\ b^{(*)} \times \mathrm{id}\ } & \mathsf{Conf}(H^*(X)_\mathbb{R} \times \Lambda^2[H(X)^\vee]^0) \\
& \searrow{\scriptstyle p} & \downarrow{\scriptstyle r} \\
& & \mathbb{H}_{g_1} \times \mathbb{H}_{g_2}.
\end{array}
$$

The map p *factors over the inclusion*

$$
\underbrace{\prod_{j=1}^q \mathbb{H}_{b_{2j-1}} \times \mathbb{H}_{\frac{1}{2}b_{2q+1}}}_{\text{odd rank}} \times \underbrace{\prod_{j=0}^q \mathbb{H}_{b_{2j}}}_{\text{even rank}} \hookrightarrow \mathbb{H}_{g_1} \times \mathbb{H}_{g_2}.
$$

The group $G \times G_\mathbb{Q}$ *acts naturally on the source of the map* r, *the subgroups* $\{(\lambda g, g) \mid \lambda \in \mathbb{R}^+, g \in G_\mathbb{Q}\}$ *and* $\mathbb{R}^+ \cdot G_\omega \times \{1\}$ *maps* $(b^{(g)}, \omega)$ *to another coherent pair.*

Remarks 7.4. 1) The left hand space is a Teichmüller space for X and p should be viewed as a substitute period map in the setting of compact smooth manifolds of dimension 2 mod 4.
2) If, moreover X is a symplectic manifold with symplectic form ω, the space $\mathcal{J}^\omega(X)$ of almost complex structures on X tamed by ω embeds in $\mathsf{Conf}(X)$ by the map $\beta(J) = g_{J,\omega}$ and this subspace is a better substitute for the moduli space in this case. The period map restricts to it and in fact, we shall make use of this remark in the Kähler setting. So, instead of the above

diagram we should use

$$\mathcal{J}^{\omega}(X) \times \{\omega\} \xrightarrow{\beta \times 1} \mathsf{Conf}(H(X)) \times \Lambda^2[H(X)^{\vee}]^0)$$

$$p \searrow \qquad \downarrow r$$

$$\mathbb{H}_{g_1} \times \mathbb{H}_{g_2}.$$

7.2. K-Groups of Compact Smooth Manifolds.

We can give a variant of the examples in § 7.1 using $K(X)$. Recall, (1) that the Chern character:

$$\mathrm{ch} : K(X) \to H^+(X)$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. So

$$\Lambda(X) = \mathrm{ch}(K(X)) \subset H^+(X)$$

is a *lattice*, i.e., a $\mathbb{Z}$-module of rank $\dim_{\mathbb{Q}} H^+(X)$. The intersection pairing ω^+ as well as the twisted pairing (21) induces $\mathbb{Q}$-bilinear pairings on Λ, i.e.,

$$(23) \qquad \omega^+(\mathrm{ch}(\xi), \mathrm{ch}(\eta)) = \int_X \mathrm{ch}(\xi \otimes \bar{\eta}).$$

In the framework of K-groups we twist by vector space isomorphisms γ defined as multiplication by a unit $\mathbf{b}$ in the ring $H^{4*}(X)$. Recall that ι is the involution on H^{2*} which on H^{4*} is the identity and on H^{4*+2} minus the identity. So $\mathbf{b}$ is invariant under ι and so multiplication with it commutes with ι. Hence the twisted pairing becomes

$$\omega_{\mathbf{a}}^+(\mathrm{ch}(\xi), \mathrm{ch}(\eta)) = \int_X \mathbf{a} \wedge \mathrm{ch}(\xi \otimes \bar{\eta}), \quad \mathbf{a} = \mathbf{b}^2.$$

It can now happen that for specific $\mathbf{a}$ such a twisted pairing becomes an integral pairing:

Definition 7.5. A *multiplier* is an element $\mathbf{a} \in H^{4*}(X)$ such that the pairing

$$\omega_{\mathbf{a}}^+ : \Lambda(X) \times \Lambda(X) \to \mathbb{Q}$$

is *integral*. If $\mathbf{a}_0 = 1$, such a multiplier is called *normalized*.

Now an interesting phenomenon occurs which is based upon Poincaré duality:

Lemma 7.6 ([AH3, 3.7])**.** *Suppose that X is* torsion free[14]*. Normalized multipliers always exist. Moreover, if $\mathbf{a} \in H^{4*}(X)$ is a normalized multiplier, the pairing $\omega_{\mathbf{a}}^+$ is unimodular on $\Lambda(X)$.*

Corollary 7.7. *Let (X, g) be a torsion free compact oriented Riemannian manifold with $\dim X \equiv 2 \bmod 4$ and let $\mathbf{a} \in H^{4*}(X; \mathbb{Q})$ be a normalized multiplier and let $\mathbf{a} = \lambda \mathbf{b}^2 \in H^*(X, \mathbb{R})$ for some $\lambda > 0$. Then (using the notation of Definition 6.5) the pair $(b_{\mathbf{b}}^{(g)}, \omega_{\mathbf{a}}^+)$ is coherent and*

$$J(\Lambda(X), b_{\mathbf{b}}^{(g)}, \omega_{\mathbf{a}}^+)$$

---

[14]This means that the integral cohomology $H^*(X; \mathbb{Z})$ has no torsion.

is a principally polarized abelian variety.

