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GEOGRAPHY OF SPIN SURFACES

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Dedicated to the memory of Boris G. Moishezon 1938–1993

1. INTRODUCTION

The chern-invariants c_1^2, χ of minimal compact complex surfaces of general type satisfy certain well-known inequalities. They are both strictly positive and furthermore

$$2\chi - 6 \leqslant c_1^2 \leqslant 9\chi$$

due to Noether's inequality (the left-most inequality) and the Bogomolov–Miyaoka–Yau inequality (the right-most inequality). See [1] for details.

It is generally believed that those are the only restrictions, i.e. "all invariants can occur". Evidence for this conjecture was supplied by [12] where it was essentially shown that all invariants with negative index (i.e. $c_1^2 \leqslant 8\chi$) can occur, and work by Chen [4, 5] essentially did the same for the harder case of positive index. In fact a few gaps still remain, but those are mainly due to technical reasons and do not reflect anything intrinsic. The combined result reads as follows:

Theorem. *For any pair of positive integers (x, y) with $y \geqslant 2(x-3)$, $y \leqslant 9x$ not on one of the lines $y = 9x - k$ with $k \leqslant 121$, there exists a minimal surface S of general type with $x = \chi(S)$ and $y = c_1^2(S)$.*

One may sharpen the question and impose restrictions on the surfaces, the most natural being simply-connectedness. Initial work for negative index was done in [12] where the following result was established:

Theorem. *For any pair of positive integers (x, y) with $2x - 6 \leqslant y \leqslant 8(x - Cx^{2/3})$ there exists a simply-connected minimal surface of general type with $x = \chi(S)$ and $y = c_1^2(S)$. And the numerical coefficient C can be taken as $9/\sqrt[3]{12}$.*

One knows, due to the work of Yau [17], that surfaces with $c_1^2 = 9\chi$ are never simply connected, but it is expected (see e.g. [8]) that there are simply-connected surfaces with c_1^2/χ arbitrarily close to 9. Inspired by a construction of G. Xiao, Chen found examples with slopes up to 8.757; see [4] where one finds the following result:

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THEOREM. *For any positive integers* (x, y) *with*

$$352/89x + C_1x^{2/3} < y < 18644/2129x - C_2x^{2/3}$$

$$x > C_3$$

where $C_1 = 140.2$, $C_2 = 365.7$ *and* C_3 *is not specified there exists a simply-connected minimal surface* S *of general type with* $x = \chi(S)$ *and* $y = c_1^2(S)$. (*Note also that* $352/89 = 3.995$ *and* $18644/2129 = 8.757$.)

Simply-connected surfaces form a specified class of simply-connected oriented smoothable compact four-manifolds and Freedman's result [6] implies that the oriented homeomorphism type is completely determined by the intersection form. For surfaces of general type, the intersection form is indefinite and thus completely characterized by the rank, signature and parity [14]. It follows that the oriented homeomorphism type is entirely determined by the pair (c_1^2, χ) and the reduction of the canonical class modulo two. Indeed, the signature of the intersection form is given by $1/3(c_1^2 - 2c_2)$, while c_2 follows from the Noether formula $\chi = 1/12(c_1^2 + c_2)$. The parity is determined by w_2, the second Stiefel–Whitney class, since Wu's formula, valid in mod 2 cohomology characterizes w_2 as the unique class for which $\langle x, x \rangle = \langle x, w_2 \rangle$ (see e.g. [7]). This indeed implies that the intersection form on integral level is even if and only if the canonical class is 2-divisible since w_2 is the modulo two restriction of c_1, or, equivalently, of the canonical class. So, given the basic pair (c_1^2, χ), in the simply-connected case at most two oriented homeomorphism types occur, one with the canonical class 2-divisible and one for which this is not the case. If the former occurs, we speak of a simply-connected *spin* surface. In fact, being spin is equivalent to the vanishing of w_2, which in turn is equivalent to the existence of a spin-structure [3, Section 26]. The moral of the above discussion is that the *a priori* complex invariants (c_1^2, parity of the canonical class) are indeed topological, and can be defined backwards via the above theorems for any 4-manifold (meaning compact, smoothable in the sequel). E.g. via the Hirzebruch signature formula one can define $c_1^2 := 2c_2 + 3\tau$.

Thus the above results of Chen and Persson can be interpreted as addressing the question whether a given compact, smoothable four-manifold has a complex structure. The inequalities of Noether, Bogomolov, Miayoka and Yau can then be thought of as obstructions for putting a complex structure. But the above authors do not explicitly determine the parity of their surfaces (i.e. whether they are spin or not), which is needed to give specific existence results. However, all of the constructed surfaces are double covers of certain surfaces whose branch-locus possibly contains triple points with one tangent. In Chen's constructions these must be present, in Persson's constructions most surfaces are forced to have these. It can be easily seen that such a singularity gives rise to an elliptic curve on the double cover with self-intersection -1. Thus one of their surfaces are spin, and their combined results suggest that apart from the standard inequalities (with the BMY inequality being strict and with the possible exception of very high slopes) there are no obstructions to putting a complex structure on a simply-connected non-spin surface (odd case). However, for spin manifolds there are additional obstructions (see Theorem A). In fact for low values of c_1^2/χ the constructions in [12] are canonical and the presence of curves with negative self-intersections is inevitable.

Since the intersection form being even imposes extra conditions, spin surfaces will only occur for pairs (c_1^2, χ) satisfying these conditions. One of these extra conditions is obvious. One should have $c_1^2 \equiv 0 \bmod 8$. The other one is far less trivial. Rohlin [13] has shown that the signature

$$\tau := \frac{c_1^2 - 2c_2}{3} = c_1^2 - 8\chi$$

for spin manifolds is divisible by 16. (A consequence of this is that one also will have the divisibility condition $c_1^2/8 \equiv \chi \pmod 2$)

There are other restrictions as well, coming from geometric considerations. We shall prove (stated in terms of the classical invariants):

THEOREM A. *Let X be a simply-connected spin surface whose Chern-invariants satisfy*

$$2\chi - 6 \leqslant c_1^2 < 3(\chi - 5)$$

then $c_1^2 = 2(\chi - 3)$ *with* $c_1^2 = 8k$ *and k odd or* $c_1^2 = 8/3(\chi - 4)$ *with* $\chi = 1 \pmod 3$. *The points on the lower line* $c_1^2 = 2(\chi - 3)$ *can be realised by spin surfaces with a fibration in genus 2-curves. The points on the other line can be realised by spin surfaces with a fibration in genus 3-curves.*

As to results for the remaining negative index range, we shall follow in the tradition of [12] and try to fill as fat a sector as possible of invariants with simply-connected surfaces with even intersection form. We find first a sector where (at least) half the allowed points are realized by spin-surfaces:

THEOREM B. *For any pair of positive integers* (x, y) *with*

$$y \equiv 0 \pmod 8$$

$$y/8 + x \equiv 2 \pmod 4$$

$$3(x - 5) \leqslant y < 16/5(x - 4)$$

there exists a simply-connected spin surface S with $c_1^2(S) = y$ *and* $\chi(S) = x$ *and which has a hyperelliptic fibration in genus 4-curves.*

At this point it should be noted that Konno, using non-hyperelliptic genus 4-curves found examples of surfaces with even canonical bundle and with first Betti number zero, see [10]. By inspecting the invariants we see that indeed all allowed pairs in the sector of the preceding theorem are realized. It seems very difficult however to see whether the surfaces constructed are simply-connected or not.