Examples 7.8. (1) If X is torsion free and spin, there is a *canonical* choice for a principally polarized abelian variety associated to the spin structure. Indeed, by the Index Theorem of Atiyah and Singer (see 3.4), in view of (23) we have:

$$\omega^+_{\hat{A}(X)} : \Lambda(X) \times \Lambda(X) \to \mathbb{Z}, \quad (\xi, \eta) \mapsto \int_X \hat{A}(X) \mathrm{ch}(\xi \otimes \bar{\eta}) = \mathrm{index}(\not{D}_{\xi \otimes \bar{\eta}}),$$

and the result follows since $\hat{A}(X)$ is normalized: it starts off with $1 \in H^0(X)$. In order to get a coherent pair (Hodge metric, twisted cup-pairing), by Cor. 7.7, we may take for **b** any multiple of $\sqrt{\hat{A}}$, such as [15]

$$\mathbf{b} = 2\pi \sqrt{\hat{A}}$$

which is the Witten-Moore choice from [Mo-Wi].

Note that one can take $\mathbf{a} = 1$ if $c_1(X) = 0$, for instance if X is a complex torus, or, more generally, any Calabi-Yau manifold.

(2) If X is a complex manifold with $c_1(X) = 0$ there is a priori *another* canonical choice for a principally polarized abelian variety. One takes $\mathbf{a} = \mathrm{td}(X)$ and for **b** one takes any multiple of $\sqrt{\mathbf{a}}$. The Todd genus is known to take values in $H^{4*}(X)$ if and only if $c_1(X) = 0$. Much more is true: by the calculations in [Hir, 1.7] and especially formula (12) in loc. cit. the Todd and $\hat{A}$–genus coincide in this case. In particular this *does not give a new example!*

These examples can also be considered with moduli, e.g. varying metrics in examples (1) and (2) and varying the complex structure in example (3). Let us formulate the final result in a setting which is common to all examples:

Theorem 7.9. *Let (X, ω) be a torsion free compact symplectic oriented manifold of dimension $4q + 2$. Let $\Lambda(X) = \mathrm{ch}(K(X)) \subset H^+(X)$. The set of normalized multipliers is denoted $H^{4*}(X)^\times_{\mathrm{norm}}$ and $g := \mathrm{rank}\, \Lambda(X)$. Recall the embedding $\mathcal{J}^\omega(X) \hookrightarrow \mathsf{Conf}(X)$ (cf. also Remark 7.4 2). Using it, we have a commutative diagram:*

$$
\begin{array}{ccc}
\mathcal{J}^\omega(X) \times H^{4*}(X)^\times_{\mathrm{norm}} & \xrightarrow{\ \beta\ } & \mathsf{Conf}(\Lambda(X)_\mathbb{R}) \times \Lambda^2[\Lambda(X)^\vee]^0 \\
 & \searrow{\scriptstyle p} & \downarrow{\scriptstyle r} \\
 & & \mathbb{H}_g.
\end{array}
$$

Here $\beta(J, \mathbf{a}) = (b^{g_J, \omega}, \omega^+_{\mathbf{a}})$.

7.3. Hodge Structures.

---

[15] The square root is unique and belongs to $H^{4*}(X)$ since $\hat{A}(X)$ starts with 1.

Odd weight. In this situation the underlying rational vector space has even dimension, say $\dim W = 2g$. Above we saw that the Weil operator C_W is a complex structure and so it then defines a weight 1 Hodge structure, say V. If W is polarized by Q it is clear that V is polarized by Q (being a polarization only depends on the Hodge structure through the Weil operator). If (W, Q) is an integral polarized Hodge structure, the corresponding polarized abelian variety is denoted $J(W, Q)$. It is the so-called *Weil jacobian* [Weil1].

Remark 7.10. The complex structures $\pm C_W$ on W are characterized by the $\pm i$-eigenspaces being the direct sums of the Hodge $W^{p,q}$-spaces with $p \equiv k$ (mod 2) and $p \equiv k+1$ (mod 2) respectively, i.e., one places the Hodge spaces alternatingly in the two different eigen-spaces. There is an obvious different choice for the complex structures $\pm\tilde{C}_W$ in which the two $\pm i$- eigenspaces are given by the sum of the $W^{p,q}$ with $p \geq k$ and $p < k$ respectively: the first (second) half of Hodge spaces form the first (second) eigenspace. This last choice gives the Griffiths intermediate jacobian $\tilde{J}(W, Q)$. For this choice the compatible metric (replacing C_W by $\tilde{C}_W$) is no longer positive definite but in general indefinite. It no longer gives a Riemann form on the torus, so $\tilde{J}(W, Q)$ need not be an abelian variety.

In terms of homogeneous spaces, the Griffiths domain for the type W-structures is $\mathrm{Sp}(g)/H$, where $g = h^{2\ell-1,0} + \cdots + h^{\ell,\ell-1}$ and

$$H = \mathrm{U}(h^{2\ell-1,0}) \times \cdots \times \mathrm{U}(h^{\ell,\ell-1}), \qquad k = 2\ell - 1$$

while the one for the V-type structures is $\mathrm{Sp}(g)/\mathrm{U}(g)$. The assocation $W \mapsto V$ is induced by the natural map

$$(24) \qquad\qquad \bar{\psi} : \mathrm{Sp}(g)/H \to \mathrm{Sp}(g)/\mathrm{U}(g) \simeq \mathfrak{h}_g$$

and is well known to be in general neither holomorphic nor anti-holomorphic [Gr, 3.21]. This is easily illustrated in weight 3 as follows. The Griffiths domain parametrizes Hodge flags $F^3 \subset F^2$ inside $W_{\mathbb{C}}$ satisfying the two Riemann conditions. The subspace $F^3 + F^1 \cap \overline{F}^2$ then is a $g = (\frac{1}{2}\dim W)$-dimensional isotropic subspace of $W_{\mathbb{C}} = V_{\mathbb{C}}$, i.e., satisfies the first Riemann condition (Lemma 4.1) and it satisfies also the second Riemann condition and we have $\bar{\psi}(F^3, F^2) = F^3 + F^1 \cap \overline{F}^2$ which obviously is non-holomorphic (consider the Plücker coordinates).