Next, there is a remaining sector where all allowed points are realized. It is bounded below by the line $y = 16/5(x - 4)$ and above by a curve which is asymptotic to the line $y = 8x$ (the index-zero surfaces).

THEOREM C. *For any pair of positive integers* (x, y) *with*

$$y \equiv 0 \pmod 8$$

$$y/8 \equiv x \pmod 2$$

$$16/5(x - 4) \leqslant y < 8x - Cx^{3/4}$$

where $C = 270.4$, *there exists a simply-connected spin surface S with* $c_1^2(S) = y$ *and* $\chi(S) = x$.

As a corollary we get

COROLLARY D. *Given any rational number* α *such that* $16/5 \leqslant \alpha < 8$ *there is a simply-connected spin surface* X *with* $c_1^2(X)/\chi(X) = \alpha$.

In fact, there is a direct proof of this corollary which is very elementary as we will outline in the next section.

For some reason it seems much harder to find even surfaces of positive index which are simply-connected. One should remark again that the construction of Chen cannot be adapted to such purpose as the singularities he uses destroy evenness. In fact the only known examples so far in the literature are due to Moishezon and Teicher [11] in their original construction of simply-connected surfaces with positive index. Those surfaces are admittedly simple to describe but their fundamental groups are very complicated to calculate. Furthermore these surfaces, infinite in number, all have slope near 8, i.e. the signature is small compared to c_1^2. More specifically, given any $\varepsilon > 0$ they only exhibit a finite number (may be zero of course) of surfaces with slopes greater than $2 + \varepsilon$

One of the main purposes of this paper is to try and find another infinite set of simply-connected spin surfaces if possibly with higher slopes, and somewhat simpler, using again the ideas for positive index constructions proposed by Xiao [15]. To do so it turned out that we were unable to find double covers along ingenious configurations on a fibration over $\mathbb{P}^1$ (in spite of valiant efforts) which would have allowed us to use the standard observation (see e.g. [12]) that such a fibration leads to a simply-connected surface if all fibres have multiplicity one and there is at least one simply-connected fibre (see Lemma D in the next section). To get out of this straightjacket we need a modification of Lemma D, although based on the same general idea, no longer insists on a simply connected fibre, but of a neighbourhood of many fibers that turn out to be simply-connected. Thus, as above, the lemma is just about the local contribution to the fundamental group, and will if applied to a fibration over $\mathbb{P}^1$ give the desired conclusion. See Lemma F in the next section.

Our construction of positive index will also be a bit more involved as we will consider a triple sequence of double covers. The calculation of c_1^2, τ will be straightforward, and to find a sufficient criterion for evenness will also be easy.

In the case of positive index we will be less concerned with actually filling out sectors (which appears to be an incredible mess), but only to be content with finding at least an infinite number of such examples of high index. In fact the precise result here is

THEOREM E. *Let* α *be a rational number given by* $\alpha = c_1^2(X)/\chi(X)$ *for some simply-connected spin surface* (*of positive index*). *Then the set of such* α *is dense in the interval* $[8, 8.76]$.

We believe that any rational slope in that region is actually taken, but to prove this would be rather messy.

The plan of our paper is as follows. In Section 2, we gather technical results: the invariants of double covers, specifically of double covers of the Hirzebruch surfaces and of double covers of $\mathbb{P}^1 \times \mathbb{P}^1$, and finally the lemmas on simply-connectedness. In Sections 3 and 4, we treat the cases of resp. negative and positive index.

2. TECHNIQUES

The techniques involved are those pioneered in [12]. We will start out with a skeleton of surfaces given by repeated double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ or alternatively as fiber products of virtual double covers. By imposing singularities of the branch curves one may then fill out the gaps.

We start out by giving an answer to a basic question: when is the canonical divisor of a double cover 2-divisible?

LEMMA A. *Let $f: X \to Y$ be a ramified double cover of smooth projective varieties and let $\sigma: X \to X$ be the covering involution. Suppose that* Pic(X) *has no 2-torsion. A line bundle $\mathscr{L}$ on X with $\sigma^*\mathscr{L} \cong \mathscr{L}$ is 2-divisible in* Pic(X) *if and only if $\mathscr{L} = f^*(\mathscr{M} \otimes \mathcal{O}(B'))$, for some 2-divisible line bundle $\mathscr{M}$ on Y and some divisor B' supported on the branch locus.*

Proof. Suppose $\mathscr{L} = \mathscr{N}^{\otimes 2}$. From $\sigma^*(\mathscr{N})^{\otimes 2} = \sigma^*(\mathscr{L}) = \mathscr{L} = \mathscr{N}^{\otimes 2}$ and the assumption that Pic(X) has no 2-torsion it follows that $\mathscr{N}$ is σ-invariant. Replacing $\mathscr{N}$ with $\mathscr{N} \otimes f^*\mathcal{O}_Y(k)$ with $k \gg 0$ we may assume that $\mathscr{N}$ has sections and then σ acts on its space of sections which splits in two obvious eigenspaces at least one of which is non-zero. The corresponding divisor is left pointwise invariant by the involution and hence must be supported in the ramification locus. This completes the proof in one direction. The other direction is obvious. ■

We shall need the following corollary.

COROLLARY B. *In the situation of Lemma A, suppose that X and Y are (smooth) surfaces. Let $\rho: X \to X'$ be the morphism onto the minimal model. Assume ρ is the blow-down of a non-empty collection of disjoint exceptional curve E_j, $j \in J$ on X. If $\sum_{j \in J} E_j$ is not the total transform of a divisor on Y then $K_{X'}$ cannot be 2-divisible.*

Proof. Suppose that $K_{X'}$ and hence $\rho^*K_{X'}$ would be 2-divisible. Now $K_X = \rho^*K_{X'} + \sum_{j \in J} E_j$ and K_X as well as $\sum_{j \in J} E_j$ are obviously σ-invariant, hence so is $\rho^*K_{X'}$. But the sum of the exceptional curves does not pull back from Y, so neither does $\rho^*K_{X'} = K_X - \sum_{j \in J} E_j$ contradicting the preceding lemma. ■

And finally there is the answer to our question.