Even in geometric situations when there is a holomorphic period map $p : M \to \mathrm{Sp}(g)/H$, the composition $\bar{\psi}{\circ}p$ is seldom holomorphic as shown Griffiths' calculation [Gr, Proposition 3.8, and I, 1.3 (b)] where the triple product of the universal family of elliptic curves over the upper half plane is considered in detail. Here $M = \mathfrak{h}_1^3$ and the composed map becomes $\bar{\psi}{\circ}p : \mathfrak{h}_1^3 \to \mathfrak{h}_3$.

In the case of simple polarized weight 3 Hodge structures W with $h^{3,0} = 1$ it is known that if W is CM, i.e., its Mumford–Tate group is a torus, then also the associated Weil Intermediate jacobian has CM [Bor]. This amounts to saying that $\bar{\psi}$ preserves complex multiplication. More generally, $\bar{\psi}$ preserves

additional endomorphisms of the Hodge structure W. In fact, this is obvious from a Tannakian point of view, since the Mumford–Tate group of W resp. V is the Tannaka group of the rigid tensor subcategory generated by W resp. V and its tensor powers. A similar remark applies in the following section.

Even Weights. Here we take $V = W \oplus W^\vee(-k)$. It has a natural complex structure given by the linear map

$$J(x + \hat{Q}y) = \hat{Q}C_W(x) - C_W(y)$$

and so, by the above, defines a weight one Hodge structure on V with $C_V = J$. The polarization Q defines a polarization q as follows:

$$q(x_1 + \hat{Q}y_1, x_2 + \hat{Q}y_2) = -Q(x_1, y_2) + Q(y_1, x_2).$$

By construction q is skew-symmetric and it is a standard verification that q is skew and satisfies the Riemann bilinear conditions. For instance, Lemma 4.2 shows that the first bilinear relation can be tested by showing that q is C_V-orthogonal which is the case since

$$
\begin{aligned}
q(C_V(x + \hat{Q}y), C_V(x' + \hat{Q}y')) &= q(\hat{Q}C_W x - C_W y, \hat{Q}C_W x' - C_W y') \\
&= Q(C_W x, -C_W y') + Q(C_W y, C_W x') \\
&= -Q(x, y') + Q(y, x') \\
&= q(x + \hat{Q}y, x' + \hat{Q}y').
\end{aligned}
$$

If (W, Q) is an integral polarized Hodge structure, (V, q) is integrally polarized and if Q happens to be unimodular, also q is unimodular. The corresponding abelian variety $J(V, q)$ will also be denoted $J(W, Q)$.

To interpret the construction in terms of Griffiths domains we need to introduce the homomorphism

$$
\begin{aligned}
(25) \quad \psi : \mathrm{SO}(W, Q) &\to \mathrm{Sp}(W \oplus W^\vee(-k), q) \\
f &\mapsto \psi(f), \qquad \psi(f)(x + \hat{Q}y) = f(x) + \hat{Q}(f(y)).
\end{aligned}
$$

To see this is well defined we need to verify that $\Phi := \psi(f)$ is indeed symplectic:

$$
\begin{aligned}
q(\Phi(x + \hat{Q}y), \Phi(x' + \hat{Q}y')) &= q(f(x) + \hat{Q}f(y), f(x') + \hat{Q}f(y')) \\
&= -Q(f(x), f(y')) + Q(f(y), f(x')) \\
&= -Q(x, y') + Q(y, x') = q(x + \hat{Q}y, x' + \hat{Q}y').
\end{aligned}
$$

Now write $k = 2\ell$. Let $p = \sum_j h^{2\ell - 2j, 2j}$ and $q = k - p$. Then $\mathrm{SO}(W, Q) \simeq \mathrm{SO}(p, q)$ and

$$H = \mathrm{U}(2h^{2\ell, 0}) \times \cdots \mathrm{U}(2h^{\ell+1, \ell-1}) \times \mathrm{SO}(h^{1,1})$$

Lemma 7.11. *Let* $\mathrm{SO}(W, Q)/H$ *be the Griffiths domain for polarized Hodge structures of type* (W, Q). *Suppose* $\dim W = g$. *Then the map* ψ *from* (25)

induces a diagram

$$\begin{array}{ccc} \mathrm{SO}(W,Q)/H & \xrightarrow{\quad\bar{\psi}\quad} & \mathrm{Sp}(g)/\,\mathrm{U}(g) \\ & \searrow{\scriptstyle\pi} & \nearrow \\ & \mathrm{SO}(W,Q)/K & \end{array}$$

where $K \subset \mathrm{SO}(W,Q)$ is the unique maximal compact subgroup containing H, π the natural map and $\tilde{\psi}$ the induced map.

Proof: Let $F \in \mathrm{SO}(W,Q)/H$ correspond to the given polarized Hodge structure (W,Q). Let

$$h : \mathbf{G}(\mathbb{R}) \to \mathrm{O}(W,Q)$$

be the representation which gives this Hodge structure. Then H is the commutant of the Mumford-Tate group of the given Hodge structure (the Zariski-closure of the image of h). Indeed, $H = \{g \in \mathrm{O}(W,Q) \mid gh(z) = h(z)g$ for all $z \in \mathbf{G}(\mathbb{R})\}$. Now, by construction $\psi \circ C_W = C_V \circ \psi$ and hence ψ sends the commutant of C_W to $\mathrm{U}(g)$, the commutant of C_V. But H, the commutant of the Mumford Tate group of W is contained in the commutant of C_W and so ψ sends H to $\mathrm{U}(g)$.