COROLLARY C. *Assume that* Pic(X) *has no 2-torsion. Let $f: X \to Y$ be a double cover of smooth projective varieties, branched in a smooth branch locus $C \equiv 2B$. Then K_X is 2-divisible if and only if there is a decomposition $C = C_1 + C_2$ such that $K_Y + B + C_1$ is 2-divisible.*

Proof. Apply the Lemma to (the line bundle corresponding to) $K_X \equiv f^*(K_Y + B)$. It is clearly σ-invariant and so there exists a divisor C' supported in the branch locus so that $f^*(K_Y + B + C')$ is 2-divisible. Obviously, we may assume that C' consists of components of the branch divisor with multiplicity one together defining C_1. Then C_2 is the sum of the other components with multiplicity one. ■

This motivates the following

Definition. Suppose that $f: X \to Y$ is a double cover of smooth projective varieties with branch divisor C. A spinning decomposition of C is any decomposition $C = C_1 + C_2$ which induces a decomposition as in Corollary C on some blow-up of Y on which the branch locus has become smooth. We say that it is *trivial* if $C = C_2$.

The calculations of invariants will make repeated use of the well-known formulas of a double cover between surfaces

$$g: Y \to X \quad \text{branched at} \quad C \equiv 2B$$

Although those formulas are well known (see e.g. [[1], V §22] and [[12], §1], we reproduce them here for easy reference:

$$\text{(DC)} \qquad \begin{aligned} K_Y &= g^*(K_X + B) \\ c_1^2(Y) &= 2c_1^2(X) + 4g(C) - 4 - \tfrac{3}{2}C^2 (= 2(K + B)^2) \\ c_2(Y) &= 2c_2(X) + 2g(C) - 2 \\ \chi(Y) &= 2\chi(X) + \tfrac{1}{2}(g(C) - 2) - \tfrac{3}{4}C^2 \\ \tau(Y) &= 2\tau(X) - \tfrac{1}{2}C^2. \end{aligned}$$

We apply them first to the Hirzebruch surface $\mathbb{F}_n$. The Picard group is freely generated by the class f of a fibre and a section s of self-intersection n. We let $B = as + bf$ and then

$$\text{(Hz)} \qquad \begin{aligned} K_S &= g^*((a - 2)s + (b + n - 2)f) \\ K^2 &= 2(a - 2)[an + 2b - 4] \\ \chi &= \tfrac{1}{2}(a - 1)[an + 2b - 4] + a \\ \tau(S) &= -2a(an + 2b) \end{aligned}$$

Observe that the invariants are all on the line

$$y = \frac{4(a-2)}{a-1}(x - a).$$

Next, we consider the fibre product Y of two double covers branched respectively at C_1 and C_2, where $C_1 \equiv 2B_1$ and $C_2 \equiv 2B_2$. We see that Y is spin if $K + B_1 + B_2$ is an even divisor (it is not strictly necessary of course) giving an easy sufficient criterion. Furthermore, we obtain using (DC) that

$$\text{(M)} \qquad \begin{aligned} c_1^2(Y) &= 4(K + B_1 + B_2)^2 \sim 4(B_1^2 + B_2^2 + 2B_1B_2) \\ \tau(Y) &= -4(B_1^2 + B_2^2) \end{aligned}$$

where B_1, B_2 can be thought of as vectors in the hyperbolic plane.

From this we see that to get the ratio τ/c_1^2 high we need to make B_1, B_2 as orthogonal as possible; and to make it low as parallel as possible.† This will essentially suffice to give a direct proof of corollary D and we will now supply the details.

Proof of Corollary D. For simplicity we suppose that $n = 0$, i.e. we work on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that B_1 has bidegree (a, b) and B_2 bidegree (c, d). If you fix a and c the two equations become linear in b and d and can be solved in rationals. By multiplying c_1^2 and χ one can assure that these become integers and the inequalities take care of the signs.

† This observation was communicated to the first author by Boris Moishezon during a Colloquium dinner at Columbia in the late seventies and became the starting seed for his geographical investigations.

In detail, let p and q be two co-prime natural numbers such that $p/q \in [16/5, 8)$. Fix two non-negative integers α and γ and put $a = (\gamma + 2)\alpha$, $c = \gamma\alpha$, $A = (\gamma + 1)\alpha$ and we search a spin surface with

$$c_1^2 = 32(A - 1)\alpha p$$

$$\chi = 32(A - 1)\alpha q.$$

Solving the linear equations we find

$$b = ((-3\gamma - 2)\alpha + 2)p + ((16\gamma + 16)\alpha - 16)q - \gamma$$

$$d = ((3\gamma + 4)\alpha - 2)p - ((16\gamma + 16)\alpha - 16)q + \gamma + 2.$$

Putting $\alpha = 1 = \gamma$ gives

$$b = -p + 16q - 1$$

$$d = 5p - 16q + 3$$

and this gives positive integers whenever $p/q \in [16/5, 16/3)$. For the remaining interval we simply take $\gamma = 0$ and make α go to infinity so that asymptotically

$$b \sim -2\alpha p + 16\alpha q$$

$$d \sim 4\alpha p - 16\alpha q$$

and this gives positive numbers for $p/q \in (4, 8)$. ■

We also need to impose singularities. There are certain singularities on the branch locus which do not alter the invariants. Those are called *simple* in [1], Chapter II §8 or *inessential* in [12]. There one can find further details for the discussion that follows. We recall that the simplest kind of non-simple singularities are the $T_{2,3,6}$ singularities (infinitely close triple points-ICT). But these are now fatal as they will instantly destroy evenness. To see this we determine the effect of imposing say one ordinary $2k$-fold point or one ordinary $2k + 1$-fold point on the branch-locus $C \equiv 2B$. If E is the exceptional curve on the surface blown up in that point and $\sigma: \tilde{Y} \to Y$ the blowing-up, $f: \tilde{X} \to \tilde{Y}$ the new double cover, one has

$$K_{\tilde{X}} = f^* \circ \sigma^*(K_Y + B) + (1 - k)f^*(E).$$

An ICT has to be resolved by two consecutive blowings up with $k = 1$, resp. $k = 2$. But the resulting double cover is not minimal: one has to blow down the proper transform of the first exceptional curve. One checks however that the proper transform of the second exceptional curve becomes an elliptic curve with self-intersection -1 and so destroys evenness.

So we will restrict ourselves mostly to singularities that do not locally destroy evenness. As before we will impose them through singularities of the branch curves and they will hence be given by local equations

$$z^2 = f(x, y).$$

From the preceding formula one sees that this works for odd k, i.e. one should restrict to ordinary $(4m + 2)$- or $(4m + 3)$-tuple points of $f = 0$. One should remark that although say a 4-tuple point destroys evenness locally, a clever global combination of 8-tuples of them may restore it! This is in sharp contrast to an ICT which destroys it irrevocably through an odd intersection curve in the resolution. In fact, we will need this below when we consider hyperelliptic genus-3 fibrations which realise all surfaces in the sector $8/3(x - 4) < y < 3(x - 5)$.