Finally we have to show that $\bar{\psi}$ factors over the quotient of $\mathrm{SO}(W,Q)$ by the maximal compact subgroup K. To see this consider $\psi(K) + \mathrm{U}(g) \subset \mathrm{Sp}(g)$. It is a compact Lie subgroup containing $\mathrm{U}(g)$ and since the latter is already maximally compact, $\psi(K) \subset \mathrm{U}(g)$ and hence $\bar{\psi}$ factors over the quotient map $\pi : \mathrm{SO}(W,Q)/H \to \mathrm{SO}(W,Q)/K$. $\square$

As in the odd weight case the map $\bar{\psi}$ is in general not holomorphic. We give a proof for weight 2. Abbreviate $F = H^{2,0}$. Then $H^{1,1} = (F + \bar{F})^{\perp}$ (with respect to the polarization). The Griffiths domain in this case is an open subset of the Grassmann variety of dim S-dimensional linear subspaces of $W_{\mathbb{C}}$. Define the two maps

$$\begin{aligned} \varphi_{\pm} : V_{\mathbb{C}} & \to & W_{\mathbb{C}} + W_{\mathbb{C}}^*(-2) = V_{\mathbb{C}} \\ z & \mapsto & z \pm \mathrm{i}\hat{Q}(z). \end{aligned}$$

The images are the $\pm\mathrm{i}$-eigenspaces for C_V and hence the map $\bar{\psi}$ comes from the map

$$F \mapsto \varphi_+(F + \bar{F}) + \varphi_-(F + \bar{F})^{\perp} \subset V_{\mathbb{C}}.$$

Considering Plücker coordinates, one sees that this map is neither holomorphic nor anti-holomorphic as soon as $h^{1,1} \neq 0$.

Remark. In geometric situations the composition of the period map with $\bar{\psi}$ is in general non-constant (even not holomorphic and not anti-holomorphic).

We illustrate this with the following

Example 7.12. Let $E = E_{\tau}$ be the elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and let α, β be the two cycles coming from the two lattice generators $\{1, \tau\}$. Then $E \times E$ has a natural principal polarization inducing one on $W = H^1(E) \otimes H^1(E)$.

The cycles $\alpha \otimes \alpha$, $\alpha \otimes \beta$, $\beta \otimes \alpha$, $\beta \otimes \beta$ define a lattice $W_{\mathbb{Z}} \subset W$ on which the polarization Q is integral (even unimodular). Let $\{(\alpha \otimes \alpha)^*, (\alpha \otimes \beta)^*, (\beta \otimes \alpha)^*, (\beta \otimes \beta)^*\}$ be the dual basis. Then

$$(26) \qquad \alpha \otimes \alpha \overset{\hat{Q}}{\Longleftrightarrow} (\beta \otimes \beta)^*, \qquad \beta \otimes \alpha \overset{\hat{Q}}{\Longleftrightarrow} (\alpha \otimes \beta)^*.$$

If $\omega = dz$ is the normalized 1-form on E with periods 1 and τ, the Hodge structure on W has $h^{2,0} = 1$ with basis $\omega \otimes \omega$ and period matrix

$$(\tau^2, \tau, \tau, 1) \in \mathbb{C}^4 = W_{\mathbb{C}}$$

spanning the line $F \subset W_{\mathbb{C}}$. The weight one Hodge structure on V then is given by calculating the periods of $\omega \otimes \omega$ with respect to a suitable basis for $\varphi_+(F + \bar{F}) + \varphi_-(F + \bar{F})^{\perp}$.

As a basis for V we take

$$\{\alpha \otimes \alpha, \alpha \otimes \beta, \beta \otimes \alpha, \beta \otimes \beta, (\alpha \otimes \alpha)^*, (\alpha \otimes \beta)^*, (\beta \otimes \alpha)^*, (\beta \otimes \beta)^*\}$$

Then $F + \bar{F} \subset W_{\mathbb{C}}$ is given by the matrix

$$M := \begin{pmatrix} \tau^2 & \tau & \tau & 1 \\ \bar{\tau}^2 & \bar{\tau} & \bar{\tau} & 1 \end{pmatrix}.$$

Define an involution ι on (4×2)-matrices: exchange column 1 and 4 as well as column 2 and 3. In view of (26), the subspace $\varphi_+(F) \subset V_{\mathbb{C}}$ is given by the matrix $(M, \mathrm{i}\iota(M))$.

The matrix for Q in the given basis then is the (4×4) anti-diagonal matrix with 1 on the antidiagonal and hence $(F + \bar{F})^{\perp}$ is given by calculating a basis for $\ker M$ and then applying ι. We find

$$N := \begin{pmatrix} |\tau|^2 & -(\tau + \bar{\tau}) & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Then $\varphi_+(F + \bar{F}) + \varphi_-(F + \bar{F})^{\perp}$ is given by the block matrix

$$(27) \qquad B := \begin{pmatrix} M & \mathrm{i}\iota(M) \\ N & -\mathrm{i}\iota(N) \end{pmatrix}.$$

One calculates $\det \begin{pmatrix} M \\ N \end{pmatrix} = (\tau - \bar{\tau})(\tau^2 + 6|\tau|^2 + \bar{\tau}^2)$ and similiarly, one finds among the other non-zero Plücker coordinates $-\mathrm{i}(\tau - \bar{\tau})(\tau + \bar{\tau})^2$ (e.g., in the previous determinant, replace the first column by the last column of the block matrix (27)). Since the quotient of these two equals $-\mathrm{i}\dfrac{\tau^2 + 6|\tau|^2 + \bar{\tau}^2}{(\tau + \bar{\tau})^2}$ the period map composed with $\bar{\phi}$ is a *non-constant* map which is neither holomorphic nor anti-holomorphic.

7.4. **Cohomology of Kähler manifolds.** Let (X, ω) be a projective manifold of dimension d with integral Kähler class $[\omega]$. We have seen (Example 4.3) that the primitive cohomology groups $H^k(X)_{\mathrm{prim}}$ as well as the full cohomology groups $H^k(X)$ have a natural weight k Hodge structure polarized by the form Q_{ω} defined by (5). We can then apply the two constructions

of § 7.3. In fact, for primitive k-cohomology these match exactly the constructions we described in the general setting of compact smooth manifolds of dimension 2 mod 4 as given in § 7.1.

7.4.1. *Odd Cohomology.* For odd rank $2k + 1$ we have the Weil jacobian

$$J^{k+1}(X) = J(H^{2k+1}(X)_{\text{prim}}, Q_\omega).$$

In this case (b^{g_ω}, Q_ω) is a coherent pair with the complex structure given by the Weil operator C. This also works on $H^{2k+1}(X)$ except that we now have to multiply Q_ω with a certain integer $N_{k,\omega}$ yielding a polarized abelian variety

$$J(H^{2k+1}(X)_{\mathbb{Z}}, b^{g_\omega}, N_{k,\omega}Q_\omega).$$

This torus is isogenous[16] to a product of tori coming from the primitive pieces, i.e., with $\sim$ denoting isogeny, we have

$$J(H^{2k+1}(X)_{\mathbb{Z}}, g_\omega, N_{k,\omega}Q_\omega) \sim \prod_{\ell=1}^{k+1} J^\ell(X).$$
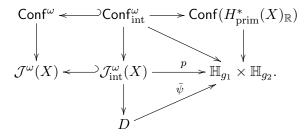
7.4.2. *Even Cohomology.* Here we have to restrict to odd dimensional complex varieties in order that the real dimension $2d$ be 2 mod 4. The even cohomology can now be given as a direct sum $W \oplus W^\vee$ where W is the sum of the first half of the even cohomology groups. Then, using Poincaré duality $W^\vee$ is the sum of the last half of the even cohomology groups. Incorporating the Hodge structure one should pair $H^{2k}(X)$ with $H^{2d-2k}(X)(d-2k)$ and the twisting cup-product pairing should be used for the symplectic form. The construction we have given for abstract Hodge structures in § 7.3 for even weight, then is the same as the third example from § 7.1 (even cohomology).

7.4.3. *Moduli.* Let (X, J, ω) be almost Kähler. This means that ω is a non-degenerate real two form, J an almost complex structure so that $g_J + i\omega$ is a hermitian metric and $d\omega = 0$. We fix ω and let the almost complex structure J vary. The coherent pairs (g_J, ω) up to conformal equivalence form the space Conf^ω. Restricting to *integrable* complex structures we get $\mathsf{Conf}^\omega_{\text{int}}$, corresponding to the true Kähler metrics. The fixed class $[\omega] \in H^2(X)_{\mathbb{R}}$ can be used to define primitive cohomology and hence we have a polarized Hodge structure on $H^*_{\text{prim}}(X)_{\mathbb{R}}$ parametrized by a Griffths period domain D and period map p. We arrive at the following moduli interpretation:

Theorem 7.13. *Let (X, ω) be a compact Kähler manifold of odd complex dimension. Let $g_1 := \frac{1}{2} \dim H^-(X)_{\text{prim}}$ and $g_2 := \dim H^+(X)_{\text{prim}}$. With*

---

[16]Two tori A and B are said to be *isogenous* if there is a surjective group homomorphism $A \twoheadrightarrow B$ with finite kernel. Despite the apparent asymmetry in the definition this does define an equivalence relation.

the map $\bar{\psi}$ the one from (24) (odd cohomology) combined with the one from Lemma 7.11 (even cohomology) we have the following commutative diagram

$$
\begin{array}{ccccc}
\mathsf{Conf}^\omega & \longleftarrow\!\!\!\longleftarrow & \mathsf{Conf}^\omega_{\mathrm{int}} & \longrightarrow & \mathsf{Conf}(H^*_{\mathrm{prim}}(X)_{\mathbb{R}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{J}^\omega(X) & \longleftarrow\!\!\!\longleftarrow & \mathcal{J}^\omega_{\mathrm{int}}(X) & \xrightarrow{\ p\ } & \mathbb{H}_{g_1} \times \mathbb{H}_{g_2}. \\
& & \downarrow & \nearrow_{\bar{\psi}} & \\
& & D &
\end{array}
$$

Moreover, if $\dim_{\mathbb{R}} X = 4q + 2$ and $q_j := \dim H^j(X)_{\mathrm{prim}}$, the map p factors over the inclusion

$$
\prod_{j=1}^{q+1} \mathbb{H}_{\frac{1}{2} q_{2j-1}} \times \prod_{j=0}^{q} \mathbb{H}_{q_{2j}} \hookrightarrow \mathbb{H}_{g_1} \times \mathbb{H}_{g_2}.
$$

In this theorem, the space $\mathcal{J}^\omega_{\mathrm{int}}$ of complex structures should be considered as a sort of Teichmüller moduli space, and the map p is a sort of period map which in this case happens to factor over the Griffiths domain D.

Remarks 7.14. (1) A similar statement holds for all of cohomology but it does not add new information since the new tori are products of the ones gotten from primitive cohomology

(2) Note that although the diagram of Theorem 7.13 is very similar to the diagram in Theorem 7.3 which applies to the cohomology of *any* compact Riemannian manifold of dimension $d = 4q + 2$, there is one crucial difference: the tori occurring in the latter situation need to be constructed by combining $H^{2k+1}(X)$ and its "dual" $H^{4q-2k+1}(X)$ since $\dim H^{2k+1}(X)$ need not be even dimensional. In the Kähler setting this is true however and the tori split in sub-tori coming from the various primitive pieces. To be precise, with $\sim$ denoting isogeny, we have

$$
J(H^{2k+1}(X)_{\mathbb{Z}} \oplus H^{4q-2k+1}(X)_{\mathbb{Z}}, b^{(g_\omega)}, \omega^-) \sim \left[\prod_{\ell=1}^{k+1} J^\ell(X)\right] \times \left[\prod_{\ell=1}^{k+1} J^\ell(X)\right].
$$