The $(4m + 2)$- or $(4m + 3)$-tuple points will modify the invariants according to the following formulas

$$\text{(Sp-vct)} \qquad \begin{aligned} c_1^2(m) &= -8m^2 \\ c_2(m) &= -4(4m^2 + 3m) \\ \chi(m) &= -(2m^2 + m) \\ \tau(m) &= 8(m + m^2). \end{aligned}$$

The case $m = 1$ will be of particular importance, (corresponding to 6-tuple points), they will be referred to as s-singularities. One may note that these singularities are nothing but even Gorenstein. Considering a resolution $\tilde{Y} \to Y$ we may write

$$K_{\tilde{Y}} = K_Y - 2mE$$

where $E^2 = -2$ and $g(E) = 2m$ (in fact, E is by construction hyperelliptic).

Next, we need a condition for simply-connectedness. In the literature we have the sufficient condition that the branch locus is very ample (Lefschetz) but this is far too stringent for our purposes. More useful is the following standard observation.

LEMMA D. *Given a fibration $Y \to \mathbb{P}^1$ with at least one simply-connected fibre, and no multiple fibres. Then Y is simply-connected.*

The idea behind this is clearly that any loop can be deformed into the simply-connected fibre and extinguished. (The presence of multiple fibres prevents local sections and stems effectively the flow of loops across the fibres)

In practice we will achieve this by insisting that the intermediate surface X_1 is a $\mathbb{P}^1$ fibration and that the final branch curve C_2 contains at least one fibral component. (This means that X_1 is in fact birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and our examples are just double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ admittedly with intricate configurations of singularities.) This criterion has been applied both by Chen and Persson (see [12] Appendix A]) to ensure the simply-connectedness of their constructions, and will be used in our construction of spin surfaces with negative index. To be precise, we have

COROLLARY E. *Let $f: S \to T$ be a double covering between surfaces and assume that a $g: T \to \mathbb{P}^1$ is a fibration in rational curves and that one of the fibres is part of the branch-locus of f. Moreover, assume that all fibres of g contain at least a reduced component which does not belong to the branch-locus of f. Then S is simply connected.*

As explained in the introduction, in the last section on positive signature we need a more sophisticated lemma: (actually stated in greater generality than we will actually need).

LEMMA F. *Let Δ be the unit disk, $Y = \Delta \times \mathbb{P}^1$, with projection $p: Y \to \Delta$. Let $\delta_i: X_i \to Y$ ($i = 1, \ldots, k$) be k cyclic covers of order r_i respectively, each δ_i being totally branched along a fibre F_i of p and l_i local sections $s_{i,1}, \ldots, s_{i,l_i}$. Suppose that $F_i \neq F_{i'}$ whenever $i \neq i'$, and that the local sections $s_{i,j}$ are all disjoint.*

Let $\delta: X \to Y$ be (the desingularization of) the fibre product of the cyclic coverings $\delta_1, \ldots, \delta_k$. Then X is simply-connected.

Proof. We may assume that the local sections $s_{i,j}$ of p are fibres of the projection $q: Y \to \mathbb{P}^1$ onto the second factor.

Let Y' be the complement in $\tilde{Y}$ of the branch locus of $\tilde{\delta}$ (which is composed of the strict transforms of the F_i's and $s_{i,j}$'s). The fundamental group of Y' is generated by classes φ_i, resp. $\sigma_{i,j}$ of small loops in Y' going around F_i resp. $s_{i,j}$. The unramified cover $\delta: X' \to Y'$ is completely described by the exact sequence

$$1 \to \pi_1(X') \xrightarrow{\delta_*} \pi_1(Y') \to G \to 1$$

where the abelian covering group G is the direct product of cyclic groups of order r_i. The classes $\varphi_i^{r_i}$ and $\sigma_{i,j}^{r_i}$ belong on the one hand to $\pi_1(X')$ (since δ ramifies totally of order r_i along the curves F_i and $s_{i,j}$), but on the other hand they represent loops in X winding once around irreducible components of the branch locus and hence are null-homotopic in X. So, if $i: X' \to X$ is the inclusion the kernel K of the surjective map $i_*: \pi_1(X') \to \pi_1(X)$ contains all of the classes $\varphi_i^{r_i}$ and $\sigma_{i,j}^{r_i}$ and hence it contains the normal group H generated by all of their conjugates (we refer to H later on).

We let $p_{i,j} = F_i \cap s_{i,j}$. Let U be a small neighbourhood of $p_{i,j}$. The resolution of the singular point $p_{i,j}$ is described for instance in [[1] Chapter III, §5]. It is done by means of Hirzebruch–Jung strings. In [7], one can find a topological description of the neighbourhood of such a string: it is a 4-manifold with boundary obtained by plumbing 2-disk-bundles over 2-spheres. In particular, the manifold is simply-connected since it has a bouquet of 2-spheres as a deformation retract. Hence, any connected component of the inverse image of U in X is simply-connected. Thus U induces a relation $\sigma_{i,j} = \varphi_i^{n_{i,j}}$ in K, with $(n_{i,j}, r_i) = 1$. As F_i and $s_{i',j}$ meet transversally, φ_i commutes with $\sigma_{i',j}$ for any i, i', j and so in $\pi_1(Y')/K$ the classes of φ_i commute with each other and so $\pi_1(Y')/K$ is abelian. Next, since $\varphi_i^{r_i} \in K$, it follows that this quotient group is isomorphic to the covering group G and hence $K \subset \pi_1(X')$ must coincide with $\pi_1(X')$. So $1 = \pi_1(X')/K \cong \pi_1(X)$. ■

3. NEGATIVE INDEX

First we will turn to the lower region below the line $y = 16/5(x - 4)$. For that purpose we use double covers $g: S \to \mathbb{F}_n$ of the Hirzebruch surfaces $\mathbb{F}_n$ branched along a curve C which is linearly equivalent to $2b$ times a fibre f and $2a$ times a section s with self-intersection n.

Let us investigate when these surfaces are spin, using Corollary C.

First we look at the case of a non-trivial spinning decomposition on $\mathbb{F}_n$. This is only possible if the branch locus is a disjoint union of the section $s_\infty \in |s - nf|$ with self-intersection $-n$ on $\mathbb{F}_n$ and another smooth curve in the linear system $|(2a-1)s|$. Here n must be even in order to be able to perform the double cover $g: S \to \mathbb{F}_n$. Applying Corollary C, we see that n must be in fact divisible by 4 and if we want $C_1 = s_\infty$ we must have a odd (a even corresponds to a trivial spinning decomposition). If we put $n = 4m$ the invariants are (using (Hz))

$$c_1^2(S) = 8(a-2)(m(a-1)-1)$$

$$\chi(S) = 2(a-1)(m(a-1)-1) + a$$

and give surfaces on the line

$$y = \frac{4(a-2)}{a-1}(x - a).$$

For $a = 3$, we get all surfaces on the Noether line with $c_1^2 = 8k$ with k odd. It is also not too hard to show that these are the only spin surfaces on the Noether line at least if we also allow double points on the branch locus (see [9]). For $a = 5$, we get spin surfaces on the line

$$y = 3(x - 5).$$

This is exactly half the allowed points (indeed, this yields only points with $c_1^2 = -24 \bmod 96$ while all points with $c_1^2 = -24 \bmod 48$ are *a priori* allowed.

By a slight extension of this construction we can give the

Proof of Theorem B. We take two integers n and t such that their sum is divisible by 4. Consider on $\mathbb{F}_n$ the section s_∞ with exactly t points on it. Choose three curves $D_j \in |3s + tf|$, $j = 1, 2, 3$, passing through these points and having no other common intersection points. We let $C_1 = s_\infty$ and $C_2 = D_1 + D_2 + D_3$ and we will check that this gives a spinning decomposition for the double cover branched in $C = C_1 + C_2$. Let $E_j, j = 1, \dots, t$, be the exceptional curves on the surface F obtained by blowing up $\mathbb{F}_n$ in the t points on the section s_∞. Let $\sigma: F \to \mathbb{F}_n$ be the blowing-up map. Let $C' = 2B'$ be the branch locus on F and let C_1' be the proper transform of s_∞ on F. We find

$$K_Y + B' + C_1' = \left(4s + \left\{\frac{3(t+n)}{2} - 2n - 2\right\}f\right) + \sigma^*\left(-2\sum_{j=1}^{t} E_j\right)$$

which indeed is even.

The invariants satisfy

$$K^2 = 24(n - 1) + 16t$$

$$\chi = 8n + 5t - 3.$$

These are indeed in the desired sector and conversely, for any two points in the sector we determine $t = y - 3(x - 5) > 0$ and $n = (16x - 5y - 72)/8$ which is an integer because y is divisible by 8 and $\geqslant 0$ since $y < 16/5(x - 4)$.

The second case when spin surfaces occur (with trivial spinning decomposition) is when $a \equiv 0 \pmod 2$ and $b + n \equiv 0 \pmod 2$. Recall (Hz)

$$K^2 = 2(a - 2)[an + 2b - 4]$$

$$\chi = \tfrac{1}{2}(a - 1)(an + 2b - 4) + a.$$

Note that the inverse images on S of the fibres of $\mathbb{F}_n \to \mathbb{P}^1$ give a pencil on S of hyperelliptic curves of genus $a - 1$. Now it is quite easy to see that on S there can be at most one pencil of curves of genus $\leqslant a - 1$ (see the proof of [15], Theorem 1]. It follows that such spin double covers of $\mathbb{F}_n$ can only admit a pencil of hyperelliptic curves of *odd* genus $a - 1$. Let us look at the case $a = 4$. This gives the invariants

$$c_1^2(S) = 16n + 8(b - 2)$$

$$\chi(S) = 6n + 3(b - 1) + 1$$

and these give all allowed points on the line $y = \frac{8}{3}(x - 4)$.

Let us now give the

Proof of Theorem A. We have given a construction for all of the allowed points on the two lines and for these we have used precisely all possible double covers branched in a 6-section or an 8-section and having at most simple singularities. So it suffices to prove the following:

PROPOSITION. *Let X be a spin surface of general type with $c_1^2 < 3(\chi - 5)$. Then X can be realized as the double cover of $\mathbb{F}_n$ branched along a curve with at most simple singularities.*

Proof of the Proposition. By [2], since $c_1^2 < 3\chi - 10$ implies that the surface is a double cover of a ruled surface and hence admits a hyperelliptic fibration. From [15], Theorem 1 and the remark following it, we conclude that X admits a unique hyperelliptic fibration into curves of genus 2 or 3 and hence X is birationally a double cover of $\mathbb{F}_n$ branched in a curve C which is a 6-section or an 8-section. (there are no even surfaces with $\chi < 22$ but $3(\chi - 5\frac{1}{8}) \leqslant K^2 < 3(\chi - 5)$).

Let us recall the general case, when C is a $2a$-section. By [16], Lemma 6 and the discussion following this lemma, we can always assume that the singularities of C are of order at most $a + 1$ and if equality holds one can furthermore assume that a is even ($= 2b$) and that $C = C' + F$, where F is a fibre through the unique singular point P of C' and that P is the only point of intersection of C' and F so that P is an a-tuple point on F with its branches tangent along F. In the latter case, we let $Y_1 \to \mathbb{F}_n$ the blow-up in P with E_1 the exceptional curve. We need to blow up once more in the resulting $(a + 2)$-tuple point on the proper transform of C', say $Y_2 \to Y_1$. Let E_2 be the new exceptional curve, $\hat{E}_1$ the proper transform on Y_2 of E_1 and $\hat{F}$ that of F. The latter two curves are -2-curves which form part of the branch locus yielding two disjoint exceptional curves on the double cover $f_2: X_2 \to Y_2$. The curve $f_2^{-1}E_2$ is a genus b-curve F' of self-intersection -2. Upon blowing down the two exceptional curves we obtain X'_2, fibred over $\mathbb{P}^1$ with the image of F' being half a fibre as one can check immediately (the total transform of F on X_2 is $\hat{F} + \hat{E}_1 + 2E_2$).

In our case, if $a = 3$ there can be at most triple points and hence at most simple singularities (the only non-simple ones locally destroy the evenness of the canonical bundle as we have seen before). If $a = 4$ there can be at most quadruple points or quintuple points. By the preceding discussion these give rise to a fibration with double fibres. Suppose that there are t of the latter. Then, by [[16], Lemma 2], the fundamental group of X has a quotient of the form $(\mathbb{Z}/2\mathbb{Z})^{t-1}$ and hence necessarily $t \leqslant 1$. Let $\rho: X \to X'$ be the blowing down to the minimal model. This minimal model admits a pencil of hyperelliptic curves as well and hence it admits an involution and the exceptional curves on X are all contracted onto fixed points of this involution. In particular, ρ consists of contracting a finite number of disjoint -1-curves. These must form part of the branch locus. By the discussion in Section 2 the only further singularities that do not destroy the evenness of the canonical bundle locally are double points (of infinitesimally near ones). These as well as the quadruple points do not introduce -1-curves on the double cover. So, if $t = 1$ there are exactly the two exceptional curves we described before. Since their sum is not a total inverse image of a divisor on Y, by Corollary B we conclude that X' cannot be spin.