8. Special Theta Functions and Ramond-Ramond Fields

8.1. **Some reminders.** Let $J = V/\Lambda$ be an abelian variety with principal polarization given by the unimodular integral and positive $(1,1)$-form ω. In this section let $g = \dim_{\mathbb{C}} V$ be the complex dimension of J. The set of line bundles L on J with $c_1(L) = \omega$ is a principal space under the Picard torus of J which is isomorphic to J. To single out a line bundle having ω as first Chern class one traditionally uses multipliers for ω:

Definition 8.1. A *multiplier for ω* is a function $\alpha : \Lambda \to \mathrm{U}(1)$ for which

$$
(28) \qquad\qquad \alpha(x + y) = (-1)^{\omega(x,y)} \cdot \alpha(x)\alpha(y).
$$

A multiplier is entirely specified by its values on a symplectic basis and any value in $U(1) \simeq S^1$ can be taken so that the above set is a topological torus of dimension $2g$ as should be the case. Indeed, line bundles with given polarization ω are in 1-1 correspondence with multipliers for ω. See for example [Mumford, Chap I.2]. A choice of a symplectic basis $\mathsf{B} := \{e_1, \ldots, e_g, f_1, \ldots, f_g\}$ for Λ with respect to ω singles out a specific line bundle: the one for which $\alpha(b) = 1$ for all $b \in \mathsf{B}$. In particular, the corresponding multiplier takes its values in the subgroup $\{\pm 1\}$ of $U(1)$. There are exactly 2^{2g} such "special" line bundles since one may choose $\alpha(b) \in \{\pm 1\}$ for every individual $b \in \mathsf{B}$ separately ; these correspond classically to *theta functions with characteristics*.

In a more explicit fashion, take a holomorphic basis for V, or, equivalently, a basis for the space of holomorphic 1-forms on J chosen in such a way that the rows in the matrix

$$\Omega = (\mathbf{1}_g, Z), \quad {}^\mathsf{T}Z = Z, \, \mathrm{Im}(Z) > 0$$

are the periods of this basis with respect to B. Choose $\theta \in \Lambda$ such that $\alpha(y) = (-1)^{\omega(\theta,y)}$ for all $y \in \Lambda$ which is possible since we have a symplectic basis. Then, setting

$$\Lambda_1 = \bigoplus \mathbb{Z}e_j \qquad \Lambda_2 := \bigoplus \mathbb{Z}f_j$$
$$\theta = \theta_1 + \theta_2, \qquad \theta_i \in \Lambda_i$$
$$u = \frac{1}{2}\theta_1 \bmod \Lambda_1 \in \frac{1}{2}\Lambda_1/\Lambda_1, \qquad v = \frac{1}{2}\theta_2 \bmod \Lambda_2 \in \frac{1}{2}\Lambda_2/\Lambda_2$$

define

$$\Theta \begin{bmatrix} u \\ v \end{bmatrix}(z) := \sum_{x \in \Lambda_1 + u} \exp[\mathrm{i}\pi\langle x, Zx\rangle] \cdot \exp[2\pi\mathrm{i}\langle x, z + v\rangle].$$

It is the classical theta function with *theta characteristic* (u, v). Here $\langle x, y \rangle = {}^\mathsf{T}x \cdot y$ is the usual euclidean inner product on $\mathbb{C}^g$. It is the non-zero holomorphic section, unique up to a multiplicative constant for the (unique) holomorphic line bundle on J with such a special multiplier α. Note that the classical theta function corresponds to $\alpha = 1$, *but this is not the one suitable for physics, according to* [Mo-Wi, § 3.1], *as we shall see in the next subsection.*

8.2. Ramond-Ramond fields.

Continue with the example 7.8 constructed from a torsion free compact spin manifold (X, g). So on $\Lambda = \Lambda(X) \simeq K(X)$ there is a natural unimodular symplectic form ω given by

$$\omega(x, y) = \omega^+_{\hat{A}(X)}(\mathrm{ch}(x), \mathrm{ch}(y)).$$

Next, one needs to assume that $\dim(X) \equiv 2 \bmod 8$. By Prop. B.2 one then has a homomorphism $j : KO(X) \to \mathbb{Z}/2\mathbb{Z}$. If $x \in K(X)$ is a virtual complex bundle $x \otimes \bar{x}$ is naturally an element of $KO(X)$ and so we get a

homomorphism

$$\alpha : K(X) \to \{\pm 1\}, \quad x \mapsto (-1)^{j(x \otimes \bar{x})}.$$

One can show that it satisfies the required transformation law (28) to make it a multiplier for the form ω. We can now formulate the main result of [Mo-Wi, §3] in mathematical terms:

Proposition 8.2 ([Mo-Wi, § 3.1])**.** *For a torsion free compact spin manifold (X, g) of dimension $2 \mod 8$ consider the principally polarized abelian variety*

$$J(\Lambda(X), b^{(g)}_{2\pi\sqrt{\mathbf{a}}}, \omega^+_{\mathbf{a}}),$$

where $\mathbf{a} = \hat{A}(X)$. The map $\alpha(x) = (-1)^{j(x \otimes \bar{x})}$ is a multiplier for ω and hence defines a unique line bundle with first Chern class ω and multiplier α. Let $\Theta \left[{u \atop v} \right](z)$ be the corresponding normalized theta function. Then the partition function for type II-A Ramond-Ramond fields on X (see § 5) is given by $\Theta \left[{u \atop v} \right](0)/\Delta$ where Δ is the determinant (10) for the non-zero modes on X.

In loc. cit. Moore and Witten argue that only this choice of the theta function gives the correct partition function. In particular, their result shows that the partition function for the Ramond-Ramond fields on, say a 10–dimensional compact space-time with spin structure can be calculated completely from a specific twist of the Riemann theta function which is canonically associated to the spin structure.