So we are left with the case of quadruple points. Assuming that X is spin we let $C = C_1 + C_2$ be a non-trivial spinning decomposition. Using Corollary C it follows easily that C_1 and C_2 meet only in 4-tuple points of C and that one of the two curves is smooth in the intersection points. Hence we have $k = (C_1 \cdot C_2)/3$ quadruple points. Since either $C_1 = 2s + b_1 f$ or $C_1 = 4s + b_1 f$, we have $k = 12n + 4(b_1 + b)$ or $k = 16n + 8b$. Substituting into the formulas for c_1^2 and χ (see (Hz) with $a = 4$) we find that $5c_1^2 - 16(\chi - 4)$ either equals $8(n + b_1) \geqslant 0$ (since $b_1 \geqslant -n$) or $80n + 56b > 0$. But this contradicts the inequality $c_1^2 < 3(\chi - 5)$ which we supposed to hold. So quadruple points cannot occur either.

It follows that only simple singularities are possible on the branch locus. ∎

We will now consider repeated covers of $\mathbb{P}^1 \times \mathbb{P}^1$, as before starting from curves C_1 of bidegree $(2a, 2b)$ and a curve C_2 of bidegree $(2c, 2d)$. We construct Y by first taking the double cover $\pi: Y' \to \mathbb{P}^1 \times \mathbb{P}^1$ branched in C_1 and then taking the double cover of Y' branched in the inverse image of C_2. We get a spin surface exactly when $a + c = 2A$, $b + d = 2B$ and, using the formulas from the previous section, we find

$$c_1^2 = 32(A-1)(B-1)$$

$$\chi = 4(A-1)(B-1) + 2(Ab + (B-b)c).$$

In order to take care of divisibility, we shall frequently use more suitable geographical invariants:

$$\Gamma = c_1^2/8$$

$$T = -\tau/16$$

and in terms of these we have

$$\text{(Ddc)} \qquad \begin{aligned} \Gamma &= 4(A-1)(B-1) \\ T &= Ab + (B-b)c \\ b &= 1, \ldots, 2B-1 \\ c &= 1, \ldots, 2A-1. \end{aligned}$$

For $b = 1$ we have a fibration over $\mathbb{P}^1$ and, by inserting a fibre in the branch-locus we can ensure simply-connectedness (see Corollary B).

As remarked before, we need to impose ordinary $(4m+2)$-, $(4m+3)$-points (or such that their infinitely close singularities are of this type and inductively) singularities on the branch curve $\pi^{-1}C_2$. Recall that this will not effect the "evenness" of the canonical divisor of the resolution, and the invariants (in our case Γ, T) are directly computable through the specialization formulas (Spc-vct)

$$\Gamma' = -m^2$$

$$T' = -\frac{m+m^2}{2}.$$

In particular, for the simplest case ($m = 1$) of the ordinary six-tuple point (s-singularities for short) we get (in terms of C, T) the specialization $(-1, -1)$. Those singularities will hence play the rôle of the infinitely close triple points of [12].

In order to construct s-singularities, we simply look at either s-singularities of C_2 away from C_1 (which pull-back to two s-singularities on Y') or at an ICT on C_2 lying on C_1 with its branches *tangent* to C_1 as well.

We will now look at the invariants given in (Ddc) restricted to the case of $B = 2$ and $d = 3$:

$$\Gamma = 4(A-1)$$

$$T = A + c$$

As c ranges from 1 to $2A - 1$, for fixed Γ with $\Gamma \equiv 0(4)$ all values T for which $\Gamma/4 + 2 \leqslant T \leqslant 3\Gamma/4 + 3$ are taken. In the (T, Γ)-plane the invariants are exactly the integral points in this sector on parallel lines at 4 units distance from each other. Clearly, if we can impose 1, 2 or 3 s-singularities on the branch-locus we would be able to fill an entire sector. Now, to get a single such singularity, one may impose a triple point with one tangent on the branch-locus C_1 of the first double curve. To get three, one may use [12, Proposition 3.1]. It implies that you can impose exactly k such singularities on a curve of bidegree $(6, 2c)$ whenever $k \leqslant 2[2c/3]$. So to get up to 3, you need $c \geqslant 3$. The proof of the proposition shows that you can arrange this with curves of bidegree $(2, 2)$ plus $2c - 6$ fibres. In order to get a simply-connected surface we need at least one fibre in the branch-locus and so we better take $c \geqslant 4$. This shows

PROPOSITION 1. *Any pair of integers* (Γ, T) *satisfying*

$$\Gamma/4 + 5 \leqslant T \leqslant 3\Gamma/4 + 2$$

$$\Gamma \geqslant 9$$

can be realized as the invariants of a simply-connected spin manifold.

We now consider an arbitrary $B > 2$, $b = 1$ and note that the invariants of T now run through $A + (B - 1)c$. So, not only is the distance between two consecutive lines $\Gamma =$ Constant equal to $4(B - 1)$, but there are gaps of width $B - 1$ on these lines as well. Now, going up from one line $\Gamma =$ Constant to the next, the values of T which occur shift one to the right. This shows that, whenever you are in a lattice point in the sector bounded by the lines corresponding to $c = 1$ and $c = 24 - 1$, which is not one of these allowed lattice points, adding some integral vector (k, k) with at most $4(B - 1)^2$ does give such an allowed point. So we must be able to assign any number k of s-singularities with $k \leqslant 4(B - 1)^2 - 1$. Note that pull-backs of the curve D will have an even number of s-singularities, to get an odd, we need only impose an extra infinitely close triple point on the branch-locus.

CLAIM. For the pull-back of $(2c, 2d)$ we can impose up to $\frac{2}{3}(c + d - 6)$ s-singularities.

To see this, note that there is a pencil of curves of bidegree (γ, δ) with base points at any number of $\kappa \leqslant \gamma + \delta$ points which are general in the sense that no two of them lie on the same fibre or the same section. Now choose γ and δ such that $2c - 5 < 6\gamma \leqslant 2c - 6$ and $2d - 5 < 6\delta \leqslant 2d - 6$ and fix $k \leqslant \delta + \gamma$ general points. Take 6 generic members of the system of curves of bidegree (γ, δ) passing simply through these points. The union of these curves has bidegree at most $(2c - 6, 2d - 6)$ and you can add three curves of bidegree $(2, 2)$, if necessary to produce an extra infinitely close triple point on the branch-locus. Since we can always insert a fibre in the preceding construction this proves the Claim.

Now $d = 2B - 1$ and if $c \geqslant 6(B - 1)^2$ we get $\frac{2}{3}(c + d - 6) \geqslant 4(B - 1)^2 + 2B - 6 \geqslant 4(B - 1)^2 - 1$ since $B \geqslant 3$. Thus we can then indeed impose $\leqslant 4(B - 1)^2 - 1$ singularities. Since $6(B - 1)^2 \leqslant c \leqslant 2A - 1$ we find $\Gamma = 4(A - 1)(B - 1) \geqslant 12(B - 1)^2$ and this gives:

PROPOSITION 2. *Let* $B \geqslant 3$ *an integer. Any pairs of invariants* (Γ, T) *satisfying*

$$\frac{\Gamma}{4(B - 1)} + 6(B - 1)^3 \leqslant T \leqslant \Gamma/2$$

$$\Gamma \geqslant 12(B - 1)^3$$

occur as invariants of a simply-connected spin surface.