Appendix A. Clifford algebras

Let k be a field of characteristic $\neq 2$ and let $V = (V, q)$ be a (finite dimensional) k-inner product space. Its tensor algebra $\mathsf{T}V$ gets a natural k-inner product, also denoted by q. The unit $1 \in k$ serves as a unit in $\mathsf{T}V$. Its *Clifford algebra* is the following quotient algebra of dimension 2^n where $n = \dim V$:

$$\mathsf{C}(V) = \mathsf{C}(V, q) := \mathsf{T}V/\text{ideal generated by } \{x \otimes x + q(x, x) \cdot 1 \mid x \in V\}.$$

There is a natural map $c : V \to \mathsf{C}(V)$, $x \mapsto$ class of x. The induced action $c(x) : \mathsf{C}(V) \to \mathsf{C}(V)$ is called the *Clifford action*. One easily shows that the pair $(\mathsf{C}(V), c)$ satisfies the following property: It is the unique pair (C, c) consisting of a k-algebra C with unit together with a k-linear map $c : V \to C$ such that[17]

$$(29) \qquad\qquad c(x) \cdot c(y) + c(y) \cdot c(x) = -2q(x, y) \cdot 1$$

which is universal with respect to this property.

Proposition-Definition A.1. *A $\mathsf{C}(V)$-Clifford module A is a k-algebra equipped with a* Clifford action *of V, i.e., a linear map $c : V \to A$ satisfying (29). If the Clifford-action is q-skew-adjoint, one says that A is* self-adjoint*:*

$$q(v \cdot x, y) + q(x, v \cdot y) = 0, \quad \text{for all } v \in V, x, y \in A.$$

---

[17]The multiplication in $\mathsf{C}(V)$ is written with dots.

The Clifford algebra is a twisted version of the exterior algebra:

Lemma A.2. $\mathsf{C}(V)$, *as a vector space is isomorphic to* ΛV, *the exterior algebra. The Clifford-action on* ΛV *is given by*

$$c(x)\alpha = x \wedge \alpha - \iota(x)\alpha$$

where the linear map $\iota(x)$ *is the contraction with* x. *This makes* ΛV *into a self-adjoint Clifford module. The bigrading given by odd and even degree in the exterior product descends:*

$$\mathsf{C}^+(V) := c\left(\Lambda^+ V\right), \quad \mathsf{C}^-(V) := c\left(\Lambda^- V\right).$$

Proof: Since $\iota(x)$ is the q-adjoint of the map $\alpha \mapsto x \wedge \alpha$ clearly ΛV is a self-adjoint Clifford module. To see that one gets an isomorphism, note that

(30) $$\sigma : \mathsf{C}(V) \xrightarrow{\sim} \Lambda V, \quad a \mapsto c(a) \cdot \mathbf{1}$$

is a bijective k-linear map whose inverse

$$c : \Lambda V \to \mathsf{C}(V)$$

can be explicitly given as follows: let $\{e_1, \ldots, e_n\}$ be an orthogonal basis for V, then send $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to the element $e_{i_1} \cdots e_{i_k}$. □

From now assume that $\dim V$ is even. Clifford modules turn out to be representations of the *spin group* which in this case can be defined as

$$\mathrm{Spin}(V) := \{x_1 \cdot \cdots \cdot x_{2k} \in \mathsf{C}^+(V) \mid \|x_j\| = 1, j = 1, \ldots, k\}.$$

To describe the basic irreducible Clifford modules extend q bilinearly to $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Then there exist maximal isotropic subspaces $H \subset V_{\mathbb{C}}$ with $\dim_{\mathbb{C}} H = \frac{1}{2} \dim_{\mathbb{R}} V$. If V is oriented with orthonormal oriented basis $\{e_1, \ldots, e_{2m}\}$ we can take for H the subspace spanned by $e_{2k-1} + \mathrm{i}e_{2k}$, $k = 1, \ldots, m$. Such H is an *oriented* maximal isotropic subspace. Next, introduce the *spinor spaces*

(31) $$\mathsf{S} = \mathsf{S}(V) := \Lambda H, \quad \mathsf{S}^+(V) = \Lambda^+ H, \quad \mathsf{S}^-(V) = \Lambda^- H.$$

The first, S, is clearly a complex Clifford module through the usual Clifford action given by Lemma A.2. It turns out to be an irreducible complex spinor representation. On the other hand, the two spinor spaces $\mathsf{S}^{\pm}(V)$ can be shown to be irreducible as *real* representations of the spinor group. They are called the *half spinor representations*.

The metric on the spinor space coming from the metric on H induced by the hermitian form $(x, y) \mapsto q(x, \bar{y})$ makes $\mathsf{S}^+(V)$ and $\mathsf{S}^-(V)$ orthogonal to each other and $\mathsf{S}(V)$ is a self-adjoint Clifford-module. One has:

Proposition A.3. *Every complex* $\mathsf{C}(V)$-*module* E *is of the form* $\mathsf{S}(V) \otimes W$ *where the* twisting space $W = \mathrm{Hom}_{\mathsf{C}(V)}(\mathsf{S}(V), E)$ *is a complex vector space with trivial* $\mathsf{C}(V)$-*action.*

The spinor space is a $\mathbb{Z}_2$-graded complex $\mathsf{C}(V)$-module. This is also the case for general Clifford modules, but here one has to consider how the *chirality operator*

$$\gamma := \mathsf{i}^m e_1 \cdots e_{2m} \in \mathsf{C}(V).$$

acts:

$$E^\pm := \{e \in E \mid \gamma \cdot v = \pm v\}.$$

This is compatible with action of γ on $\mathsf{S}(V)$ since it turns out that $\gamma = \pm \mathbf{1}$ on $\mathsf{S}^\pm(V)$. In particular, one has a $\mathbb{Z}_2$-graded action of $\mathsf{C}(V)$ on E.