Theorem C from the introduction will be seen to follow from

THEOREM 3. *Any pair of invariants* (Γ, T) *satisfying*

$$K\Gamma^{3/4} \leqslant T \leqslant 3\Gamma/4 + 2$$

with $K = 21 \cdot 2^3/(3^2 \cdot 19)^{3/4} = 3.616$ *can be realized by simply-connected spin surfaces.*

In particular, the ratio c_1^2/χ can be arbitrarily close to 8. Furthermore, the constant K can of course with care be improved, as could the exponent of Γ (although this is of course much harder).

Proof of Theorem 3. Combine Proposition 1 and 2. Consider the line L_B with equation $y = x/(4(B-1)) + 6(B-1)^3$. This line has slope $1/4(B-1)$ which decreases when B increases. The line L_B meets L_{B+1} at the point (x_B, y_B) with

$$x_B = 24B(B-1)(3B^2 - 3B + 1)$$

$$y_B = 6B(3B^2 - 3B + 1) + 6(B-1)^3.$$

Now $x_B \geqslant 12B^3$ and therefore, the region described by Proposition 2 by letting B vary is certainly above the piecewise linear function $f(x)$ defined by the segments of the lines L_B in the interval $[x_{B-1}, x_B]$. Now the function $f(x)$ is monotone increasing and convex and we bound it above by Kx^t with $t \in (0, 1)$ and $K > 0$ suitably chosen. Since x_B is quartic in B and y_B cubic it is easy to see that $t = 3/4$ will do and a calculation shows that $K = 21 \cdot 2^3/(3^2 \cdot 19)^{3/4} = 3.616$. ∎

Proof of Theorem B. This follows almost immediately from the previous theorem. With $K := 21 \cdot 2^3/(3^2 \cdot 19)^{3/4}$ it translates first into the inequality in the $(x, y) = (\chi, c_1^2)$-plane

$$y + 16K \cdot y^{3/4} \leqslant 8x \leqslant 5/2y + 32$$

and then you have to use that the inequality $8x \geqslant y + 16Ky^{3/4}$ is in turn implied by the reverted inequality $y \leqslant 8x - 8^{3/4} \cdot 16K \cdot x^{3/4}$. ∎

4. SPIN MANIFOLDS OF POSITIVE INDEX

We will once again use the standard configurations of (1, 1) curves on $\mathbb{P}^1 \times \mathbb{P}^1$ stemming from infinite group actions on $\mathbb{P}^1 \times \mathbb{P}^1$. Specifically, let G be a finite group action on $\mathbb{P}^1$, for each $g \in G$ we consider its graph $\gamma(g) \subset \mathbb{P}^1 \times \mathbb{P}^1$ and the ensuing configuration $\Gamma = \bigcup_{g \in G} \gamma(g)$. If $|G| = N$ then Γ is a curve of bidegree (N, N) on $\mathbb{P}^1 \times \mathbb{P}^1$.

Recall that Γ is determined by three integers (p, q, r) satisfying $(1/p) + (1/q) + (1/r) > 1$† and Γ will have three block of singularities, of ordinary p-tuple, q-tuple points etc., each block being a square of side N/p, N/q ...

By considering the diagonalization of the standard $x:1$ covering of $\mathbb{P}^1$ to itself, we can pull back Γ from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and consider $\Gamma(x)$ as configuration of type (Nx, Nx) with $(N/p)x^2$ ordinary p-tuple points etc.

† We are skipping the A_n cases.

We will construct our desired surface Y in a number of steps. First we will consider the sequence of double covers

$$Y = Y_3 \xrightarrow{\pi_3} Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} \mathbb{P}^1 \times \mathbb{P}^1$$

branched at B_1, $\pi_1^* B_2$ and $'B_3 = (\pi_2\pi_1)^* B_3$. Where Y_2 is simply given a fibre product of double covers, but the effectual branch-locus of $'B_3$ will not come from downstairs, although it will be linearly equivalent to something that is a pull-back. (This is the main technical complication in the construction, which calls for some caution, but will in the end turn out to be insignificant).

Now we need three things,

(I) A formula for K to check evenness. This is easy, it is given by $\pi^*(\frac{1}{2}(B_1 + B_2 + B_3) + K_{\mathbb{P}^1 \times \mathbb{P}^1})$. Incidentally it gives of course a formula for $K_Y^2 \sim 2(B_1 + B_2 + B_3)^2$.

(II) A condition to ensure local simply-connectedness. This is furnished by Lemma F.

(III) A formula to compute the index. This is straightforward from the index formula for double covers, and we get

$$\tau(Y) = -2(B_1^2 + B_2^2 + B_3^2).$$

Now Y will be singular, with well-controlled singularities, and our interest will be in the desingularization $\tilde{Y}$. Our strategy of choosing the branch-loci B_i will ensure that only even singularities occur, thus (I) will not be a problem. For (II) there will be no problem, although the singularities are not simply-connected. The strategy of choosing a simply connected fibral neighbourhood makes it irrelevant what happens outside it. As for (III) we have already established the formulas for the modification of the index.

Strategies

Given an n-tuple point on $\Gamma(x)$, we can do five different things: (I) We can make it disjoint from both B_1 and B_2 and add either (a) nothing, (b) a vertical (or horizontal) fibre or (c) both, in the construction of $'B_3$; Or (II) we can let a horizontal fibre of B_1 pass through it as well as a vertical fibre of B_2 and for the case of $'B_3$ either add (a) nothing or (b) both horizontal and vertical fibres.

In any case the ordinary n-tuple point becomes an ordinary point of multiplicity according to the following diagram:

$$n \mapsto \left\{ \begin{array}{rll} n & \text{in case } n \equiv 2 \bmod 4 & \text{(Ia)} \\ n+1 & \text{in case } n \equiv 1 \bmod 4 & \text{(Ib)} \\ n+2 & \text{in case } n \equiv 0 \bmod 4 & \text{(Ic)} \\ 2n & \text{in case } n \equiv 1 \bmod 2 & \text{(IIa)} \\ 2n+2 & \text{in case } n \equiv 0 \bmod 2 & \text{(IIb)} \end{array} \right\}$$

where our aim is obviously to make the singularities even. In particular (unless $n \equiv 3 \bmod 4$), we have two choices what to do with a singularity of $\Gamma(x)$.

Minor complications

Of course one has to be careful. First to each B_1, B_2 one must add some transverse fibres, to keep the surfaces fibred over $\mathbb{P}^1 \times \mathbb{P}^1$. One may also have to add additional vertical fibres, so as to allow Lemma F to be used, or to keep the total count divisible by four. Whenever $'B_3$ contains a fibre already used, it cannot be a pull-back, although it should be equivalent to one. (So we can use criteria (I) and (III)) Whenever we include such a fibre, we will also be forced to include the transversal fibres that have been used in the B_1 and B_2 branch-loci, as well as the exceptional divisor that comes from desingularization at the first stage on Y_2. This is all to ensure that the locus $'B_3$ is indeed even, and that it comes from a B_3 downstairs. In fact, as the reader can convince himself of, locally the cover is given by $z^4 = xy$ which can be split up in two subsequent double covers.

Now these minor complications do affect (marginally) the calculations. The point is of course that we think of x in $\Gamma(x)$ to be sufficiently large, in fact in a sense as an indeterminate, so they can be ignored as the forthcoming calculations will illustrate.

Calculations

Consider a dihedral Γ of type $(2, 2, n)$ where say $n \equiv 2 \bmod 4$ to fix ideas.

We have two options, either leaving the n-tuple points alone, or soup them up into $(2n + 2)$-tuple points. Let us do the first.

We will anyway have to deal with the two blocks of double points, thus we will choose B_1 to consist of $2nx$ horizontal fibres passing through all the double points and 2 generic vertical, similarly B_2 will consist of $2nx$ vertical fibres through the double points and 2 generic horizontal. As to B_3 we will make its pull-back to X_2 have vertical and horizontal fibres, passing through the double points; but as they are already part of the branch-loci B_1 and B_2 they will only count half, and the actual branch curve will not be physical pull-back so to speak, only linearly equivalent to one, thus B_3 will be equivalent to $\Gamma(x) \cup (nx, nx)$ rather than $\cdots \cup (2nx, 2nx)$. Thus B_3 has bidegree $(3nx, 3nx)$.

In this way we get $2n^2x^2$ ordinary 6-tuple points and $16x^2n$-tuple points. The first ones give contributions of 16 and more generally the n-tuple give contributions of $(n^2 - 4)/2$.

The asymptotic calculation, ignoring anything but the x^2 terms then gives us an index of

$$-2(2(3nx)^2) + 32n^2x^2 + 16x^2(n^2 - 4)/2 = (4n^2 - 32)x^2$$

which is positive if $n \geqslant 6$ (recall $n \equiv 2 \bmod 4$).

Using the second method we would instead get $4x^2(2n + 2)$-tuple points, and the calculations would come out as

$$-2(2(3n + 1)x)^2 + 32n^2x^2 + 4x^2 2(n^2 + n) = (4n^2 - 24n + 8)x^2.$$

Which is positive again if $n \geqslant 6$ but not quite as good.

The most interesting thing is to compare the relative size of the index τ compared to c_1^2. For the latter case we get the asymptotic calculation

$$c_1^2 = 2(5nx, 5nx) - 8(2n^2x^2) - 16x^2(8((n - 2)/4)^2) = 100n^2x^2 - 16n^2x^2 - 8n^2x^2 + o(n)x^2$$

and the ratio

$$c_1^2/c_2 = \frac{2c_1^2}{c_1^2 - 3\tau} = \frac{152 + o(n)}{64 + o(n)}$$

which for large n gives in the limit $2.375(= 2\frac{3}{8})$. Or in terms of $c_1^2/\chi = 8.44. \ldots$

Now it would be interesting to systematically compute those ratios for various configurations Γ.

Let us ignore the x and x^2 factors for shorthand and let us write down the types of B_1, B_2 and B_3 respectively. Furthermore computing τ, c_1^2 and finally the ratios c_1^2/c_2, c_1^2/χ.

	type	p, q, r	B_1	B_2	B_3	τ	c_1^2	c_1^2/c_2	c_1^2/χ
Tetrahedral	i	2, 3, 3	(14, *)	(*, 14)	(15, 15)	188	2820	2.5	8.57..
Octahedral	i	2, 3, 4	(20, *)	(*, 20)	(36, 36)	448	9728	2.32..	8.39..
	ii		(26, *)	(*, 26)	(33, 33)	700	11,108	2.46..	8.54..
Icosahedral	i	2, 3, 5	(50, *)	(*, 50)	(75, 87)	3916	53,492	2.56..	8.63..
	ii		(62, *)	(*, 62)	(75, 75)	5212	60,068	2.70..	8.76..

Thus we see that the highest slope is attained for the second icosahedral construction (rendering the others in a way superfluous).

Souping up

Now we may try variations of the same theme. Instead of sticking to double covers for the maps π_1, π_2 we may consider arbitrary cyclic covers. This has the advantage that by choosing the right degree we may not repeat fibres in $'B_3$ and get a *bona fide* fibre product of simple coverings. One should also then realize that repeated use of n-fold covers on a k-tuple point yields a point of multiplicity kn. This we have tried in a variety of combinations, however none of them has exceeded the icosahedral constructions in slope. One may also in the final cover (over B_3) consider arbitrary cyclic covers, the problem is now to get a list of even singularities and to compute how they affect index and euler characteristic etc. One example is given by $z^4 = f$, where f has an ordinary 5-tuple point. Its specialization is possible to compute ($c_1^2 = -16, c_2 = -32, \tau = 16$). But none unfortunately seem to yield any higher slopes.

Dense ratios

In order to prove the claim of Theorem C we only need to observe that we may not use all the double points, but only a suitable fraction of them. Now the double points that will not receive full treatment will give rise to harmless rational double points. The actual calculations are routine but we supply them for the convenience of the reader. So let us out of the $30x$ vertical and horizontal fibres only use $2\alpha x$ where we have $0 \leqslant \alpha \leqslant 15$. This will produce $4(\alpha x)^2$ ordinary six-tuple points and a certain (unspecified) number of a_1 and a_3 singularities which will have no influence on the invariants.

Applying the second icosahedral construction we hence get branch-curves

$$B_1 = (32 + 2\alpha, *), \qquad B_2 = (*, 32 + 2\alpha), \qquad B_3 = (60 + \alpha, 60 + \alpha)$$

where we suppress x as usual.

Straightforward computations give (ignoring all but the x^2 terms)

$$\tau = -4(60 + \alpha)^2 + (4\alpha^2 + 20^2)16 + 12^2 \times 48$$

which simplifies to

$$\tau = 60\alpha(\alpha - 8) - 1088$$

for the index (note that if $\alpha \geqslant 10$ we have positive index).

While for the c_1^2 we get by the same asymptotic calculation

$$c_1^2 = 4(92 + 3\alpha)^2 - (4\alpha^2 + 20^2)8 - 12^2 \times 32$$

which simplifies to

$$4\alpha^2 + 2208\alpha + 26048$$

we get

$$c_2 = -88\alpha^2 + 1824\alpha + 14656$$

Looking at the quotient c_1^2/c_2 we get a rational function $R(\alpha)$ which has a local minimum α_0 in the interval $[0, 15]$ which is a positive solution to the quadratic

$$201{,}600\alpha^2 + 470{,}1696\alpha - 15{,}151{,}104.$$

We easily compute $R(0) = 1.77$ and $R(15) = 2.70$ which gives dense ratios in the interval $[2, 2.7]$ in which we were interested.

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