Appendix B. K-theory of Real Vector Bundles

A reference for this appendix is [At], also contained as an appendix in [At1].

The Grothendieck group of real vector bundles on a manifold X is denoted by $\mathrm{KO}(X)$. As in the complex case one sets $\mathrm{KO}^{-n}(X) = \mathrm{KO}(S^n X)$ and now there is periodicity of order 8.

There is still another K-group defined for pairs (X, ι) where X is a manifold and ι is an involution. One defines $\mathrm{KR}(X)$ as the K-group for complex bundles E on X admitting involutions covering ι and which are $\mathbb{C}$ anti-linear on the fibres. Again, there is a Bott-periodicity result, namely $\mathrm{KR}^*(X) \simeq \mathrm{KR}^{*+8}(X)$. The standard example is the total space of a *real* vector bundle E with involution ι given by id on the fibres. This gives back $KO(X)$. Another example is the Thom space (BV, SV) of a Riemannian vector bundle V. Here the involution is the antipodal map. This space figures in a very general form of the Thom isomorphism theorem which can be deduced from [At2, Theorem 6.2]. We explain the latter theorem in a simplified situation. Let X be a compact differentiable manifold, G a compact Lie-group acting trivially on X and suppose we have a group homomorphism $\rho : G \to \mathrm{Spin}^c(8r)$. Moreover let V be a vector bundle on X of rank $8r$ with spin^c-structure. Put a G-module structure on V through ρ. Then there is a natural isomorphism

$$\varphi : \mathrm{KR}(X) \to \mathrm{KR}((BV, SV) \times_G X).$$

Specialize this to the case where X is a spin manifold of dimension $(8r - m)$ so that TX gets a spin^c-structure, let $G = \mathrm{Spin}^c(8r - m)$ and let $\rho : \mathrm{Spin}^c(8r - m) \hookrightarrow \mathrm{Spin}^c(8r)$ the embedding. Put $V = TX \oplus \mathbb{R}^m$. Then $(BV, SV) \times_G X = (BX, SX) \oplus (B^m, S^m)$ where the involution on the second summand is not the identity *but the antipodal map*. Applying the periodicity [At, Theorem 2.3] we deduce:

Theorem B.1 (Thom isomorphism theorem)**.** *Suppose X is a spin manifold of dimension $(8r - m)$. There is a natural isomorphism*

$$\varphi : \mathrm{KO}(X) = \mathrm{KR}(X) \xrightarrow{\sim} \mathrm{KR}^m(B(X), S(X)).$$

Now it is time to pass to index theory. It can be shown that the symbol of a real elliptic operator belongs to $\mathrm{KR}(BX, SX)$ where one complexifies

the operator first; the involution covering ι comes then from complex conjugation. So, by construction, there is a forgetful map $\mathrm{KR}(BX, SX) \to \mathrm{K}(X)$ and the index theorem for complex bundles can be applied, but this gives nothing extra. However, for families $X \to T$ the situation becomes different. The (analytic) index can be extended to a homomorphism

$$\mathrm{ind} : \mathrm{KR}(B(X/T), S(X/T)) \to \mathrm{KO}(T)$$

covering the complex index map. But since the covering maps are in general not injective one gets extra information from the Index theorem for families of real elliptic operators [AS4]. It states that (complexified) analytic index equals an explicit expression in terms of Chern classes and which can be called the topological index, ind_τ.

In the special case of a product family $X \times S^m \to S^m$ with X a spin manifold of dimension $8r + m$, these two maps together with the above Thom isomorphism theorem induce a commutative diagram[18]

$$
\begin{array}{ccc}
\mathrm{KR}((BX, SX) \times S^m) & \to & \mathrm{KO}(S^m) \\
\downarrow & & \downarrow \\
\mathrm{KR}^{-m}(BX, SX) & \to & \mathrm{KO}^{-m}(\text{point}) \\
\| & & \| \\
\mathrm{KO}(X) & \to & \mathrm{KO}^{-m}(\text{point}).
\end{array}
$$

In particular, if $d = 8r + 2$ and $m = 2$ Bott periodicity gives two maps

$$\mathrm{ind}, \mathrm{ind}_\tau : \mathrm{KO}(X) \to \mathrm{KO}^{-2}(\text{point}) = \mathbb{Z}/2\mathbb{Z}.$$

These are equal and called the *mod-2-index* for a family over S^2. This can in particular be applied to real bundles E; the Dirac operator $\slashed{D}_E$ on X with values in E has index 0 (see Cor. 3.6) and a priori one does not expect information. But from such E one can canonically construct a family of Diracs depending on a complex parameter which then extends to the Riemann sphere S^2; hence, the above considerations with $m = 2$ apply. It turns out (see [AS4] for details) that the analytic index is the mod-2 dimension of the bundle $\ker(\slashed{D}_E)$ and the topological index comes from the Gysin map associated to $X \to$ point. Recall at this point that for any map $f : X \to Y$ between compact *spin* manifolds, there are Gysin maps

$$f_! : \mathrm{KO}^*(X) \to \mathrm{KO}^{*-c}(Y), \quad c = \dim X - \dim Y.$$

The upshot is

Theorem B.2. *Let X be a compact spin manifold of dimension* 2 mod 8. *Let $a_X : X \to \{\text{point}\}$ be the constant map. For $x \in \mathrm{KO}(X)$, let $j(x)$ be the mod–2 index of the Dirac operator with values in x. Then there we have an equality of maps*

$$j = (a_X)_! : \mathrm{KO}(X) \to \mathrm{KO}^{-2}(\text{point}) = \mathbb{Z}/2.$$

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[18]Observe the change of sign in front of m.

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